# Stability of the bipartite matching model 

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#### Abstract

We consider the bipartite matching model of customers and servers introduced by Caldentey, Kaplan, and Weiss (Adv. Appl. Probab., 2009). Customers and servers play symmetrical roles. There is a finite set $C$, resp. $S$, of customer, resp. server, classes. Time is discrete and at each time step, one customer and one server arrive in the system according to a joint probability measure $\mu$ on $C \times S$, independently of the past. Also, at each time step, pairs of matched customer and server, if they exist, depart from the system. Authorized matchings are given by a fixed bipartite graph $(C, S, E \subset C \times S)$. A matching policy is chosen, which decides how to match when there are several possibilities. Customers/servers that cannot be matched are stored in a buffer. The evolution of the model can be described by a discrete time Markov chain. We study its stability under various admissible matching policies including: ML (Match the Longest), MS (Match the Shortest), FIFO (match the oldest), priorities. There exist natural necessary conditions for stability (independent of the matching policy) defining the maximal possible stability region. For some bipartite graphs, we prove that the stability region is indeed maximal for any admissible matching policy. For the ML policy, we prove that the stability region is maximal for any bipartite graph. For the MS and priority policies, we exhibit a bipartite graph with a non-maximal stability region.


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## Contents

1 Introduction ..... 2
2 The bipartite matching model ..... 3
2.1 State space description ..... 4
2.2 Admissible matching policies ..... 6
3 Necessary conditions for stability ..... 7
3.1 Complexity of verifying NCoND . ..... 9
4 Connectivity properties of the Markov chain ..... 12
4.1 Stable structures ..... 12
4.2 Back to property UTC ..... 15
5 Models that are stable for all admissible policies ..... 16
6 Priorities and MS are not always stable ..... 18
7 ML is always stable ..... 22

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## 1 Introduction

In queueing theory, customers and servers play different roles. Customers arrive in the system, accumulate in a buffer, get served by a server, and eventually depart. Servers on the other hand alternate between idle and busy periods but remain forever in the system.

Within this framework, many variations and refinements are possible. For instance, we may consider a model with multi-class customers and distinguishable servers. A customer of a given class $c$ must choose its server from a specified subset $S(c)$ of the servers. And of course, the subsets $S(c)$ may intersect, see Figure 1


Figure 1: Queueing model of a call center.

In this paper, we consider a model with the same multi-class flavor, but in which, by contrast, customers and servers play completely identical roles. We now argue that this simple symmetry requirement leads in a natural and ineluctable way to the bipartite matching model.

By symmetry, both customers and servers should arrive into the system and depart from it. More specifically, upon completion of a service, both the customer and the server should depart simultaneously. To model arrivals, we have a priori more flexibility, but there is basically one non-trivial choice which is to assume that time is discrete and that customers and servers arrive in pairs.

Consider indeed the simplest possible model with continuous-time arrivals: (i) there is only one class of customers and one class of servers; (ii) customers, resp. servers, arrive according to a Poisson process of rate $\lambda$, resp. $\mu$; (iii) services have duration 0 . Let us describe the state by $Z=X-Y$, where $X$ is the number of unmatched customers and $Y$ the number of unmatched servers. The process $Z$ is a birth-and-death continuous-time Markov process on $\mathbb{Z}$ with drift $\lambda-\mu$. It is either transient (if $\lambda \neq \mu$ ) or null recurrent (if $\lambda=\mu$ ), but it is never positive recurrent.

Let us switch to discrete-time i.i.d. arrivals. At each time step, a batch of customers and a batch of servers arrive into the system. If the size of the batches are allowed to be different for customers and servers, then we are back to the continuous-time situation, and even the simplest model is never positive recurrent.

Therefore to get a non-trivial model, the natural assumption is that exactly one customer and one server arrive into the system at each time step. The resulting model is symmetric in another respect: both arrivals and departures occur in pairs.

For simplicity, we always assume that the service durations are null. So the model is specified by: (i) the finite set $C$ of customer classes and the finite set $S$ of server classes; (ii) the probability law $\mu$ on $C \times S$ for the arrivals in pairs; (iii) the bipartite graph $(C, S, E \subset C \times S)$ giving the possible matchings between customers and servers (hence the possible departures in pairs); (iv) the matching policy to decide how to match when several choices are possible. We consider so called admissible policies which depend only on the current state of the system. Under these assumptions, the buffer content evolves as a discrete-time Markov chain.

We call this model the bipartite matching model.
The bipartite matching model has been introduced by Caldentey, Kaplan, and Weiss 2], under an additional assumption of independence between arriving customers and servers $(\forall c, s, \mu(c, s)=$ $\mu(c, S) \mu(C, s)$ ), and for the FIFO policy. In their paper, the authors mention several possible domains of applications ranging from call centers to crossbar data switches. They also provide references to papers on related models. We refer the interested reader to [2] for details.

In the bipartite matching model, there is an equal number of customers and servers at any time. But the matching constraints may result in instability with unmatched customers and servers accumulating. It turns out that proving stability, i.e. positive recurrence of the Markov chain, is highly non-trivial. Given a bipartite graph $(C, S, E)$, there exist natural necessary conditions on $\mu$ for stability to hold true. When these conditions are also sufficient, we say that the stability region is maximal.

Caldentey \& al conjecture that any bipartite graph has a maximal stability region for the FIFO policy [2, Conjecture 4.2]. They prove the conjecture for some specific models (under the additional assumption of independence of customers and servers): (i) the N model defined by $C=\{1,2\}, S=\left\{1^{\prime}, 2^{\prime}\right\}, E=C \times S-\left\{\left(2,2^{\prime}\right)\right\}$, (ii) the W model, i.e. the matching model version of Figure 1, (iii) the NN model of Figure 2, In the first two cases, they are also able to compute explicitly the stationary distribution. For the last case, the proof of stability is already intricate.

In the present paper, we consider the stability issue for various admissible matching policies: ML (Match the Longest), MS (Match the Shortest), FIFO (match the oldest), random (match uniformly), priorities. The irreducibility of the Markov chain describing the model is not granted, and we first study this question in detail (Section 4). Then we obtain the following results:

- sufficient conditions under which any admissible policy is stable (Section 5);
- for the NN model, the MS policy and some priority policies do not have a maximal stability region (Section 6);
- for any bipartite graph, the ML policy has a maximal stability region (Section 7 ).

We do not know if the stability region is always maximal for the FIFO and random policies.
Notations. Denote by $\mathbb{N}=\{0,1,2, \ldots\}$ the set of non-negative integers. Let $A^{*}$ be the free monoid generated by $A$. For any word $w \in A^{*}$ and any $B \subset A$, set $|w|_{B}=\#\left\{i \mid w_{i} \in B\right\}$, the number of occurrences in $w$ of letters from $B$. For $B=\{b\}$, we shorten the notation to $|w|_{b}$. Furthermore, for any $w \in A^{*}$, set $[w]:=\left(|w|_{a}\right)_{a \in A}$ (the commutative image of $w$ ).

## 2 The bipartite matching model

We now proceed to a more formal definition of the model.
Definition 2.1. A bipartite matching structure is a quadruple $(C, S, E, F)$ where

- $C$ is the non-empty and finite set of customer types;
- $S$ is the non-empty and finite set of server types;
- $E \subset C \times S$ is the set of possible matchings;
- $F \subset C \times S$ is the set of possible arrivals.

The bipartite graph $(C, S, E)$ is called the matching graph. It is assumed to be connected. The bipartite graph $(C, S, F)$ is called the arrival graph. It is assumed to have no isolated vertices.

The two assumptions in Def. 2.1 are made without loss of generality, see Remarks 1 and 2
In Figure 2 we give an example of a matching graph with 3 customer and 3 server types, called the "NN graph" in the following.


Figure 2: NN graph.

Customers and servers play symmetrical roles in the model. Also $E$ and $F$ play dual roles. The graph $(C, S, E)$ defines the pairs that may depart from the system, while the graph $(C, S, F)$ defines the pairs that may arrive into the system.
Definition 2.2. A bipartite matching model is a triple $[(C, S, E, F), \mu, \mathrm{POL}]$, where

- $(C, S, E, F)$ is a bipartite matching structure;
- $\mu$ is a probability measure on $C \times S$ satisfying

$$
\begin{equation*}
\operatorname{supp}(\mu)=F, \operatorname{supp}\left(\mu_{C}\right)=C, \operatorname{supp}\left(\mu_{S}\right)=S \tag{1}
\end{equation*}
$$

where $\mu_{C}$ and $\mu_{S}$ are the $C$ and $S$ marginals of $\mu$.

- Pol is an admissible matching policy (to be defined in 2.2).

Observe that we can simplify the notation to $[(C, S, E), \mu, \mathrm{POL}]$. We say that the model $[(C, S, E), \mu, \mathrm{PoL}]$ is associated with the structure $(C, S, E, F)$.
Remark 1. For (1) to have solutions, $(C, S, F)$ must be without isolated vertices, the assumption made in Definition 2.1. This is not a real restriction: if it is not satisfied, we can consider a new model without such customer or server classes.

A realization of the model is as follows. Consider an i.i.d. sequence of random variables of law $\mu$, representing the arrival stream of pairs of customer/server. A state of the buffer consists of an equal number of customers and servers with no possible matchings between the classes. Upon arrival of a new ordered pair $(c, s)$, two situations may occur: if neither $c$ nor $s$ match with the servers/customers already present in the buffer, then $c$ and $s$ are simply added to the buffer; if $c$, resp. $s$, can be matched then it departs the buffer with its match. If several matchings are possible for $c$, resp. s, then it is the role of the matching policy to select one. An admissible policy selects according to the current state of the buffer (and not according to the whole history of the buffer contents, for instance). The resulting evolution of the buffer is described by a discrete-time Markov chain.

### 2.1 State space description

Depending on the matching policy, we consider either a commutative (e.g. for Random) or a non-commutative (e.g. for FIFO) state space description. The different policies considered in the paper will be formally defined in 2.2 .

Let us choose a matching graph $(C, S, E)$. We introduce the following convenient notations: $C(s)$ is the set of customer classes that can be matched with an $s$-server; $S(c)$ is the set of server classes that can be matched with a $c$-customer:

$$
S(c)=\{s \in S:(c, s) \in E\}, \quad C(s)=\{c \in C:(c, s) \in E\} .
$$

For any subsets $A \subset C$, and $B \subset S$, we define

$$
S(A)=\cup_{c \in A} S(c), \quad C(B)=\cup_{s \in B} C(s)
$$

Commutative state space. A state of the system is given by $(x, y), x=\left(x_{c}\right)_{c \in C}$ and $y=$ $\left(y_{s}\right)_{s \in S}$, where $x_{c}$ denotes the number of customers of type $c$ and $y_{s}$ the number of servers of type $s$. The commutative state space is:

$$
\begin{equation*}
\mathcal{E}=\left\{(x, y) \in \mathbb{N}^{C} \times \mathbb{N}^{S}: \sum_{c \in C} x_{c}=\sum_{s \in S} y_{s} ; \forall(c, s) \in E, x_{c} y_{s}=0\right\} \tag{2}
\end{equation*}
$$

Non-commutative state space. A state of the system is given by two finite words of the same size $k \geq 0$, respectively on the alphabets $C$ and $S$, describing unmatched customers and servers. The non-commutative state space is:

$$
\begin{equation*}
\mathcal{E}=\left\{(u, v) \in \cup_{k \geq 0}\left(C^{k} \times S^{k}\right):([u],[v]) \text { belongs to } \sqrt{2}\right\} . \tag{3}
\end{equation*}
$$

Facet. Both the commutative and the non-commutative state space can be decomposed into facets, defined only by the non-zero classes.

Definition 2.3. $A$ facet is an ordered pair $(U, V)$ such that: $U \subset C, V \subset S$ and $U \times V \subset$ $(C \times S-E)$. The zero-facet is the facet $(\emptyset, \emptyset)$, we denote it shortly by $\emptyset$.

For a facet $\mathcal{F}=(U, V)$, define:

$$
\begin{array}{ll}
C_{\bullet}(\mathcal{F})=U, & S_{\bullet}(\mathcal{F})=V \\
C_{\odot}(\mathcal{F})=C(V), & S_{\odot}(\mathcal{F})=S(U), \\
C_{\circ}(\mathcal{F})=C-\left(C_{\bullet}(\mathcal{F}) \cup C_{\odot}(\mathcal{F})\right), & S_{\circ}(\mathcal{F})=S-\left(S_{\bullet}(\mathcal{F}) \cup S_{\odot}(\mathcal{F})\right) .
\end{array}
$$

We alleviate the notations to $C_{\bullet}, S_{\bullet}, C_{\odot}, \ldots$, when there is no possible confusion. The symbol - stands for the non-zero classes, the symbol © for the classes that are forced to be at zero (since they are matched with non-zero classes), and the symbol $\circ$ for the classes that happen to be at zero.

The following notion will play an important role later on.
Definition 2.4. A facet $\mathcal{F}$ is called saturated if $C_{\circ}(\mathcal{F})=\emptyset$ or $S_{\circ}(\mathcal{F})=\emptyset$.
In Figure 3, the facet on the left is non-saturated, while the one on the right is saturated.


Figure 3: NN graph: facets $\left(\{3\},\left\{3^{\prime}\right\}\right)$ and $\left(\{2\},\left\{3^{\prime}\right\}\right)$.

Graphical convention. A facet $\mathcal{F}$ can be represented graphically by coloring the nodes of the bipartite graph according to the above convention (see Figure 3 for an illustration):

- nodes in $C_{\bullet}(\mathcal{F})$ and $S_{\bullet}(\mathcal{F})$ are represented as filled circles;
- nodes in $C_{\odot}(\mathcal{F})$ and $S_{\odot}(\mathcal{F})$ are represented as double circles;
- nodes in $C_{\circ}(\mathcal{F})$ and $S_{\circ}(\mathcal{F})$ are represented as simple circles.

In Figure 4, we have represented the facets of the NN graph. The more complex case of the NNN graph will be given in Section 5 . Figure 13


Figure 4: Facets for the NN graph.
Algorithm 1 takes as input a matching graph and returns as output the set of facets. The termination and correctness of the algorithm are easily proved.

```
Algorithm 1: Computation of the facets
    Data: A bipartite graph \(G=(C, S, E)\).
    Result: Facets - set of all facets of \(G\).
    begin
        Facets \(\leftarrow \emptyset ;\) New \(\leftarrow \emptyset\);
        foreach \((i, j) \in C \times S-E\) do \(N e w \leftarrow N e w \cup\{(\{i\},\{j\})\}\);
        while \(N e w \neq \emptyset\) do
            Facets \(\leftarrow\) Facets \(\cup\) New;
            Old \(\leftarrow N e w ; N e w \leftarrow \emptyset\);
            forall the \(\mathcal{H}, \mathcal{K} \in\) Old such that \(\mathcal{H} \neq \mathcal{K}\) do
                if \(C_{\bullet}(\mathcal{H})=C_{\bullet}(\mathcal{K})\) or \(S_{\bullet}(\mathcal{H})=S_{\bullet}(\mathcal{K})\) then
                    \(Z \leftarrow\left(C_{\bullet}(\mathcal{H}) \cup C_{\bullet}(\mathcal{K}), S_{\bullet}(\mathcal{H}) \cup S_{\bullet}(\mathcal{K})\right) ;\)
                \(N e w \leftarrow N e w \cup\{Z\} ;\)
        Facets \(\leftarrow\) Facets \(\cup\{\emptyset\} ;\)
        return Facets;
```


### 2.2 Admissible matching policies

Informally, a matching policy is admissible if:

- only the current state of the buffer is taken into account;
- priority is given to customers/servers that are already present in the buffer: if the state is $(u, v)$ and the new arrival is $(c, s) \in E$, then $c$ and $s$ are matched together iff there are no servers from $S(c)$ in $v$ and no customers from $C(s)$ in $u$.
It results from the first point that an admissible matching policy can be described as a mapping $\odot: \mathcal{E} \times(C \times S) \rightarrow \mathcal{E}$ which returns the new state of the system after an arrival. The second point is called the buffer-first assumption. It is not a real restriction: a matching policy that always gives priority to new arrivals can be seen as a special case of the above with an arrival probability $\mu$ such that $\mu(E)=0$.

We now define admissible policies formally, distinguishing between the non-commutative and commutative state spaces.

For a word $w \in A^{k}$ and $i \in\{1, \ldots, k\}$, we denote by $w_{[i]}:=w_{1} \ldots w_{i-1} w_{i+1} \ldots w_{k}$ the subword of $w$ obtained by deleting $w_{i}$.

Definition 2.5 (Non-commutative case). A matching policy is admissible if there are functions $\Phi$ and $\Psi$ such that:

$$
(u, v) \odot(c, s)= \begin{cases}(u c, v s), & \text { if }|u|_{C(s)}=0,|v|_{S(c)}=0,(c, s) \notin E \\ (u, v), & \text { if }|u|_{C(s)}=0,|v|_{S(c)}=0,(c, s) \in E \\ \left(u_{[\Phi(u, s)]}, v v_{[\Psi(v, c)]}\right), & \text { if }|u|_{C(s)} \neq 0,|v|_{S(c)} \neq 0 \\ \left(u_{[\Phi(u, s)]} c, v\right), & \text { if }|u|_{C(s)} \neq 0,|v|_{S(c)}=0 \\ \left(u, v_{[\Psi(v, c)]} s\right), & \text { if }|u|_{C(s)}=0,|v|_{S(c)} \neq 0\end{cases}
$$

The FIFO and LIFO policies are admissible matching policies with functions $\Phi$ and $\Psi$ as follows:

- FIFO : $\Phi(u, s)=\arg \min \left\{u_{k} \in C(s)\right\}, \Psi(v, c)=\arg \min \left\{v_{k} \in S(c)\right\}$.
- LIFO : $\Phi(u, s)=\arg \max \left\{u_{k} \in C(s)\right\}, \Psi(v, c)=\arg \max \left\{v_{k} \in S(c)\right\}$.

For $c \in C$, let $e_{c} \in \mathbb{N}^{C}$ be defined by $\left(e_{c}\right)_{c}=1$ and $\left(e_{c}\right)_{d}=0, d \neq c$. For $s \in S$, let $e_{s}$ be defined accordingly.

Definition 2.6 (Commutative case). A matching policy is admissible if there are functions $\Phi$ and $\Psi$ such that:

$$
(x, y) \odot(c, s)= \begin{cases}\left(x+e_{c}, y+e_{s}\right), & \text { if } x_{C(s)}=0, y_{S(c)}=0,(c, s) \notin E \\ (x, y), & \text { if } x_{C(s)}=0, y_{S(c)}=0,(c, s) \in E \\ \left(x-e_{\Phi(x, s)}, y-e_{\Psi(y, c)}\right), & \text { if } x_{C(s)} \neq 0, y_{S(c)} \neq 0 \\ \left(x-e_{\Phi(x, s)}+e_{c}, y\right), & \text { if } x_{C(s)} \neq 0, y_{S(c)}=0 \\ \left(x, y-e_{\Psi(y, c)}+e_{s}\right), & \text { if } x_{C(s)}=0, y_{S(c)} \neq 0\end{cases}
$$

The following commutative matching policies are admissible (for RANDOM, ML, and MS policies $\Phi(u, s)$ and $\Psi(v, c)$ are random variables):

- PR (Priorities). For each customer type $c \in C$, we define a priority function $\alpha_{c}: S(c) \rightarrow$ $\{1, \ldots,|S(c)|\}$. Similarly, for each server type $s \in S$, we define $\beta_{s}: C(s) \rightarrow\{1, \ldots,|C(s)|\}$. In the case of several matching options, a customer/server is matched with the server/customer that has the highest priority (greatest value of the priority function). It is convenient to specify the priorities by two $|C| \times|S|$ matrices $A$ and $B$ defined by:

$$
A_{c s}=\left\{\begin{array}{ll}
\alpha_{c}(s), & (c, s) \in E \\
0, & \text { otherwise }
\end{array} \quad B_{c s}=\left\{\begin{array}{ll}
\beta_{s}(c), & (c, s) \in E \\
0, & \text { otherwise }
\end{array} .\right.\right.
$$

Then $\Phi(x, s)=\arg \max \left\{\beta_{s}(c): c \in C(s), x_{c}>0\right\}$ and $\Psi(y, c)=\arg \max \left\{\alpha_{c}(s): s \in\right.$ $\left.S(c), y_{s}>0\right\}$.

- RANDOM : $\Phi(x, s)$, resp. $\Psi(y, c)$, is a random variable valued in $C(s)$, resp. $S(c)$, and distributed as $\left(x_{i} / \sum_{j \in C(s)} x_{j}\right)_{i \in C(s)}$, resp. $\left(y_{i} / \sum_{j \in S(c)} y_{j}\right)_{i \in S(c)}$. Intuitively, the match is chosen uniformly among all possible ones.
- ML: $\Phi(x, s)$, resp. $\Psi(y, c)$, is a random variable uniformly distributed on $\arg \max \left\{x_{i}: i \in\right.$ $C(s)\}$, resp. $\arg \max \left\{y_{i}: i \in S(c)\right\}$.
- MS: $\Phi(x, s)$, resp. $\Psi(y, c)$, is a random variable uniformly distributed on $\arg \min \left\{x_{i}>0\right.$ : $i \in C(s)\}$, resp. $\arg \min \left\{y_{i}>0: i \in S(c)\right\}$.


## 3 Necessary conditions for stability

To introduce the main ideas, consider first a simpler finite and deterministic problem. Let ( $C, S, E$ ) be a matching graph. Consider a batch of customers $x \in \mathbb{N}^{C}$ and a batch of servers $y \in \mathbb{N}^{S}$ of equal size: $\sum_{c} x_{c}=\sum_{s} y_{s}$. A perfect matching of $x$ and $y$ is a tuple $m \in \mathbb{N}^{E}$ such that:

$$
\forall c \in C, x_{c}=\sum_{s \in S(c)} m_{c s}, \quad \forall s \in S, y_{s}=\sum_{c \in C(s)} m_{c s}
$$

By Hall's Theorem (aka the "marriage Theorem"), there exists a perfect matching if and only if:

$$
\begin{array}{ll}
\sum_{c \in U} x_{c} \leq \sum_{s \in S(U)} y_{s}, & \forall U \subset C  \tag{4}\\
\sum_{s \in V} y_{s} \leq \sum_{c \in C(V)} x_{c}, & \forall V \subset S
\end{array}
$$

A perfect matching, if there is one, can be obtained by restating the model as a flow network and by solving the maximum flow problem for which efficient algorithms exist [4, 3].

The bipartite matching model is much more complicated: first it is random, and second the matchings have to be performed on the fly, at each time step. However the two ingredients of the simpler model will play an instrumental role in the analysis: (i) the conditions NCond, to be defined in (5), are related to (4); (ii) the restatement as a flow problem is used in most of the proofs.

Consider now a bipartite matching model $[(C, S, E), \mu, \mathrm{POL}]$. We identify the model with the Markov chain on the state space $\mathcal{E}$ describing the evolution of the buffer content.

Let $P$ be the transition matrix of the Markov chain. A probability measure $\pi$ on $\mathcal{E}$ is stationary if $\pi P=\pi$. It is attractive if for any probability measure $\nu$ on $\mathcal{E}$, the sequence of Cesaro averages of $\nu P^{n}$ converges weakly to $\pi$.
Definition 3.1. The model is said to be stable if the Markov chain has a unique and attractive stationary probability measure.

It implies in particular that the graph of the Markov chain has a unique terminal strongly connected component with all states leading to it.

Let $\mu_{C}$ be a probability measure on $C$ and $\mu_{S}$ a probability measure on $S$. Define the following conditions on $\left(\mu_{C}, \mu_{S}\right)$ :

$$
\text { NConD : } \quad \begin{cases}\mu_{C}(U)<\mu_{S}(S(U)), & \forall U \subsetneq C  \tag{5}\\ \mu_{S}(V)<\mu_{C}(C(V)), & \forall V \subsetneq S\end{cases}
$$

The above conditions appear in [2]. They have a natural interpretation. Let $\mu_{C}$ and $\mu_{S}$ be the marginals of the arrival probability $\mu$. Customers from $U$ need to be matched with servers from $S(U)$. The first line in NCond asks for strictly more servers in average from $S(U)$ than customers from $U$. The second line has a dual interpretation.

Using the Strong Law of Large Numbers, we also see that the arrivals up to time $n$ satisfy (4) for all values of $n$ large enough if and only if NCond is satisfied.

Lemma 3.2. The conditions NCond are necessary stability conditions: if the Markov chain is stable then the conditions NCOND are satisfied by the marginals of $\mu$.
Proof. We suppose that the conditions NCond are not satisfied.
Assume first that there exists $U \subset C$ such that $\mu_{C}(U)>\mu_{S}(S(U))$. Let $A_{n}$, resp. $B_{n}$, be the total numbers of customers of type $U$, resp. servers of type $S(U)$, to arrive in the system up to time $n$. Let $X_{n}$ be the number of customers of type $U$ present in the system at time $n$. By definition, $X_{n} \geq A_{n}-B_{n}$. By the Strong Law of Large Numbers, we have, a.s.,

$$
\lim _{n} \frac{A_{n}}{n}=\mu_{C}(U), \lim _{n} \frac{B_{n}}{n}=\mu_{S}(S(U)), \quad \lim _{n} \frac{X_{n}}{n} \geq \mu_{C}(U)-\mu_{S}(S(U))>0
$$

So the Markov chain is transient. Similarly, if there exists $V \subset S$ such that $\mu_{S}(V)>\mu_{C}(C(V))$, the model is unstable. (This part of the argument appears in [2, Prop. 3.4].)

Assume now that there exists $U \subset C, U \neq C$, such that

$$
\begin{equation*}
\mu_{C}(U)=\mu_{S}(S(U)) . \tag{6}
\end{equation*}
$$

Observe that $S(U) \neq S$, otherwise we would have $\mu_{C}(U)=\mu_{S}(S)=1$ which would contradict $U \neq C$. Set $V=S-S(U)$. Eqn (6) is equivalent to: $\mu_{S}(V)=\mu_{C}(C-U)$. The bipartite matching graph $(C, S, E)$ is represented in Figure 5 By assumption, $U \times V \cap E=\emptyset$.


Figure 5: The bipartite graph $(C, S, E)$.
We have

$$
\begin{aligned}
\mu(U \times V) & =\mu_{C}(U)-\mu(U \times S(U)) \\
\mu((C-U) \times S(U)) & =\mu_{S}(S(U))-\mu(U \times S(U)) .
\end{aligned}
$$

Using Eqn (6), we get

$$
\begin{equation*}
\mu(U \times V)=\mu((C-U) \times S(U)) \tag{7}
\end{equation*}
$$

Let $D_{n}$ be the number of departures of type $(C-U) \times S(U)$ up to time $n$. Let $A_{n}, B_{n}$, and $X_{n}$ be defined as above. Set $Z_{n}=A_{n}-B_{n}$. For an arrival of type $U \times V$, the $Z$-process makes a +1 jump, for an arrival of type $(C-U) \times S(U)$, the $Z$-process makes a -1 jump, otherwise the $Z$-process remains constant. We have $X_{n} \geq A_{n}-\left(B_{n}-D_{n}\right) \geq Z_{n}$. We have two cases:

- If $\mu(U \times V)>0$, then, according to $\sqrt{7}$, the $Z$-process is null recurrent.
- If $\mu(U \times V)=0$, then for any initial condition such that $X_{0}>0$, a.s. $X_{n} \geq X_{0}, \forall n$. Hence, in both cases, the model cannot be stable.

Remark 2. Consider a non-connected matching graph $(C, S, E)$. Consider a probability $\mu$ and an admissible matching policy such that the bipartite matching model is stable. Let $\left(C^{\prime}, S^{\prime}, E^{\prime}\right)$ be a connected subgraph of $(C, S, E)$. Following the exact same steps as in the proof of Lemma 3.2. we prove that

$$
\mu_{C}\left(C^{\prime}\right)=\mu_{S}\left(S^{\prime}\right), \quad \mu\left(C^{\prime} \times\left(S-S^{\prime}\right)\right)=0, \quad \mu\left(\left(C-C^{\prime}\right) \times S^{\prime}\right)=0
$$

(otherwise the Markov chain is either transient or null recurrent). Therefore, we can decompose the model into connected components and treat them separately. Hence the assumption of connectedness of $(C, S, E)$ in Def. 2.1 was made without loss of generality.

### 3.1 Complexity of verifying NCOND

Let us fix $(C, S, E)$ and the probability measures $\left(\mu_{C}, \mu_{S}\right)$ such that $\operatorname{supp}\left(\mu_{C}\right)=C, \operatorname{supp}\left(\mu_{S}\right)=S$. We want an efficient algorithm to decide if the conditions NCond are satisfied.

The number of inequalities in NCond is exponential in $|C|+|S|$. So checking directly if all the inequalities are satisfied is a method whose time complexity is exponential in $|C|+|S|$. To go beyond, we need additional material.

We use the standard terminology of network flow theory, see for instance 4. Consider the directed graph

$$
\begin{equation*}
\mathcal{N}=(C \cup S \cup\{i, f\}, E \cup\{(i, c), c \in C\} \cup\{(s, f), s \in S\}) . \tag{8}
\end{equation*}
$$

Endow the arcs of $E$ with infinite capacity, an arc of type $(i, c)$ with capacity $\mu_{C}(c)$, and an arc of type $(s, f)$ with capacity $\mu_{S}(s)$.


Figure 6: The graph $\mathcal{N}$ associated with the NN model of Figure 2
Recall that a cut is a subset of the arcs whose removal disconnects $i$ and $f$. The capacity of a cut is the sum of the capacities of the arcs. Set $A=E \cup\{(i, c), c \in C\} \cup\{(s, f), s \in S\}$. Recall that $T: A \rightarrow \mathbb{R}_{+}$is a flow if: (i) $\forall c, T(i, c)=\sum_{s \in S(c)} T(c, s), \forall s, \sum_{c \in C(s)} T(c, s)=T(s, f)$; (ii) $\forall(x, y) \in E, T(x, y)$ is less or equal to the capacity of $(x, y)$. The value of $T$ is $\sum_{c} T(i, c)=$ $\sum_{s} T(s, f)$.

Let $\operatorname{NConD} \leq$ be the set of inequalities obtained from NCond by replacing the strict inequalities by large inequalities.

Lemma 3.3. There exists a flow of value 1 in $\mathcal{N}$ iff $\left(\mu_{C}, \mu_{S}\right)$ satisfies $\mathrm{NCOND}_{\leq}$. There exists a flow $T$ of value 1 such that $T(c, s)>0$ for all $(c, s) \in E$ iff $\left(\mu_{C}, \mu_{S}\right)$ satisfies NCond.

The first part of Lemma 3.3 is proved in [2, Prop. 3.7]. We repeat the argument for completeness.

Proof. The celebrated Max-flow Min-cut Theorem [4] states that the maximal value of a flow is equal to the minimal capacity of a cut. Observe that the set of $\operatorname{arcs}\{(i, c), c \in C\}$ forms a cut of capacity 1 . Therefore the maximal flow is $\leq 1$ and it is 1 iff all cuts have a capacity $\geq 1$.

To be of finite capacity, a cut must contain only customer arcs $\{(i, c), c \in C\}$ and server arcs $\{(s, f), s \in S\}$. Consider a subset $\mathcal{A}=\left\{(i, c), c \in C_{1}\right\} \cup\left\{(s, f), s \in S_{1}\right\}$. Set $C_{2}=C-C_{1}$ and $S_{2}=S-S_{1}$. The set $\mathcal{A}$ is a cut iff $C_{2} \times S_{2} \cap E=\emptyset$, equivalently iff $S\left(C_{2}\right) \subset S_{1}$ and $C\left(S_{2}\right) \subset C_{1}$. Also the capacity of $\mathcal{A}$ is $\mu_{C}\left(C_{1}\right)+\mu_{S}\left(S_{1}\right)$.


Figure 7: Illustration of the proof of Lemma 3.3 .
Assume that the cut $\left\{(i, c), c \in C_{1}\right\} \cup\left\{(s, f), s \in S_{1}\right\}$ is of capacity strictly less than 1 . We have

$$
\mu_{C}\left(C_{1}\right)+\mu_{S}\left(S_{1}\right)<1 \Longleftrightarrow \mu_{C}\left(C_{1}\right)<\mu_{S}\left(S_{2}\right)
$$

But $C\left(S_{2}\right) \subset C_{1}$ so, if $\mathrm{NCOND}_{\leq}$is satisfied, we must have:

$$
\mu_{S}\left(S_{2}\right) \leq \mu_{C}\left(C\left(S_{2}\right)\right) \leq \mu_{C}\left(C_{1}\right)
$$

So we have proved that $\mathrm{NCOND}_{\leq}$is not satisfied.
The other way round, if $\mathrm{NCOND}_{\leq}$is not satisfied, then there exist $C_{1}, S_{2}, C\left(S_{2}\right)=C_{1}$ such that $\mu_{C}\left(C_{1}\right)<\mu_{S}\left(S_{2}\right)$. Set $C_{2}=\bar{C}-C_{1}$ and $S_{1}=S-S_{2}$. By definition, $C_{2} \times S_{2} \cap E=\emptyset$, therefore $\left\{(i, c), c \in C_{1}\right\} \cup\left\{(s, f), s \in S_{1}\right\}$ is a cut. Its capacity is $\mu_{C}\left(C_{1}\right)+\mu_{S}\left(S_{1}\right)<1$.

By contrapposing the above, we get that:

$$
\left[\mathrm{NCOND}_{\leq} \text {satisfied }\right] \Longleftrightarrow[\text { all cuts have a capacity } \geq 1] \Longleftrightarrow[\text { maximal flow is } 1] .
$$

We now prove the second part of the lemma. Assume that the conditions NCond are not satisfied. If the conditions $\mathrm{NCOND}_{\leq}$are not satisfied either, then by the first part of the proof there exists no flow of value 1. Assume now that the conditions NCOND $\leq$ are satisfied. Then there exists $U \subset C, U \neq C$, such that $\mu_{C}(U)=\mu_{S}(S(U))$. Let $T$ be any flow of value 1 . Using the flow relation for $U$, we get:

$$
\sum_{(c, s) \in U \times S(U)} T(c, s)=\sum_{(i, c) \in\{i\} \times U} T(i, c)=\mu_{C}(U) .
$$

Using $\mu_{C}(U)=\mu_{S}(S(U))$ and the flow relation for $S(U)$, we deduce that:

$$
\left[\sum_{(c, s) \in U \times S(U)} T(c, s)=\mu_{S}(S(U))\right] \Longrightarrow\left[\sum_{(c, s) \in(C-U) \times S(U)} T(c, s)=0\right]
$$

Now it follows from the connectedness of $(C, S, E)$ that $(C-U) \times S(U) \cap E \neq \emptyset$. We conclude that the flow $T$ is such that $T(c, s)=0$ for some $(c, s) \in E$.

Assume now that the conditions NCond are satisfied. Fix $\eta$ such that $0<\eta<1 /|E|$. Consider the function $T_{\eta}: A \rightarrow \mathbb{R}_{+}$defined by

$$
T_{\eta}(x, y)= \begin{cases}\eta & \text { for }(x, y)=(c, s) \in E \\ |S(c)| \eta & \text { for }(x, y)=(i, c) \\ |C(s)| \eta & \text { for }(x, y)=(s, f)\end{cases}
$$

By construction $T_{\eta}$ is a flow. Set

$$
\begin{equation*}
\widetilde{\mu}_{C}(c)=\frac{\mu_{C}(c)-|S(c)| \eta}{1-|E| \eta}, \quad \widetilde{\mu}_{S}(s)=\frac{\mu_{S}(s)-|C(s)| \eta}{1-|E| \eta} \tag{9}
\end{equation*}
$$

For $\eta$ small enough, observe that $\widetilde{\mu}_{C}$, resp. $\widetilde{\mu}_{S}$, is a probability measure on $C$, resp. on $S$. Choose $\eta$ small enough such that $\left(\widetilde{\mu}_{C}, \widetilde{\mu}_{S}\right)$ satisfies NCond. This is possible since the conditions NCond are open conditions.

Consider the directed graph $\mathcal{N}$, see (8), with new capacities on the customer and server arcs defined by $\widetilde{\mu}_{C}$ and $\widetilde{\mu}_{S}$. By applying the first part of the proof, there exists a flow $\widetilde{T}: A \rightarrow \mathbb{R}_{+}$of value 1. Define

$$
T: A \rightarrow \mathbb{R}_{+}, \quad T=T_{\eta}+(1-|E| \eta) \widetilde{T}
$$

By construction $T$ is a flow for the graph $\mathcal{N}$ with the original capacity constraints ( $\mu_{C}$ for the customer arcs and $\mu_{S}$ for the server arcs). The value of $T$ is 1 and it satisfies $T(x, y)>0$ for all $(x, y) \in E$. This completes the proof.

There exist algorithms to find the maximal flow which are polynomial in the size of the underlying graph, independent of the arc capacities. For instance, the classical "augmenting path algorithm" of Edmonds \& Karp [3] operates in $O\left((|C|+|S|)|E|^{2}\right)$ time, and there exist more sophisticated algorithms operating in $O\left((|C|+|S|)^{3}\right)$ time.

Take one of these polynomial algorithms, call it MaxFlow and consider it as a blackbox. We build on this to design a polynomial algorithm to check NCond. Let us detail the construction.
Lemma 3.4. Define $\left(\widetilde{\mu}_{C}, \widetilde{\mu}_{S}\right)$ as in (9). The pair $\left(\mu_{C}, \mu_{S}\right)$ satisfies NCond iff the pair $\left(\widetilde{\mu}_{C}, \widetilde{\mu}_{S}\right)$ satisfies NCOND for $\eta$ strictly positive and small enough.
Proof. Assume that $\left(\mu_{C}, \mu_{S}\right)$ satisfies NCond. Since we are dealing with open conditions, any small enough perturbation of $\left(\mu_{C}, \mu_{S}\right)$ still satisfies NCond.

Assume now that $\left(\mu_{C}, \mu_{S}\right)$ does not satisfy NCond. There exists $U \subset C, U \neq C$, such that $\mu_{C}(U) \geq \mu_{S}(S(U))$. By using (9), we get

$$
\begin{aligned}
(1-|E| \eta) \widetilde{\mu}_{C}(U)+\left(\sum_{c \in U}|S(c)|\right) \eta & \geq(1-|E| \eta) \widetilde{\mu}_{S}(S(U))+\left(\sum_{s \in S(U)}|C(s)|\right) \eta \\
(1-|E| \eta) \widetilde{\mu}_{C}(U)+|E \cap(U \times S(U))| \eta & \geq(1-|E| \eta) \widetilde{\mu}_{S}(S(U))+|E \cap(C(S(U)) \times S(U))| \eta
\end{aligned}
$$

By definition, we have $U \subset C(S(U))$. We conclude that $\widetilde{\mu}_{C}(U) \geq \widetilde{\mu}_{S}(S(U))$. So the pair $\left(\widetilde{\mu}_{C}, \widetilde{\mu}_{S}\right)$ does not satisfy NCond.

Using Lemmas 3.3 and 3.4. NCond is satisfied iff $\operatorname{MaxFlow}\left(\mathcal{N}, \widetilde{\mu}_{C}, \widetilde{\mu}_{S}\right)$ returns 1 for $\eta$ small enough. So the trick is to run MaxFlow on the input $\left(\mathcal{N}, \widetilde{\mu}_{C}, \widetilde{\mu}_{S}\right)$ by considering $\eta$ as a formal parameter made "as small as needed".

The precise meaning is the following. If $x_{1}, x_{2}, y_{1}, y_{2} \in \mathbb{R}$, then: $\left(x_{1}+y_{1} \eta\right)+\left(x_{2}+y_{2} \eta\right)=$ $\left(x_{1}+x_{2}\right)+\left(y_{1}+y_{2}\right) \eta$. Furthermore,

$$
\begin{align*}
{\left[x_{1}+y_{1} \eta=x_{2}+y_{2} \eta\right] } & \Longleftrightarrow\left[x_{1}=x_{2}, y_{1}=y_{2}\right] \\
{\left[x_{1}+y_{1} \eta<x_{2}+y_{2} \eta\right] } & \Longleftrightarrow\left[\left(x_{1}<x_{2}\right) \text { or }\left(x_{1}=x_{2}, y_{1}<y_{2}\right)\right] \tag{10}
\end{align*}
$$

So $\eta$ is small enough not to reverse any strict inequality. When running MaxFlow on $\left(\mathcal{N}, \widetilde{\mu}_{C}, \widetilde{\mu}_{S}\right)$, the algorithm deals with values of the type $(x+y \eta)$, and adds and compare them according to the above rules. Now observe that the algorithm stops in finite time, so it will have performed only a finite number of operations. Therefore, it would be possible, a posteriori, to assign to $\eta$ a value which would be small enough to enforce 10 .

```
Algorithm 2: Checking the necessary stability conditions
    Data: \((C, S, E),\left(\mu_{C}, \mu_{S}\right)\) such that \(\operatorname{supp}\left(\mu_{C}\right)=C, \operatorname{supp}\left(\mu_{S}\right)=S\).
    Result: "Yes" if NCond, "No" if \(\neg\) (NCond)
    begin
        Compute \(\mathcal{N}, \widetilde{\mu}_{C}, \widetilde{\mu}_{S}\);
        if \(\operatorname{MaxFlow}\left(\mathcal{N}, \widetilde{\mu}_{C}, \widetilde{\mu}_{S}\right)=1\) then
            Result \(\leftarrow\) Yes;
        else
            Result \(\leftarrow \mathrm{No}\);
    return Result;
```

The termination is obvious and the correctness follows from Lemmas 3.3 and 3.4
Proposition 3.5. Given a bipartite model $[(C, S, E), \mu]$, there exists an algorithm of time complexity $O\left((|C|+|S|)^{3}\right)$ to decide if NCond is satisfied.

## 4 Connectivity properties of the Markov chain

Define the following property for the transition graph of the Markov chain:
UTC : a unique (terminal) strictly connected component with all states leading to it.
Property UTC is necessary for stability as defined in Def. 3.1. But property UTC is not granted in bipartite matching models and counterexamples are given below (Examples 2 and 33). In fact, we will see that we are in an unusual situation: the necessary stability conditions NCOND turn out to be sufficient conditions for the property UTC (Theorem 4.3)! Observe also that property UTC is weaker than irreducibility, and we will give an example of a model satisfying NConD and UTC without being irreducible (Example 4).

### 4.1 Stable structures

To establish property UTC, we make a detour by introducing and studying a notion of independent interest: stable structures.

Definition 4.1. A bipartite matching structure $(C, S, E, F)$ is stable if there exists a probability measure $\mu$ satisfying (1) and whose marginals $\mu_{C}$ and $\mu_{S}$ satisfy NCond.

The justification for this terminology will appear in $\$ 7$, we prove there that under the ML policy, any model satisfying NCoND is stable. So a structure is stable iff there exists an associated model which is stable.

First of all, there exist stable structures.
Example 1. Consider $(C, S, E, C \times S)$, where $(C, S, E)$ is the NN bipartite graph of Figure 2 Let

$$
\mu_{C}: \mu_{C}(1)=\mu_{C}(2)=2 / 5, \mu_{C}(3)=1 / 5, \quad \mu_{S}: \mu_{S}\left(1^{\prime}\right)=\mu_{S}\left(2^{\prime}\right)=2 / 5, \mu_{S}\left(3^{\prime}\right)=1 / 5
$$

The product measure $\mu=\mu_{C} \times \mu_{S}$ has marginals $\mu_{C}$ and $\mu_{S}$ and we check that $\left(\mu_{C}, \mu_{S}\right)$ satisfy NCond. Also it is easily proved that for any admissible matching policy, the graph of the Markov chain is irreducible.

On the other hand, there exist unstable structures. We illustrate this on two examples.
Example 2. Consider the structure $(C, S, E, F)$ where $(C, S, E)$ is the NN graph of Figure 2, and where

$$
F=\left\{\left(1,3^{\prime}\right),\left(2,2^{\prime}\right),\left(3,1^{\prime}\right)\right\} .
$$

Consider any $\mu$ with $\operatorname{supp}(\mu)=F$. We have $\mu_{C}(1)=\mu_{S}\left(3^{\prime}\right)=\mu\left(1,3^{\prime}\right)$ which violates NCond for $V=\left\{3^{\prime}\right\}$. We can also prove that the property UTC is not satisfied. Consider a state of the type $(x, y)$ with $x=y=(0,0, k)$, for some $k \geq 0$. Any one of the three possible arrivals leave the state unchanged. In particular, there is an infinite number of terminal components.
Example 3. Consider the bipartite matching structure defined in Figure 8 . The graph $(C, S, E)$ is represented on the left of the figure, while the graph $(C, S, F)$ is represented on the right.


Figure 8: The matching graph $(C, S, E)$ on the left, and the arrival graph $(C, S, F)$ on the right.
Consider any $\mu$ with $\operatorname{supp}(\mu)=F$. We have

$$
\mu_{S}\left(\left\{1^{\prime}, 2^{\prime}\right\}\right)=\mu\left(3,1^{\prime}\right)+\mu\left(4,2^{\prime}\right) \leq \mu_{C}(\{3,4\})
$$

which contradicts NCond for $U=\{3,4\}$. We can also prove that the property UTC is not satisfied. Consider a state $(x, y)$ with $x_{3}+x_{4}=k>0$. Reducing the number of customers of types $3 / 4$ would require an arrival of type $\left(1,1^{\prime}\right)$ or $\left(1,2^{\prime}\right)$ or $\left(2,1^{\prime}\right)$ or $\left(2,2^{\prime}\right)$. But none of these pairs belong to $F$. Therefore it is impossible to reach a state ( $x^{\prime}, y^{\prime}$ ) with $x_{3}^{\prime}+x_{4}^{\prime}<x_{3}+x_{4}$. On the other hand an arrival of type $\left(3,3^{\prime}\right)$ or $\left(3,4^{\prime}\right)$ or $\left(4,3^{\prime}\right)$ or $\left(4,4^{\prime}\right)$ strictly increases the number of customers of types $3 / 4$. Hence all the states are transient, and there is no terminal strongly connected component.

Stability of a structure is a decidable property. There exists a probability measure $\mu$ with the requested properties iff the following system of linear inequalities in the indeterminates $\mu(c, s), c \in$ $C, s \in S$, have a solution:

$$
\begin{cases}\sum_{(c, s) \in C \times S} \mu(c, s)=1, &  \tag{11}\\ \mu(c, s)>0, & \forall(c, s) \in F, \\ \mu(c, s)=0, & \forall(c, s) \in C \times S-F, \\ \mu_{C}(c)=\sum_{s \in S} \mu(c, s), & \forall c \in C, \\ \mu_{S}(s)=\sum_{c \in C} \mu(c, s), & \forall s \in S, \\ \text { NCond . } & \end{cases}
$$

However, the number of inequalities is exponential in $|C|+|S|$. We are going to propose a criterion which is much simpler, both conceptually and algorithmically.

Consider a bipartite matching structure $(C, S, E, F)$. Define $\widetilde{F}=\{(s, c) \mid(c, s) \in F\}$. Associate with the structure the directed graph $(C \cup S, E \cup \widetilde{F})$, in other words the nodes are $C \cup S$ and the arcs are

$$
c \longrightarrow s, \quad \text { if }(c, s) \in E, \quad s \longrightarrow c, \quad \text { if }(c, s) \in F
$$

We have represented in Figure 9 the directed graph associated with the structure of Example 3


Figure 9: The directed graph associated with the structure of Figure 8

The graph of Figure 9 is not strongly connected: the four nodes on the right form a strongly connected component. Similarly, the directed graph associated with the structure of Example 2 is not strongly connected. On the other hand, the directed graph associated with the structure of Example 1 is strongly connected. This is not a coincidence.

Theorem 4.2. Let $(C, S, E, F)$ be a bipartite matching structure. The following two properties are equivalent:

1. $(C, S, E, F)$ is a stable structure;
2. $(C \cup S, E \cup \widetilde{F})$ is strongly connected.

In particular, one can decide if a structure is stable with an algorithm of time complexity $O(|C||S|)$ by testing the strong connectivity of $(C \cup S, E \cup \widetilde{F})$.

Proof of Theorem 4.2. Assume that $(C, S, E, F)$ is a stable structure. Let $\mu$ be a probability measure satisfying (1) and NCond. Suppose that there exist $c \in C, s \in S$, with no directed path from $c$ to $s$ in $(C \cup S, E \cup \widetilde{F})$. Let $\operatorname{succ}(c)$ be the set of nodes that can be reached starting from $c$ in $(C \cup S, E \cup \widetilde{F})$. Set

$$
C_{1}=C \cap \operatorname{succ}(c), S_{1}=S \cap \operatorname{succ}(c), C_{2}=C-C_{1}, S_{2}=S-S_{1}
$$

By assumption, $s \in S_{2}$. The following two properties hold:

$$
\mu\left(C_{2}, S_{1}\right)=0, \quad\left(C_{1} \times S_{2}\right) \cap E=\emptyset
$$

Using $\mu\left(C_{2}, S_{1}\right)=0$, we get

$$
\mu_{S}\left(S_{1}\right)=\mu\left(C_{1}, S_{1}\right) \leq \mu_{C}\left(C_{1}\right)
$$

But using $\left[\left(C_{1} \times S_{2}\right) \cap E=\emptyset\right]$ and NCond for $U=C_{1}$, we get

$$
\mu_{C}\left(C_{1}\right)<\mu_{S}\left(S\left(C_{1}\right)\right)=\mu_{S}\left(S_{1}\right)
$$

From this contradiction, we deduce that for all $c \in C, s \in S$, there exists a directed path from $c$ to $s$ in $(C \cup S, E \cup \widetilde{F})$. Similarly, we can prove that for all $s \in S, c \in C$, there exists a directed path from $s$ to $c$ in $(C \cup S, E \cup \widetilde{F})$.

Assume now that $(C \cup S, E \cup \widetilde{F})$ is strongly connected. Consider the matrices $A \in \mathbb{R}_{+}^{C \times S}$ and $B \in \mathbb{R}_{+}^{S \times C}$ defined by

$$
A_{c s}=\left\{\begin{array}{ll}
1 /|S(c)| & \text { if }(c, s) \in E \\
0 & \text { otherwise }
\end{array}, \quad B_{s c}= \begin{cases}1 / \#\{d,(s, d) \in \widetilde{F}\} & \text { if }(s, c) \in \widetilde{F} \\
0 & \text { otherwise }\end{cases}\right.
$$

Consider the matrix $A B \in \mathbb{R}_{+}^{C \times C}$. By construction, we have $(A B)_{c d}>0$ if and only if there is a path of length 2 from $c$ to $d$ in the graph $(C \cup S, E \cup \widetilde{F})$. Since $(C \cup S, E \cup \widetilde{F})$ is strongly connected, we deduce that $A B$ is irreducible. Clearly the spectral radius of $A B$ is 1 . Applying
the Perron-Frobenius Theorem [5], we obtain the existence of a line vector $x \in \mathbb{R}_{+}^{C}$ such that: $\forall c, x_{c}>0, \sum_{c} x_{c}=1$, and $x A B=x$. Set $y=x A$. Define the probability measure $\mu$ on $C \times S$ by $\mu(c, s)=y_{s} B_{s c}$. By construction, we have $\mu_{C}=x, \mu_{S}=y$. Also, by construction, $\operatorname{supp}(\mu)=F$. Define the function $T: A \rightarrow \mathbb{R}_{+}$by

$$
\forall c \in C, T(i, c)=x_{c}, \quad \forall s \in S, T(s, f)=y_{s}, \quad \forall(c, s) \in E, T(c, s)=x_{c} A_{c s}
$$

By construction, $T$ is a flow of value 1 such that $T(c, s)>0$ for all $(c, s) \in E$. Using Lemma 3.3, we get that $(x, y)=\left(\mu_{C}, \mu_{S}\right)$ satisfies NCond.

### 4.2 Back to property UTC

We now have all the ingredients needed to prove the following result.
Theorem 4.3. Consider a bipartite matching model $[(C, S, E), \mu, \mathrm{POL}]$. Assume that the structure $(C, S, E, F)$ is stable, equivalently that $(C \cup S, E \cup \widetilde{F})$ is strongly connected. Then the transition graph of the Markov chain of the bipartite matching model satisfies the property UTC.

Proof of Theorem 4.3. We are going to prove that the empty state can be reached starting from any state. This is a sufficient condition for property UTC to hold. The unique terminal strongly connected component is the set of states that can be reached from the empty state.

We carry out the proof in the commutative case, but it works unchanged in the non-commutative case (the only information needed is the number of customer/server of each class). Consider a non-empty state $(X, Y)$, with $X=\left(x_{c}\right)_{c \in C}$ and $Y=\left(y_{s}\right)_{s \in S}$. It is sufficient to prove that we can always reach a state $\left(X^{\prime}, Y^{\prime}\right)$ such that $\left|X^{\prime}\right|<|X|$.

If there exists $(c, s) \in C_{\odot} \times S_{\odot}$ such that $\mu(c, s)>0$, then the proof is completed. Assume now that $\mu\left(C_{\odot} \times S_{\odot}\right)=0$. Choose $(c, s) \in C_{\odot} \times S_{\odot}$. By assumption, there exists a path from $s$ to $c$ in $(C \cup S, E \cup \widetilde{F})$. Let us denote it by $\left(s=s_{1}, c_{1}, s_{2}, c_{2}, \ldots, s_{k}, c_{k}=c\right)$. Assume that $c_{1}, \ldots, c_{k-1} \notin C_{\odot}$ and $s_{2}, \ldots, s_{k} \notin S_{\odot}$. (If not, consider a subpath with this property.) Assume further that $c_{1} \in C_{\circ}$. (If $c_{1} \in C_{\bullet}$, then $\left(c_{1}, s_{2}\right) \in E$ implies $s_{2} \in C_{\odot}$, so we can consider the subpath $\left(s=s_{2}, c_{2}, \ldots, s_{k}, c_{k}=c\right)$.) Since $\left(s_{i}, c_{i}\right) \in \widetilde{F}$ and $\left(c_{i}, s_{i+1}\right) \in E$ by construction and since $c_{1} \in C_{\circ}$, we get that $c_{1}, \ldots, c_{k-1} \in C_{\circ}$ and $s_{2}, \ldots, s_{k} \in S_{\circ}$.


Figure 10: The path $\left(s=s_{1}, c_{1}, s_{2}, c_{2}, \ldots, s_{k}, c_{k}=c\right)$ in $(C \cup S, E \cup \widetilde{F})$.
By definition of the graph $(C \cup S, E \cup \widetilde{F})$, we have $\mu\left(c_{i}, s_{i}\right)>0$ for all $i$. Choose the sequence of arrivals $\left(c_{1}, s_{1}\right), \ldots,\left(c_{k}, s_{k}\right)$. Consider the effect of the arrival of $\left(c_{1}, s_{1}\right)$. Since $s_{1} \in S_{\odot}$, it will be matched with a customer of $C \bullet$ (and not with $c_{1}$, even if $\left(c_{1}, s_{1}\right) \in E$, since an admissible matching policy is always buffer-first, see $\$ 2.2$. Since $c_{1} \notin C_{\odot}$, it will remain unmatched. Let $\left(X^{(1)}, Y^{(1)}\right)$ be the new state. We have $\left|X^{(1)}\right|=|X|$. Also, in the new state, we have $c_{1} \in C_{\bullet}^{(1)}$, which implies that $s_{2} \in S_{\odot}^{(1)}$. So we can repeat the argument inductively. After the arrivals of $\left(c_{1}, s_{1}\right), \ldots,\left(c_{k-1}, s_{k-1}\right)$, we are in a state $\left(X^{(k-1)}, Y^{(k-1)}\right)$ satisfying:

$$
\left|X^{(k-1)}\right|=|X|, \quad s_{k} \in S_{\odot}^{(k-1)}, \quad c_{k} \in C_{\odot}^{(k-1)} .
$$

Therefore, after the arrival of $\left(c_{k}, s_{k}\right)$, we end up in a state $\left(X^{(k)}, Y^{(k)}\right)$ such that $\left|X^{(k)}\right|=|X|-1$. This completes the proof.

Example 4. Consider a bipartite matching model associated with the structure ( $C, S, E, F$ ) where $(C, S, E)$ is the NN graph of Figure 2, and where

$$
F=\left\{\left(1,1^{\prime}\right),\left(2,2^{\prime}\right),\left(3,3^{\prime}\right)\right\}
$$

The graph $(C \cup S, E \cup \widetilde{F})$ is strongly connected. According to Theorem 4.2, the graph satisfies property UTC. But it is not irreducible. Indeed, it is impossible to reach the state $((0,1,0) ;(0,0,1))$ starting from the empty state. More generally, none of the states of the facet $\left(\{2\},\left\{3^{\prime}\right\}\right)$ belong to the terminal strongly connected component.

Below, we study the stability of bipartite matching models. Therefore, we always assume that the necessary conditions NCond are satisfied. So we get the property UTC for the Markov chain as a consequence of Theorem 4.3.

## 5 Models that are stable for all admissible policies

Definition 5.1. Consider a bipartite graph $(C, S, E)$ and an admissible matching policy Pol. The stability region is the set of values of $\mu$ for which the bipartite matching model $[(C, S, E), \mu, \mathrm{POL}]$ is stable.

The stability region is included in the polyhedron defined by NCond. The stability region is maximal if it is equal to this polyhedron.

Denote by $\mathfrak{F}$ the set of facets. Define the following conditions on $\mu$ :

$$
\begin{equation*}
\text { SConD : } \quad \mu_{C}\left(C_{\odot}(\mathcal{F})\right)+\mu_{S}\left(S_{\odot}(\mathcal{F})\right)>1-\mu\left(E \cap C_{\circ}(\mathcal{F}) \times S_{\circ}(\mathcal{F})\right), \quad \forall \mathcal{F} \in \mathfrak{F}-\{\emptyset\} \tag{12}
\end{equation*}
$$

Let $\mathcal{F}$ be a saturated facet, see Definition 2.4 Assume for instance that $C_{\circ}(\mathcal{F})=\emptyset$. Then $E \cap C_{\circ}(\mathcal{F}) \times S_{\circ}(\mathcal{F})=\emptyset$ and $C_{\odot}(\mathcal{F})=C-C_{\bullet}(\mathcal{F})$. So 12$)$ implies:

$$
\mu_{S}\left(S_{\odot}(\mathcal{F})\right)>\mu_{C}\left(C_{\bullet}(\mathcal{F})\right)
$$

Since $S_{\odot}(\mathcal{F})=S(C \bullet(\mathcal{F}))$, we recognize exactly (5) for $U=C \bullet(\mathcal{F})$. Conversely, consider $U \subsetneq C$ and the associated condition in NCond: $\mu_{C}(U)<\mu_{S}(S(U))$. Choose a state with a strictly positive number of customers/servers for the classes $U$ and $S-S(U)$. Let $\mathcal{F}$ be the corresponding facet. The facet $\mathcal{F}$ is saturated: $S_{\odot}(\mathcal{F})=S(U), S_{\bullet}(\mathcal{F})=S-S(U), S_{\circ}(\mathcal{F})=\emptyset$. Let us apply (12) to the facet $\mathcal{F}$, we get:

$$
\begin{aligned}
\mu_{C}\left(C_{\odot}(\mathcal{F})\right)+\mu_{S}(S(U)) & >1 \\
\mu_{S}(S(U)) & >\mu_{C}\left(C-C_{\odot}(\mathcal{F})\right) \geq \mu_{C}(U)
\end{aligned}
$$

To summarize, the subset of the inequalities (12) obtained by considering only the saturated facets gives precisely the inequalities NCond.

We now show that the conditions SCond are sufficient stability conditions.
Proposition 5.2. A bipartite model with probability $\mu$ satisfying SCond is stable under any admissible matching policy.
Proof. Consider the linear Lyapunov function:

$$
L(u, v)=|u|, \quad(u, v) \in \mathcal{E},
$$

the number of unmatched customers (servers). Let $\left(U_{n}, V_{n}\right)_{n}$ be the Markov chain of the buffercontent. Let $\mathcal{F} \neq \emptyset$ be an arbitrary and fixed facet. Then for any $(u, v) \in \mathcal{F}$ we have (see Table 1):

$$
\begin{aligned}
& \mathrm{E}\left[L\left(U_{n+1}, V_{n+1}\right) \mid\left(U_{n}, V_{n}\right)=(u, v)\right]-L(u, v)=-\mu\left(C_{\odot}(\mathcal{F}), S_{\odot}(\mathcal{F})\right)+\mu\left(C_{\circ}(\mathcal{F}), S_{\bullet}(\mathcal{F})\right) \\
& +\mu\left(C_{\bullet}(\mathcal{F}), S_{\bullet}(\mathcal{F})\right)+\mu\left(C_{\bullet}(\mathcal{F}), S_{\circ}(\mathcal{F})\right)+\mu\left(C_{\circ}(\mathcal{F}) \times S_{\circ}(\mathcal{F}) \cap E^{c}\right) \\
= & 1-\mu_{C}\left(C_{\odot}(\mathcal{F})\right)-\mu_{S}\left(S_{\odot}(\mathcal{F})\right)-\mu\left(C_{\circ}(\mathcal{F}) \times S_{\circ}(\mathcal{F}) \cap E\right) .
\end{aligned}
$$

The inequality 12 implies directly that:

$$
\begin{equation*}
\mathrm{E}\left[L\left(U_{n+1}, V_{n+1}\right) \mid\left(U_{n}, V_{n}\right)=(u, v)\right]-L(u, v)<\epsilon<0 . \tag{13}
\end{equation*}
$$

By application of the Lyapunov-Foster Theorem, see for instance [1, §5.1], we conclude that the model is stable.

|  | $C_{\odot}$ | $C_{\circ}$ | $C_{\bullet}$ |
| :---: | :---: | :---: | :---: |
| $S_{\odot}$ | -1 | 0 | 0 |
| $S_{\circ}$ | 0 | 0 or 1 | 1 |
| $S_{\bullet}$ | 0 | 1 | 1 |

Table 1: Variation of the linear Lyapunov function.

Corollary 5.3. Consider a bipartite graph in which any non-zero facet is saturated. For any admissible matching policy, the stability region is maximal.

The bipartite graph $\left(C=\{1,2\}, S=\left\{1^{\prime}, 2^{\prime}\right\}, C \times S-\left\{\left(2,2^{\prime}\right)\right\}\right)$ is such that any non-zero facet is saturated. Therefore, its stability region is maximal for any admissible policy. The same is true for the "almost complete graphs" $\left(C=\{1, \ldots, k\}, S=\left\{1^{\prime}, \ldots, k^{\prime}\right\}, C \times S-\left\{\left(i, i^{\prime}\right), \forall i\right\}\right)$.

Example 5. Consider the NN graph from Figure 2. The graph has only one non-zero facet that is non-saturated, facet $\left(\{3\},\left\{3^{\prime}\right\}\right)$. For any admissible policy, the stability region is at least the polyhedron SCond, Proposition 5.2, which is defined by:

$$
\begin{equation*}
\text { NCond, } \quad \mu_{C}(1)+\mu_{S}\left(1^{\prime}\right)>1-\mu\left(2,2^{\prime}\right) . \tag{14}
\end{equation*}
$$

Assume now $\mu=\mu_{C} \times \mu_{S}$ and $\mu_{C}=\mu_{S}$. Set $x=\mu_{C}(1)=\mu_{S}\left(1^{\prime}\right)$ and $y=\mu_{C}(2)=\mu_{S}\left(2^{\prime}\right)$. Then:

$$
\text { NConD : }\left\{\begin{array} { l } 
{ x < 0 . 5 } \\
{ 2 x + y > 1 }
\end{array} \quad \text { SConD : } \quad \left\{\begin{array}{l}
\text { NConD } \\
2 x+y^{2}>1
\end{array}\right.\right.
$$

In Figure 11, the light (yellow) region corresponds to SCond, and the union of the light and


Figure 11: NCond and SCond for the NN-graph with $\mu=\mu_{C} \times \mu_{S}$ and $\mu_{C}=\mu_{S}$.
dark (red) regions corresponds to NCond.
Unfortunately, for some bipartite graphs, the polyhedron SCond is empty. This is illustrated by the following example.
Example 6. Consider the NNN graph of Figure 12. The condition SCond for facet ( $\{1\},\left\{4^{\prime}\right\}$ ) gives:

$$
\begin{equation*}
\mu_{C}(\{3,4\})+\mu_{S}\left(\left\{1^{\prime}, 2^{\prime}\right\}\right)>1-\mu\left(2,3^{\prime}\right) \tag{15}
\end{equation*}
$$

and for facet ( $\left.\{4\},\left\{1^{\prime}\right\}\right)$ :

$$
\begin{equation*}
\mu_{C}(1)+\mu_{S}\left(4^{\prime}\right)>1-\mu\left(2,2^{\prime}\right)-\mu\left(2,3^{\prime}\right)-\mu\left(3,3^{\prime}\right) . \tag{16}
\end{equation*}
$$

The inequality 15$\}$ is equivalent to: $\mu_{C}(1)+\mu_{S}\left(4^{\prime}\right)<1-\mu_{C}(2)-\mu\left(\{1,3,4\}, 3^{\prime}\right)$. Together with (16) this gives:

$$
\mu_{C}(1)+\mu_{S}\left(4^{\prime}\right)<1-\mu_{C}(2)-\mu\left(\{1,3,4\}, 3^{\prime}\right)<1-\mu\left(2,2^{\prime}\right)-\mu\left(2,3^{\prime}\right)-\mu\left(3,3^{\prime}\right)<\mu_{C}(1)+\mu_{S}\left(4^{\prime}\right)
$$

which is impossible.


Figure 12: NNN graph: facets $\left(\{1\},\left\{4^{\prime}\right\}\right)$ and $\left(\{4\},\left\{1^{\prime}\right\}\right)$.


Figure 13: Facets for the NNN graph. Saturated facets are encircled (13 among 25 facets).

## 6 Priorities and MS are not always stable

Consider the NN bipartite graph of Figure 2 and Example 5 For this model, Proposition 5.2 does not allow to decide if the stability region is maximal (see Figure 11). In Figure 14, we give simulation results for the average buffer size up to time $n=1000000$ for the NN-graph with $\mu=\mu_{C} \times \mu_{S}, \mu_{C}=\mu_{S}$, and MS policy. We can see that the average buffer size is growing rapidly


Figure 14: Average buffer size for the NN-graph with $\mu=\mu_{C} \times \mu_{S}, \mu_{C}=\mu_{S}$, and MS policy.
near the $2 x+y=1$ line. This does not necessarily imply unstability, as even for stable models we could have the mean stationary buffer size that is growing unboundedly as we approach the boundary of the stability region. In fact, we show below that for the PR and MS matching policies, the stability region is not maximal.

Proposition 6.1. Consider the $N N$ model with either the $M S$ policy or the $P R$ policy given by:

$$
A=\left[\begin{array}{lll}
0 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{array}\right] \text { and } \quad B=\left[\begin{array}{lll}
0 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{array}\right]
$$

For both policies, the stability region is not maximal.
Proof. We carry out the proof for the PR policy. The idea of the proof is to play with two different Markov chains: the one describing the evolution of the buffer content, and an auxiliary one which mimicks the evolution of the customers/servers of type $2 / 2^{\prime}$ on some of the facets.

Consider an auxiliary Markov chain on $\mathbb{Z}$ with transition probabilities:

$$
\begin{array}{c|ccc} 
& x \rightarrow x-1 & x \rightarrow x & x \rightarrow x+1 \\
\hline x<0 & a_{-1} & a_{0} & a_{1} \\
x=0 & b_{-1} & b_{0} & b_{1} \\
x>0 & c_{-1} & c_{0} & c_{1}
\end{array}
$$

Assume that $a_{-1}, a_{1}, b_{-1}, b_{1}, c_{-1}, c_{1}$, are all different from 0 . The chain is positive recurrent iff: $\left[a_{-1}<a_{1}, c_{1}<c_{-1}\right]$. The stationary distribution is then equal to:

$$
\begin{aligned}
& \pi(0)=\left(1+\frac{b_{-1}}{a_{1}-a_{-1}}+\frac{b_{1}}{c_{-1}-c_{1}}\right)^{-1}, \\
& \pi(x)=\pi(0) \frac{b_{1}}{c_{1}}\left(\frac{c_{1}}{c_{-1}}\right)^{x} \quad \text { if } x>0, \\
& \pi(x)=\pi(0) \frac{b_{-1}}{a_{-1}}\left(\frac{a_{-1}}{a_{1}}\right)^{|x|} \quad \text { if } x<0 .
\end{aligned}
$$

Consider a NN-model with a probability $\mu$ such that $\operatorname{supp}(\mu)=C \times S$. Let $(X, Y)=$ $(X(n), Y(n))_{n}$ be the Markov chain of the buffer content, where $X(n)=\left(X_{1}(n), X_{2}(n), X_{3}(n)\right)$ and $Y(n)=\left(Y_{1}(n), Y_{2}(n), Y_{3}(n)\right)$. Assume wlog that $(X, Y)$ is given under the form of a Stochastic Recursive Sequence, that is:

$$
(X(n+1), Y(n+1))=\Phi\left(X(n), Y(n), \theta_{n}\right),
$$

where $\left(\theta_{n}\right)_{n}$ is an i.i.d. sequence of r.v.'s distributed according to $\mu$, and $\Phi$ is a deterministic function.

Consider now the process $\left(X_{2}(n)-Y_{2}(n)\right)_{n}$. This is not a Markov chain. However, if $X_{2}(n)+$ $X_{3}(n)>0$, then "it becomes one". More precisely, if $X_{2}(n)+X_{3}(n)>0$, then

$$
\begin{equation*}
X_{2}(n+1)-Y_{2}(n+1)=\Phi_{2}\left(X_{2}(n)-Y_{2}(n), \theta_{n}\right), \tag{17}
\end{equation*}
$$

where $\Phi_{2}$ is a deterministic function. This can be checked by direct inspection. Moreover, the transition kernel on $\mathbb{Z}$ defined by the recursion 17 is of the type of the above auxiliary chain with parameters:

$$
\begin{aligned}
a_{1} & =\mu\left(1,1^{\prime}\right)+\mu\left(1,3^{\prime}\right)+\mu\left(2,1^{\prime}\right)+\mu\left(2,3^{\prime}\right) \\
a_{0} & =\mu\left(1,2^{\prime}\right)+\mu\left(2,2^{\prime}\right)+\mu\left(3,1^{\prime}\right)+\mu\left(3,3^{\prime}\right) \\
a_{-1} & =\mu\left(3,2^{\prime}\right) \\
b_{1} & =\mu\left(2,1^{\prime}\right)+\mu\left(2,3^{\prime}\right), \\
b_{0} & =\mu\left(1,1^{\prime}\right)+\mu\left(1,3^{\prime}\right)+\mu\left(2,2^{\prime}\right)+\mu\left(3,1^{\prime}\right)+\mu\left(3,3^{\prime}\right), \\
b_{-1} & =\mu\left(1,2^{\prime}\right)+\mu\left(3,2^{\prime}\right) \\
c_{1} & =\mu\left(2,3^{\prime}\right) \\
c_{0} & =\mu\left(1,3^{\prime}\right)+\mu\left(2,1^{\prime}\right)+\mu\left(2,2^{\prime}\right)+\mu\left(3,3^{\prime}\right) \\
c_{-1} & =\mu\left(1,1^{\prime}\right)+\mu\left(1,2^{\prime}\right)+\mu\left(3,1^{\prime}\right)+\mu\left(3,2^{\prime}\right) .
\end{aligned}
$$

Let us justify for instance the values of $c_{-1}, c_{0}, c_{1}$. We are in the case $X_{2}(n)+X_{3}(n)>0$ and $X_{2}(n)-Y_{2}(n)>0$ which implies:

$$
X_{1}(n)=0, X_{2}(n)>0, \quad Y_{1}(n)=Y_{2}(n)=0, Y_{3}(n)>0
$$

In Table 2 below, we show the effect of the different possible types of arrivals, restricting to the ones which may affect $X_{2}$, i.e. when the customer class is 2 or the server class is $1^{\prime}$ or $2^{\prime}$. To simplify, we have assumed in Table 2 that $X_{3}>0$. For $X_{3}=0$, the "Possible matchings" column would be affected, but not the "Selected matchings" and " $\Delta X_{2}$ " columns.

| Arrival | Possible matchings | Selected matchings | $\Delta X_{2}$ |
| :---: | :---: | :---: | :---: |
| $\left(1,1^{\prime}\right)$ | $\left(1,3^{\prime}\right),\left(2,1^{\prime}\right),\left(3,1^{\prime}\right)$ | $\left(1,3^{\prime}\right),\left(2,1^{\prime}\right)$ | -1 |
| $\left(1,2^{\prime}\right)$ | $\left(1,3^{\prime}\right),\left(1,2^{\prime}\right),\left(2,2^{\prime}\right)$ | $\left(1,3^{\prime}\right),\left(2,2^{\prime}\right)$ | -1 |
| $\left(2,1^{\prime}\right)$ | $\left(2,1^{\prime}\right),\left(3,1^{\prime}\right)$ | $\left(2,1^{\prime}\right)$ | 0 |
| $\left(2,2^{\prime}\right)$ | $\left(2,2^{\prime}\right)$ | $\left(2,2^{\prime}\right)$ | 0 |
| $\left(2,3^{\prime}\right)$ | $\emptyset$ | $\emptyset$ | +1 |
| $\left(3,2^{\prime}\right)$ | $\left(2,2^{\prime}\right)$ | $\left(2,2^{\prime}\right)$ | -1 |
| $\left(3,1^{\prime}\right)$ | $\left(2,1^{\prime}\right),\left(3,1^{\prime}\right)$ | $\left(2,1^{\prime}\right)$ | -1 |

Table 2: Effect of arrivals.

Let us comment on a couple of cases. If the arrival is of type $\left(1,1^{\prime}\right)$, then the selected matching is $\left(2,1^{\prime}\right)$ rather than $\left(3,1^{\prime}\right)$ due to the PR policy $\left(B_{2,1^{\prime}}>B_{3,1^{\prime}}\right)$. If the arrival is of type $\left(1,2^{\prime}\right)$, the selected matching is $\left(2,2^{\prime}\right)$ rather than $\left(1,2^{\prime}\right)$ according to the buffer-first property of admissible policies, see 2.2 . The other cases are argued similarly.

Let us introduce a new Markov chain $\left(W_{n}\right)_{n}$ on $\mathbb{Z}$ defined by:

$$
W_{n+1}=\Phi_{2}\left(W_{n}, \theta_{n}\right)
$$

(The process $\left(W_{n}\right)_{n}$ is different from the process $\left(X_{2}(n)-Y_{2}(n)\right)_{n}$. The former is always defined according to the recursion (17) while the latter is defined according to (17) only for the $n$ 's such that $X_{2}(n)+X_{3}(n)>0$. The former is Markovian while the latter is not.)

Condition $c_{1}<c_{-1}$ becomes $\mu_{C}(2)<\mu_{S}\left(1^{\prime}\right)+\mu_{S}\left(2^{\prime}\right)$ and $a_{-1}<a_{1}$ becomes $\mu_{S}\left(2^{\prime}\right)<\mu_{C}(1)+$ $\mu_{C}(2)$. Both conditions follow from NCond. So the auxiliary chain $\left(W_{n}\right)_{n}$ is ergodic and its stationary distribution $\pi$ satisfies:

$$
\pi(0)=\left(1+\frac{b_{-1}}{a_{1}-a_{-1}}+\frac{b_{1}}{c_{-1}-c_{1}}\right)^{-1}, \quad \pi\left(\mathbb{Z}_{+}^{*}\right)=\pi(0) \frac{b_{1}}{c_{-1}-c_{1}}, \quad \pi\left(\mathbb{Z}_{-}^{*}\right)=\pi(0) \frac{b_{-1}}{a_{1}-a_{-1}},
$$

where $\mathbb{Z}_{+}^{*}=\{1,2, \ldots\}$ is the set of strictly positive integers and $\mathbb{Z}_{-}^{*}=\{-1,-2, \ldots\}$ is the set of strictly negative integers. From now on, we fix an initial condition $W_{0}$ satisfying

$$
W_{0} \sim \pi, \quad W_{0} \Perp\left(\theta_{n}\right)_{n} .
$$

Let us switch back to the Markov chain $(X, Y)$. Set

$$
L(n)=X_{2}(n)+X_{3}(n), \quad \Delta L(n)=L(n+1)-L(n)
$$

If $L(n)>0$, then we check by direct inspection that:

$$
\Delta L(n)=\Psi\left(X_{2}(n)-Y_{2}(n), \theta_{n}\right)
$$

where $\Psi$ is a deterministic function. We have in particular

$$
\begin{aligned}
& \alpha \\
& \stackrel{\text { def }}{=} \mathrm{E}\left[\Delta L(n) \mid L(n)>0, Y_{2}(n)>0\right]=\mu\left(3,2^{\prime}\right)+\mu\left(3,3^{\prime}\right)-\mu\left(1,1^{\prime}\right)-\mu\left(2,1^{\prime}\right), \\
& \beta \\
& \stackrel{\text { def }}{=} \mathrm{E}\left[\Delta L(n) \mid L(n)>0, X_{2}(n)=Y_{2}(n)=0\right]=\mu\left(2,3^{\prime}\right)+\mu\left(3,2^{\prime}\right)+\mu\left(3,3^{\prime}\right)-\mu\left(1,1^{\prime}\right), \\
& \gamma \\
& \stackrel{\text { def }}{=} \mathrm{E}\left[\Delta L(n) \mid L(n)>0, X_{2}(n)>0\right]=\mu\left(2,3^{\prime}\right)+\mu\left(3,3^{\prime}\right)-\mu\left(1,1^{\prime}\right)-\mu\left(1,2^{\prime}\right) .
\end{aligned}
$$

Let us turn again to the auxiliary chain $\left(W_{n}\right)_{n}$. By performing the computation, we get

$$
\mathrm{E}\left[\Psi\left(W_{n}, \theta_{n}\right)\right]=\pi\left(\mathbb{Z}_{-}^{*}\right) \alpha+\pi(0) \beta+\pi\left(\mathbb{Z}_{+}^{*}\right) \gamma
$$

The Ergodic Theorem for Markov Chains, see for instance [1, §3.4], gives:

$$
\lim _{n} \frac{1}{n} \sum_{i=0}^{n-1} \Psi\left(W_{n}, \theta_{n}\right)=\pi\left(\mathbb{Z}_{-}^{*}\right) \alpha+\pi(0) \beta+\pi\left(\mathbb{Z}_{+}^{*}\right) \gamma \quad \text { a.s. }
$$

Assume that $\pi\left(\mathbb{Z}_{-}^{*}\right) \alpha+\pi(0) \beta+\pi\left(\mathbb{Z}_{+}^{*}\right) \gamma>0$. Then we have:

$$
\begin{equation*}
\lim _{n} \sum_{i=0}^{n-1} \Psi\left(W_{n}, \theta_{n}\right)=+\infty \quad \text { a.s. } \tag{18}
\end{equation*}
$$

Therefore, for each $\varepsilon>0$, there exists $K_{\varepsilon} \geq 0$ such that

$$
\begin{equation*}
P\left(\min _{n \geq 1} \sum_{i=0}^{n-1} \Psi\left(W_{n}, \theta_{n}\right)>-K_{\varepsilon}\right) \geq 1-\varepsilon \tag{19}
\end{equation*}
$$

Let us switch back to the Markov chain $(X(n), Y(n))_{n}$. Choose the initial condition $(X(0), Y(0))$ such that

$$
\begin{equation*}
X_{2}(0)-Y_{2}(0)=W_{0} \quad \min \left(X_{3}(0), Y_{3}(0)\right)=K_{\varepsilon} \tag{20}
\end{equation*}
$$

where $K_{\varepsilon}$ is defined in 19). By construction, on the event $\mathcal{A}=\left\{\min _{n \geq 1} \sum_{i=0}^{n-1} \Psi\left(W_{n}, \theta_{n}\right)>-K_{\varepsilon}\right\}$, we have

$$
\forall n, \quad L(n)>0, \quad X_{2}(n)-Y_{2}(n)=W_{n} .
$$

So, on the event $\mathcal{A}$, we have

$$
\begin{equation*}
L(n)=K_{\varepsilon}+\sum_{i=0}^{n-1} \Delta L(i)=K_{\varepsilon}+\sum_{i=0}^{n-1} \Psi\left(W_{i}, \theta_{i}\right) \longrightarrow+\infty \tag{21}
\end{equation*}
$$

We conclude that the Markov chain $(X, Y)$ of the NN-model is transient.
We now show that the stability region is not maximal, by giving an example such that $\pi\left(\mathbb{Z}_{-}^{*}\right) \alpha+$ $\pi(0) \beta+\pi\left(\mathbb{Z}_{+}^{*}\right) \gamma>0$. Consider $\mu_{C}=(1 / 3,2 / 5,4 / 15), \mu_{S}=\mu_{C}$, and $\mu=\mu_{C} \times \mu_{S}$. Thus conditions NCond are satisfied. However, we have:

$$
\begin{array}{lll}
a_{1}=\frac{11}{25}, & a_{0}=\frac{34}{75}, & a_{-1}=\frac{8}{75} \\
b_{1}=\frac{6}{25}, & b_{0}=\frac{13}{25}, & b_{-1}=\frac{6}{25} \\
c_{1}=\frac{8}{75}, & c_{0}=\frac{34}{75}, & c_{-1}=\frac{11}{25}
\end{array}
$$

and

$$
\pi(0)=\frac{25}{61}, \quad \pi\left(\mathbb{Z}_{+}^{*}\right)=\frac{18}{61}, \quad \pi\left(\mathbb{Z}_{-}^{*}\right)=\frac{18}{61}
$$

This gives $\alpha=-1 / 15, \beta=13 / 75, \gamma=-1 / 15$, and,

$$
\pi\left(\mathbb{Z}_{-}^{*}\right) \alpha+\pi(0) \beta+\pi\left(\mathbb{Z}_{+}^{*}\right) \gamma=\frac{29}{915}>0
$$

This completes the proof.
Consider now the MS policy. Set $L(n)=\min \left(X_{3}(n)-X_{2}(n), Y_{3}(n)-Y_{2}(n)\right)$. The initial distribution can be taken such that

$$
X_{2}(0)-Y_{2}(0) \sim \pi, \quad \begin{cases}X_{3}(0)-X_{2}(0)=K_{\varepsilon} & \text { if } X_{2}(0)>0 \\ Y_{3}(0)-Y_{2}(0)=K_{\varepsilon} & \text { otherwise }\end{cases}
$$

Modulo these modifications, the proof carries over unchanged.

## 7 ML is always stable

In this section, we show that the ML policy has a maximal stability region.
The idea of the proof is as follows. Consider the quadratic Lyapunov function:

$$
\begin{equation*}
L(x, y)=\sum_{c \in C} x_{c}^{2}+\sum_{s \in S} y_{s}^{2}, \quad(x, y) \in \mathcal{E} \tag{22}
\end{equation*}
$$

Observe that the ML policy minimizes the value of this Lyapunov function at each step. We introduce an alternate policy that depends on the arrival distribution $\mu$. For this policy, we manage to prove that the quadratic Lyapunov function has a negative drift outside a finite region.
Theorem 7.1. For any bipartite graph, the ML policy has a maximal stability region.
Proof. We introduce an alternate matching policy. This policy is admissible, corresponds to a commutative state space, but does not belong to the policies listed in $\S 2.2$. It is a random policy and its specificity is to be facet dependent.

Let us describe the alternate policy on a non-empty facet $\mathcal{F}$. Set $C_{\bullet}=C_{\bullet}(\mathcal{F}), S_{\bullet}=S_{\bullet}(\mathcal{F}), C_{\odot}=$ $C_{\odot}(\mathcal{F})$, etc. To describe the matching policy, the only thing we have to describe is the way to match an arriving customer of class $c \in C_{\odot}$, resp. server of class $s \in S_{\odot}$. Let us concentrate first on a server of class $s \in S_{\odot}$.

From NCond:

$$
\mu_{C}\left(C_{\bullet}\right)<\mu_{S}\left(S_{\odot}\right), \quad \mu_{S}\left(S_{\bullet}\right)<\mu_{C}\left(C_{\odot}\right) .
$$

We build a directed graph as in (8) but restricted to the nodes in $C_{\bullet}$ and $S_{\odot}$. Formally,

$$
\begin{equation*}
\mathcal{N}_{\mathcal{F}}=\left(C_{\bullet} \cup S_{\odot} \cup\{i, f\},\left\{E \cap C_{\bullet} \times S_{\odot}\right\} \cup\left\{(i, c), c \in C_{\bullet}\right\} \cup\left\{(s, f), s \in S_{\odot}\right\}\right) . \tag{23}
\end{equation*}
$$

Endow the arcs of $E \cap C_{\bullet} \times S_{\odot}$ with infinite capacity, an arc of type $(i, c)$ with capacity $\mu_{C}(c)$, and an arc of type $(s, f)$ with capacity $\mu_{S}(s)$.

As in Lemma 3.3. NCond implies that the minimal cut of $\mathcal{N}_{\mathcal{F}}$ has capacity $\mu_{C}\left(C_{\bullet}\right)$. Any maximal flow $T$ is such that: $\forall c \in C \bullet, T(i, c)=\mu_{C}(c), \forall s \in S_{\odot}, T(s, f) \leq \mu_{S}(s)$. Let us prove that there exists a maximal flow $T$ such that:

$$
\begin{array}{ll}
\forall c \in C_{\bullet}, & T(i, c)=\mu_{C}(c) \\
\forall s \in S_{\odot}, & T(s, f)<\mu_{S}(s) .
\end{array}
$$

Define $\widetilde{\mu}_{S}$ on $S_{\odot}$ by $\widetilde{\mu}_{S}(s)=\mu_{S}(s)-\eta$. Here $\eta>0$ is chosen to be small enough so that: $\forall U \subset C \bullet, \mu_{C}(U)<\widetilde{\mu}_{S}(S(U)), \forall V \subset S_{\odot}, \widetilde{\mu}_{S}(V)<\mu_{C}(C(V))$. This is possible since NCond are open conditions. Consider the same network as above but with the capacities $\widetilde{\mu}_{S}(s)$ on the $\operatorname{arcs}(s, f)$. The minimal cut still has capacity $\mu_{C}\left(C_{\bullet}\right)$. A maximal flow $T$ is such that: $\forall c \in$ $C_{\bullet}, T(i, c)=\mu_{C}(c), \forall s \in S_{\odot}, T(s, f) \leq \widetilde{\mu}_{S}(s)<\mu_{S}(s)$. Clearly, $T$ is also a flow for the original network.

The server $s \in S_{\odot}$ is matched to $c \in C \bullet \cap C(s)$ randomly, independently of the past, with probability:

$$
P_{s c}^{\mathcal{F}}=\frac{1}{\mu_{S}(s)}\left[T(c, s)+\frac{\mu_{S}(s)-T(s, f)}{|C \bullet \cap C(s)|}\right] .
$$

Let us check that this defines indeed a probability:

$$
\begin{aligned}
\sum_{c \in C \bullet \cap C(s)} P_{s c}^{\mathcal{F}} & =\frac{1}{\mu_{S}(s)}\left[\mu_{S}(s)-T(s, f)+\sum_{c \in C \cdot \cap C(s)} T(c, s)\right] \\
& =\frac{1}{\mu_{S}(s)}\left[\mu_{S}(s)-T(s, f)+T(s, f)\right]=1
\end{aligned}
$$

For $c \in C_{\bullet}, s \in S(c)$, set $\varepsilon_{s c}=\left(\mu_{S}(s)-T(s, f)\right) /\left|C_{\bullet} \cap C(s)\right|$. For $c \in C_{\bullet}$, set $\varepsilon_{c}=\sum_{s \in S(c)} \varepsilon_{s c}$. We have $\varepsilon_{c}>0$. Observe that:

$$
\begin{equation*}
\forall c \in C_{\bullet}, \quad \sum_{s \in S(c)} \mu_{S}(s) P_{s c}^{\mathcal{F}}=\mu_{C}(c)+\varepsilon_{c} \tag{24}
\end{equation*}
$$

Symmetrically, we define the directed graph of type (23) but on the nodes $C_{\odot}$ and $S_{\bullet}$. We build a maximal flow on this new graph as above, and based on this flow, we define the probability $P_{c s}^{\mathcal{F}}$ that a customer $c \in C_{\odot}$ is matched to a server $s \in S_{\bullet} \cap S(c)$. For $s \in S_{\bullet}$, we define $\varepsilon_{c s}, c \in C(s)$, and $\varepsilon_{s}$ accordingly. We have $\varepsilon_{s}>0$.

Let $(X(n), Y(n))_{n}$ be the Markov chain of the buffer-content of the model. Assume that $(X(n), Y(n))=(x, y) \in \mathcal{F}$ and let $c \in C_{\bullet}$. We have:
(i) $X(n+1)_{c}=X(n)_{c}-1$ iff:

- the arriving customer is not of class $c$;
- the arriving server is of class $s \in S(c)$;
- the arriving server is matched with $c$ (probability $P_{s c}^{\mathcal{F}}$ ).

This case happens with probability $\alpha_{c}=\sum_{s \in S(c)} \mu(C-c, s) P_{s c}^{\mathcal{F}}$.
(ii) $X(n+1)_{c}=X(n)_{c}+1$ iff:

- the arriving customer is of class $c$;
- the arriving server is not matched with $c$. This may occur in two possible ways: either the arriving server is of class $s \notin S(c)$, or the arriving server is of class $s \in S(c)$ but is not matched with $c$ (probability $1-P_{s c}^{\mathcal{F}}$ ).
This case happens with probability $\beta_{c}=\mu(c, S-S(c))+\sum_{s \in S(c)} \mu(c, s)\left(1-P_{s c}^{\mathcal{F}}\right)$.
(iii) Otherwise, $X(n+1)_{c}=X(n)_{c}$.

Using (24), we get:

$$
\begin{aligned}
\sum_{s \in S(c)} \mu(C-c, s) P_{s c}^{\mathcal{F}} & =\sum_{s \in S(c)} \mu_{S}(s) P_{s c}^{\mathcal{F}}-\sum_{s \in S(c)} \mu(c, s) P_{s c}^{\mathcal{F}} \\
& =\mu_{C}(c)+\varepsilon_{c}-\sum_{s \in S(c)} \mu(c, s) P_{s c}^{\mathcal{F}} \\
& =\mu(c, S-S(c))+\mu(c, S(c))+\varepsilon_{c}-\sum_{s \in S(c)} \mu(c, s) P_{s c}^{\mathcal{F}} \\
& =\mu(c, S-S(c))+\sum_{s \in S(c)} \mu(c, s)\left(1-P_{s c}^{\mathcal{F}}\right)+\varepsilon_{c}
\end{aligned}
$$

Thus, $\alpha_{c}=\beta_{c}+\varepsilon_{c}$. Observe that $\beta_{c}<\mu_{C}(c)$. We get, for $(x, y) \in \mathcal{F}$ and $c \in C_{\bullet}$ :

$$
\begin{aligned}
\mathrm{E}\left[X(n+1)_{c}^{2}-X(n)_{c}^{2} \mid(X(n), Y(n))=(x, y)\right] & =\beta_{c}\left(2 x_{c}+1\right)-\alpha_{c}\left(2 x_{c}-1\right) \\
& =2 \beta_{c}-\varepsilon_{c}\left(2 x_{c}-1\right) \\
& <2 \mu_{C}(c)-\varepsilon_{c} x_{c}
\end{aligned}
$$

Let $L$ be the quadratic Lyapunov function 22. Define $\Delta L(n)=L\left(X_{n+1}, Y_{n+1}\right)-L\left(X_{n}, Y_{n}\right)$. Set $\varepsilon=\min _{v \in C \bullet \cup S} \cup \varepsilon_{v}>0$. Then (the first term in the sum takes care of the vertices in $C-C \bullet$ and $S-S_{\bullet}$ ):

$$
\begin{aligned}
\mathrm{E}\left[\Delta L(n) \mid\left(X_{n}, Y_{n}\right)=(x, y)\right] & <2+\sum_{c \in C_{\bullet}}\left(2 \mu_{C}(c)-\varepsilon_{c} x_{c}\right)+\sum_{s \in S_{\bullet}}\left(2 \mu_{S}(s)-\varepsilon_{s} y_{s}\right) \\
& <2+2 \mu_{C}\left(C_{\bullet}\right)+2 \mu_{S}\left(S_{\bullet}\right)-\varepsilon\left(\sum_{c \in C} x_{c}+\sum_{s \in S_{\bullet}} y_{s}\right) \\
& <6-2 \varepsilon \sum_{c \in C} x_{c} .
\end{aligned}
$$

Fix $\delta>0$. If $\sum_{c \in C} x_{c}>(6+\delta) / 2 \varepsilon$, then $\mathrm{E}[\Delta L(n)]<-\delta$. There are finitely many facets, so there is a finite set $A \subset \mathcal{E}$ such that

$$
\begin{equation*}
\forall(x, y) \notin A, \quad \mathrm{E}[\Delta L(n)]<-\delta . \tag{25}
\end{equation*}
$$

By the Lyapunov-Foster's Theorem, see for instance [1, §5.1], the alternate matching policy is stable.

Since the ML matching policy minimizes the value of the quadratic Lyapunov function, we have a fortiori that 25 holds for it. Therefore, the ML policy is also stable.

Conclusion. Many open questions remain. First, we do not know if the stability region is always maximal for the FIFO and Random policies. Numerical experiments seem to indicate that it is indeed the case. Second, for the MS and priority policies, we know that the stability region is not always maximal, but we do not know how to compute it. Last, we would like to obtain sufficient conditions for stability, valid for all admissible policies, and which are better than the ones of $\$ 5$

The program used to carry out the numerical experiments is available on request from Ana Bušić.

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