

# Data Cleaning and Query Answering with Matching Dependencies and Matching Functions\*

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## ABSTRACT

Matching dependencies were recently introduced as declarative rules for data cleaning and entity resolution. Enforcing a matching dependency on a database instance identifies the values of some attributes for two tuples, provided that the values of some other attributes are sufficiently similar. Assuming the existence of matching functions for making two attributes values equal, we formally introduce the process of cleaning an instance using matching dependencies, as a chase-like procedure. We show that matching functions naturally introduce a lattice structure on attribute domains, and a partial order of semantic domination between instances. Using the latter, we define the semantics of clean query answering in terms of certain/possible answers as the greatest lower bound/least upper bound of all possible answers obtained from the clean instances. We show that clean query answering is intractable in some cases. Then we study queries that behave monotonically wrt semantic domination order, and show that we can provide an under/over approximation for clean answers to monotone queries. Moreover, non-monotone positive queries can be relaxed into monotone queries.

## 1. INTRODUCTION

Matching dependencies (MDs) in relational databases were recently introduced in [16] as a means of codifying a domain expert's knowledge that is used in improving data quality. They specify that a pair of attribute values in two database tuples are to be matched, i.e., made equal, if similarities hold between other pairs of values in the same tuples. This is a generalization of entity resolution [15], where basically full tuples have to be merged or identified since they seem to refer to the same entity of the outside reality. This form of data fusion [10] is important in data quality assessment

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and in data cleaning.

Matching dependencies were formally studied in [17], as semantic constraints for data cleaning and were given a model-theoretic semantics. The main emphasis in that paper was on the problem of entailment of MDs and on the existence of a formal axiom system for that task.

MDs as presented in [17] do not specify *how* the matching of attribute values is to be done. In data cleaning, the user, on the basis of his or her experience and knowledge of the application domain, may have a particular methodology or heuristics for enforcing the identifications. *In this paper we investigate MDs in the context of matching functions.* These are functions that abstract the implementation of value identification. Rather than investigate specific matching functions, we explore a class of matching functions satisfying certain natural and intuitive axioms. With these axioms, matching functions impose a lattice-theoretic structure on attribute domains. Intuitively, given two input attribute values that need to be made equal, the matching function produces a value that *contains* the information present in the two inputs and *semantically dominates* them. We show this semantic domination partial order can be naturally lifted to tuples of values as well as database instances as sets of tuples.

EXAMPLE 1. Consider the following database instance  $D_0$ . Assume there is a matching dependency stating that if for two tuples the values of name and phone are similar, then the value of address should be made identical. Consider a similarity relation that indicates the values of name and phone are similar for the two tuples in this instance. To enforce the matching dependency, we create another instance  $D_1$  in which the value of address for two tuples is the result of applying a matching function on the two previous addresses. This function combines the information in those address values.

$D_0$	name	phone	address
	John Doe	(613)123 4567	Main St., Ottawa
	J. Doe	123 4567	25 Main St.
↓			
$D_1$	name	phone	address
	John Doe	(613)123 4567	25 Main St., Ottawa
	J. Doe	123 4567	25 Main St., Ottawa

We can continue this process in a chase-like manner if there are still other MD violations in  $D_1$ . ■

The framework of [17] leaves the implementation details of data cleaning process with MDs completely unspecified and

implicitly leaves it to the application on hand. We point out some limitations of the proposal in [17] for purposes of cleaning dirty instances in the presence of multiple MDs, and show that a formulation of the formal semantics of the satisfaction and enforcement of MDs, incorporating matching functions, remedies this problem. In giving such a formulation, we revisit the original semantics for MDs proposed in [17], propose some changes and investigate their consequences. More precisely, we define *intended clean instances*, those that are obtained through the application of the MDs in a chase-like procedure. We further investigate properties of this procedure in relation to the properties of the matching functions, and show that, in general, the chase procedure produces several different clean instances, each of which semantically dominates the original dirty instance.

We then address the problem of query answering over a dirty instance, where the MDs do *not* hold. We take advantage of the semantic domination order between instances, and define *clean answers* by specifying a tight lower bound (corresponding to certain answers) and a tight upper bound (corresponding to possible answers) for all answers that can be obtained from any of the possibly many clean instances. We show that computing the exact bounds is intractable in general. However, in polynomial time we can generate an under-approximation for certain answers as well as an over-approximation for possible answers for queries that behave *monotonically* w.r.t. the semantic domination order.

We argue that monotone queries provide more informative answers on instances that have been cleaned with MDs and matching functions. We therefore introduce new relational algebra operators that make use of the underlying lattice structure on the domain of attribute values. These operators can be used to *relax* a regular positive relational algebra query and make it monotone w.r.t. the semantic domination order.

Recently, Swoosh [8] has been proposed as generic framework for entity resolution. In entity resolution, whole tuples are identified, or merged into a new tuple, whenever similarities hold between the tuples on some attributes. Accordingly, the similarity and matching functions work at the tuple level. Given their similarity of purpose, it is interesting to ask what is the relationship between the frameworks of MDs and of Swoosh. We address this question in this paper.

In summary, we make the following contributions:

- We identify the limitations of the original proposal of MDs [17] wrt the application of data cleaning in the presence of multiple MDs and show that the limitations can be overcome by considering MDs along with matching functions.
- We study matching functions in terms of their properties, which are certain intuitive and natural axioms. Matching functions induce a lattice framework on attribute domains which can be lifted to a partial order over instances, that we call semantic domination.
- We formally characterize answering a query given a dirty instance and a set of MDs, and capture it using certain and possible answers. Computing these answers is intractable in general. For queries that are monotone wrt the semantic domination relation, we develop a polynomial time heuristic procedure for ob-

taining under- and over-approximations of query answers.

- We demonstrate the power of the framework of MDs and of our chase procedure for MD application by reconstructing the most common case for Swoosh, the so-called *union and merge* case, in terms of matching dependencies with matching functions.

The paper is organized as follows. In Section 2, we provide necessary background on matching dependencies as originally introduced. We introduce matching functions and the notion of semantic domination in Section 3. Then we define the data cleaning process with MDs in Section 4. We explore the semantic of query answering in Section 5. In Section 6, we study monotone queries and show how clean answers can be approximated. We establish a connections to an important related work, Swoosh, in Section 7, and present concluding remarks in Section 8.

## 2. BACKGROUND

A database schema  $\mathcal{R}$  is a set  $\{R_1, \dots, R_n\}$  of relation names. Every relation  $R_i$  is associated with a set of attributes, written as  $R_i(A_1, \dots, A_m)$ , where each attribute  $A_j$  has a domain  $Dom_{A_j}$ . We assume that attribute names are different across relations in the schema, but two attributes  $A_j, A_k$  can be *comparable*, i.e.,  $Dom_{A_j} = Dom_{A_k}$ . An instance  $D$  of schema  $\mathcal{R}$  assigns a finite set of tuples  $t^D$  to every relation  $R_i$ , where  $t^D$  can be seen as a function that maps every attribute  $A_j$  in  $R_i$  to a value in  $Dom_{A_j}$ . We write  $t^D[A_j]$  to refer to this value. When  $X$  is a list of attributes, we may write  $t^D[X]$  to refer to the corresponding list of attribute values. A tuple  $t^D$  for a relation name  $R \in \mathcal{R}$  is called an  $R$ -tuple. We deal with queries  $Q$  that are expressed in relational algebra, and treat them as operators that map an instance  $D$  to an instance  $Q(D)$ .

For every attribute  $A$  in the schema, we assume a binary similarity relation  $\approx_A \subseteq Dom_A \times Dom_A$ . Notice that whenever  $A$  and  $A'$  are comparable, the similarity relations  $\approx_A$  and  $\approx_{A'}$  are identical. We assume that each  $\approx_A$  is symmetric and subsumes equality, i.e.,  $=_{Dom_A} \subseteq \approx_A$ . When there is no confusion, we simply use  $\approx$  for the similarity relation. In particular, for lists of pairwise comparable attributes,  $X_i = A_1^i, \dots, A_n^i$ ,  $i = 1, 2$ , we write  $X_1 \approx X_2$  to mean  $A_1^1 \approx_1 A_1^2 \wedge \dots \wedge A_n^1 \approx_n A_n^2$ , where  $\approx_i$  is the similarity relation applicable to attributes  $A_i^1, A_i^2$ .

Given two pairs of pairwise comparable attribute lists  $X_1, X_2$  and  $Y_1, Y_2$  from relations  $R_1, R_2$ , resp., a *matching dependency* (MD) [17] is a sentence of the form

$$\varphi: R_1[X_1] \approx R_2[X_2] \rightarrow R_1[Y_1] = R_2[Y_2].^1 \quad (1)$$

This dependency intuitively states that if for an  $R_1$ -tuple  $t_1$  and an  $R_2$ -tuple  $t_2$  in instance  $D$ , the attribute values in  $t_1^D[X_1]$  are similar to attribute values in  $t_2^D[X_2]$ , then we need to make the values  $t_1^D[Y_1]$  and  $t_2^D[Y_2]$  pairwise identical.

Enforcing MDs may cause a database instance  $D$  to be changed to another instance  $D'$ . To keep track of every single change, we assume that every tuple in an instance has a unique identifier  $t$ , which could identify it in both instance  $D$  and its changed version  $D'$ . We use  $t^D$  and  $t^{D'}$  to refer to

<sup>1</sup>All the variables in  $X_i, Y_j$  are implicitly universally quantified in front of the formula.

a tuple and its changed version in  $D'$  that has resulted from enforcing an MD. For convenience, we may use the terms tuple and tuple identifier interchangeably.

Fan et al. [17] give a *dynamic semantics* for matching dependencies in terms of a pair of instances: one where the similarities hold, and a second where the specified identifications have been enforced:

**DEFINITION 1.** [17] A pair of instances  $(D, D')$  satisfies the MD  $\varphi : R_1[X_1] \approx R_2[X_2] \rightarrow R_1[Y_1] \rightleftharpoons R_2[Y_2]$ , denoted  $(D, D') \models \varphi$ , if for every  $R_1$ -tuple  $t_1$  and  $R_2$ -tuple  $t_2$  in  $D$  that match the left-hand side of  $\varphi$ , i.e.,  $t_1^P[X_1] \approx t_2^P[X_2]$ , the following holds in the instance  $D'$ :

- (a)  $t_1^{D'}[Y_1] = t_2^{D'}[Y_2]$ , i.e., the values of the right-hand side attributes of  $\varphi$  have been identified in  $D'$ ; and
- (b)  $t_1, t_2$  in  $D'$  match the left-hand side of  $\varphi$ , that is,  $t_1^{D'}[X_1] \approx t_2^{D'}[X_2]$ .

For a set  $\Sigma$  of MDs,  $(D, D') \models \Sigma$  iff  $(D, D') \models \varphi$  for every  $\varphi \in \Sigma$ . An instance  $D'$  is called *stable* if  $(D', D') \models \Sigma$ . ■

Notice that a stable instance satisfies the MDs by itself, in the sense that all the required identifications are already enforced in it. So, whenever we say that an instance is *dirty*, we mean that it is not stable w.r.t. the given set of MDs.

While this definition may be sufficient for the implication problem of MDs considered by Fan et al. [17], it does not specify how a dirty database should be updated to obtain a clean instance, especially when several *interacting updates* are required in order to enforce all the MDs. Thus, it does not give a complete prescription for the purpose of cleaning dirty instances. Moreover, from a different perspective, the requirements in the definition may be too strong, as the following example shows.

**EXAMPLE 2.** Consider the set of MDs  $\Sigma$  consisting of  $\varphi_1: R[A] \approx R[A] \rightarrow R[B] \rightleftharpoons R[B]$  and  $\varphi_2: R[B, C] \approx R[B, C] \rightarrow R[D] \rightleftharpoons R[D]$ . The similarities are:  $a_1 \approx a_2$ ,  $b_2 \approx b_3$ ,  $c_2 \approx c_3$ . Instance  $D_0$  below is not a stable instance, i.e., it does not satisfy  $\varphi_1, \varphi_2$ . We start by enforcing  $\varphi_1$  on  $D_0$ .

$D_0$	$A$	$B$	$C$	$D$	$D_1$	$A$	$B$	$C$	$D$
	$a_1$	$b_1$	$c_1$	$d_1$		$a_1$	$\langle b_1, b_2 \rangle$	$c_1$	$d_1$
	$a_2$	$b_2$	$c_2$	$d_2$		$a_2$	$\langle b_1, b_2 \rangle$	$c_2$	$d_2$
	$a_3$	$b_3$	$c_3$	$d_3$		$a_3$	$b_3$	$c_3$	$d_3$

Let  $\langle b_1, b_2 \rangle$  in instance  $D_1$  denote the value that replaces  $b_1$  and  $b_2$  to enforce  $\varphi_1$  on instance  $D_0$ , and assume that  $\langle b_1, b_2 \rangle \not\approx b_3$ . Now,  $(D_0, D_1) \models \varphi_1$ . However,  $(D_0, D_1) \not\models \varphi_2$ .

If we identify  $d_2, d_3$  via  $\langle d_2, d_3 \rangle$  producing instance  $D_2$ , the pair  $(D_0, D_2)$  satisfies the condition (a) in Definition 1 for  $\varphi_2$ , but not condition (b). Notice that making more updates on  $D_1$  (or  $D_2$ ) to obtain an instance  $D'$ , such that  $(D_0, D') \models \Sigma$ , seems hopeless as  $\varphi_2$  will not be satisfied because of the broken similarity that existed between  $b_2$  and  $b_3$ . ■

Definition 1 seems to capture well the one-step enforcement of a single MD. However, as shown by the above example, the definition has to be refined in order to deal with sets of interacting MDs and to capture an iterative process of MD enforcement. We address this problem in Section 4.

Another issue worth mentioning is that stable instances  $D'$  for  $D$  and  $\Sigma$  are not subject to any form of minimality criterion on  $D'$  in relation with  $D$ . We would expect such an instance to be obtained via the enforcement of the MDs, without unnecessary changes. Unfortunately, this is not the case here: If in Example 2 we keep only  $\varphi_1$ , and in instance  $D_1$  we change  $a_3$  by an arbitrary value  $a_4$  that is not similar to either  $a_1$  or  $a_2$ , we obtain a stable instance with origin in  $D_0$ , but the change of  $a_3$  is unjustified and unnecessary. We will also address this issue.

Following [17], we assume in the rest of this paper that each MD is of the form  $R_1[X_1] \approx R_2[X_2] \rightarrow R_1[A_1] \rightleftharpoons R_2[A_2]$ . That is, the right-hand side of the MDs contains a pair of single attributes. We also assume that the sets  $\Sigma$  of MDs we consider are always finite.

### 3. MATCHING FUNCTIONS AND SEMANTIC DOMINATION

In order to enforce a set of MDs (cf. Section 4) we need an operation that identifies two values whenever necessary. With this purpose in mind, we will assume that for each comparable pair of attributes  $A_1, A_2$  with domain  $Dom_A$ , there is a binary *matching function*  $m_A : Dom_A \times Dom_A \rightarrow Dom_A$ , such that the value  $m_A(a, a')$  is used to replace two values  $a, a' \in Dom_A$  whenever the two values need to be made equal. Here are a few natural properties to expect of the matching function  $m_A$  (similar properties were considered in [8], cf. Section 7): For  $a, a', a'' \in Dom_A$ ,

- I** (Idempotency):  $m_A(a, a) = a$ ,
- C** (Commutativity):  $m_A(a, a') = m_A(a', a)$ ,
- A** (Associativity):  $m_A(a, m_A(a', a'')) = m_A(m_A(a, a'), a'')$ .

It is reasonable to assume that any matching function satisfies at least these three axioms. Under this assumption, the structure  $(Dom_A, m_A)$  forms a *join semilattice*,  $L_A$ , that is, a partial order with a least upper bound (*lub*) for every pair of elements. The induced partial order  $\preceq_A$  on the elements of  $Dom_A$  is defined as follows: For every  $a, a' \in Dom_A$ ,  $a \preceq_A a'$  whenever  $m_A(a, a') = a'$ . The *lub* operator with respect to this partial order coincides with  $m_A$ :  $lub_{\preceq_A} \{a, a'\} = m_A(a, a')$ .

A natural interpretation for the partial order  $\preceq_A$  in the context of data cleaning would be the notion of *semantic domination*. Basically, for two elements  $a, a' \in Dom_A$ , we say that  $a'$  *semantically dominates*  $a$  if we have  $a \preceq_A a'$ . In the process of cleaning the data by enforcing matching dependencies, we always replace two values  $a, a'$ , whenever certain similarities hold, by the value  $m_A(a, a')$  that semantically dominates both  $a$  and  $a'$ . This notion of domination is also related to relative information contents [11, 21, 22].

To define the semantics of query answering on instances that have been cleaned with matching dependencies, we might, in addition, need the existence of the greatest lower bound (*glb*) for any two elements in the domain of an attribute. We therefore assume that  $(Dom_A, m_A)$  is a lattice (i.e., both *lub* and *glb* exist for every pair of elements in  $Dom_A$  w.r.t.  $\preceq_A$ ). Moreover, there is a special element  $\perp \in Dom_A$  such that  $m_A(a, \perp) = a$ , for every  $a \in Dom_A$ . Notice that if we add an additional assumption to the semilattice, which requires that for every element  $a \in Dom_A$ , the set  $\{c \in Dom_A \mid c \preceq_A a\}$  (the set of elements  $c$  with  $m_A(a, c) = a$ ), is finite, then  $glb_{\preceq_A} \{a, a'\}$  does exist for ev-

ery two elements  $a, a' \in \text{Dom}_A$  and is equal to  $\text{lub}_{\preceq_A}\{c \in \text{Dom}_A \mid c \preceq_A a \text{ and } c \preceq_A a'\}$ . We could also assume the existence of another special element  $\top \in \text{Dom}_A$  such that  $m_A(a, \top) = \top$ , for every  $a \in \text{Dom}_A$ . This element could represent the existence of inconsistency in data whenever matching dependencies force to match two completely unrelated elements  $a, a'$  from the domain, in which case  $m_A(a, a') = \top$ . However, *the existence of  $\top$  is not essential in our framework*.

**EXAMPLE 3.** We give a few concrete examples of matching functions for different attribute domains. Our example functions have all the properties **I**, **C**, and **A**.

**Name, Address, Phone** Each atomic value  $s$  of these string domains could be treated as a singleton set  $\{s\}$ . Then a matching function  $m(S_1, S_2)$  for sets of strings  $S_i$  could return  $S_1 \cup S_2$ . E.g., when matching addresses,  $m(\{\text{'2366 Main Mall'}\}, \{\text{'Main Mall, Vancouver'}\})$  could return  $\{\text{'2366 Main Mall'}, \text{'Main Mall, Vancouver'}\}$ . (This union matching function is further investigated in Section 7.) Alternatively, a more sophisticated matching function could merge two input strings into a third string that contains both of the inputs. E.g., the match of the two input strings above could instead be ‘2366 Main Mall, Vancouver’.

**Age, Salary, Price** Each atomic value  $a$  in these numerical domains could be treated as an interval  $[a, a]$ . Then the matching function  $m([a_1, b_1], [a_2, b_2])$  would return the smallest interval containing both  $[a_1, b_1]$  and  $[a_2, b_2]$ , i.e.,  $m([a_1, b_1], [a_2, b_2]) = [\min\{a_1, a_2\}, \max\{b_1, b_2\}]$ .

**Boolean Attributes** For attributes which take either a 0 or 1 value, the matching function would return  $m(0, 1) = \top$ , where  $\top$  shows inconsistency in the data, and furthermore  $m(0, \top) = \top$  and  $m(1, \top) = \top$ . In this case, the purpose of applying the matching function is to *record* the inconsistency in the data and still conduct sound reasoning in presence of inconsistency.<sup>2</sup> ■

An additional property of matching functions worthy of consideration is *similarity preservation*, that is, the result of applying a matching function preserves the similarity that existed between the old value being replaced and other values in the domain. More formally:

**S (Similarity Preservation):** If  $a \approx a'$ , then  $a \approx m_A(a', a'')$ , for every  $a, a', a'' \in \text{Dom}_A$ .

Unlike the previous properties (**I**, **C**, **A**), property **S** turns out to be a strong assumption, and we must consider both matching functions with **S** and without it. Indeed, notice that since  $\approx$  subsumes equality, and  $m_A$  is commutative, assumption **S** implies  $a \approx m_A(a, a')$  and  $a' \approx m_A(a, a')$ , i.e., similarity preserving matching always results in a value similar to the value being replaced. In the rest of the paper, we assume that for every comparable pair of attributes  $A_1, A_2$ , there is an idempotent, commutative, and associative binary matching function  $m_A$ . Unless otherwise specified, we do not assume that these functions are similarity preserving.

<sup>2</sup>Matching of boolean attributes requires the existence of the top element  $\top$ .

**DEFINITION 2.** Let  $D_1, D_2$  be instances of schema  $\mathcal{R}$ , and  $t_1, t_2$  be two  $R$ -tuples in  $D_1, D_2$ , respectively, with  $R \in \mathcal{R}$ . We write  $t_1^{D_1} \preceq t_2^{D_2}$  if  $t_1^{D_1}[A] \preceq_A t_2^{D_2}[A]$  for every attribute  $A$  in  $R$ . We write  $D_1 \sqsubseteq D_2$  if for every tuple  $t_1$  in  $D_1$ , there is a tuple  $t_2$  in  $D_2$ , such that  $t_1^{D_1} \preceq t_2^{D_2}$ . ■

Clearly, the relation  $\preceq$  on tuples can be applied to tuples in the same instance. The ordering  $\sqsubseteq$  on sets has been used in the context of complex objects [6, 24] and also powerdomains, where it is called *Hoare ordering* [11]. It is also used in [8] for entity resolution (cf. Section 7). It is known that for  $\sqsubseteq$  to be a partial order, specifically to be antisymmetric, we need to deal with *reduced* instances [6], i.e., instances  $D$  in which there are no two different tuples  $t_1, t_2$ , such that  $t_1^D \preceq t_2^D$ . We can obtain the *reduced version* of every instance  $D$  by removing every tuple  $t_1$ , such that  $t_1^D \prec t_2^D$  for some tuple  $t_2$  in  $D$ .

Next we will show that the set of reduced instances with the partial order  $\sqsubseteq$  forms a lattice: the least upper bound and the greatest lower bound for every finite set of reduced instances exist. This result will be used later for query answering. We adapt some of the results from [6], where they prove a similar result for a lattice formed by the set of complex objects and the sub-object partial order.

**DEFINITION 3.** Let  $D_1, D_2$  be instances of schema  $\mathcal{R}$ , and  $t_1, t_2$  be two  $R$ -tuples in  $D_1, D_2$ , respectively, for  $R \in \mathcal{R}$ .

- We define  $D_1 \vee D_2$  to be the reduced version of  $D_1 \cup D_2$ , where  $D_1 \cup D_2$  refers to the instance that takes the union of  $R$ -tuples from  $D_1$  and  $D_2$  for every  $R \in \mathcal{R}$ .
- We define  $t_1 \wedge t_2$  to be tuple  $t$ , such that  $t[A] = \text{glb}_{\preceq_A}\{t_1^{D_1}[A], t_2^{D_2}[A]\}$  for every attribute  $A$  in  $R$ .
- We define  $D_1 \wedge D_2$  to be the reduced version of the instance that assigns the set of tuples  $\{t_1 \wedge t_2 \mid t_1 \in D_1, t_2 \in D_2, t_1, t_2 \text{ } R\text{-tuples}\}$  to every  $R \in \mathcal{R}$ . ■

Next we show that the operations defined in Definition 3 are equivalent to the greatest lower bound and least upper bound of instances w.r.t. the partial order  $\sqsubseteq$ .

**LEMMA 1.** For every two instances  $D_1, D_2$  and  $R$ -tuples  $t_1, t_2$  in  $D_1, D_2$ , the following holds:

- $D_1 \vee D_2$  is the least upper bound of  $D_1, D_2$  w.r.t.  $\sqsubseteq$ .
- $t_1 \wedge t_2$  is the greatest lower bound of  $t_1, t_2$  w.r.t.  $\preceq$ .
- $D_1 \wedge D_2$  is the greatest lower bound of  $D_1, D_2$  w.r.t.  $\sqsubseteq$ . ■

In particular, we can see that  $\preceq$  imposes a lattice structure on  $R$ -tuples. Using Lemma 1, we immediately obtain the following result.

**THEOREM 1.** The set of reduced instances for a given schema with the  $\sqsubseteq$  ordering forms a lattice. ■

## 4. ENFORCEMENT OF MDS AND CLEAN INSTANCES

In this section, we define *clean* instances that can be obtained from a dirty instance by iteratively enforcing a set of MDs in a chase-like procedure. Let  $D, D'$  be two database instances with the same set of tuple identifiers, and  $t_1, t_2$  be



an  $R_1$ -tuple and an  $R_2$ -tuple, respectively, in both  $D$  and  $D'$ . Let  $\Sigma$  be a set of MDs, and  $\varphi : R_1[X_1] \approx R_2[X_2] \rightarrow R_1[A_1] \approx R_2[A_2]$  be an MD in  $\Sigma$ .

DEFINITION 4. Instance  $D'$  is the immediate result of enforcing  $\varphi$  on  $t_1, t_2$  in instance  $D$ , denoted by  $(D, D')_{[t_1, t_2]} \models \varphi$ , if

1.  $t_1^D[X_1] \approx t_2^D[X_2]$ , but  $t_1^D[A_1] \neq t_2^D[A_2]$ ;
2.  $t_1^{D'}[A_1] = t_2^{D'}[A_2] = m_A(t_1^D[A_1], t_2^D[A_2])$ ; and
3.  $D, D'$  agree on every other tuple and attribute value.

■

Definition 4 captures a single step in a chase-like procedure that starts from a dirty instance  $D_0$  and enforces MDs step by step, by applying matching functions, until the instance becomes stable. We propose that the output of this chase should be treated as a clean version of the original instance, given a set of MDs, formally defined as follows.

DEFINITION 5. For an instance  $D_0$  and a set of MDs  $\Sigma$ , an instance  $D_k$  is  $(D_0, \Sigma)$ -clean if  $D_k$  is stable, and there exists a finite sequence of instances  $D_1, \dots, D_{k-1}$  such that, for every  $i \in [1, k]$ ,  $(D_{i-1}, D_i)_{[t_1^i, t_2^i]} \models \varphi^i$ , for some  $\varphi^i \in \Sigma$  and tuple identifiers  $t_1^i, t_2^i$ .

■

Notice that if  $(D_0, D_0) \models \Sigma$ , i.e., it is already stable, then  $D_0$  is its only  $(D_0, \Sigma)$ -clean instance. Moreover, we have  $D_{i-1} \sqsubseteq D_i$ , for every  $i \in [1, k]$ , since we are using matching functions to identify values. In particular, we have  $D_0 \sqsubseteq D_k$ . In other words, clean instance  $D_k$  semantically dominates dirty instance  $D_0$ , and we might say  $D_k$  is *more informative* than  $D_0$ .

THEOREM 2. Let  $\Sigma$  be a set of matching dependencies and  $D_0$  be an instance. Then every sequence  $D_1, D_2, \dots$  such that  $(D_{i-1}, D_i)_{[t_1^i, t_2^i]} \models \varphi^i$ , for some  $\varphi^i \in \Sigma$  and tuple identifiers  $t_1^i, t_2^i$  in  $D_{i-1}$ , is finite and computes a  $(D_0, \Sigma)$ -clean instance  $D_k$  in polynomial number of steps in the size of  $D_0$ .

■

In other words, the sequence of instances obtained by chasing MDs reaches a fixpoint after polynomial number of steps. That is, it is not possible to generate a new instance, because condition 1 in Definition 4 is not satisfied by the last generated instance, i.e., the last instance is stable w.r.t. all MDs. This is the consequence of assuming that matching functions are idempotent, commutative, and associative.

Observe that, for a given instance  $D_0$  and set of MDs  $\Sigma$ , multiple clean instances may exist, each resulting from a different order of application of MDs on  $D_0$  and from different selections of violating tuples. It is easy to show that the number of clean instances is finite.

Notice also that for a  $(D_0, \Sigma)$ -clean instance  $D_k$ , we may have  $(D_0, D_k) \not\models \Sigma$  (cf. Definition 1). Intuitively, the reason is that some of the similarities that existed in  $D_0$  could have been broken by iteratively enforcing the MDs to produce  $D_k$ . We argue that this is a price we may have to pay if we want to enforce a set of interacting MDs. However, each  $(D_0, \Sigma)$ -clean instance is stable and captures the persistence of attribute values that are not affected by MDs. The following example illustrates these points. For simplicity, we write  $\langle a_1, \dots, a_l \rangle$  to represent  $m_A(a_1, m_A(a_2, m_A(\dots, a_l)))$ , which is allowed by the associativity assumption.

EXAMPLE 4. Consider the set of MDs  $\Sigma$  consisting of  $\varphi_1: R[A] \approx R[A] \rightarrow R[B] \approx R[B]$  and  $\varphi_2: R[B] \approx R[B] \rightarrow R[C] \approx R[C]$ . We have the similarities:  $a_1 \approx a_2, b_2 \approx b_3$ . The following sequence of instances leads to a  $(D_0, \Sigma)$ -clean instance  $D_2$ .

$D_0$	A	B	C	$D_1$	A	B	C
	$a_1$	$b_1$	$c_1$		$a_1$	$\langle b_1, b_2 \rangle$	$c_1$
	$a_2$	$b_2$	$c_2$		$a_2$	$\langle b_1, b_2 \rangle$	$c_2$
	$a_3$	$b_3$	$c_3$		$a_3$	$b_3$	$c_3$
$D_2$	A	B	C				
	$a_1$	$\langle b_1, b_2 \rangle$	$\langle c_1, c_2 \rangle$				
	$a_2$	$\langle b_1, b_2 \rangle$	$\langle c_1, c_2 \rangle$				
	$a_3$	$b_3$	$c_3$				

However,  $(D_0, D_2) \not\models \Sigma$ , and the reason is that  $\langle b_1, b_2 \rangle \approx b_3$  does not necessarily hold. We can enforce the MDs in another order and obtain a different  $(D_0, \Sigma)$ -clean instance:

$D_0$	A	B	C	$D'_1$	A	B	C
	$a_1$	$b_1$	$c_1$		$a_1$	$b_1$	$c_1$
	$a_2$	$b_2$	$c_2$		$a_2$	$b_2$	$\langle c_2, c_3 \rangle$
	$a_3$	$b_3$	$c_3$		$a_3$	$b_3$	$\langle c_2, c_3 \rangle$
$D'_2$	A	B	C	$D'_3$	A	B	C
	$a_1$	$\langle b_1, b_2 \rangle$	$c_1$		$a_1$	$\langle b_1, b_2 \rangle$	$\langle c_1, c_2, c_3 \rangle$
	$a_2$	$\langle b_1, b_2 \rangle$	$\langle c_2, c_3 \rangle$		$a_2$	$\langle b_1, b_2 \rangle$	$\langle c_1, c_2, c_3 \rangle$
	$a_3$	$b_3$	$\langle c_2, c_3 \rangle$		$a_3$	$b_3$	$\langle c_2, c_3 \rangle$

Again,  $D'_3$  is a  $(D_0, \Sigma)$ -clean instance, but  $(D_0, D'_3) \not\models \Sigma$ .

■

It would be interesting to know when there is only one  $(D_0, \Sigma)$ -clean instance  $D_k$ , and also when, for a clean instance  $D_k$ ,  $(D_0, D_k) \models \Sigma$  holds. The following two propositions establish sufficient conditions for these properties to hold.

PROPOSITION 1. Suppose that for every pair of comparable attributes  $A_1, A_2$ , the matching function  $m_A$  is similarity preserving. Then, for every set of MDs  $\Sigma$  and every instance  $D_0$ , there is a unique  $(D_0, \Sigma)$ -clean instance  $D_k$ . Furthermore,  $(D_0, D_k) \models \Sigma$ .

■

We say that a set of matching dependencies  $\Sigma$  is *interaction-free* if for every two MDs  $\varphi_1, \varphi_2 \in \Sigma$ , the two sets of attributes on the right-hand side of  $\varphi_1$  and left-hand side of  $\varphi_2$  are disjoint. The two sets of MDs in Examples 2 and 4 are not interaction-free.

PROPOSITION 2. Let  $\Sigma$  be an interaction-free set of MDs. Then for every instance  $D_0$ , there is a unique  $(D_0, \Sigma)$ -clean instance  $D_k$ . Furthermore,  $(D_0, D_k) \models \Sigma$ .

■

The chase-like procedure that produces a  $(D_0, \Sigma)$ -clean instance makes only those changes to instance  $D_0$  that are necessary, and are imposed by the dynamic semantics of MDs. In this sense, we can say that the chase implements minimal changes necessary to obtain a clean instance.

Another interesting question is whether  $(D_0, \Sigma)$ -clean instances are at a minimal distance to  $D_0$  w.r.t. the partial order  $\sqsubseteq$ . This is not true in general. For instance in Example 4, observe that for the two  $(D_0, \Sigma)$ -clean instances  $D_2$  and  $D'_3$ ,  $D_2 \sqsubseteq D'_3$ , but  $D'_3 \not\sqsubseteq D_2$ , which means  $D'_3$  is not at a minimal distance to  $D_0$  w.r.t.  $\sqsubseteq$ . However, both of these clean instances may be useful in query answering, because,

informally speaking, they can provide a lower bound and an upper bound for the possible clean interpretations of the original dirty instance w.r.t. the semantic domination. This issue is discussed in the next section.

## 5. CLEAN QUERY ANSWERING

Most of the literature on data cleaning has concentrated on producing a clean instance starting from a dirty one. However, the problem of characterizing and retrieving the data in the original instance that can be considered to be clean has been neglected. In this section we study this problem, focusing on query answering. More precisely, given an instance  $D$ , a set  $\Sigma$  of MDs, and a query  $Q$  posed to  $D$ , we want to characterize the answers that are consistent with  $\Sigma$ , i.e., that would be returned by an instance where all the MDs have been enforced. Of course, we have to take into account that there may be several such instances.

This situation is similar to the one encountered in *consistent query answering* (CQA) [3, 9, 13], where query answering is characterized and performed on database instances that may fail to satisfy certain classic integrity constraints (ICs). For such a database instance, a *repair* is an instance that satisfies the integrity constraints and minimally differs from the original instance. For a given query, a *consistent answer* (a form of certain answer) is defined as the set of tuples that are present in the intersection of answers to the query when posed to every repair. A less popular alternative is the notion of *possible answer*, that is defined as the union of all tuples that are present in the answer to the query when posed to every repair.

A similar semantics for clean query answering under matching dependencies can be defined. However, the partial order relationship  $\sqsubseteq$  between a dirty instance and its clean instances establishes an important difference between clean instances w.r.t. matching dependencies and repairs w.r.t. traditional ICs.

Intuitively, a clean instance has improved the information that already existed in the dirty instance and made it more informative and consistent. We would like to carefully take advantage of this partial order relationship and use it in the definition of certain and possible answers. We do this by taking the greatest lower bound (glb) and least upper bound (lub) of answers of the query over multiple clean instances, instead of taking the set-theoretic intersection and union.

Let  $\Sigma$  be a set of MDs,  $D_0$  be a database instance, and  $Q$  be a query posed to instance  $D_0$ . We define *certain* and *possible answers* as follows.

$$Cert_Q(D_0) = glb_{\sqsubseteq} \{Q(D) \mid D \text{ is a } (D_0, \Sigma)\text{-clean instance}\}. \quad (2)$$

$$Poss_Q(D_0) = lub_{\sqsubseteq} \{Q(D) \mid D \text{ is a } (D_0, \Sigma)\text{-clean instance}\}. \quad (3)$$

The glb and lub above are defined on the basis of the partial order  $\sqsubseteq$  on sets of tuples. Since there is a finite number of clean instances for  $D_0$ , these glb and lub exist (cf. Theorem 1). In Eq. (2) and (3) we are assuming that each of the  $Q(D)$  is reduced (cf. Section 3). By Definition 3,  $Cert_Q(D_0)$  and  $Poss_Q(D_0)$  are also reduced. Moreover, we clearly have  $Cert_Q(D_0) \sqsubseteq Poss_Q(D_0)$ .

The following example motivates these choices. It also shows that, unlike some cases of inconsistent databases and consistent query answering, certain answers could be quite informative and meaningful for databases with matching dependencies.

EXAMPLE 5. Consider relation  $R(name, phone, address)$ , and set  $\Sigma$  consisting of the following MDs:

$$\begin{aligned} \varphi_1 : R[name, phone, address] &\approx R[name, phone, address] \rightarrow R[address] \Rightarrow R[address], \\ \varphi_2 : R[phone, address] &\approx R[phone, address] \rightarrow R[phone] \Rightarrow R[phone]. \end{aligned}$$

Suppose that in the dirty instance  $D_0$ , shown below, the following similarities hold:

$$\begin{aligned} \text{"John Doe"} &\approx \text{"J. Doe"}, \\ \text{"Jane Doe"} &\approx \text{"J. Doe"}, \\ \text{"(613)123 4567"} &\approx \text{"123 4567"}, \\ \text{"(604)123 4567"} &\approx \text{"123 4567"}, \\ \text{"25 Main St."} &\approx \text{"Main St., Ottawa"}, \\ \text{"25 Main St."} &\approx \text{"25 Main St., Vancouver"} \end{aligned}$$

Other non-trivial similarities that are not mentioned do not hold. Moreover, the matching functions act as follows:

$$\begin{aligned} m_{phone}(\text{"(613)123 4567"}, \text{"123 4567"}) &= \text{"(613)123 4567"}, \\ m_{phone}(\text{"123 4567"}, \text{"(604)123 4567"}) &= \text{"(604)123 4567"}, \\ m_{address}(\text{"Main St., Ottawa"}, \text{"25 Main St."}) &= \text{"25 Main St., Ottawa"}, \\ m_{address}(\text{"25 Main St."}, \text{"25 Main St., Vancouver"}) &= \text{"25 Main St., Vancouver"} \end{aligned}$$

$D_0$	<i>name</i>	<i>phone</i>	<i>address</i>
	John Doe	(613)123 4567	Main St., Ottawa
	J. Doe	123 4567	25 Main St.
	Jane Doe	(604)123 4567	25 Main St., Vancouver

Observe that from  $D_0$  we can obtain two different  $(D_0, \Sigma)$ -clean instances  $D, D'$ , depending on the order of enforcing MDs.

$D$	<i>name</i>	<i>phone</i>	<i>address</i>
	John Doe	(613)123 4567	25 Main St., Ottawa
	J. Doe	(613)123 4567	25 Main St., Ottawa
	Jane Doe	(604)123 4567	25 Main St., Vancouver

$D'$	<i>name</i>	<i>phone</i>	<i>address</i>
	John Doe	(613)123 4567	Main St., Ottawa
	J. Doe	(604)123 4567	25 Main St., Vancouver
	Jane Doe	(604)123 4567	25 Main St., Vancouver

Now consider the query  $Q : \pi_{address}(\sigma_{name=\text{"J. Doe"}}R)$ , asking for the residential address of J. Doe. We are interested in a certain answer. It can be obtained by taking the greatest lower bound of the two answer sets:

$$\begin{aligned} Q(D) &= \{\text{"25 Main St., Ottawa"}\}, \\ Q(D') &= \{\text{"25 Main St., Vancouver"}\}. \end{aligned}$$

In this case, and according to [6], and using Lemma 1,

$$\begin{aligned} glb_{\sqsubseteq} \{Q(D), Q(D')\} &= \{a \wedge a' \mid a \in Q(D), a' \in Q(D')\} \\ &= \{\text{"25 Main St., Ottawa"} \wedge \text{"25 Main St., Vancouver"}\} \\ &= \{glb_{\sqsubseteq_{Address}} \{\text{"25 Main St., Ottawa"}, \text{"25 Main St., Vancouver"}\}\} \\ &= \{\text{"25 Main St."}\}. \end{aligned}$$

We can see that, no matter how we clean  $D_0$ , we can say for sure that J. Doe is at 25 Main St. Notice that the set-theoretic intersection of the two answer sets is empty. If we

were interested in all possible answers, we could take the least upper bound of two answer sets, which would be the union of the two in this case. ■

We define *clean answer* to be an upper and lower bound of query answers over all possible clean interpretations of a dirty database instance. This definition is inspired by the same kind of approximations used in the contexts of partial and incomplete information [29, 1], inconsistent databases [3, 9, 13], and data exchange [28]. These upper and lower bounds could provide useful information about the value of aggregate functions, such as sum and count [5, 18, 2].

DEFINITION 6. For a query  $Q$  posed to a database instance  $D_0$  and a set of MDs  $\Sigma$ , a *clean answer* is specified by two bounds as

$$\text{Clean}_Q(D_0) = (\text{Cert}_Q(D_0), \text{Poss}_Q(D_0)). \quad \blacksquare$$

Notice, from the results in Section 4, that in the case of having similarity-preserving matching functions or non-interacting matching dependencies, these bounds would collapse into a single set, which is obtained by running the query on the unique clean instance.

### Complexity of Computing Clean Answers

Here we study the complexity of computing clean answers over database instances in presence of MDs. As with incomplete and inconsistent databases, this problem easily becomes intractable for simple MDs and queries, which motivates the need for developing approximate solutions to these problems. We explore approximate solutions for queries that behave monotonically w.r.t. the partial order  $\sqsubseteq$  in Section 6.2.

THEOREM 3. There are a schema with two interacting MDs and a relational algebra query, for which deciding whether a tuple belongs to the certain answer set for an instance  $D_0$  is coNP-complete (in the size of  $D_0$ ). ■

## 6. MONOTONE QUERIES

So far we have seen that clean instances are a more informative view of a dirty instance obtained by enforcing matching dependencies. That is,  $D_0 \sqsubseteq D$ , for every  $(D_0, \Sigma)$ -clean instance  $D$ . From this perspective, it would be natural to expect that for a positive query, we would obtain a more informative answer if we pose it to a clean instance instead of to the dirty one. We can translate this requirement into a monotonicity property for queries w.r.t. the partial order  $\sqsubseteq$ .

DEFINITION 7. A query  $Q$  is  $\sqsubseteq$ -monotone if, for every pair of instances  $D, D'$ , such that  $D \sqsubseteq D'$ , we have  $Q(D) \sqsubseteq Q(D')$ . ■

Monotone queries have an interesting behavior when computing clean answers. For these queries, we can under-approximate (over-approximate) certain answers (possible answers) by taking the greatest lower bound (least upper bound) of all clean instances and then running the query on the result. Notice that we are not claiming that these are polynomial-time approximations.

PROPOSITION 3. If  $\mathcal{D}$  is a finite set of database instances and  $Q$  is a  $\sqsubseteq$ -monotone query, the following holds:

$$Q(\text{glb}_{\sqsubseteq}\{D \mid D \in \mathcal{D}\}) \sqsubseteq \text{glb}_{\sqsubseteq}\{Q(D) \mid D \in \mathcal{D}\}, \quad (4)$$

$$\text{lub}_{\sqsubseteq}\{Q(D) \mid D \in \mathcal{D}\} \sqsubseteq Q(\text{lub}_{\sqsubseteq}\{D \mid D \in \mathcal{D}\}). \quad (5)$$

As is well known, positive relational algebra queries composed of selection, projection, Cartesian product, and union, are monotone. However, the following example shows that monotonicity does not hold for very simple positive queries involving selections.

EXAMPLE 6. Consider instance  $D_0$  in Example 5 and two  $(D_0, \Sigma)$ -clean instances  $D$  and  $D'$ . Let  $Q$  be a query asking for names of people residing at “25 Main St.”, expressed as relational algebra expression  $\pi_{\text{name}}(\sigma_{\text{address}=\text{“25 Main St.”}}(R))$ . Observe that  $Q(D_0) = \{(\text{“J. Doe”})\}$ , and  $Q(D) = Q(D') = \emptyset$ . Query  $Q$  is therefore not monotone, because we have  $D_0 \sqsubseteq D$ ,  $D_0 \sqsubseteq D'$ , but  $Q(D_0) \not\sqsubseteq Q(D)$ ,  $Q(D_0) \not\sqsubseteq Q(D')$ . ■

It is not surprising that  $\sqsubseteq$ -monotonicity is not captured by usual relational queries, in particular, by queries that are monotone w.r.t. set inclusion. After all, the queries we have considered so far do not even mention the  $\preceq$ -lattice that is at the basis of the  $\sqsubseteq$  order. Next we will consider queries expressing conditions in term of the semantic domination lattice.

### 6.1 Query relaxation

As shown in Example 6, we may not get the answer we expect by running a usual relational algebra query on an instance that has been cleaned using matching dependencies. We therefore propose to *relax* the queries, by taking advantage of the underlying  $\preceq$ -lattice structure obtained from matching functions, to make them  $\sqsubseteq$ -monotone. In this way, we achieve two goals: First, the resulting queries provide more informative answers; and second, we can take advantage of Proposition 3 to approximate clean answers from below and from above.

We introduce the (negation free) language *relaxed relational algebra*,  $\mathcal{RA}_{\preceq}$ , by providing two selection operators  $\sigma_{a \preceq A}$  and  $\sigma_{A_1 \bowtie_{\preceq} A_2}$  (for comparable attributes  $A_1, A_2$ ), defined as follows.

DEFINITION 8. The language  $\mathcal{RA}_{\preceq}$  is composed of relational operators  $\pi, \times, \cup$  (with usual definitions), plus  $\sigma_{a \preceq A}$ , and  $\sigma_{A_1 \bowtie_{\preceq} A_2}$ , defined by:

$$\begin{aligned} \sigma_{a \preceq A}(D) &= \{t^D \mid a \preceq_A t^D[A]\} \quad (\text{here } a \in \text{Dom}_A), \\ \sigma_{A_1 \bowtie_{\preceq} A_2}(D) &= \{t^D \mid \exists a \in \text{Dom}_A \text{ s.t. } a \preceq_A t^D[A_1], \\ &\quad a \preceq_A t^D[A_2], a \neq \perp\}. \end{aligned} \quad \blacksquare$$

For string attributes, for instance, the selection operator  $\sigma_{a \preceq A}$  checks whether the value of attribute  $A$  dominates the substring  $a$ , and the join selection operator  $\sigma_{A_1 \bowtie_{\preceq} A_2}$  checks whether the values of attributes  $A_1, A_2$  dominate a common substring. Notice that queries in the language  $\mathcal{RA}_{\preceq}$  are not domain independent: The result of posing a query to an instance depends not only on the values in the active domain of the instance but also on the domain lattices. In other words, query answering depends on how data cleaning is being implemented.

It can be easily observed that all operators in the language  $\mathcal{RA}_{\preceq}$  are  $\sqsubseteq$ -monotone, and therefore every query expression in  $\mathcal{RA}_{\preceq}$  that is obtained by composing these operators is also  $\sqsubseteq$ -monotone.

**PROPOSITION 4.** Let  $Q$  be a query in  $\mathcal{RA}_{\preceq}$ . For every two instances  $D, D'$  such that  $D \sqsubseteq D'$ , we have  $Q(D) \sqsubseteq Q(D')$ . ■

Now suppose that we have a query  $Q$ , written in positive relational algebra, i.e., composed of  $\pi, \times, \cup, \sigma_{A=a}, \sigma_{A_1=A_2}$ , the last two being *hard* selection conditions, which is to be posed to an instance  $D_0$ . After cleaning  $D_0$  by enforcing a set of MDs  $\Sigma$  to obtain a  $(D_0, \Sigma)$ -clean instance  $D$ , running query  $Q$  on  $D$  may no longer provide the expected answer, because some of the values have changed in  $D$ , i.e., they have semantically grown w.r.t.  $\preceq$ . In order to capture this semantic growth, our query relaxation framework proposes the following *query rewriting* methodology: Given a query  $Q$  in positive RA, transform it into a query  $Q_{\preceq}$  in  $\mathcal{RA}_{\preceq}$  by simply replacing the selection operators  $\sigma_{A=a}$  and  $\sigma_{A_1=A_2}$  by  $\sigma_{a \preceq A}$  and  $\sigma_{A_1 \bowtie_{\preceq} A_2}$ , respectively.

**EXAMPLE 7.** Consider again instance  $D_0$  in Example 5 and  $(D_0, \Sigma)$ -clean instances  $D$  and  $D'$ , and query  $Q$  asking for names of people residing at “25 Main St.”, expressed as  $\pi_{name}(\sigma_{address=\text{“25 Main St.”}}(R))$ . We obtain the empty answer from each of  $D, D'$ . So, in this case the certain and the possible answers are empty, a not very informative outcome.

However, after the relaxation rewriting of  $Q$ , we obtain the query  $Q_{\preceq} : \pi_{name}(\sigma_{\text{“25 Main St.”} \preceq address}(R))$ . If we pose  $Q_{\preceq}$  to the clean instances, we obtain

$$Q_{\preceq}(D) = \{(\text{“John Doe”}), (\text{“J. Doe”}), (\text{“Jane Doe”})\}, \\ Q_{\preceq}(D') = \{(\text{“J. Doe”}), (\text{“Jane Doe”})\},$$

and thus  $Cert_{Q_{\preceq}}(D_0) = \{(\text{“J. Doe”}), (\text{“Jane Doe”})\}$ . This outcome is much more informative; and, above all, is sensitive to the underlying information lattice. ■

**PROPOSITION 5.** For every positive relational algebra query  $Q$  and every instance  $D$ , we have  $Q(D) \sqsubseteq Q_{\preceq}(D)$ , where  $Q_{\preceq}$  is the relaxed rewriting of  $Q$ . ■

## 6.2 Approximating Clean Answers

Given the high computational cost of clean query answering when there are multiple clean instances, it would be desirable to provide an approximation to clean answers that is computable in polynomial time. In this section, we are interested in approximating clean answers by producing an under-approximation of certain answers and an over-approximation of possible answers for a given monotone query  $Q$ . That is, we would like to obtain  $(Q_{\downarrow}(D_0), Q_{\uparrow}(D_0))$ , such that  $Q_{\downarrow}(D_0) \sqsubseteq Cert_Q(D_0)$  and  $Poss_Q(D_0) \sqsubseteq Q_{\uparrow}(D_0)$ .

Since  $Q$  is a monotone query, by Proposition 3, we have  $Q(glb_{\sqsubseteq}\{D \mid D \text{ is } (D_0, \Sigma)\text{-clean}\}) \sqsubseteq Cert_Q(D_0)$ , and moreover,  $Poss_Q(D_0) \sqsubseteq Q(lub_{\sqsubseteq}\{D \mid D \text{ is } (D_0, \Sigma)\text{-clean}\})$ . In consequence, it is good enough to find under- and over-approximations for the greatest lower bound and the least upper bound, resp., of the set of all  $(D_0, \Sigma)$ -clean instances, and then pose  $Q$  to these approximations to obtain  $Q_{\downarrow}(D_0)$  and  $Q_{\uparrow}(D_0)$ .

The reason for having multiple clean instances is that matching dependencies are not necessarily interaction-free and the matching functions are not necessarily similarity

preserving. Intuitively speaking, we can under-approximate the greatest lower bound of clean instances by not enforcing some of the interacting MDs. On the other side, we can over-approximate the least upper bound by assuming that the matching functions are similarity preserving. This would lead us to keep applying MDs on the assumption that unresolved similarities still persist. We present two chase-like procedures to compute  $D_{\downarrow}$  and  $D_{\uparrow}$  corresponding to these approximations.

### Under-approximating the greatest lower bound.

To provide an under-approximation for the greatest lower bound of all clean instances, we provide a new chase-like procedure, which enforces only MDs that are enforced in every clean instance. These MDs are applicable to those initial similarities that exist in the original dirty instance, which are never broken by enforcing other MDs during any chase procedure of producing a clean instance.

Let  $\Sigma$  be a set of MDs, and  $\varphi, \varphi' \in \Sigma$ . We say that  $\varphi$  *precedes*  $\varphi'$  if the set of attributes on the left-hand side of  $\varphi'$  contains the attribute on the right-hand side of  $\varphi$ . We say that  $\varphi$  *interacts with*  $\varphi'$  if there are MDs  $\varphi_1, \dots, \varphi_k \in \Sigma$ , such that  $\varphi$  precedes  $\varphi_1$ ,  $\varphi_k$  precedes  $\varphi'$ , and  $\varphi_i$  precedes  $\varphi_{i+1}$  for  $i \in [1, k-1]$ , i.e., the interaction relationship can be seen as the transitive closure of precedence relationship.

Let  $D_0$  be a dirty database instance. Let  $\varphi : R_1[X_1] \approx R_2[X_2] \rightarrow R_1[A_1] \approx R_2[A_2]$  be an MD in  $\Sigma$ . We say  $\varphi$  is *freshly applicable* on  $t_1, t_2$  in  $D_0$  if  $t_1^{D_0}[X_1] \approx t_2^{D_0}[X_2]$ , and  $t_1^{D_0}[A_1] \neq t_2^{D_0}[A_2]$ . We say  $\varphi$  is *safely applicable* on  $t_1, t_2$  in  $D_0$  if  $\varphi$  is freshly applicable on  $t_1, t_2$  in  $D_0$ , and for every  $\varphi' \in \Sigma$  that interacts with  $\varphi$ ,  $\varphi'$  is not freshly applicable on  $t_1, t_3$  or  $t_2, t_3$  in  $D_0$  for any tuple  $t_3$  (see Example 8).

**DEFINITION 9.** For an instance  $D_0$  and a set of MDs  $\Sigma$ , an instance  $D_k$  is  $(D_0, \Sigma)$ -*under clean* if there exists a finite sequence of instances  $D_1, \dots, D_{k-1}$ , such that

1. For every  $i \in [1, k]$ ,  $(D_{i-1}, D_i)_{[t_1^i, t_2^i]} \models \varphi^i$ , for some  $\varphi^i \in \Sigma$  and tuple identifiers  $t_1^i, t_2^i$ , such that  $\varphi^i$  is safely applicable on  $t_1^i, t_2^i$  in  $D_0$ .
2. For every MD  $\varphi : R_1[X_1] \approx R_2[X_2] \rightarrow R_1[A_1] \approx R_2[A_2]$  in  $\Sigma$  and tuples  $t_1, t_2$ , such that  $\varphi$  is safely applicable on  $t_1, t_2$  in  $D_0$ , we have  $t_1^{D_k}[A_1] = t_2^{D_k}[A_2]$ . ■

Definition 9 characterizes a chase-based procedure that keeps enforcing MDs that are safely applicable in the original dirty instance until all such MDs are enforced. Notice that an under clean instance may not be stable. Moreover, safely applicable MDs never interfere with each other, in the sense that enforcing one of them never breaks the initial similarities in the dirty instance that are needed for enforcing other safely applicable MDs.

**PROPOSITION 6.** For every instance  $D_0$  and every set of MDs  $\Sigma$ , there is a unique  $(D_0, \Sigma)$ -under clean instance  $D_{\downarrow}$ . ■

Clearly, an under clean instance  $D_{\downarrow}$  can be computed in polynomial time in the size of the dirty instance  $D_0$ . To construct it, we first need to identify safely applicable MDs in  $D_0$ , and then enforce them in any arbitrary order until no such MDs can be enforced. Next we show that  $D_{\downarrow}$  is an



under-approximation to every  $(D_0, \Sigma)$ -clean instance. Intuitively, this is because  $D_\downarrow$  is obtained by enforcing MDs that are enforced in every chase-based procedure of producing a clean instance.

**PROPOSITION 7. (Soundness of under-approximation)** For every  $(D_0, \Sigma)$ -under clean instance  $D_\downarrow$  and every  $(D_0, \Sigma)$ -clean instance  $D$ , we have  $D_\downarrow \sqsubseteq D$ . ■

Notice that an arbitrary  $(D_0, \Sigma)$ -clean instance  $D$  may not be a sound under-approximation for every other  $(D_0, \Sigma)$ -clean instances  $D'$ , because  $D \sqsubseteq D'$  may not hold.

Let  $D_\downarrow$  be a  $(D_0, \Sigma)$ -under clean instance. Then from Propositions 7 and 3, we immediately obtain the following result.

**THEOREM 4.** For every monotone query  $Q$ , we have  $Q(D_0) \sqsubseteq Q(D_\downarrow) \sqsubseteq \text{Cert}_Q(D_0)$ . ■

**EXAMPLE 8.** Consider the instance  $D_0$  and set of MDs  $\Sigma$  in Example 4. Observe that MD  $\varphi_1$  is safely applicable on the first and second tuples in  $D_0$ . Moreover,  $\varphi_2$  is freshly applicable, but not safely applicable on the second and third tuples. Accordingly, we obtain  $(D_0, \Sigma)$ -under clean instance  $D_\downarrow$ , shown below, by enforcing  $\varphi_1$  on the first two tuples.

$D_\downarrow$	$A$	$B$	$C$
	$a_1$	$\langle b_1, b_2 \rangle$	$c_1$
	$a_2$	$\langle b_1, b_2 \rangle$	$c_2$
	$a_3$	$b_3$	$c_3$

Notice that for the two  $(D_0, \Sigma)$ -clean instances  $D_2, D'_3$  in Example 4, we have  $D_\downarrow \sqsubseteq D_2$  and  $D_\downarrow \sqsubseteq D'_3$ . Also notice that  $D_\downarrow$  is not a stable instance. Now consider the query  $Q : \pi_C(\sigma_{A=a_2} R)$ . This query behaves monotonically for our purpose, because the values of attribute  $A$  are not changing by enforcing MDs. If we pose  $Q$  to  $D_\downarrow$ , we obtain  $Q(D_\downarrow) = \{c_2\}$ . Observe that  $\text{Cert}_Q(D_0) = \{\langle c_1, c_2 \rangle\}$ , and thus  $Q(D_\downarrow)$  provides an under-approximation for  $\text{Cert}_Q(D_0)$ . This example also shows that an arbitrary clean instance,  $D'_3$  here, may not provide a sound approximation to certain answer since  $Q(D'_3) = \{\langle c_1, c_2, c_3 \rangle\} \not\sqsubseteq \text{Cert}_Q(D_0)$ . ■

### Over-approximating the least upper bound.

To provide an over-approximation for the least upper bound of all clean instances, we *modify* every similarity relation so that the corresponding matching function becomes similarity preserving. For a similarity relation  $\approx_A$  and the corresponding matching function  $m_A$ , we define  $\approx_A^*$  as follows: For every  $a, a' \in \text{Dom}_A$ ,  $a \approx_A^* a'$  iff there is  $a'' \in \text{Dom}_A$ , such that  $a \approx_A a''$  and  $m_A(a', a'') = a'$ . Given a set of MDs  $\Sigma$ , we obtain  $\Sigma^*$  by replacing every similarity relation  $\approx_A$  in the MDs by  $\approx_A^*$ .

**DEFINITION 10.** For an instance  $D_0$  and a set of MDs  $\Sigma$ , an instance  $D_\uparrow$  is  $(D_0, \Sigma)$ -over clean if  $D_\uparrow$  is  $(D_0, \Sigma^*)$ -clean. ■

By Proposition 1, for every instance  $D_0$  and set of MDs  $\Sigma$ , there is a unique  $(D_0, \Sigma)$ -over clean instance  $D_\uparrow$ , and moreover, it can be computed in polynomial time in the size of  $D_0$ . To construct  $D_\uparrow$ , we first need to obtain  $\Sigma^*$ , as described above, and enforce MDs in  $\Sigma^*$  in any arbitrary order until we get a stable instance w.r.t.  $\Sigma^*$ . Next we

show that  $D_\uparrow$  is an over-approximation for every  $(D_0, \Sigma)$ -clean instance. Intuitively, this is because  $D_\uparrow$  is obtained by enforcing (at least) all MDs that are present in any chase-like procedure of producing a clean instance.

**PROPOSITION 8. (Completeness of over-approximation)** For every  $(D_0, \Sigma)$ -over clean instance  $D_\uparrow$  and every  $(D_0, \Sigma)$ -clean instance  $D$ , we have  $D \sqsubseteq D_\uparrow$ . ■

Notice again that an arbitrary  $(D_0, \Sigma)$ -clean instance  $D$  may not be an over-approximation for every other  $(D_0, \Sigma)$ -clean instance  $D'$ , because  $D' \sqsubseteq D$  may not hold.

Let  $D_\uparrow$  be a  $(D_0, \Sigma)$ -over clean instance. Then from Propositions 8 and 3, we immediately obtain the following result.

**THEOREM 5.** For every monotone query  $Q$ , we have  $\text{Poss}_Q(D_0) \sqsubseteq Q(D_\uparrow)$ . ■

**EXAMPLE 9.** (Example 8 continued.) By assuming that old similarities hold after applying matching functions (e.g.,  $\langle b_1, b_2 \rangle \approx^* b_3$ ), we obtain the  $(D_0, \Sigma)$ -over clean instance  $D_\uparrow$  shown below.

$D_\uparrow$	$A$	$B$	$C$
	$a_1$	$\langle b_1, b_2 \rangle$	$\langle c_1, c_2, c_3 \rangle$
	$a_2$	$\langle b_1, b_2 \rangle$	$\langle c_1, c_2, c_3 \rangle$
	$a_3$	$b_3$	$\langle c_1, c_2, c_3 \rangle$

Notice that for the two  $(D_0, \Sigma)$ -clean instances  $D_2, D'_3$  in Example 4, we have  $D_2 \sqsubseteq D_\uparrow$  and  $D'_3 \sqsubseteq D_\uparrow$ . If we pose query  $Q : \pi_C(\sigma_{A=a_2} R)$  to  $D_\uparrow$ , we obtain  $Q(D_\uparrow) = \{\langle c_1, c_2, c_3 \rangle\}$ . Observe that  $\text{Poss}_Q(D_0) = \{\langle c_1, c_2, c_3 \rangle\}$ , and thus  $Q(D_\uparrow)$  provides an over-approximation for  $\text{Poss}_Q(D_0)$ . It can be seen that an arbitrary  $(D_0, \Sigma)$ -clean instance, say  $D_2$  for instance, may not provide a complete approximation to possible answer since  $\text{Poss}_Q(D_0) \not\sqsubseteq Q(D_2) = \{\langle c_1, c_2 \rangle\}$ . ■

## 7. A CASE FOR SWOOSH'S ENTITY RESOLUTION

In [8], a generic conceptual framework for entity resolution is introduced. It considers a general match relation  $M$ , which is close to our similarity predicates  $\approx$ , and a general merge function,  $\mu$ , which is close to our  $m$  functions. In this section we establish a connection between our MD framework and Swoosh. However, a full comparison is problematic, for several reasons, among them: (a) Swoosh works at the record (tuple) level, and we concentrate on the attribute level. (b) Swoosh does not use tuple identifiers and some tuples may be discarded at the end, those that are dominated by others in the instance. The main problem is (a). However, to ease the comparison, we consider a particular (but still general enough) case of Swoosh, namely the combination of the *union case* with *merge domination*. In the following we embed this case of Swoosh into our MD framework, thus showing the power of the latter.

Although it is not explicitly said in [8], it is safe to say that the conceptual framework is applied to ground tuples of a single relational predicate, say  $R$ , which are called *records* there. In consequence,  $\text{Rec}$  denotes the set of ground tuples of the form  $R(\bar{s})$ . If the attributes of  $R$  are  $A_1, \dots, A_n$ , then the component  $s_i$  of  $\bar{s}$  belongs to an underlying domain  $\text{Dom}_{A_i}$ .

Relation  $M$  maps  $\text{Rec} \times \text{Rec}$  into  $\{\text{true}, \text{false}\}$ . When two tuples are similar and have to be merged,  $M$  takes the value

true. Moreover,  $\mu$  is a partial function from  $Rec \times Rec$  into  $Rec$ . It produces the merge of the two tuples into a single tuple, and is defined only when  $M$  takes the value true.

Now, the union case for Swoosh arises when the merge function  $\mu$  produces the union of the records, defined as the component-wise union of attribute values. This latter union makes sense if the attribute values are sets of values from an even deeper data domain.

More precisely, for each of the  $n$  attributes  $A_i$  of  $R$ , we consider  $n$  possibly denumerable domains  $D_{A_i}$ . (Repeated attributes in  $R$  share the same domain, but it is conceptually simpler to assume that attributes are all different.) Each  $D_{A_i}$  has a similarity relation  $\approx_{A_i}$ , which is reflexive and symmetric. Now, for each attribute  $A_i$  of  $R$ , its domain becomes  $Dom_{A_i} := \cup_{k \in \mathbb{N}} \mathcal{P}^k(D_{A_i})$ , where  $k > 0$  and  $\mathcal{P}^k(D_{A_i})$  denotes the set of subsets of  $D_{A_i}$  of cardinality  $k$ . In consequence, the elements of  $Rec$  are of the form  $R(s_1, \dots, s_n)$ , with each  $s_i$  being a set that belongs to  $Dom_{A_i}$ . An initial instance  $D$ , before any entity resolution, will be a finite subset of  $Rec$ , and each attribute value in a record, say  $s_i$  for  $A_i$ , will be a singleton of the form  $\{a_i\}$ , with  $a_i \in D_{A_i}$ .

The  $\approx_{A_i}$  relation on  $D_{A_i}$  induces a similarity relation  $\approx_{\{A_i\}}$  on  $Dom_{A_i}$ , as follows:  $s_1 \approx_{\{A_i\}} s_2$  holds iff there exist  $a_1 \in s_1, a_2 \in s_2$  with  $a_1 \approx_{A_i} a_2$ . Each  $\approx_{\{A_i\}}$  is reflexive and symmetric. ( $s \approx_{\{A_i\}} s$ , because there is  $a \in s$  and  $\approx_{A_i}$  is reflexive; and symmetry follows from the symmetry of  $\approx_{A_i}$ .) We also consider matching functions  $m_{\{A_i\}} : Dom_{A_i} \times Dom_{A_i} \rightarrow Dom_{A_i}$  defined by  $m_{\{A_i\}}(s_1, s_2) := s_1 \cup s_2$ . The structures  $\langle Dom_{A_i}, \approx_{\{A_i\}}, m_{\{A_i\}} \rangle$  have all the properties described in Sections 2 and 3.

PROPOSITION 9. Each matching function  $m_{\{A_i\}}$  is total, idempotent, commutative and associative. It is also similarity preserving w.r.t. the  $\approx_{\{A_i\}}$  similarity relation. ■

Now, based on [8], we are ready to define the “union match and merge case” for Swoosh. Consider two elements of  $Rec$ , say  $r_1 = R(\bar{s}^1), r_2 = R(\bar{s}^2)$ : (a)  $M(r_1, r_2) := true$  iff for some  $i$ ,  $s_i^1 \approx_{\{A_i\}} s_i^2$ . (b) When  $M(r_1, r_2) := true$ ,  $\mu(r_1, r_2) := R(m_{\{A_1\}}(s_1^1, s_1^2), \dots, m_{\{A_n\}}(s_n^1, s_n^2))$ .

Function  $M$  is reflexive and commutative, which follows from the reflexivity and symmetry of the  $\approx_{\{A_i\}}$ . From [8, Prop. 2.4] we obtain that the combination of  $M$  and  $\mu$  has Swoosh’s ICAR properties, namely:<sup>3</sup>

$I^s$ : Idempotency:  $\forall r \in Rec, M(r, r)$  holds, and  $\mu(r, r) = r$ .

$C^s$ : Commutativity:  $\forall r_1, r_2 \in Rec, M(r_1, r_2)$  iff  $M(r_2, r_1)$ . Also  $M(r_1, r_2)$  implies  $\mu(r_1, r_2) = \mu(r_2, r_1)$ .

$A^s$ : Associativity:  $\forall r_1, r_2, r_3 \in Rec$ , if  $\mu(r_1, \mu(r_2, r_3))$  and  $\mu(\mu(r_1, r_2), r_3)$  exist, then they are equal.

$R^s$ : Representativity:  $\forall r_1, r_2, r_3, r_4 \in Rec$ , if  $r_3 = \mu(r_1, r_2)$  and  $M(r_1, r_4)$  holds, then  $M(r_3, r_4)$  also holds.

Now, Swoosh framework with  $M$  and  $\mu$  on  $Dom_A$  can be reconstructed by means of the following set  $\Sigma^S$  of MDs: For  $1 \leq i, j \leq n$ ,

$$R[A_i] \approx_{\{A_i\}} R[A_i] \longrightarrow R[A_j] \approx R[A_j]. \quad (6)$$

<sup>3</sup>We use the superscript  $s$ , for Swoosh, to distinguish them from the properties listed in Section 3.

The RHS of (6) has to be applied, as expected, with the matching functions  $m_{\{A_j\}}$ . From Propositions 1 and 9, we obtain that there is a single  $(D, \Sigma^S)$ -clean instance  $D^m$ . Consistently with our MD framework, we will assume that records have tuple identifiers. Actually, in order to make more clear the comparison between the two frameworks, in this section and for MDs, we will use explicit tuple ids. They will be positioned in the first, extra attribute of each relation. When the MDs are applied, only the new version of a tuple is kept.

In the case of Swoosh, the application of  $\mu$  generates a new, merged tuple, but the old ones may stay. However, Swoosh applies a pruning process based on an abstract domination partial order between records,  $\preceq^S$ . The framework concentrates mostly on the *merge domination relation*  $\leq$ , which is defined by:

$$r_1 \leq r_2 :\iff M(r_1, r_2) = true \text{ and } \mu(r_1, r_2) = r_2. \quad (7)$$

The  $I^s C^s A^s R^s$  properties make  $\leq$  a partial order with some pleasant and expected monotonicity properties [8].

According to Section 3, we may consider each of the partial orders  $\preceq_{\{A_i\}}$  on the  $Dom_{A_i}$ :  $s \preceq_{\{A_i\}} s' :\iff m_{\{A_i\}}(s, s') = s'$ . They induce a  $\preceq$  relation on  $Rec$  (cf. Definition 2).

PROPOSITION 10. The general dominance relation  $\preceq$  on  $Rec$  coincides with the merge domination relation  $\leq$  obtained from  $M$  and  $\mu$ . ■

Given a dirty instance  $D$ , it is a natural question to ask about the relationship between the clean instance  $D^m$  obtained under our approach, by enforcing the above MDs, and the *entity resolution* instance  $D^s$  obtained directly via Swoosh. The entity resolution  $D^s$  is defined in [8, Def. 2.3] through the conditions: 1.  $D^s \subseteq \bar{D}$ . 2.  $\bar{D} \leq D^s$ . 3.  $D^s$  is  $\subseteq$ -minimal for the two previous conditions. Here, the partial-order  $\leq$  between instances is induced by the partial order  $\leq$  between records as in Definition 2. Instance  $\bar{D}$  is the *merge closure* of  $D$ , i.e., the  $\subseteq$ -minimal instance that includes  $D$  and is closed under  $M$ :  $r_1, r_2 \in \bar{D}$  and  $M(r_1, r_2) = true \Rightarrow \mu(r_1, r_2) \in \bar{D}$ .

Notice that, in order to obtain  $D^m$ , tuple identifiers are introduced and kept, whereas under Swoosh, there are no tuple identifiers and new tuples are generated (via  $\mu$ ) and some are deleted (those  $\leq$ -dominated by other tuples). In consequence, the elements of  $D$  and  $D^m$  under the MD framework are of the form  $R(t, s_1, \dots, s_n)$ , and those in  $D$  and  $D^s$  under Swoosh are the records  $r$  of the form  $R(s_1, \dots, s_n)$ . Since  $t$  is a tuple identifier, for every  $R(t, s_1, \dots, s_n)$ ,  $r(t)$  denotes the record  $R(s_1, \dots, s_n)$ .

PROPOSITION 11. (a) For every  $r$  in  $D^s$  there is a tuple in  $D^m$  with tuple identifier  $t$ , such that  $r(t) = r$ . (b) For every tuple  $t \in D^m$ , there is a record  $r \in D^s$ , such that  $r(t) \leq r$ . ■

## 8. CONCLUSIONS

The introduction of matching dependencies (MDs) in [16] has been a valuable addition to data quality and data cleaning research. They can be regarded as *data quality constraints* that are declarative in nature and are based on a precise model-theoretic semantics. They are bound to play an important role in database research and practice, together (and in combination) with classical integrity constraints.

In this work we have made several contributions to the semantics of matching dependencies. We have refined the original semantics introduced in [17], addressing some important open issues, but we have also introduced into the semantic framework the notion of matching function. This is an important ingredient in entity resolution since matching functions indicate how attribute values have to be merged or identified.

The matching functions, under certain natural assumptions, induce lattice-theoretic structures in the attributes' domains. We also investigated their interaction with the similarity relations in those same domains. This led us to introduce a partial order of domination between instances. This allows us to compare them in terms of information content. This same notion was then applied to sets of query answers.

On the basis of all these notions, we defined the class of clean instances for a given dirty instance. They are the intended and admissible instances that could be obtained after enforcing the matching dependencies. The clean instances were defined by means of a chase-like procedure that enforces the MDs, while not making unjustified changes on other attribute values. The chase procedure stepwise improves the information content as related to the domination order.

The notion of clean answer to a query posed to the dirty database was defined as a pair formed by a lower and an upper bound in terms of information content for the query answers. In this context we studied the notion of monotone query w.r.t. to the domination order and how to relax a query into a monotone one that provides more informative answer than the original one.

The domination-monotone relational query language introduced uses the lattice-theoretic structure of the domain, and is interesting in its own right. It certainly deserves further investigation, independently of MDs. It is an interesting open question to explore its connections with querying databases over partially ordered domains, with incomplete or partial information [30, 22, 27, 26]. with query relaxation in general [25, 19], and with relational languages based on similarity relations [23].

We addressed some problems around the enforcement of a set of matching dependencies for purposes of data cleaning based on the original proposal of [16, 17], by explicitly making use of matching functions. We studied issues such as the existence and uniqueness of clean instances, the computational cost of computing them, and the complexity of computing clean answers. We identified cases where clean query answering is tractable, e.g., when there is a single clean instance. However, we established that this problem is intractable in general. We proposed polynomial time approximations.

Identifying other tractable cases and more efficient approaches to the intractable ones is part of ongoing research. We are currently investigating the use of logic programs with stable model semantics in the specification of clean instances and in clean query answering. This idea has been investigated in consistent query answering and has led to useful insights and implementations [4, 20, 7, 14, 12]. This route would avoid the explicit computation of the clean instances, and clean query answering could be done on top of the program. However, the lattice-theoretic structure of the domains and the domination order create a scenario that is substantially different from the one encountered in database

repairs w.r.t. classical integrity constraints.

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## APPENDIX

### A. SOME PROOFS

**Proof of Lemma 1:** (1) Let  $D$  be the instance  $D_1 \curlyvee D_2$ . Clearly,  $D_1 \sqsubseteq D$  and  $D_2 \sqsubseteq D$ . Now let  $D'$  be an arbitrary instance such that  $D_1 \sqsubseteq D'$  and  $D_2 \sqsubseteq D'$ , and let  $t$  be a tuple in  $D$ . Then, by definition,  $t$  is in  $D_1$  or in  $D_2$ , and hence there should be a tuple  $t'$  in  $D'$  such that  $t^D \preceq t'^{D'}$ . Therefore, we have  $D \sqsubseteq D'$ , and thus  $D$  is the least upper bound of  $D_1, D_2$ .

(2) Let  $t$  be the tuple  $t_1 \wedge t_2$ . Clearly,  $t \preceq t_1^{D_1}$  and  $t \preceq t_2^{D_2}$ . Let  $t'$  be an arbitrary tuple such that  $t' \preceq t_1^{D_1}$  and  $t' \preceq t_2^{D_2}$ . Then  $t'[A] \preceq t_1^{D_1}[A]$  and  $t'[A] \preceq t_2^{D_2}[A]$  for every attribute  $A$  in the schema. Thus,  $t'[A] \preceq \text{glb}(t_1^{D_1}[A], t_2^{D_2}[A])$  for every attribute  $A$ , and hence  $t' \preceq t$ .

(3) Let  $D$  be the instance  $D_1 \wedge D_2$ . Let  $t$  be a tuple in  $D$ . Then there exist tuples  $t_1$  in  $D_1$  and  $t_2$  in  $D_2$ , such that  $t = t_1 \wedge t_2$ , and thus  $t^D \preceq t_1^{D_1}$  and  $t^D \preceq t_2^{D_2}$ . Therefore, it holds  $D \sqsubseteq D_1$  and  $D \sqsubseteq D_2$ .

Let  $D'$  be an arbitrary instance such that  $D' \sqsubseteq D_1$  and

$D' \sqsubseteq D_2$ , and let  $t'$  be a tuple in  $D'$ . Then there exist tuples  $t_1$  in  $D_1$  and  $t_2$  in  $D_2$ , such that  $t'^{D'} \preceq t_1^{D_1}$  and  $t'^{D'} \preceq t_2^{D_2}$ , and thus  $t'^{D'} \preceq \text{glb}(t_1^{D_1}, t_2^{D_2})$ , which exists in  $D$ . We thus have  $D' \sqsubseteq D$ . ■

**Proof of Theorem 2:** (sketch) It is easy to see that for every  $i \in [1, k]$ , we have  $D_{i-1} \sqsubseteq D_i$  and  $D_i \not\sqsubseteq D_{i-1}$ . That is,  $D_i$  strictly dominates  $D_{i-1}$  (for simplicity, assume that the new generated tuples are not completely dominated by other tuples). Consider a database instance  $D_{max}$  that has a single tuple in every relation, for which the value of every attribute  $A$  is the result of matching all values of  $A$  (and other attributes comparable with  $A$ ) in the active domain of  $D_0$ . Clearly,  $D_{max}$  provides an upper bound for the instances in the sequence, and thus the sequence stops after finite number of steps. Furthermore, the number of matching applications needed to reach  $D_{max}$  is polynomial in the size of  $D_0$ . ■

**Proof of Proposition 1:** Let  $D, D'$  be two  $(D_0, \Sigma)$ -clean instances. We use two lemmas.

LEMMA 2. Let  $m_A$  be a similarity preserving function, and  $a_1, a_2, a_3, a_4$  be values in the domain  $Dom_A$ , such that  $a_1 \preceq a_3$  and  $a_2 \preceq a_4$ . If  $a_1 \approx_A a_2$ , then  $a_3 \approx_A a_4$ .

LEMMA 3. Let  $D_1, \dots, D_k$  be a sequence of instances such that  $D = D_k$ , and for every  $i \in [1, k]$ ,  $(D_{i-1}, D_i)_{[t_1^i, t_2^i]} \models \varphi^i$ , for some  $\varphi^i \in \Sigma$  and tuple identifiers  $t_1^i, t_2^i$ . Then for every  $i \in [0, k]$ , the following holds:

1.  $t_1^{D_i}[A_1] \preceq t_1^{D'}[A_1]$ , for every tuple identifier  $t_1$  and every attribute  $A_1$ .
2. if  $t_1^{D_i}[A_1] \approx t_2^{D_i}[A_2]$ , then  $t_1^{D'}[A_1] \approx t_2^{D'}[A_2]$ , for every two tuple identifiers  $t_1, t_2$  and two comparable attributes  $A_1, A_2$ .

We prove Lemma 3 by an induction on  $i$ . For  $i = 0$ , we clearly have  $t_1^{D_0}[A_1] \preceq t_1^{D'}[A_1]$  since  $D'$  is a  $(D_0, \Sigma)$ -clean instance. Moreover, if  $t_1^{D_0}[A_1] \approx t_2^{D_0}[A_2]$ , then  $t_1^{D'}[A_1] \approx t_2^{D'}[A_2]$  by Lemma 2.

Suppose 1 and 2 hold for every  $i < j$ . If 1 holds for  $i = j$ , then 2 also holds for  $i = j$  by Lemma 2. Suppose 1 does not hold for  $i = j$ :  $t_1^{D_j}[A_1] \not\preceq t_1^{D'}[A_1]$ . Since 1 holds for every  $i < j$ , the value of  $t_1^{D_j}[A_1]$  should be different from  $t_1^{D_{j-1}}[A_1]$ . Therefore, there should be an MD  $\varphi^j : R_1[X_1] \approx R_2[X_2] \rightarrow R_1[A_1] \approx R_2[A_2]$  in  $\Sigma$  and a tuple identifier  $t_2$ , such that  $D_j$  is the immediate result of enforcing  $\varphi^j$  on  $t_1, t_2$  in  $D_{j-1}$ . That is,  $t_1^{D_{j-1}}[X_1] \approx t_2^{D_{j-1}}[X_2]$ ,  $t_1^{D_{j-1}}[A_1] \neq t_2^{D_{j-1}}[A_2]$ , and  $t_1^{D_j}[A_1] = t_2^{D_j}[A_2] = m_A(t_1^{D_{j-1}}[A_1], t_2^{D_{j-1}}[A_2])$ . Since  $t_1^{D_{j-1}}[X_1] \approx t_2^{D_{j-1}}[X_2]$ , by induction assumption, we have  $t_1^{D'}[X_1] \approx t_2^{D'}[X_2]$ , and thus,  $t_1^{D'}[A_1] = t_2^{D'}[A_2]$ , because  $D'$  is a stable instance. Again by induction assumption,  $t_1^{D_{j-1}}[A_1] \preceq t_1^{D'}[A_1]$  and  $t_2^{D_{j-1}}[A_2] \preceq t_2^{D'}[A_2] = t_1^{D'}[A_1]$ . Therefore,  $t_1^{D_j}[A_1] = m_A(t_1^{D_{j-1}}[A_1], t_2^{D_{j-1}}[A_2]) \preceq t_1^{D'}[A_1]$  since  $m_A$  takes the least upper bound, which leads to a contradiction.

To prove the first part of Proposition 1, notice that, from Lemma 3, we obtain  $t_1^D[A_1] \preceq t_1^{D'}[A_1]$  and  $t_1^{D'}[A_1] \preceq t_1^D[A_1]$  for every tuple identifier  $t_1$  and every attribute  $A_1$ . Thus, the two  $(D_0, \Sigma)$ -clean instances  $D, D'$  should be identical.



To prove the second part of the proposition, let  $\varphi : R_1[X_1] \approx R_2[X_2] \rightarrow R_1[A_1] \rightleftharpoons R_2[A_2]$  be an MD in  $\Sigma$ . By Lemma 3, if  $t_1^{D_0}[X_1] \approx t_2^{D_0}[X_2]$ , then  $t_1^D[X_1] \approx t_2^D[X_2]$ , for every two tuple identifiers  $t_1, t_2$ . Since  $D$  is a stable instance,  $t_1^D[A_1] = t_2^D[A_2]$ , and thus  $(D_0, D) \models \varphi$  and  $(D_0, D) \models \Sigma$ . ■

**Proof of Proposition 2:** Let  $D, D'$  be two  $(D_0, \Sigma)$ -clean instances. It is enough to prove a lemma similar to Lemma 3.

LEMMA 4. Let  $D_1, \dots, D_k$  be a sequence of instances such that  $D = D_k$ , and for every  $i \in [1, k]$ ,  $(D_{i-1}, D_i)_{[t_1^i, t_2^i]} \models \varphi^i$ , for some  $\varphi^i \in \Sigma$  and tuple identifiers  $t_1^i, t_2^i$ . Then for every  $i \in [0, k]$ , it holds

1.  $t_1^{D_i}[X_1] = t_1^{D_0}[X_1]$  and  $t_2^{D_i}[X_2] = t_2^{D_0}[X_2]$ , where  $X_1, X_2$  are lists of attributes on the left-hand side of  $\varphi^i$ .
2.  $t_1^{D_i}[A_1] \leq t_1^{D'}[A_1]$ , for every tuple identifier  $t_1$  and every attribute  $A_1$ .

Notice that 1 trivially holds: since MDs are interaction free, there is no MD  $\varphi \in \Sigma$ , such that the attributes on the right-hand side of  $\varphi$  has an intersection with  $X_1, X_2$ , and therefore no MD enforcement could change the values in  $t_1^{D_i}[X_1]$  or  $t_2^{D_i}[X_2]$  into something different from the original values in  $D_0$ .

We prove 2 by an induction on  $i$ . For  $i = 0$ , we clearly have  $t_1^{D_0}[A_1] \leq t_1^{D'}[A_1]$  since  $D'$  is a  $(D_0, \Sigma)$ -clean instance. Now suppose 2 holds for  $i < j$ , and it does not hold for  $i = j$ :  $t_1^{D_j}[A_1] \not\leq t_1^{D'}[A_1]$ . Then there should be an MD  $\varphi^j : R_1[X_1] \approx R_2[X_2] \rightarrow R_1[A_1] \rightleftharpoons R_2[A_2]$  in  $\Sigma$  and a tuple identifier  $t_2$ , such that  $D_j$  is the immediate result of enforcing  $\varphi^j$  on  $t_1, t_2$  in  $D_{j-1}$ . That is,  $t_1^{D_{j-1}}[X_1] \approx t_2^{D_{j-1}}[X_2]$ ,  $t_1^{D_{j-1}}[A_1] \neq t_2^{D_{j-1}}[A_2]$ , and  $t_1^{D_j}[A_1] = t_2^{D_j}[A_2] = m_A(t_1^{D_{j-1}}[A_1], t_2^{D_{j-1}}[A_2])$ . Since  $t_1^{D_{j-1}}[X_1] \approx t_2^{D_{j-1}}[X_2]$ , by part 1 we have  $t_1^{D'}[X_1] \approx t_2^{D'}[X_2]$ , and thus  $t_1^{D'}[A_1] = t_2^{D'}[A_2]$ , because  $D'$  is a stable instance. By induction assumption,  $t_1^{D_{j-1}}[A_1] \leq t_1^{D'}[A_1]$  and  $t_2^{D_{j-1}}[A_2] \leq t_2^{D'}[A_2] = t_1^{D'}[A_1]$ . Therefore,  $t_1^{D_j}[A_1] = m_A(t_1^{D_{j-1}}[A_1], t_2^{D_{j-1}}[A_2]) \leq t_1^{D'}[A_1]$ , since  $m_A$  takes the least upper bound, which leads to a contradiction. ■

**Proof of Theorem 3:** Consider relation schema  $R(C, V, L)$ , query  $\mathcal{Q} : \pi_L(R)$ , and set  $\Sigma$  consisting of two MDs  $\varphi_1 : R[C] \approx R[C] \rightarrow R[C] \rightleftharpoons R[C]$  and  $\varphi_2 : R[CV] \approx R[CV] \rightarrow R[L] \rightleftharpoons R[L]$ . The domains of attributes, similarity relations, and matching functions are as follows:  $Dom_C = \{\perp, c, c_1, d_1, c_2, d_2, \dots\}$ ,  $Dom_V = \{\perp, y, x_1, x_2, \dots\}$ ,  $Dom_L = \{\perp, \top, +, -\}$ . For every  $c_i, d_i \in Dom_C$ ,  $c_i \approx d_i$  and  $m_C(c_i, d_i) = m_C(d_i, c_i) = c$ . We also have  $m_L(+, -) = m_L(-, +) = \top$ . Notice that similarity relations and match functions are not fully described here. The full descriptions can be derived using the reflexivity and symmetry of similarity relations and idempotency, commutativity, and associativity of match functions.

To prove membership in coNP, it is easy to see that given a certificate, which is a  $(D_0, \Sigma)$ -clean instance  $D$ , we can check whether  $\top \notin \mathcal{Q}(D)$  in polynomial time. To prove hardness, we reduce from 3SAT. Let  $C_1 \wedge \dots \wedge C_N$  be CNF formula, where each clause  $C_i$ ,  $i \in [1, N]$ , is a disjunction of three literals  $l_{i1} \vee l_{i2} \vee l_{i3}$ , and each literal  $l_{ik}$ ,  $k \in [1, 3]$ , is either  $x_j$  or  $\neg x_j$  for some variable  $x_j \in Dom_V$ . We create

an instance  $D_0$  of  $R$  as follows. For every clause  $C_i$  and every literal  $l_{ik}$  of variable  $x_j$  in  $C_i$ , there is a tuple  $t$  with  $t^{D_0}[C] = c_i$ ,  $t^{D_0}[V] = x_j$ ,  $t^{D_0}[L] = +$  if  $l_{ik} = x_j$  (a positive literal), and  $t^{D_0}[L] = -$  if  $l_{ik} = \neg x_j$  (a negative literal). Moreover, for every clause  $C_i$ , there is another tuple  $t$  with  $t^{D_0}[C] = d_i$ ,  $t^{D_0}[V] = y$ , and  $t^{D_0}[L] = +$ .

We show that the CNF formula  $C$  is satisfiable if and only if  $\top \notin \text{Cert}_{\mathcal{Q}}(D_0)$ . Let  $C$  be a satisfiable formula. For each clause  $C_i$ , we pick a tuple corresponding to the literal that is made true in the satisfying assignment and also the only tuple with  $t^{D_0}[C] = d_i$ , and enforce the MD  $\varphi_1$  on these two tuples. It is easy to see that the result would be a stable instance  $D$ . In particular,  $(D, D) \models \varphi_2$  since for each variable the satisfying assignment has picked only one of the positive or negative literals to be true. Therefore, we do not need to enforce  $\varphi_2$ , which means that  $\top$  does not appear for any value of attribute  $L$ , and hence  $\top \notin \mathcal{Q}(D)$ , and  $\top \notin \text{Cert}_{\mathcal{Q}}(D_0)$ .

Conversely, if  $\top \notin \text{Cert}_{\mathcal{Q}}(D_0)$ , there is a  $(D_0, \Sigma)$ -clean instance  $D$  in which  $\top$  does not appear for any value of attribute  $L$ . To obtain the clean instance  $D$  starting from  $D_0$ , we need to enforce  $\varphi_1$  once for each clause  $C_i$ , as described above, before we can enforce  $\varphi_2$  on any tuple corresponding to  $C_i$ . Moreover, for every two tuples in  $D$  that match the left-hand side of  $\varphi_2$ , we should have identical values for attribute  $L$  (either  $+$  or  $-$ ), otherwise we would get  $\top$  when enforcing  $\varphi_2$ . Therefore, for each clause, we can make true the literal corresponding to the tuple on which  $\varphi_1$  has been enforced, and obtain a correct satisfying assignment. ■

**Proof of Proposition 3:** For every instance  $D' \in \mathcal{D}$ , we clearly have  $\text{glb}_{\sqsubseteq}\{D \mid D \in \mathcal{D}\} \sqsubseteq D'$ , since  $\mathcal{Q}$  is a monotone query, it holds  $\mathcal{Q}(\text{glb}_{\sqsubseteq}\{D \mid D \in \mathcal{D}\}) \sqsubseteq \mathcal{Q}(D')$ . Consequently,  $\mathcal{Q}(\text{glb}_{\sqsubseteq}\{D \mid D \in \mathcal{D}\}) \sqsubseteq \text{glb}_{\sqsubseteq}\{\mathcal{Q}(D') \mid D' \in \mathcal{D}\}$ . With a similar argument, we can show that the second equation of Proposition 3 holds. ■

**Proof of Proposition 4:** (sketch) We can prove the proposition by an structural induction on the relational algebra expression. It is enough to show that every operation in  $\mathcal{RA}_{\leq}$  is monotone. Projection, Cartesian product, and union are clearly monotone operators w.r.t.  $\sqsubseteq$ . Now let  $D, D'$  be two instances such that  $D \sqsubseteq D'$ . Consider query  $\mathcal{Q} : \sigma_{a \leq A} R$  for relation  $R$  in the schema. Let  $t$  be an  $R$ -tuple in  $\mathcal{Q}(D)$ . Clearly  $t$  is an  $R$ -tuple in  $D$ . Therefore, there is an  $R$ -tuple  $t'$  in  $D'$  with  $t \leq t'$ . Now it holds  $a \leq t^D[A] \leq t'^{D'}[A]$ , and thus  $t'$  is in  $\mathcal{Q}(D')$ .

Now consider the query  $\mathcal{Q}' : \sigma_{A_1 \bowtie_{\leq} A_2} R$ , and let  $t$  be an  $R$ -tuple in  $\mathcal{Q}'(D)$ . Then there is  $a \in Dom_A$  s.t.  $a \leq t^D[A_1]$ ,  $a \leq t^D[A_2]$ , and  $a \neq \perp$ . Since  $D \sqsubseteq D'$ , there should be an  $R$ -tuple  $t'$  in  $D'$  with  $t \leq t'$ . Now it holds  $a \leq t^D[A_1] \leq t'^{D'}[A_1]$  and  $a \leq t^D[A_2] \leq t'^{D'}[A_2]$ . Therefore,  $t'$  is in  $\mathcal{Q}'(D')$ . ■

**Proof of Proposition 6:** The proof of this proposition is very similar to that of Proposition 2. Let  $D_{\downarrow}, D'_{\downarrow}$  be two  $(D_0, \Sigma)$ -under clean instances. It is enough to prove the following lemma.

LEMMA 5. Let  $D_1, \dots, D_k$  be a sequence of instances for deriving  $D_{\downarrow}$  as described in Definition 9. Then for every  $i \in [0, k]$ , it holds

1.  $t_1^{D_i}[X_1] = t_1^{D_0}[X_1]$  and  $t_2^{D_i}[X_2] = t_2^{D_0}[X_2]$ , where  $X_1, X_2$  are lists of attributes on the left-hand side of  $\varphi^i$ .
2.  $t_1^{D_i}[A_1] \preceq t_1^{D_0}[A_1]$ , for every tuple identifier  $t_1$  and every attribute  $A_1$ .

Suppose that for some  $i \in [0, k]$ ,  $t_1^{D_i}[X_1] \neq t_1^{D_0}[X_1]$ . Then there exists  $j < i$ , tuple  $t_3$ , and MD  $\varphi^j \in \Sigma$ , such that  $(D_{j-1}, D_j)_{[t_1, t_3]} \models \varphi^j$ , with attribute  $B_1 \in X_1$  on the right-hand side of  $\varphi^j$ . MD  $\varphi^j$  has to be safely applicable on  $t_1, t_3$  in  $D_0$ , which means that  $\varphi^j$  cannot be safely applicable on  $t_1, t_2$  in  $D_0$ , a contradiction. The proof of part 2 is similar to the proof of part 2 in Lemma 4. ■

**Proof of Proposition 7:** Let  $D_\downarrow$  be a  $(D_0, \Sigma)$ -under clean instance, and  $D$  be a  $(D_0, \Sigma)$ -clean instance. The proof follows from the following two lemmas.

LEMMA 6. For every two tuples  $t_1, t_2$  and MD  $\varphi : R_1[X_1] \approx R_2[X_2] \rightarrow R_1[A_1] \Leftarrow R_2[A_2]$  in  $\Sigma$ , such that  $\varphi$  is safely applicable on  $t_1, t_2$  in  $D_0$ , it holds  $t_1^D[X_1] = t_1^{D_0}[X_1]$  and  $t_2^D[X_2] = t_2^{D_0}[X_2]$ .

LEMMA 7. Let  $D_1, \dots, D_k$  be a sequence of instances for deriving  $D_\downarrow$  as described in Definition 9. Then for every  $i \in [0, k]$ , it holds  $t_1^{D_i}[A_1] \preceq t_1^D[A_1]$ , for every tuple identifier  $t_1$  and every attribute  $A_1$ .

The proof of lemma is by induction on  $i$  and is very similar to the proof part 2 in Lemma 4. ■

**Proof of Proposition 9:** In fact: If  $s_1 \approx_{\{A\}} s_2$ , then there are  $a_1 \in s_1, a_2 \in s_2$  with  $a_1 \approx_A a_2$ . Since  $a_2$  also belongs to  $s_2 \cup s_3$ , for every  $s_3 \in \text{Dom}_A$ , it holds  $s_2 \cup s_3 = m_{\{A\}}(s_2, s_3) \approx_{\{A\}} s_1$ . ■

**Proof of Proposition 10:** For  $\preceq_{\{A\}}$  on  $\text{Dom}_A$  it holds:  $s \preceq_{\{A\}} s' \Leftrightarrow m_{\{A\}}(s, s') = s' \Leftrightarrow s \cup s' = s' \Leftrightarrow s \subseteq s'$ . Now, for records  $r_1 = R(s_1^1, \dots, s_n^1), r_2 = R(s_1^2, \dots, s_n^2)$ , it holds  $r_1 \preceq r_2 \Leftrightarrow$  for every  $i, s_i^1 \preceq_{\{A\}} s_i^2 \Leftrightarrow$  for every  $i, s_i^1 \subseteq s_i^2$ .

On the other side, from (7) we obtain that, for records  $r_1 = R(s_1^1, \dots, s_n^1), r_2 = R(s_1^2, \dots, s_n^2)$ , it holds:  $r_1 \leq r_2 \Leftrightarrow M(r_1, r_2) = \text{true}$  and for every  $i, s_i^1 \subseteq s_i^2$ . Since the  $s_i^2$  are non-empty, the first condition on the RHS is implied by the second one. ■

**Proof of Proposition 11:** (sketch) As a preliminary and useful remark, let us mention that the  $I^s C^s A^s R^s$  properties make  $\leq$  a partial order with the following monotonicity properties [8]: (A)  $M(r_1, r_2) = \text{true} \implies r_1 \leq \mu(r_1, r_2)$  and  $r_2 \leq \mu(r_1, r_2)$ . (B)  $r_1 \leq r_2$  and  $M(r_1, r) = \text{true} \implies M(r_2, r) = \text{true}$ . (C)  $r_1 \leq r_2$  and  $M(r_1, r) = \text{true} \implies \mu(r_1, r) \leq \mu(r_2, r)$ . (D)  $r_1 \leq s, r_2 \leq s$  and  $M(r_1, r_2) = \text{true} \implies \mu(r_1, r_2) \leq s$ .

More specifically for our proof, first notice that every application of  $\mu$  can be simulated by a finite sequence of enforcement of the MDs in (6). More precisely, given two tuples  $R(t_1, \bar{s}^1), R(t_2, \bar{s}^2)$  in an instance  $D$ , such that  $M(r(t_1), r(t_2))$  holds, then  $\mu(r(t_1), r(t_2)) = r(t)$  for some tuple  $R(t, r(t))$  of the form  $m_{\{A_{i_1}\}} \dots m_{\{A_{i_n}\}}(R(t_1, \bar{s}^1), R(t_2, \bar{s}^2))$ , i.e., obtained by enforcing the MDs. Furthermore, it holds  $r(t_1) \leq r(t)$  and  $r(t_2) \leq r(t)$ .

Conversely, every enforcement of an MD in (6) is dominated by a tuple obtained through the application of  $\mu$ . More precisely, for tuples  $R(t_1, \bar{s}^1), R(t_2, \bar{s}^2)$  in an instance  $D$  for which  $s_j^1 \approx_{\{A_j\}} s_j^2$  holds, it also holds  $M(t_1(r), t_2(r))$ , and  $m_{\{A_j\}}(R(t_1, \bar{s}^1), R(t_2, \bar{s}^2)) \leq R(t, \mu(t_1(r), t_2(r)))$  for some tuple id  $t$  (actually,  $t_1$  or  $t_2$ ).

Now, for (a), consider  $D^m \downarrow := \{r(t) \mid R(t, \bar{s}) \in D^m\}$  (from where duplicates are eliminated). It is good enough to prove that  $D^s \subseteq D^m \downarrow$ . For this it suffices to prove that  $D^m \downarrow$  satisfies conditions 1. and 2. on the entity resolution instance, namely: 1.  $D^m \downarrow \subseteq \bar{D}$  and 2.  $\bar{D} \leq D^m \downarrow$ . The first condition follows from the definition (or construction) of  $D^m$  as a stable instance obtained by minimally applying the MDs and when justified only. The second condition follows from the simulation and properties of  $\mu$  as a finitely long enforcement of the MDs.

Now (b) follows from the domination of a tuple obtained by applying one MD by a tuple obtained applying  $\mu$  as described above. ■