Characteristic classes for irregular singularities

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Abstract

For an endomorphism in a finite dimensional vector space, one can define its characteristic polynomial and rational Jordan normal form. In this article something analogous is done for differential operators in a finite dimensional vector space. An overview of (partial) algorithms to compute these invariants is also given. Proofs and more results and details can be found in [8] on which this article is based.

1 Introduction and notation

In this article k is an arbitrary field of characteristic zero, $K = k((x))$ is the field of formal Laurent series and $\tau = x\frac{d}{dx}$ is the derivation on K with field of constants k.

A *differential operator* in a finite dimensional vector space over K is an additive function $D: V \longrightarrow V$ which satisfies the Leibniz rule:

$$D(av) = aDv + \tau(a)v \quad \text{for all } a \in K \text{ and } v \in V.$$

A differential operator $D: V \longrightarrow V$ gives V the structure of a $K[\tau]$-module by the action $\tau \cdot v := Dv$ for all $v \in V$. Here $K[\tau]$ denotes the K-algebra generated by τ with relation $\tau \cdot a = a\tau + \tau(a)$ for all a in K.

Two differential operators $D: V \longrightarrow V$ and $D': V' \longrightarrow V'$ are called *equivalent* or *isomorphic* if their corresponding $K[\tau]$-modules are isomorphic, i.e., if there exists an invertible linear transformation $f: V \longrightarrow V'$ such that $D'f = fD$.

If $D: V \longrightarrow V$ is a differential operator and $\vec{e} = (e_1, \ldots, e_n)$ a K-basis of the vector space V then $M(D, \vec{e}) \in \mathrm{Mat}_n(K)$ is the matrix $(a_{ij})_{i,j\in\{1,\ldots,n\}}$ defined by

$$D(e_j) = \sum_{i=1}^{n} a_{ij} e_i \quad \text{for all } j \in \{1, \ldots, n\}.$$

Notice that if $\vec{f} = (f_1, \ldots, f_n)$ is another K-basis of V and T is the transformation matrix from $\vec{e}$ to $\vec{f}$, i.e., $\vec{f}T = \vec{e}$, then

$$TM(D, \vec{f})T^{-1} - \tau(T)T^{-1} = M(D, \vec{e}).$$

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We therefore introduce the following notation: for a matrix $A \in \mathrm{Mat}_n(K)$ and $g \in \mathrm{Gl}_n(K)$ define $g[A]$ by

$$g[A] := gAg^{-1} - \tau(g)g^{-1}.$$

An example of a differential operator is

$$D: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \begin{pmatrix} \tau y_1 \\ \tau y_2 \end{pmatrix} + \begin{pmatrix} 1 & 1/x^2 \\ x & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

One can also conceive this differential operator as the following system of first order linear differential equations:

$$\begin{cases} x\dot{y}_1(x) + y_1(x) + y_2(x)/x^2 = 0, \\ x\dot{y}_2(x) + xy_1(x) - \frac{1}{2}y_2(x) = 0. \end{cases}$$

The general solution of this system is

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \lambda e^{2/\sqrt{x}} \begin{pmatrix} 1/x \\ \sqrt{x} \end{pmatrix} + \mu e^{-2/\sqrt{x}} \begin{pmatrix} 1/x \\ -\sqrt{x} \end{pmatrix} \qquad (\lambda, \mu \in k).$$

It has a so-called *irregular singularity* in the origin.

For a differential operator $D: V \longrightarrow V$ with a regular singularity in the origin it is well-known that $C := M(D, \vec{v}) \in \mathrm{Mat}(k)$ for some K-basis $\vec{v}$ of V and that the isomorphism class of D is determined by the conjugacy class of the *monodromy* matrix $\exp(2\pi i C)$. So in this case we have both the notion of a "characteristic polynomial" (the equivalence class of the characteristic polynomial of C) and a rational normal form (the rational Jordan normal form of C).

For a differential operator with an irregular singularity Levelt defined a characteristic polynomial and a Jordan normal form in [7]. However the characteristic polynomial does not contain enough information to determine the isomorphism class of the semisimple part of the differential operator and for the Jordan normal form a field extension is necessary. It is possible that two non-isomorphic differential operators have the same Jordan normal form.

The characteristic class and rational normal form described in the rest of this article do not have these deficiencies.

2 The characteristic class

In this section we will define an invariant of a differential operator which we will call its *characteristic class* since it resembles the characteristic polynomial of a linear transformation. The characteristic class of a differential operator is defined using eigenvalues.

Theorem 2.1 (Levelt) *Let $D: V \to V$ be a differential operator with $V \neq 0$. Then there exists a finite field extension L of K, a non-zero element $v \in V_L := L \otimes_K V$ and an element $a \in L$ such that $D_L v = av$.*

A differential operator can have many related different eigenvalues. In general, if $a \in L$ is an eigenvalue of the differential operator D_L then $a + \tau(f)/f$ is also an eigenvalue of D_L for all $f \in L^*$.

We will therefore introduce the notion of a "normalized" eigenvalue.

Definition 2.2 Let $D: V \to V$ be a differential operator and let L be a finite field extension of K. Write $L = k'K(t)$ where k' is the field of constants of L and $t \in L$ satisfies $t^e/x \in k' \setminus \{0\}$ for some $e \in \mathbb{N}$. An element $a \in L$ is called a *normalized eigenvalue* of the differential operator D_L if

1. $a \in k'[1/t]$;
2. $D_L v = av$ for some non-zero $v \in K(a) \otimes V$.

Theorem 2.3 *Let $D: V \to V$ be a differential operator with $V \neq 0$. Then D_L has a normalized eigenvalue over some finite field extension L of K.*

Normalized eigenvalues determine simple differential operators up to isomorphism, more precisely:

Theorem 2.4 *Let $D: V \to V$ and $D': V' \to V'$ be two simple differential operators, i.e., there does not exist a proper subspace of V (resp. V') that is invariant under D (resp. D'). Let L be a finite field extension of K and let $a \in L$ be a normalized eigenvalue of D_L with minimal polynomial $f(X) \in K[X]$ and $a' \in L$ a normalized eigenvalue of D'_L with minimal polynomial $g(X) \in K[X]$. Then $\deg(f) = \dim V$ and $\deg(g) = \dim V'$ and the following two statements are equivalent:*

1. *$D: V \to V$ and $D': V' \to V'$ are isomorphic.*
2. *There exists an $r \in \mathbb{Z}$ such that $g(X) = f(X + r/e)$.*

Here the integer e is the ramification index of the field extension $K(a)/K$, i.e., there exists a non-zero $t \in K(a)$ such that $K(a) = k'K(t)$ and $t^e/x \in k'$, where k' is the field of constants in $K(a)$.

Using theorem 2.4 we can now define the characteristic class of a simple differential operator:

Definition 2.5 Let $D: V \to V$ be a simple differential operator. Then the *characteristic class* of D, notation $c(D)$, is defined as the set of minimal polynomials of normalized eigenvalues. Theorem 2.4 shows that

$$c(D)(\lambda) = \{ f(\lambda + \frac{r}{e}) \mid r \in \mathbb{Z} \}, \qquad (1)$$

where f is a monic irreducible polynomial of degree $\dim V$ with coefficients in K and e is the ramification index of a splitting field of f over K. We will write $[f]$ for the right hand side of (1).

Example 2.6 Let

$$D = x\frac{d}{dx} + \begin{pmatrix} 1 & 1/x^2 \\ x & -\frac{1}{2} \end{pmatrix}.$$

Then $D: K^2 \to K^2$ is a simple differential operator and

$$D\begin{pmatrix} 1/x \\ \sqrt{x} \end{pmatrix} = \frac{1}{\sqrt{x}}\begin{pmatrix} 1/x \\ \sqrt{x} \end{pmatrix},$$

so $1/\sqrt{x}$ is a normalized eigenvalue. Hence

$$c(D)(\lambda) = [\lambda^2 - 1/x] = \{ (\lambda + r/2)^2 - 1/x \mid r \in \mathbb{Z} \}.$$

For a general differential operator $D: V \to V$ the characteristic class of a differential operator is defined using a composition series of the $K[\tau]$-module V.

Definition 2.7 Let $D: V \to V$ be a differential operator and let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$$

be a composition series of the $K[\tau]$-module V. Define the *characteristic class* of D, notation $c(D)$, as the set of all products of the elements in the characteristic classes of the $D_i: V_i/V_{i-1} \to V_i/V_{i-1}$. By the Jordan-Hölder theorem, this does not depend on the composition series of the $K[\tau]$-module V.

The characteristic class is a refinement of the characteristic polynomial of a differential operator defined in [7].

Using the definition of the characteristic class and theorem 2.4 one can show that the characteristic class has the following universal property:

Theorem 2.8 *Let $\mathcal{V}$ denote the Abelian category of differential operators. Let M be an Abelian monoid and let $\psi: \mathcal{V} \to M$ be a map satisfying*

$$\psi\big((V, D)\big) = \psi\big((V', D')\big) + \psi\big((V'', D'')\big)$$

for all short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle D'} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D''} & & \\ 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \end{array}.$$

Then there exists a homomorphism $\varphi: \operatorname{im} c \to M$ such that the following triangle commutes:

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\psi"] \arrow[dr, "c"'] & M \\ & \operatorname{im} c \arrow[u, "\varphi"'] \end{tikzcd}.$$

One can define the semisimple part of a differential operator (see theorem 3.1 or [7]). The characteristic class completely determines the isomorphism class of the semisimple part of a differential operator. Notice that the characteristic polynomial of an endomorphism in a finite dimensional vector space has the same property.

The Newton-Puiseux polygon of a differential operator follows from its characteristic class. If

$$\lambda^n + a_{n-1}(x)\lambda^{n-1} + \cdots + a_0(x)$$

is an element in $k[1/x][\lambda]$ representing the characteristic class of a differential operator $D: V \to V$, then the Newton-Puiseux polygon of D is the same as the Newton-Puiseux polygon of

$$(x\frac{d}{dx})^n + a_{n-1}(x)(x\frac{d}{dx})^{n-1} + \cdots + a_0(x).$$

3 Rational normal form

Theorem 3.1 *Let $D: V \to V$ be a differential operator. Then there exists a K-basis $\vec{v} = (v_1, \dots, v_n)$ of V such that*

$$M(D, \vec{v}) = \operatorname{diag}(B_1, B_2, \dots, B_s),$$

where $s \in \mathbb{N}^*$ and every B_i is of the form

$$\begin{pmatrix} S_i & I & 0 & \dots & 0 \\ 0 & S_i & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I \\ 0 & \dots & \dots & 0 & S_i \end{pmatrix}.$$

Here I stands for the identity matrix and S_i is a square matrix such that $\tau + S_i$ is a simple differential operator. This form is unique in the sense that any other such form can be obtained by permuting the blocks B_i and replacing the S_i by $g_i[S_i]$ with $g_i \in \mathrm{Gl}(K)$.

The semisimple and nilpotent part of the differential operator follow directly from this form.

The proof of theorem 3.1 is based on lemma 3.2 and 3.3, the theorem of Jordan for commutative fields and the fact that

$$\ker \operatorname{ad}(D) = \{ f \in \operatorname{End}_K(V) \mid Df = fD \}$$

is a commutative field if $D: V \to V$ is a simple differential operator (see [8, section 3.3]).

Lemma 3.2 *Let $D: V \to V$ be a semisimple differential operator. Then*

$$\operatorname{End}_K(V) = \ker \operatorname{ad}(D) \oplus \operatorname{im} \operatorname{ad}(D)$$

as vector spaces over k, where

$$\operatorname{ad}(D): \operatorname{End}_K(V) \to \operatorname{End}_K(V)$$

is defined by $f \mapsto Df - fD$ for all $f \in \operatorname{End}_K(V)$.

Lemma 3.3 *Let $D_1: V \to V$ and $D_2: W \to W$ be two non-isomorphic simple differential operators. Define the function $\varphi: \operatorname{Hom}_K(V, W) \to \operatorname{Hom}_K(V, W)$ by $g \mapsto D_2 g - g D_1$ for all $g \in \operatorname{Hom}_K(V, W)$. Then φ is an isomorphism of vector spaces over k.*

For the simple components a cyclic form representation is not satisfactory, since the uniqueness is not clear and the characteristic class cannot be obtained easily from it. The form described in the next theorem does not have these deficiencies.

Theorem 3.4 *Let $D: V \to V$ be a differential operator. Then the following statements hold:*

1. *If $D: V \to V$ is a simple differential operator, there exist $e, f \in \mathbb{N}^*$ with $\dim V = ef$, a K-basis $\vec{v}$ of V, $C \in \mathrm{Gl}_f(k)$ and $f \times f$ matrices $A_0, A_1, \dots, A_{e-1}$ with entries in $k[1/x]$ such that*

   (a) C and all $A_i^{(j)}$ are semisimple and all commute pairwise, where $A_i^{(j)}$ are the constant $f \times f$ matrices satisfying

   $$A_i = \sum_{j \le 0} A_i^{(j)} x^j \quad \text{for all } i \in \{0, \dots, e-1\}.$$

   (b) The matrix $M(D, \vec{v})$ equals

   $$\begin{pmatrix} A_0 & CA_{e-1}/x & \dots & \dots & CA_1/x \\ A_1 & A_0 - \frac{1}{e}I & \ddots & & \vdots \\ \vdots & A_1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & CA_{e-1}/x \\ A_{e-1} & A_{e-2} & \dots & A_1 & A_0 - \frac{e-1}{e}I \end{pmatrix}.$$

   (c) The characteristic polynomial h of the matrix

   $$M(D, \vec{v}) + \operatorname{diag}(0, \frac{1}{e}I, \dots, \frac{e-1}{e}I)$$

   is irreducible.

2. *Conversely, let $e, f \in \mathbb{N}^*$ satisfy $\dim V = ef$ and let $\vec{v}$ be a K-basis of V, $C \in \mathrm{Gl}_f(k)$ and $f \times f$ matrices $A_0, A_1, \dots, A_{e-1}$ with entries in $k[1/x]$ such that (a), (b) and (c) hold, then $D: V \to V$ is a simple differential operator with characteristic class $c(D) = [h]$.*

3. *If $D: V \to V$ is a simple differential operator and $g \in \mathrm{Gl}_{ef}(K)$ a transformation matrix such that the matrix $g[M(D, \vec{v})]$ is of the form*

   $$\begin{pmatrix} A'_0 & C'A'_{e-1}/x & \dots & \dots & C'A'_1/x \\ A'_1 & A'_0 - \frac{1}{e}I & \ddots & & \vdots \\ \vdots & A'_1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & C'A'_{e-1}/x \\ A'_{e-1} & A'_{e-2} & \dots & A'_1 & A'_0 - \frac{e-1}{e}I \end{pmatrix}$$

   with $C' \in \mathrm{Gl}_f(k)$ and $A'_0, A'_1, \dots, A'_{e-1}$ $f \times f$ matrices with entries in $k[1/x]$. Then the following conditions are equivalent:

   - *C', A'_0, A'_1, $\dots$, A'_{e-1} satisfy condition (a).*
   - *There exists an integer $q \in \mathbb{Z}$ and matrices $B, U \in \mathrm{Gl}_f(k)$ such that B is semisimple and commutes with C and with all A_i and*

     $$g = \operatorname{diag}(U, UB, \dots, UB^{e-1}) x^{q/e}.$$

   If $C' = C$ and $A'_i = A_i$ for all $i \in \{0, \dots, e-1\}$ then $g = \operatorname{diag}(T, T, \dots, T)$ for some semisimple $T \in \mathrm{Gl}_f(k)$ which commutes with C and all A_i.

The proof of this theorem, which can be found in [8, section 4.1], is based on Galois descend starting with a normalized eigenvalue. The details of the proof of part 2 and 3 are cumbersome.

Theorem 3.4 generalizes a result found in [2] in which a rational normal form of a simple differential operator over a algebraically closed field of constants appears.

A representation of a differential operator as given in theorem 3.1 with the simple components in the form given by theorem 3.4 is called a *rational normal form*. One can show that a rational normal form is super-irreducible, a notion introduced by Hilali (see [4] and [6]).

4 Some algorithms

The algorithm we will describe in this section leads to a matrix of the form

$$A_1x^{r_1} + \cdots + A_mx^{r_m} + C, \tag{2}$$

where $r_1, \ldots, r_m \in \mathbb{Q}$ with $r_1 < \cdots < r_m < 0$ and A_1, A_2, $\ldots$, A_m, C are square matrices with entries in k satisfying:

- $A_1, A_2, \ldots, A_m, C$ commute pairwise;
- $A_1, A_2, \ldots, A_m$ are semisimple and non-zero.

The characteristic polynomial of (2) is the characteristic class of the differential operator in some finite field extension of K.

Moreover if the so-called nilpotent case described in section 4.6 does not occur during execution of the algorithm then one can obtain the characteristic class and a rational normal form of the differential operator.

In [8] more details and additional complete algorithms to compute the characteristic class and a rational normal form for the nilpotent case of differential operators in two and three dimensional vector spaces are described.

Throughout this section we will assume that A is an $n \times n$ matrix with entries in K and

$$A = A_rx^r + A_{r+1}x^{r+1} + A_{r+2}x^{r+2} + \cdots$$

with all A_i in $\mathrm{Mat}_n(k)$ and $r \leq 0$ and $A_r \neq 0$ (unless $r = 0$). Let D be the differential operator $\tau + A$ on K^n.

By first applying a constant transformation matrix one may assume that the matrix A_r is in the rational Jordan normal form.

We will now distinguish six different cases.

4.1 $r = 0$

If $r = 0$ then the differential operator D has a *regular singularity* in the origin. This type of differential operator has been studied at length in the literature, see [9, Ch. II and Ch. V, sec. 17]. In this case one can give an algorithm to compute a rational normal form of the differential operator D and also the transformation matrix up to a given order.

Step 1. Construct a transformation matrix $g \in \mathrm{Gl}_n(k[[x]])$ such that

$$g[A] = A'_0 + A'_1x + A'_2x^2 + \cdots$$

with all A'_i in $\mathrm{Mat}_n(k)$ and $\gcd\big(f(\lambda), f(\lambda-1)\big) = 1$ where $f(\lambda) := \det(\lambda I - A'_0)$.

Step 2. Construct (up to a given order) a transformation matrix $\widetilde{g} \in \mathrm{Gl}_n(k[[x]])$ such that $(\widetilde{g}g)[A] = A'_0$.

The construction of $\widetilde{g}$ consists of repeatedly solving X from $(A_0 + iI)X - XA_0 = M$ for given $M \in \mathrm{Mat}_n(k)$ and $i \in \mathbb{N}^*$.

Characteristic class.

$$c(D)(\lambda) = [\det(\lambda I - A'_0)] = [\det(\lambda I - A_0)].$$

Rational normal form. Is given by a rational Jordan normal form of the matrix A'_0.

4.2 Splitting lemma

Assume that $r < 0$ and the matrix A_r is of the form

$$\begin{pmatrix} P_r & 0 \\ 0 & Q_r \end{pmatrix},$$

where P_r and Q_r are square matrices whose characteristic polynomials are relatively prime.

Step 1. Construct (up to a given order) a transformation matrix $g \in \mathrm{Gl}_n(k[[x]])$ such that

$$g[A] = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where P and Q are square matrices with entries in K with $P = P_rx^r + \cdots$ and $Q = Q_rx^r + \cdots$. This construction is known under the name *splitting lemma.* It requires repeatedly solving X from $P_rX - XQ_r = M$ for given constant matrix M.

Step 2. Recursively apply the algorithm to the differential operators $\tau + P$ and $\tau + Q$.

Characteristic class. $c(D) = c(\tau + P)c(\tau + Q)$.

Rational normal form. $\begin{pmatrix} \widetilde{P} & 0 \\ 0 & \widetilde{Q} \end{pmatrix}$ where $\widetilde{P}$ and $\widetilde{Q}$ are rational normal forms of the differential operators $\tau + P$ resp. $\tau + Q$.

4.3 Squarefree case

Assume that $r < 0$ and $\det(\lambda I - A_r)$ is squarefree.

Step 1. Construct (up to a given order) a transformation matrix $g \in \mathrm{Gl}_n(k[[x]])$ such that

$$g[A] = A'_rx^r + A'_{r+1}x^{r+1} + A'_{r+2}x^{r+2} + \cdots$$

with $A'_i \in \mathrm{Mat}_n(k)$, $A'_r = A_r$ and $A'_rA'_i = A'_iA'_r$ for all $i \geq r$. The construction of this g is based on the following fact. There exists an algorithm that for a given $B \in \mathrm{Mat}_n(k)$ constructs $b_0, b_1, \ldots, b_{n-1} \in k$ and $Y \in \mathrm{Mat}_n(k)$ such that

$$B = b_0I + b_1A_r + \cdots + b_{n-1}A_r^{n-1} + A_rY - YA_r.$$

Step 2. Construct (up to a given order) a transformation matrix $\widetilde{g} \in \mathrm{Gl}_n(k[[x]])$ such that $(\widetilde{g}g)[A] = A'_rx^r + A'_{r+1}x^{r+1} + \cdots + A'_0$.

Characteristic class.

$$c(D)(\lambda) = \left[\det\left(\lambda I - \sum_{i \leq 0} A'_ix^i\right)\right].$$

Rational normal form. If the matrix A_r is in rational Jordan normal form, then a rational normal form of D is given by $A'_rx^r + A'_{r+1}x^{r+1} + \cdots + A'_0$.

4.4 One eigenvalue

Assume $r < 0$ and $\det(\lambda I - A_r) = (\lambda - a)^n$ for some $a \in k^*$. Recursively apply the algorithm to the differential operator $D - ax^rI$.

Characteristic class. $c(D)(\lambda) = c(D - ax^rI)(\lambda - a)$.

Rational normal form. If $\widetilde{A}$ is a rational normal form of $D - ax^rI$, then $\widetilde{A} + ax^rI$ is a rational normal form of D.

4.5 Field extension case

Assume that $r < 0$ and $\det(\lambda I - A_r) = f(\lambda)^s$ where $f(\lambda)$ is an irreducible polynomial of degree $m > 1$ in $k[\lambda]$ and $s > 1$. Then (the rational normal form of) the matrix A_r is of the form

$$\begin{pmatrix} P & * & 0 & \dots & 0 \\ 0 & P & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & P & * \\ 0 & \dots & \dots & 0 & P \end{pmatrix},$$

where P is a square matrix such that $\det(\lambda I - P) = f(\lambda)$ and every asterisk stands for either the identity matrix or zero.

Step 1. Construct (up to a given order) a transformation matrix $g \in \mathrm{Gl}_n(k[[x]])$ such that

$$g[A] = \begin{pmatrix} f_{11}(P) & \dots & f_{1s}(P) \\ \vdots & \ddots & \vdots \\ f_{s1}(P) & \dots & f_{ss}(P) \end{pmatrix},$$

where all f_{ij} are univariate polynomials of degree less than m with coefficients in K. This can be done by a generalization of the algorithm needed in the squarefree case of the algorithm.

Step 2. Let $k' = k(\alpha)$ be a finite field extension of k where α is an eigenvalue of the matrix P. Define the matrix $A' \in \mathrm{Mat}_s(k'K)$ by

$$A' := \begin{pmatrix} f_{11}(\alpha) & \dots & f_{1s}(\alpha) \\ \vdots & \ddots & \vdots \\ f_{s1}(\alpha) & \dots & f_{ss}(\alpha) \end{pmatrix}$$

and recursively apply the algorithm on the differential operator $D' := \tau + A'$.

Characteristic class.

$$\begin{aligned} c(D)(\lambda) &= N_{k'K(\lambda)/K(\lambda)}\big(c(D')(\lambda)\big) \\ &= \mathrm{Res}_y\big(f(y),\, c(D')(\lambda)\big|_{\alpha=y}\big), \end{aligned}$$

where $N_{k'K(\lambda)/K(\lambda)}$ is the norm function of the field extension $k'K(\lambda)/K(\lambda)$ and Res_y denotes the resultant w.r.t. the variable y.

Rational normal form. Replacing every α by P in the transformation matrix and in the resulting matrix gives a rational normal form of the differential operator D.

4.6 Nilpotent case

If $r < 0$ and A_r is nilpotent, then there are (at least) two ways to proceed. For both methods ramification might be necessary, i.e., replacing x by t^m for some $m \in \mathbb{N}$ with $m > 1$.

The first method consists of computing a super-irreducible form. If for the resulting matrix A one still has $r < 0$ and A_r is nilpotent, then compute the principal level (or equivalently the invariant of Katz) by one of the methods described in [3] or [5]. Now replace x by t^m where $m \in \mathbb{N}^*$ is the smallest integer such that the principal level is an element of $(1/m)\mathbb{Z}$. Using the algorithm of Hilali over the field $K(t)$ to make the matrix 1-irreducible yields a matrix A with r equal to the principal level and A_r not nilpotent.

Another method described in [1, § 4] uses the theory of orbits of the adjoint representation of the algebraic group $\mathrm{Gl}_n(k)$. We will describe this method in some more detail.

Let $g\ell_n(k)$ denote the general linear algebra $\mathrm{Mat}_n(k)$ with bracket operation $[x, y] = xy - yx$ for all $x, y \in g\ell_n(k)$ and let $s\ell_n(k)$ denote the special linear algebra: the subalgebra of $g\ell_n(k)$ consisting of the matrices having trace zero.

Step 1. Compute matrices H and X in $s\ell_n(k)$ such that

$$[H, X] = 2X, \quad [H, A_r] = -2A_r, \quad [X, A_r] = H.$$

In other words $\{X, A_r, H\}$ span a Lie subalgebra of $g\ell_n(k)$ which is isomorphic to $s\ell_2(k)$. The triplet (A_r, H, X) is called a *standard triplet.*

It follows from the representation theory of $s\ell_2(k)$ that

$$\begin{aligned} g\ell_n(k) = \{\, [A_r, Y] \mid Y \in g\ell_n(k) \,\} \oplus \\ \oplus \{\, Y \in g\ell_n(k) \mid [X, Y] = 0 \,\}. \end{aligned} \tag{3}$$

Step 2. Construct a transformation matrix $g \in \mathrm{Gl}_n(k[[x]])$ such that

$$g[A] = A_r x^r + A'_{r+1} x^{r+1} + \dots + A'_{r-rn-1} x^{r-rn-1} + \dots$$

with $A'_i X = X A'_i$ for all $i \in \{r+1, \dots, r-rn-1\}$. Construction of g uses the direct sum splitting (3).

Step 3. Apply a transformation matrix $\widetilde{g}$ of the form

$$x^{-\frac{\delta}{2}H}$$

for some $\delta \in \mathbb{Q}$ with $0 < \delta \le |r|$ such that

$$(\widetilde{g}g)[A] = \widetilde{A}_{r+\delta} x^{r+\delta} + \cdots$$

with $0 \ne \widetilde{A}_{r+\delta} \ne A_r$ or $r + \delta = 0$.

Step 4. Now recursively apply the algorithm to $(\widetilde{g}g)[A]$. One can prove that if $r + \delta < 0$ and $\widetilde{A}_{r+\delta}$ is nilpotent, then $d(\widetilde{A}_{r+\delta}) > d(A_r)$, where $d(\cdot)$ denotes the dimension of the orbit, i.e., for any M in $g\ell_n(k)$, $d(M)$ is the dimension of

$$\{\, gMg^{-1} \mid g \in \mathrm{Gl}(k) \,\}.$$

This guarantees termination in this case.

References

[1] Donald G. Babbitt and V. S. Varadarajan, *Formal reduction of meromorphic differential equations: a group theoretic view*, Pacific Journal of Mathematics **109** (1983), no. 1, 1–80.

[2] Arno van den Essen and A. H. M. Levelt, *An explicit description of all simple $k[[x]][\partial]$-modules*, Abelian Groups and Noncommutative Rings (Providence, Rhode Island) (L. Fuchs et al., ed.), Contemporary Mathematics, vol. 130, American Mathematical Society, Providence, Rhode Island, 1992, pp. 121–131.

[3] A. Hilali, *On the algebraic and differential Newton-Puiseux polygons*, Journal of Symbolic Computation **4** (1987), no. 3, 335–349.

[4] ———, *Solutions formelles de systèmes différentielles linéaires au voisinage d'un singulier*, Ph.D. thesis, l'Université Scientifique, Technologique et Médicale de Grenoble, 1987.

[5] A. Hilali and A. Wazner, *Un algorithme de calcul de l'invariant de Katz d'un système différentiel linéaire*, Annales de l'Institut Fourier **36** (1986), no. 3, 67–81.

[6] ———, *Formes super-irréductibles des systèmes différentiels linéaires*, Numerische Mathematik **50** (1987), no. 4, 429–449.

[7] A. H. M. Levelt, *Jordan decomposition for a class of singular differential operators*, Arkiv för matematik **13** (1975), no. 1, 1–27.

[8] Ron Sommeling, *Characteristic classes for irregular singularities*, Ph.D. thesis, University of Nijmegen, June 1993.

[9] Wolfgang Wasow, *Asymptotic expansions for ordinary differential equations*, Pure and Applied Mathematics (Interscience), vol. 14, Interscience Publishers, New York, London, Sydney, 1965.