

A General Framework for Graph Sparsification

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Abstract

Given a weighted graph G and an error parameter $\varepsilon > 0$, the *graph sparsification* problem requires sampling edges in G and giving the sampled edges appropriate weights to obtain a sparse graph G_ε with the following property: the weight of every cut in G_ε is within a factor of $(1 \pm \varepsilon)$ of the weight of the corresponding cut in G . Benczúr and Karger [2] showed how to obtain G_ε with $O(n \log n / \varepsilon^2)$ edges in time $O(m \log^3 n)$ for weighted graphs and $O(m \log^2 n)$ for unweighted graphs using a combinatorial approach based on strong connectivity. Spielman *et al* [22] showed how to obtain G_ε with $O(n \log n / \varepsilon^2)$ edges in time $O(m \log^c n)$ for some (large) constant c using an algebraic approach based on effective resistances. Our contributions are as below (all for weighted graphs G with n vertices and m edges having polynomial-sized weights, unless otherwise stated):

- Benczúr and Karger [2] conjectured that using standard connectivity instead of strong connectivity for sampling would simplify the result substantially, and posed this as an open question. In this correspondence, we resolve this question by showing that sampling using standard connectivities also preserves cut weights and yields a G_ε with $O(n \log^2 n / \varepsilon^2)$ edges.
- We provide a very simple strictly linear time algorithm (i.e. $O(m)$ time) for graph sparsification that yields a G_ε with $O(n \log^2 n / \varepsilon^2)$ edges.
- We provide another algorithm for graph sparsification that yields a G_ε with $O(n \log n / \varepsilon^2)$ edges in $O(m \log^2 n)$ time (for unweighted graphs, this reduces to $O(m \log n)$ time).
- Combining the above two results, we obtain the fastest known algorithm for obtaining a G_ε with $O(n \log n / \varepsilon^2)$ edges; this algorithm runs in time $O(m + n \log^4 n / \varepsilon^2)$ whereas the previous best bound is $O(m \log^3 n)$.
- If G has arbitrary edge weights, we give an $O(m \log^2 n)$ -time algorithm that yields a G_ε containing $O(n \log^2 n / \varepsilon^2)$ edges. The previous best bound is $O(m \log^3 n)$ time for a G_ε with $O(n \log n / \varepsilon^2)$ edges.
- Most importantly, we provide a generic framework that sets out sufficient conditions for any particular sampling scheme to result in good sparsifiers; all the above results can be obtained by simple instantiations of this framework, as can known results on sampling by strong connectivity and sampling by effective resistances¹.

Our algorithms are Monte-Carlo, i.e. work with high probability, as are all efficient algorithms for graph sparsification.

A key ingredient of our proofs is a generalization of bounds on the number of small cuts in an undirected graph due to Karger [8]; this generalization might be of independent interest.

¹ with a G_ε that is slightly denser than the best-known result for the effective resistance case.

1 Introduction

A *cut* of an undirected graph is a partition of its vertices into two disjoint sets. The *weight* of a cut is the sum of weights of the edges crossing the cut, i.e. edges having one endpoint each in the two vertex subsets of the partition. For unweighted graphs, each edge is assumed to have unit weight. Cuts play an important role in many problems in graphs: e.g., the maximum flow between a pair of vertices is equal to the minimum weight cut separating them.

A *skeleton* G' of an undirected graph G is a subgraph of G on the same set of vertices where each edge in G' can have an arbitrary weight. In a series of results, Karger [9, 10] showed that an appropriately weighted sparse skeleton generated by *random sampling* of edges approximately preserves the weight of *every* cut in an undirected graph. This series of results culminated in a seminal work by Benczúr and Karger [2] that showed the following theorem. Throughout this paper, for any undirected graph G and any $\varepsilon \in (0, 1]$, $(1 \pm \varepsilon)G$ is the set of all appropriately weighted subgraphs of G where the weight of every cut in the subgraph is within a factor of $(1 \pm \varepsilon)$ of the weight of the corresponding cut in G .

Theorem 1 (Benczúr-Karger [2]). *For any undirected graph G with m edges and n vertices, and for any error parameter $\varepsilon \in (0, 1]$, there exists a skeleton G_ε containing $O(\frac{n \log n}{\varepsilon^2})$ edges such that $G_\varepsilon \in (1 \pm \varepsilon)G$ with high probability.² Further, such a skeleton can be found in $O(m \log^2 n)$ time if G is unweighted and $O(m \log^3 n)$ time otherwise.*

Besides its combinatorial ramifications, the importance of this result stems from its use as a pre-processing step in several graph algorithms, e.g. to obtain an $\tilde{O}(n^{3/2} + m)$ -time algorithm for approximate maximum flow using the $\tilde{O}(m\sqrt{m})$ -time algorithm for exact maxflow due to Goldberg and Rao [6]; and more recently, $\tilde{O}(n^{3/2} + m)$ -time algorithms for approximate sparsest cut [12, 20].

Subsequent to Benczúr and Karger's work, Spielman and Teng [23, 24] extended their results to preserving all quadratic forms, of which cuts are a special case; however, the size of the skeleton constructed was $O(n \log^c n)$ for some large constant c . Spielman and Srivastava [22] improved this result by constructing skeletons of size $O(\frac{n \log n}{\varepsilon^2})$ in $O(m \log^{O(1)} n)$ time, while continuing to preserve all quadratic forms. Recently, this result was further improved by Batson *et al* [1] who gave a deterministic algorithm for constructing skeletons of size $O(\frac{n}{\varepsilon^2})$. While their result is optimal in terms of the size of the skeleton constructed, the time complexity of their algorithm is $O(\frac{mn^3}{\varepsilon^2})$, rendering it somewhat useless in terms of applications.

Benczúr and Karger [2], and Spielman *et al* [23, 24, 22, 1] use contrasting techniques to obtain their respective results; the former use combinatorial graph techniques while the latter use algebraic graph techniques. In each case, the goal is to obtain a probability value p_e for each edge e so that sampling each edge e independently with probability p_e and giving each sampled edge e a weight $1/p_e$ yields $G_\varepsilon \in (1 \pm \varepsilon)G$. Benczúr and Karger [2] choose p_e inversely proportional to the *strong connectivity* of e while Spielman *et al* [23, 24, 22, 1] choose p_e proportional to the *effective resistance* of e (both concepts are defined below).

Definition 1. *The strong connectivity of an edge (u, v) in an undirected graph G is the maximum value of k such that there is an induced subgraph G' of G containing both u and v , and every cut in G' has weight at least k .*

Definition 2. *The effective resistance of an edge (u, v) in an undirected graph G is the effective electrical resistance between u and v if each edge in G is replaced by an electrical resistor between its endpoints whose electrical resistance is equal to the weight of the edge.*

²We say that a property holds *with high probability* (or whp) for a graph on n vertices if its failure probability can be bounded by the inverse of a fixed polynomial in n .

1.1 Our Results

We obtain the following results.

The Generic Framework. We provide a general proof framework as follows. For any given sampling scheme (i.e., assignment to the p_e 's), we show that if this assignment satisfies two sufficient conditions, then the sampling scheme results in good sparsifiers. All of the results stated below are then simple instantiations of the above framework, i.e. we show that the sufficient conditions hold. The resulting algorithms are also much simpler than those in [2] or in [22, 1].

Faster Algorithms. Our first result is an efficient algorithm for constructing a sparse skeleton.

Theorem 2. *Suppose G is an undirected graph with n vertices and m edges. Then, for any fixed $\varepsilon \in (0, 1]$, there is an efficient algorithm for finding a skeleton G_ε of G having $O(\frac{n \log n}{\varepsilon^2})$ edges in expectation such that $G_\varepsilon \in (1 \pm \varepsilon)G$ whp. The time complexity of the algorithm is $O(m + n \log^4 n / \varepsilon^2)$ if the weights of all edges are bounded by a fixed polynomial in n (including all unweighted graphs).*

This is the first sampling algorithm that runs in time strictly linear in m ; all previous algorithms had a time bound of at least $O(m \log^2 n)$ for unweighted graphs, and $O(m \log^3 n)$ for weighted graphs. This algorithm improves the time complexity of several problems, where creating a graph sparsifier in the first step. We mention some of these applications.

- This yields an $O(m) + \tilde{O}(n^{3/2}/\varepsilon^3)$ -time algorithm for finding the ε -approximate maximum flow between two vertices of an undirected graph using the exact maxflow algorithm in [6]. The previous best algorithm had a running time of $O(m \log^3 n) + \tilde{O}(n^{3/2}/\varepsilon^3)$.
- This yields an $O(m) + \tilde{O}(n^{3/2})$ -time algorithm for finding an $O(\log n)$ -approximate sparsest cut [12, 20], and an $O(m) + \tilde{O}(n^{3/2+\delta})$ -time algorithm for finding an $O(\sqrt{\log n})$ -approximate sparsest cut for any constant δ [20]. The previous best algorithms had running time of $O(m \log^3 n) + \tilde{O}(n^{3/2})$ and $O(m \log^3 n) + \tilde{O}(n^{3/2+\delta})$ respectively.

The sampling algorithm in Theorem 2 is obtained by composing two different algorithms described below. The first algorithm is fast but generates a slightly denser skeleton. The second (slower) algorithm then operates on this skeleton to obtain a smaller skeleton.

Theorem 3. *Suppose G is an undirected graph with n vertices and m edges. Then, for any fixed $\varepsilon \in (0, 1]$, there is an efficient algorithm for finding a skeleton G_ε of G having $O(\frac{n \log^2 n}{\varepsilon^2})$ edges in expectation such that $G_\varepsilon \in (1 \pm \varepsilon)G$ whp. The time complexity of the algorithm is $O(m)$ if the weights of all edges are bounded by a fixed polynomial in n (including all unweighted graphs), and $O(m \log^2 n)$ if the edges have arbitrary weights.*

Theorem 4. *Suppose G is an undirected graph with n vertices and m edges. Then, for any fixed $\varepsilon \in (0, 1]$, there is an algorithm for finding a skeleton G_ε of G having $O(\frac{n \log n}{\varepsilon^2})$ edges in expectation such that $G_\varepsilon \in (1 \pm \varepsilon)G$ whp. The time complexity of the algorithm is $O(m \log n)$ for unweighted graphs and $O(m \log^2 n)$ if the weights of all edges are bounded by a fixed polynomial in n (including all unweighted graphs).*

Sampling by Standard Connectivity, Effective Resistances and Strong Connectivity. In proving Theorem 1, the authors had to use strong connectivity because the more natural notion of *standard connectivities* seemed to pose complications.

Definition 3. *The standard connectivity, or simply connectivity, of an edge (u, v) in an undirected graph G is the maximum flow between u and v in G .*

The authors conjectured that using standard connectivity instead of strong connectivity for sampling would simplify the result substantially, and posed this as their main open question. In this correspondence, we resolve this question by showing that sampling using standard connectivities also preserves cut weights.

Theorem 5. *Suppose G is an undirected graph on n vertices. For any fixed $\varepsilon \in (0, 1]$, let G_ε be a skeleton of G formed by sampling edge e in G with probability³ $p_e = \min(\frac{96(3+\lg n)\ln n}{0.38k_e\varepsilon^2}, 1)$, where k_e is the standard connectivity of edge e in G . If selected in the sample, edge e is given a weight of $1/p_e$ in the skeleton. Then, G_ε has $O(\frac{n\log^2 n}{\varepsilon^2})$ edges in expectation and $G_\varepsilon \in (1 \pm \varepsilon)G$ whp.*

Observe that the size of the skeleton constructed using standard connectivity has an extra $\log n$ factor compared to that constructed using strong connectivity. We conjecture that this factor can indeed be removed by more careful analysis.

We show that exactly the same proof as above holds if we replace standard connectivity with *effective resistance* of an edge. Thus, we show that sampling edges using effective resistances also produces a sparse skeleton that approximately preserves all cut weights, a result independently obtained by Spielman and Srivastava recently for the larger class of all quadratic forms (cuts are a special type of quadratic forms) with a tighter bound on the size of the skeleton [22]. Our result, though weaker, has a much simpler proof.

We also show that the results obtained in [2] using strong connectivity can be obtained as a simple instantiation of our general sampling framework.

Generalizations of Cut Counting. The *edge connectivity* of an undirected graph is the minimum weight of a cut in the graph. A key ingredient in the proof of Theorem 1 is a celebrated theorem due to Karger [8]) that gives tight bounds on the number of distinct cuts of a fixed weight in an undirected graph in terms of the ratio of the weight of the cuts to the edge connectivity of the graph.

Theorem 6 (Karger [8]). *For an undirected graph with edge connectivity c and for any $\alpha \geq 1$, the number of cuts of weight at most αc is at most $O(n^{2\alpha})$.*

While this theorem is extremely useful in bounding the number of *small* cuts in an undirected graph (e.g. in sampling [9, 10, 2], network reliability [11], etc.), it does not shed any light on the distribution of edges according to their connectivities in cuts. We generalize the above theorem and show that though there may be many distinct cuts of a fixed large weight in a graph, there are a small number of distinct sets of edges in these cuts if we restrict our attention to only edges with large (standard) connectivity. To state our theorem precisely, we need to introduce the notion of *k-heavy* and *k-light* edges, and that of the *k-projection* of a cut.

Definition 4. *An edge is said to be k-heavy if it has connectivity at least k , and k-light otherwise. The k-projection of a cut is the set of k-heavy edges in the cut.*

Since every edge has connectivity at least c , Theorem 6 can be interpreted as bounding the number of distinct *k-projections* of cuts of size αk by $O(n^{2\alpha})$ for $k = c$. We generalize this result to arbitrary values of k .

³ $\ln n = \log_e n; \lg n = \log_2 n.$

Theorem 7. *For any undirected graph with edge connectivity c and for any $k \geq c$ and any $\alpha \geq 1$, the number of distinct k -projections of cuts of weight at most αk is at most $n^{2\alpha}$.*

We believe this theorem will be of independent interest.

Roadmap. In section 2, we describe our generic sampling framework, and provide one example of instantiating this framework that proves Theorem 3 for the unweighted case. In section 3, we prove Theorem 7 and use it to prove Theorem 8, the main framework theorem stated in section 2. In section 4, we give two sampling algorithms for graphs with polynomial edge weights: the first algorithm constructs skeletons containing $O(\frac{n \log^2 n}{\varepsilon^2})$ edges in expectation and has time complexity $O(m)$, thus proving Theorem 3 for the polynomial weights case; the second algorithm constructs skeletons containing $O(\frac{n \log n}{\varepsilon^2})$ edges in expectation and has time complexity $O(m \log n)$ for unweighted graphs, and $O(m \log^2 n)$ for graphs with polynomial edge weights, thus proving Theorem 4. Combining these two theorems proves Theorem 2. In section 5, we prove Theorem 5 and show that results on sampling by effective resistances and sampling by strong connectivities can also be derived from our framework. Finally, in section 6, we give a sampling algorithm for graphs with arbitrary edge weights that constructs skeletons containing $O(\frac{n \log^2 n}{\varepsilon^2})$ edges in expectation and has time complexity $O(m \log^2 n)$, thus proving Theorem 3 for the arbitrary weights case.

2 The Generic Framework

We describe a generic sampling framework—each of our individual sampling schemes is obtained by a particular setting of parameters of this generic framework.

Suppose $G = (V, E)$ is an undirected graph where edge $e \in E$ has weight w_e . We will assume throughout that w_e is a positive integer. Let $G_M = (V, E_M)$ denote the multi-graph constructed by replacing each edge e by w_e unweighted parallel edges e_1, e_2, \dots, e_{w_e} . Consider any $\varepsilon \in (0, 1]$. We construct a skeleton G_ε where each edge $e_\ell \in E_M$ is present in graph G_ε independently with probability p_e , and if present, it is given a weight of $1/p_e$. (For algorithmic efficiency, observe that an identical skeleton can be created by assigning to edge e a weight of R_e/p_e where R_e is generated from the binomial distribution $B(w_e, p_e)$; this can be done in time $O(w_e p_e)$ rather than time $O(w_e)$ (see e.g. [7])).

What values of p_e result in a sparse G_ε that satisfies $G_\varepsilon \in (1 \pm \varepsilon)G$? Let $p_e = \min(\frac{96\alpha \ln n}{0.38\lambda_e \varepsilon^2}, 1)$, where α is independent of e and λ_e is some parameter of e satisfying $\lambda_e \leq 2^n - 1$. The exact choice of values for α and the λ_e 's will vary from application to application. However, we describe below a sufficient condition that characterizes a good choice of α and λ_e 's.

To describe this sufficient condition, partition the edges in G_M according to the value of λ_e into sets F_0, F_1, \dots, F_k where $k = \lceil \lg \max_{e \in E} \{\lambda_e\} \rceil \leq n - 1$ and $e_i \in F_j$ iff $2^j \leq \lambda_e \leq 2^{j+1} - 1$. Now, let $\mathcal{G} = G_0, G_1, G_2, \dots, G_i = (V, E_i), \dots, G_k$ be a set of subgraphs of G_M (we allow edges of G_M to be replicated multiple times in the G_i s) such that $F_i \subseteq E_i$ for every i . \mathcal{G} is said to be a (π, α) -certificate corresponding to the above choice of α and λ_e 's if the following properties are satisfied:

π -connectivity For $i \geq 0$, any edge $e_\ell \in F_i$ is π -heavy in G_i .

α -overlap For any cut C containing c edges in G_M , let $e_i^{(C)}$ be the number of edges that cross C in G_i . Then, for all cuts C , $\sum_{i=0}^k \frac{e_i^{(C)} 2^{i-1}}{\pi} \leq \alpha c$.

Theorem 8 describes the sufficient condition; its proof appears later in section 3. The intuition for this proof is as follows. Consider all cuts C in G_M ; restrict each cut to just the edges in F_i (we do this because

edges in F_i have roughly the same sampling probabilities, which enables an easy application of Chernoff bounds). How many such distinct F_i -restricted cuts are there? Organize all cuts C in G_M into doubling categories, each comprising cuts with roughly equal values of $e_i^{(C)}$; now using Theorem 7 as applied to G_i and the π -connectivity property above, we can conclude that this count is $n^{O(e_i^{(C)}/\pi)}$ per category. Next, for a particular cut C and its F_i -restriction, we need to apply an appropriate Chernoff bound with a carefully chosen deviation-from-expectation parameter so that this deviation has probability at most $n^{-\Omega(e_i^{(C)}/\pi)}$; this probability offsets the above count, thereby allowing us to claim that this deviation holds for all cuts in one doubling category (and the number of categories is not too many, so the same fact extends across categories as well). The actual value of this deviation comes out to be $O(\varepsilon) \cdot \frac{e_i^{(C)}}{\pi} \cdot \frac{2^{i-1}}{\alpha}$. The α -overlap property now allows us to bound the sum of this deviation over all i , $0 \leq i \leq k$, by εc , as required.

Theorem 8. *If there exists a (π, α) -certificate for a particular choice of α and λ_e 's, then the skeleton $G_\varepsilon \in (1 \pm \varepsilon)G$ with probability at least $1 - 4/n$. Further G_ε has $O(\frac{\alpha \log n}{\varepsilon^2} \sum_{e \in E} \frac{w_e}{\lambda_e})$ edges in expectation.*

2.1 A Simple Algorithm for Unweighted Graphs

We show how we can instantiate the above framework with specific values of α , λ_e 's to obtain a very simple sampling algorithm that runs in $O(m)$ time and obtains a skeleton of size $O(\frac{n \log^2 n}{\varepsilon^2})$. This proves Theorem 3 for the unweighted case.

In order to present our sampling algorithm, we need to define the notion of *spanning forests*. As earlier, G denotes a graph with integer edge weights w_e for edge e and G_M is the unweighted multi-graph where e is replaced with w_e parallel unweighted edges.

Definition 5. *A spanning forest T of G_M (or equivalently of G) is an (unweighted) acyclic subgraph of G satisfying the property that any two vertices are connected in T if and only if they are connected in G .*

We partition the set of edges in G_M into a set of forests T_1, T_2, \dots using the following rule: T_i is a spanning forest of the graph formed by removing all edges in T_1, T_2, \dots, T_{i-1} from G_M such that for any edge $e \in G$, all its copies in G_M appear in a set of contiguous forests $T_i, T_{i+1}, \dots, T_{i+w_e-1}$. This partitioning technique was introduced by Nagamochi and Ibaraki in [19], and these forests are known as *Nagamochi-Ibaraki forests* (or NI forests). The following is a basic property of NI forests.

Lemma 1 (Nagamochi-Ibaraki [19, 18]). *For any pair of vertices u, v , they are connected in NI forests $T_1, T_2, \dots, T_{k(u,v)}$ for some $k(u, v)$ and not connected in any forest T_j , for $j > k(u, v)$.*

Nagamochi and Ibaraki also gave an algorithm for constructing NI forests that runs in $O(m+n)$ time if G_M is a simple graph (i.e. G is unweighted) and $O(m+n \log n)$ time otherwise [19, 18]. Note that our sampling schemes are relevant only when $m > n \log n$; therefore, the NI forests can be constructed in $O(m)$ time for all relevant input graphs.

We set λ_e to the index of the NI forest that e appears in, and set $\alpha = 2$ and $\pi = 2^{i-1}$. For any $i > 0$, let G_i contain all edges in NI forests $T_{2^{i-1}}, T_{2^{i-1}+1}, \dots, T_{2^i-1}$; let $G_0 = F_0 = T_1$. Each edge in F_i appears exactly once in G_i , once in G_{i+1} , and does not appear at all in any of the other G_j 's, $j \neq i, i+1$. This proves α -overlap. Further, for any edge $e \in F_i$, $i > 0$, Lemma 1 ensures that the endpoints of e are connected in each of $T_{2^{i-1}}, T_{2^{i-1}+1}, \dots, T_{2^i-1}$. It follows that e is 2^{i-1} -heavy in G_i , thereby proving π -connectivity. We can now invoke Theorem 8 and conclude that this sampling scheme results in $G_\varepsilon \in (1 \pm \varepsilon)G$ with probability at least $1 - 4/n$. It remains to bound the number of edges in G_ε , as follows.

Since $w_e = 1$ for each edge e and the total number of NI forests K is at most n^2 , we have

$$\sum_{e \in E} \frac{w_e}{\lambda_e} = \sum_{e \in E} \frac{1}{\lambda_e} = \sum_{j=1}^K \sum_{e \in T_j} \frac{1}{\lambda_e} = \sum_{j=1}^K \sum_{e \in T_j} \frac{1}{j} \leq (n-1) \sum_{j=1}^K \frac{1}{j} = O(n \log K) = O(n \log n).$$

It follows from Theorem 8 that G_ε has $O(\frac{n \log^2 n}{\varepsilon^2})$ edges.

The time complexity for constructing the NI forests is $O(m)$ and that for sampling is $O(1)$ per edge giving another $O(m)$; so overall, the algorithm takes $O(m)$ time.

3 Proofs of Main Theorems

In this section, we will first prove Theorem 7, and then use it to prove Theorem 8. Let us start by defining k -heavy and k -light vertices.

Definition 6. A vertex in an undirected graph is said to be k -heavy if at least one edge incident on the vertex is k -heavy; otherwise, the vertex is said to be k -light.

We need the following property of k -heavy vertices.

Lemma 2. The sum of weights of edges incident on a k -heavy vertex is at least k .

Proof. For any k -heavy vertex v , there exists some other vertex u such that the maxflow between u and v is at least k . Thus, any cut separating u and v must have weight at least k ; in particular, this holds for the cut containing only v on one side. \square

Suppose G is an any weighted undirected graph. We scale up the weights of all edges in G uniformly until the weight of every edge is an even integer; call this graph G_S . We replace each edge $e = (u, v)$ of weight w_e in G_S with w_e parallel unweighted edges between u and v to form an unweighted multi-graph G_M . Clearly, any cut in G_M has an even number of edges. Theorem 7 holds for any value of k in G if and only if it holds for any even integer k in G_M . Therefore, it suffices to prove Theorem 7 for all even integers k on unweighted multigraphs where the weight of every cut is even. We also assume that G_M is connected; if not, the theorem holds for the entire graph since it holds for each connected component.

We introduce two operations on undirected multigraphs: *splitting-off* and *edge contraction*. The splitting-off operation was introduced by Lovász in [13, 14] (ex. 6.53):

Definition 7. A pair of edges (s, u) and (u, t) are said to be split-off in an undirected multigraph if they are replaced by a single edge (s, t) .

Various properties of the splitting-off operation have been explored [15, 16, 5, 25]. We need the following property.

Definition 8. For any $k > 0$, a splitting-off operation is said to be k -preserving if all edges in the graph (except those being split-off) that were k -heavy before the splitting-off continue to be k -heavy after the splitting-off.

The following lemma is a corollary of a deep result of Mader [15] for splitting-off edges while maintaining the maxflows of pairs of vertices; however, we give a much simpler direct proof of this lemma here.

Lemma 3. *Suppose G_M is an undirected multigraph where every cut contains an even number of edges. Let $k > 0$ be any even integer. Then, for any k -light non-isolated vertex u in G_M , there exists a pair of edges (s,u) and (u,t) such that splitting-off this pair is k -preserving.*

Proof. We will prove that for every edge (s,u) , there exists an edge (u,t) such that splitting-off this pair of edges retains the following property: *any pair of vertices x,y that were k -connected (i.e. had a maxflow of at least k) before the splitting-off continue to be so after the splitting-off.* We define a k -separator to be any cut that separates at least one pair of k -connected vertices, and call a k -separator with exactly k edges a *tight* cut. Since all cuts have even number of edges and the weight of a cut can decrease by at most 2 due to a splitting-off operation, we only need to ensure that we do not decrease the number of edges in any tight cut when we split-off a pair of edges.

Suppose there exists no edge (u,t) such that splitting-off (s,u) and (u,t) retains the k -heavy property for all k -heavy edges. Then, for every neighbor t (other than s) of u , there exists at least one tight cut having s,t on one side and u on the other. Consider a minimum-sized collection of tight cuts X_1, X_2, \dots, X_ℓ , where X_i is the subset of vertices on the side of the cut not containing u . If $\ell = 1$, moving u to the side of X_1 produces a k -separator containing less than k edges, which is a contradiction. Thus $\ell \geq 2$. Now, let

$$A = X_1 \cap X_2; B = X_1 \setminus X_2; C = X_2 \setminus X_1; D = V \setminus (X_1 \cup X_2).$$

Then, $s \in A$ and $u \in D$. Since X_1 and X_2 are k -separators, either (1) A and D are k -separators, or (2) B and C are k separators. In either case, this pair of k -separators must be tight cuts since they contain at least k edges each being k -separators and at most k edges each because their total number of edges is at most that of X_1 and X_2 . If A and D are tight cuts, we can replace cuts X_1 and X_2 by D in the collection of tight cuts, contradicting minimality of this collection. On the other hand, if B and C are tight cuts, the counting argument also shows that there is no edge between A and D , contradicting the existence of edge (s,u) . \square

Let us now extend the notion of splitting-off to vertices.

Definition 9. *A vertex with even degree in an undirected graph is said to be split-off if a pair of edges incident on it is repeatedly split-off until the vertex becomes isolated. Splitting-off of a vertex is said to be k -preserving if each constituent edge splitting-off is k -preserving.*

Note that the number of edges in a cut either stays unchanged or decreases by 2 after a splitting-off operation. Thus, if every cut in the graph had an even number of edges to start with, then each cut continues to have an even number of edges after a sequence of splitting-off operations. Therefore, the following lemma is obtained by repeatedly applying Lemma 3 to a k -light vertex.

Lemma 4. *Suppose G_M is an undirected multigraph where the number of edges in every cut is even. Let k be an even integer. Then, there exists a k -preserving splitting-off of any non-isolated k -light vertex u in G_M .*

Our second operation is *edge contraction*.

Definition 10. *Contraction of edge $e = (u,v)$ in an undirected multigraph G is defined as merging u and v into a single vertex (i.e. all edges incident on either u or v are now incident on the new vertex instead). Any self-loops produced by edges between u and v are discarded.*

We will now prove Theorem 7.

Proof of Theorem 7. We run the following randomized algorithm on multigraph G_M :

1. Split-off all k -light vertices ensuring the k -preserving property (Lemma 4).
2. Contract an edge chosen uniformly at random in the resulting graph.
3. If the contraction produces a k -light vertex, split it off.⁴
4. If $\leq 2\alpha$ vertices are left, output a random cut; otherwise, go to step 2.

Consider a cut C that has at most αk edges; let its k -projection be S . In any of the splitting-off operations, no edge in S can be split-off since these edges continue to be k -heavy throughout the execution of the algorithm. So, if no edge crossing cut C (either an edge in G_M or one produced by the splitting-off operations) is contracted during the execution of the algorithm, then all edges in S survive till the end. To estimate the probability that no edge crossing cut C is contracted, let h_j be the number of vertices left at the beginning of the j th iteration. Thus, h_1 is the number of k -heavy vertices in G_M (note that all k -light vertices are split-off initially), and h_{j+1} is either $h_j - 1$ or $h_j - 2$ depending on whether a vertex was split-off in step 3 of iteration j . Observe that the number of edges crossing C cannot increase due to the splitting-off operations. Further, Lemma 2 asserts that at the beginning of iteration j , there are at least $h_j k/2$ edges in the graph. Thus, the probability that no edge in C is selected for random contraction in step 2 of iteration j is at least $1 - \frac{\alpha k}{h_j k/2} = 1 - \frac{2\alpha}{h_j}$. Then, the probability that no edge crossing C is contracted in the entire execution of the algorithm is at least

$$\prod_j \left(1 - \frac{2\alpha}{h_j}\right) \geq \prod_{i=n}^{2\alpha+1} \left(1 - \frac{2\alpha}{i}\right) = \binom{n}{2\alpha}^{-1}.$$

Since there are $2^{2\alpha-1}$ cuts in a graph with 2α vertices, the probability that the random cut output by the algorithm contains only edges crossing cut C (and therefore S is exactly the set of k -heavy edges in G_M output by the algorithm) is at least $\binom{n}{2\alpha}^{-1} 2^{1-2\alpha} \geq n^{-2\alpha}$. This is true for every distinct k -projection of cuts having at most αk edges; hence, the total number of such k -projections is at most $n^{2\alpha}$. \square

In addition to the above theorem, we need the following non-uniform version of Chernoff bounds (for Chernoff bounds, see e.g. [17]) to prove Theorem 8. (A proof of this theorem is given in the appendix.)

Theorem 9. *Consider any subset C of unweighted edges, where each edge $e \in C$ is sampled independently with probability p_e for some $p_e \in [0, 1]$ and given weight $1/p_e$ if selected in the sample. Let the random variable X_e denote the weight of edge e in the sample; if e is not selected in the sample, then $X_e = 0$. Then, for any p such that $p \leq p_e$ for all edges e , any $\varepsilon \in (0, 1]$, and any $N \geq |C|$, the following bound holds:⁵*

$$\mathbb{P} \left[\left| \sum_i X_e - |C| \right| > \varepsilon N \right] < 2e^{-0.38\varepsilon^2 p N}.$$

We will now use Theorem 7 to prove Theorem 8. (We re-use the notation defined in section 2.) For any cut C in G_M , let $F_i^{(C)} = F_i \cap C$ and $E_i^{(C)} = E_i \cap C$ for $0 \leq i \leq k$;⁶ let $f_i^{(C)} = |F_i^{(C)}|$ and $e_i^{(C)} = |E_i^{(C)}|$. Also, let $\widehat{f}_i^{(C)}$ be the expected weight of all edges in $F_i^{(C)}$ in the skeleton graph G_ε . We first prove a key lemma.

⁴If an edge between u and v is contracted in step 2, all edges that were previously k -heavy continue to be so after the contraction, except the edges between u and v . So, at most one vertex (the new vertex) becomes k -light as a result of this contraction.

⁵For any event \mathcal{E} , $\mathbb{P}[\mathcal{E}]$ represents the probability of event \mathcal{E} .

⁶For any cut C and any set of edges Z , $Z \cap C$ denotes the set of edges in Z that cross cut C .

Lemma 5. For any fixed i , with probability at least $1 - \frac{4}{n^2}$,

$$|f_i^{(C)} - \widehat{f_i^{(C)}}| \leq \frac{\varepsilon}{2} \max \left(\frac{e_i^{(C)} 2^{i-1}}{\pi \alpha}, f_i^{(C)} \right)$$

for all cuts C in G_M .

Proof. By the π -connectivity property, any edge $e \in F_i$ is π -heavy in G_i for any $i \geq 0$. Therefore, $e_i^{(C)} \geq \pi$. Let \mathcal{C}_{ij} be the set of all cuts C such that $\pi 2^j \leq e_i^{(C)} \leq \pi 2^{j+1} - 1$, $j \geq 0$. We will prove that with probability at least $1 - 2n^{-2^{j+1}}$, all cuts in \mathcal{C}_{ij} satisfy the property of the lemma. Then, the lemma follows by using the union bound over j (keeping i fixed) since $2n^{-2} + 2n^{-4} + \dots + 2n^{-2^j} + \dots \leq 4n^{-2}$.

We now prove the above claim for cuts $C \in \mathcal{C}_{ij}$. Let $X_i^{(C)}$ denote the set of edges in $F_i^{(C)}$ that are sampled with probability strictly less than 1; correspondingly, let $x_i^{(C)} = |X_i^{(C)}|$ and let $\widehat{x_i^{(C)}}$ be the total weight of edges in $X_i^{(C)}$ in the skeleton graph G_ε . Since edges in $F_i^{(C)} \setminus X_i^{(C)}$ have a weight of exactly 1 in G_ε , it is sufficient to show that with probability at least $1 - 2n^{-2^{j+1}}$, $|x_i^{(C)} - \widehat{x_i^{(C)}}| \leq \left(\frac{\varepsilon}{2}\right) \max \left(\frac{e_i^{(C)} 2^{i-1}}{\pi \alpha}, x_i^{(C)} \right)$ for all cuts $C \in \mathcal{C}_{ij}$. Since each edge $e \in X_i^{(C)}$ has $\lambda_e < 2^{i+1}$, we can use Theorem 9 with the lower bound on probabilities $p = \frac{96\alpha \ln n}{0.38 \cdot 2^{i+1} \varepsilon^2}$. There are two cases. In the first case, suppose $x_i^{(C)} \leq \frac{e_i^{(C)} 2^{i-1}}{\pi \alpha}$. Then, for any $X_i^{(C)}$ where $C \in \mathcal{C}_{ij}$, by Theorem 9, we have

$$\mathbb{P} \left[\left| x_i^{(C)} - \widehat{x_i^{(C)}} \right| > \left(\frac{\varepsilon}{2} \right) \frac{e_i^{(C)} 2^{i-1}}{\pi \alpha} \right] < 2e^{-0.38 \frac{\varepsilon^2}{4} \left(\frac{96\alpha \ln n}{0.38 \cdot 2^{i+1} \varepsilon^2} \right) \frac{e_i^{(C)} 2^{i-1}}{\pi \alpha}} \leq 2e^{-\frac{6e_i^{(C)} \ln n}{\pi}} \leq 2e^{-6 \cdot 2^j \ln n},$$

since $e_i^{(C)} \geq \pi 2^j$ for any $C \in \mathcal{C}_{ij}$. In the second case, suppose $x_i^{(C)} > \frac{e_i^{(C)} 2^{i-1}}{\pi \alpha}$. Then, for any $X_i^{(C)}$ where $C \in \mathcal{C}_{ij}$, by Theorem 9, we have

$$\mathbb{P} \left[\left| x_i^{(C)} - \widehat{x_i^{(C)}} \right| > \left(\frac{\varepsilon}{2} \right) x_i^{(C)} \right] < 2e^{-0.38 \frac{\varepsilon^2}{4} \left(\frac{96\alpha \ln n}{0.38 \cdot 2^{i+1} \varepsilon^2} \right) x_i^{(C)}} < 2e^{-\frac{6e_i^{(C)} \ln n}{\pi}} \leq 2e^{-6 \cdot 2^j \ln n},$$

since $x_i^{(C)} > \frac{e_i^{(C)} 2^{i-1}}{\pi \alpha} \geq \frac{2^{i+j-1}}{\alpha}$ for any $C \in \mathcal{C}_{ij}$. Thus, we have proved that

$$\mathbb{P} \left[\left| x_i^{(C)} - \widehat{x_i^{(C)}} \right| > \left(\frac{\varepsilon}{2} \right) \max \left(\frac{e_i^{(C)} 2^{i-1}}{\pi \alpha}, x_i^{(C)} \right) \right] < 2e^{-6 \cdot 2^j \ln n} = 2n^{-6 \cdot 2^j}$$

for any cut $C \in \mathcal{C}_{ij}$. Now, by the π -connectivity property, we know that edges in $F_i^{(C)}$, and therefore those in $X_i^{(C)}$, are π -heavy in G_i . Therefore, by Theorem 7, the number of distinct $X_i^{(C)}$ sets for cuts $C \in \mathcal{C}_{ij}$ is at most $n^{2 \left(\frac{\pi 2^{j+1}}{\pi} \right)} = n^{4 \cdot 2^j}$. Using the union bound over these distinct $X_i^{(C)}$ edge sets, we conclude that with probability at least $1 - 2n^{-2^{j+1}}$, all cuts in \mathcal{C}_{ij} satisfy the property of the lemma. \square

We now use the above lemma to prove Theorem 8.

Proof of Theorem 8. For any cut C in G_M , let c be the number of edges in C ; correspondingly, let \hat{c} be the total weight of the edges crossing cut C in the skeleton graph G_ε . Since $k \leq n - 1$, we apply the union bound

to the property from Lemma 5 over the different values of i to conclude that with probability at least $1 - \frac{4}{n}$, we have $\sum_{i=0}^k |\widehat{f}_i^{(C)} - f_i^{(C)}| \leq \sum_{i=0}^k \left(\frac{\varepsilon}{2}\right) \max\left(\frac{e_i^{(C)} 2^{i-1}}{\pi\alpha}, f_i^{(C)}\right)$ for all cuts C in G_M . Then, with probability at least $1 - \frac{4}{n}$,

$$|\hat{c} - c| = \left| \sum_{i=0}^k \widehat{f}_i^{(C)} - \sum_{i=0}^k f_i^{(C)} \right| \leq \sum_{i=0}^k |\widehat{f}_i^{(C)} - f_i^{(C)}| \leq \frac{\varepsilon}{2} \sum_{i=0}^k \max\left(\frac{e_i^{(C)} 2^{i-1}}{\pi\alpha}, f_i^{(C)}\right) \leq \frac{\varepsilon}{2} \left(\sum_{i=0}^k \frac{e_i^{(C)} 2^{i-1}}{\pi\alpha} + \sum_{i=0}^k f_i^{(C)} \right) \leq \varepsilon c,$$

since $\sum_{i=0}^k \frac{e_i^{(C)} 2^{i-1}}{\pi\alpha} \leq c$ by the α -overlap property and $\sum_{i=0}^k f_i^{(C)} \leq c$ since $F_i^{(C)}$'s form a partition of the edges in C .

We now prove the size bound on G_ε . The expected number of distinct edges in G_ε is

$$\sum_{e \in E} 1 - (1 - p_e)^{w_e} \leq \sum_e w_e p_e.$$

The bound follows by substituting the value of p_e . □

4 Sampling in Graphs with Polynomial Edge Weights

In this section, we will give an algorithm for sampling in undirected weighted graphs, where the weight of every edge is an integer bounded by n^d for a fixed constant $d > 0$. The algorithm constructs a skeleton graph containing $O\left(\frac{n \log n}{\varepsilon^2}\right)$ edges in expectation and has time complexity $O\left(m + \frac{n \log^4 n}{\varepsilon^2}\right)$. Our strategy, as outlined in the introduction, has two steps: first we run an algorithm that constructs a skeleton graph with $O\left(\frac{n \log^2 n}{\varepsilon^2}\right)$ edges in expectation and has time complexity $O(m)$; then, we run a different algorithm that constructs a sparser skeleton containing $O\left(\frac{n \log n}{\varepsilon^2}\right)$ edges in expectation on the skeleton graph constructed in the first step. The second algorithm takes time $O(m \log^2 n)$ on a graph with m edges and therefore $O\left(\frac{n \log^4 n}{\varepsilon^2}\right)$ time on the skeleton graph produced in the first step. To ensure that the final skeleton graph is in $(1 \pm \varepsilon)G$, we choose $\varepsilon/3$ as the error parameter for each algorithm. As an additional observation, we show that the time complexity of the second algorithm improves to $O(m \log n)$ if its input graph is unweighted.

We will describe both these algorithms for an input graph G , where the weight w_e of every edge e is an integer bounded by n^d for a fixed constant $d > 0$. Note that the input graph to the second algorithm in the above two-step sampling scheme may have fractional weights. However, we can scale up all weights uniformly until they are integral, and the scaled weights continue to be bounded by some fixed polynomial in n . Once the skeleton graph is obtained, we scale all weights down uniformly to obtain the final skeleton graph. The unweighted multigraph constructed by replacing each edge e with w_e parallel unweighted edges e_1, e_2, \dots, e_{w_e} between u and v is denoted by G_M . Also, T_1, T_2, \dots denotes a set of NI forests of G_M ; edge e_j appears in forest T_{i_e+j-1} , where $1 \leq j \leq w_e$. Thus, the copies of edge e appear in NI forests $T_{i_e}, T_{i_e+1}, \dots, T_{i_e+w_e-1}$. For both algorithms, we will use the generic sampling scheme described in section 2.

Algorithm for Step 1. For any edge $e = (u, v)$, we choose $\lambda_e = i_e + w_e - 1$, i.e. the index of the last NI forest where a copy of e appears; also set $\alpha = 2$ and $\pi = 2^{i-1}$. For any $i \geq 1$, define G_i to be the graph containing all edges in NI forests $T_{2^{i-1}}, T_{2^{i-1}+1}, \dots, T_{2^i-1}$ (call this set of edges Y_i) and all edges in F_i , i.e. all edges e with $2^i \leq \lambda_e \leq 2^{i+1} - 1$. Let G_0 only contain edges in F_0 . For any $i \neq j$, $F_i \cap F_j = Y_i \cap Y_j = \emptyset$; thus, each edge appears in G_i for at most two different values of i , proving α -overlap. Further, for any edge

$e \in F_i$, Lemma 1 ensures that the endpoints of e are connected in each of $T_{2^{i-1}}, T_{2^{i-1}+1}, \dots, T_{2^i-1}$. It follows that e is 2^{i-1} -heavy in G_i , thereby proving π -connectivity.

We now prove the size bound. For any edge $e' \in E_M$, let $t(e')$ be the index of the NI forest it appears in. Then,

$$\sum_{e \in E} \frac{w_e}{\lambda_e} = \sum_{e \in E} \sum_{j=1}^{w_e} \frac{1}{i_e + w_e - 1} \leq \sum_{e \in E} \sum_{j=1}^{w_e} \frac{1}{i_e + j - 1} = \sum_{e' \in E_M} \frac{1}{t(e')} = \sum_{\ell=1}^K \sum_{e' \in T_\ell} \frac{1}{\ell} \leq (n-1) \sum_{\ell=1}^K \frac{1}{\ell} = O(n \log K) = O(n \log n),$$

where the last step follows from the observation that the total number of NI forests K is at most n^{d+2} , where d is a constant. Using Theorem 8, we conclude that the skeleton graph G_ε constructed by the above algorithm has $O(\frac{n \log^2 n}{\varepsilon^2})$ edges in expectation and is in $(1 \pm \varepsilon)G$ whp.

Time Complexity. The time complexity for constructing the NI forests, and therefore figuring out p_e values is $O(m + n \log n)$. We sample each edge e by setting its weight in the skeleton G_ε to r_e/p_e , where r_e is drawn randomly from the Binomial distribution with parameters w_e and p_e . This is clearly equivalent to the sampling scheme described above, and can be done in $w_e p_e$ expected time for each edge e (see e.g. [7]), and therefore $O(\frac{n \log^2 n}{\varepsilon^2})$ time overall. Since $m > \frac{n \log^2 n}{\varepsilon^2}$ for this algorithm to be invoked, the overall time complexity of the algorithm is $O(m)$.

Algorithm for Step 2. Before describing our second sampling algorithm, we define the following operation on graphs. (Recall the definition of edge contraction given in section 3.)

Definition 11. Let $G = (V, E)$ be an undirected graph, and let V_1, V_2, \dots, V_k be a partition of the vertices in G such that for each V_i , the induced graph of G on V_i is connected. Then, shrinking G with respect to V_1, V_2, \dots, V_k produces the graph formed by contracting all edges between vertices in the same V_i for all i .

Our sampling algorithm uses our generic sampling scheme where λ_e is determined using the following algorithm. Here $H_c = (V_c, E_c)$ is a graph variable representing a weighted graph. The algorithm is described recursively; we call $\text{SetLambda}(G, 0)$ to execute it.

$\text{SetLambda}(H, i)$

1. Set $H_c = H$
2. If total weight of edges in E_c is at most $|V_c| \cdot 2^{i+1}$, then
 - (a) Set $\lambda_e = 2^i$ for all edges $e \in E_c$
 - (b) Remove all edges in E_c from H ; suppose H splits into connected components H_1, H_2, \dots, H_k
 - (c) For each H_j containing at least 2 vertices, call $\text{SetLambda}(H_j, i+1)$

Else,

- (a) Construct $2^i + 1$ NI forests $T_1, T_2, \dots, T_{2^i+1}$ for H_c
- (b) Shrink H_c wrt the connected components in T_{2^i+1} ; update V_c and E_c accordingly
- (c) Go to step 2

Also, set $\alpha = 4$ and $\pi = 2^k$ where $k = \lfloor \lg \max_{e \in E} \{\lambda_e\} \rfloor$. For any r , recall that F_r contains all w_e unweighted copies of edge e from G_M , where e satisfies $2^r \leq \lambda_e \leq 2^{r+1} - 1$. For any $i \geq 1$, let G_i contain all edges in F_r for all $r \geq i - 1$, where each edge in F_r is replicated 2^{k-r+1} times in G_i ; let G_0 contain edges of F_0 where each edge is replicated 2^k times. We need the following lemma to prove that π -connectivity is satisfied.

Lemma 6. *For any $j \geq 1$, consider any edge $e \in F_j$, i.e. an edge e for which the above algorithm sets $\lambda_e = 2^j$. Then, e is 2^{j-1} -heavy in the graph $\cup_{r \geq j-1} F_r$.*

Proof. For any edge e in F_j , let $G_e = (V_e, E_e)$ be the component of G containing e such that $\text{SetLambda}(G_e, j - 1)$ was executed. We will show that e is 2^{j-1} -heavy in G_e ; since G_e is a subgraph of G , the lemma follows. In the execution of $\text{SetLambda}(G_e, j - 1)$, there are multiple shrinking operations, each of them comprising the contracting of a set of edges. We claim that any such contracted edge is 2^{j-1} -heavy in G_e ; it follows that any two vertices u and v that got shrunk into a single vertex are 2^{j-1} -connected in G_e .

Let G_e have k shrinking phases; let the graph produced after shrinking phase r be $G_{e,r}$. We now prove that all edges contracted in phase r must be 2^{j-1} -connected in G_e by induction on r . For $r = 1$, since e appears in the $(2^{j-1} + 1)$ st NI forest of phase 1, e is 2^{j-1} -connected in G_e . For the inductive step, assume that the property holds for phases $1, 2, \dots, r$. Any edge that is contracted in phase $r + 1$ appears in the $(2^{j-1} + 1)$ st NI forest of phase $r + 1$; therefore, e is 2^{j-1} -connected in $G_{e,r}$. By the inductive hypothesis, all edges of G_e contracted in previous phases are 2^{j-1} -heavy in G_e ; therefore, an edge that is 2^{j-1} -heavy in $G_{e,r}$ must have been 2^{j-1} -heavy in G_e . \square

Consider any cut C in G containing an edge $e \in F_i$ for any $i \geq 0$. Let the corresponding cut (i.e. with the same bipartition of vertices) in G_i be C_i . We need to show that the number of edges in C_i is at least 2^k to prove π -connectivity. If $i = 0$, e is replicated 2^k times in G_0 thereby proving the property. For $i \geq 1$, let the maximum λ_a of an edge a in C be k_C , where $2^j \leq k_C \leq 2^{j+1} - 1$ for some $j \geq i$. By the above lemma, C_i contains at least 2^{j-1} distinct edges of G , each of which is replicated at least 2^{k-j+1} times. Thus, C_i contains at least 2^k edges.

We now prove α -overlap. For any cut C , recall that $f_i^{(C)}$ and $e_i^{(C)}$ respectively denote the number of edges in $F_i \cap C$ and in C_i (where C_i is as defined in the previous paragraph) respectively. Then,

$$\begin{aligned} \sum_{i=0}^k \frac{e_i^{(C)} 2^{i-1}}{\pi} &= \frac{e_0^{(C)}}{2\pi} + \sum_{i=1}^k \frac{e_i^{(C)} 2^{i-1}}{\pi} = \frac{f_0^{(C)} 2^k}{2^{k+1}} + \sum_{i=1}^k \frac{f_i^{(C)} 2^{k-r+1} 2^{i-1}}{2^k} = \frac{f_0^{(C)}}{2} + \sum_{i=1}^k \sum_{r=i-1}^k \frac{f_r^{(C)}}{2^{r-i}} \\ &\leq f_0^{(C)} + \sum_{r=0}^k \sum_{i=1}^{r+1} \frac{f_r^{(C)}}{2^{r-i}} \leq 3f_0^{(C)} + \sum_{r=1}^k f_r^{(C)} \sum_{i=1}^{r+1} \frac{1}{2^{r-i}} \leq 4f_0^{(C)} + 4 \sum_{r=1}^k f_r^{(C)} \leq 4c. \end{aligned}$$

Define D_i to be the set of connected components in the graph $G \setminus (F_0 \cup F_1 \cup \dots \cup F_{i-1})$ for any $i \geq 1$; let D_0 be the single connected component in G . For any $i \geq 0$, if any connected component in D_i remains intact in D_{i+1} , then there is no edge from that connected component in F_i . On the other hand, if a component in D_i splits into η components in D_{i+1} , then the algorithm explicitly ensures that the number of edges in F_i from that connected component is at most $\eta 2^{i+1}$. Since each such edge has $\lambda_e = \frac{1}{2^i}$, the contribution of these edges to the sum $\sum_{e \in E} \frac{w_e}{\lambda_e}$ is at most $2\eta \leq 4(\eta - 1)$ (since $\eta \geq 2$). But, $\eta - 1$ is the increase in the number of components arising from this single component. Therefore, if $d_i = |D_i|$, then

$$\sum_e \frac{w_e}{\lambda_e} \leq \sum_{i=0}^k 4(d_{i+1} - d_i) \leq 4n$$

since ultimately we have n singleton components. Using Theorem 8, we conclude that the skeleton graph G_ε constructed by the above algorithm has $O(\frac{n \log n}{\varepsilon^2})$ edges in expectation and is in $(1 \pm \varepsilon)G$ whp.

Time Complexity. We show below that the algorithm to find values of λ_e can be implemented in $O(m \log n)$ time for unweighted graphs, and $O(m \log^2 n)$ time for graphs with polynomial edge weights. Once we have obtained the sampling probabilities, we use the same trick as in the previous algorithm, i.e. sample from a Binomial distribution, to produce the skeleton in $O(\frac{n \log n}{\epsilon^2})$ additional time. Since the algorithm is invoked only if $m > \frac{n \log n}{\epsilon^2}$, the total running time is $O(m \log n)$ if G is unweighted and $O(m \log^2 n)$ otherwise.

We now determine the time complexity for finding the values of λ_e . Consider one call to `SetLambda(H, i)` which begins with $H = (V, E)$ and let $H_c = (V_c, E_c)$ denote the graph H as it evolves over the various iterations in this procedure. Each iteration of steps (a) and (b) in the else block takes $O(|V_c| \log n + |E_c|)$ time. We show that the number of vertices halves in each iteration (save the last) and therefore the total time over all iterations is $O(|V| \log n + |E| \log n)$. Since we are dealing with the case of polynomial edge weights, the depth of recursion is $O(\log n)$. Therefore, over all recursive calls, the time comes to $O(n \log^2 n + m \log^2 n) = O(m \log^2 n)$.

To see that the number of vertices halves from one iteration to the next, consider an iteration that begins with E_c having weight at least $|V_c| \cdot 2^{i+1}$. E_c for the next iteration (denoted by E'_c) comprises only edges in the first 2^i NI forests constructed in the current iteration. So the total weight of edges in E'_c is at most $|V_c| \cdot 2^i$. If this is not the last iteration, then this weight exceeds $|V'_c| \cdot 2^{i+1}$. It follows that $|V'_c| \leq |V_c|/2$, as required.

From the above description, note that for the unweighted case, $|E'_c| \leq |E_c|/2$, and therefore the time taken over all iterations in one recursive call is $O(|V| + |E|)$. Over all recursive calls this comes to $O(m \log n)$.

5 Sampling Schemes using various Connectivity Parameters

In this section, we present several sampling schemes using various measures of connectivity. Some of these results were previously known; however, we will show that these results follow as simple corollaries of our generic sampling scheme whereas the original proofs were specific to each scheme and substantially more complicated. The algorithms for implementing these schemes are less efficient than the algorithms that we have previously presented; therefore we restrict ourselves to structural results in this section. As earlier, G is the weighted input graph (with arbitrary integer weights); G_M is the corresponding unweighted multigraph; T_1, T_2, \dots, T_k is a set of NI forests of G_M .

5.1 Sampling using Standard Connectivities

For any edge $e = (u, v)$, set λ_e to the standard connectivity of the edge; also set $\alpha = 3 + \lg n$ and $\pi = 2^{i-1}$. F_i is defined as the set of all edges e with $2^i \leq \lambda_e \leq 2^{i+1} - 1$ for any $i \geq 0$. For any $i \geq 1 + \lg n$, let G_i contain all edges in NI forests $T_{2^{i-1-\lg n}}, T_{2^{i-1-\lg n}+1}, \dots, T_{2^{i+1}-1}$ and all edges in F_i . For $i \leq \lg n$, G_i contains all edges in T_1, T_2, \dots, T_i and all edges in F_i . For any $i \geq 0$, let Y_i denote the set of edges in G_i but not in F_i . For any $i \neq j$, $F_i \cap F_j = \emptyset$ and each edge appears in Y_i for at most $2 + \log n$ different values of i ; this proves α -overlap. To prove π -connectivity, we note that Lemma 1 ensures that for any pair of vertices u, v with maximum flow $f(u, v)$ and for any $k \geq 1$, u, v are at least $\min(f(u, v), k)$ -connected in the union of the first k NI forests, i.e. in $T_1 \cup T_2 \cup \dots \cup T_k$. Thus, any edge $e \in F_i$ is at least 2^i -heavy in the union of the NI forests $T_1, T_2, \dots, T_{2^{i+1}-1}$. Since there are at most 2^{i-1} edges overall in $T_1, T_2, \dots, T_{2^{i-1-\lg n}-1}$, any edge $e \in F_i$ is 2^{i-1} -heavy in G_i . This proves π -connectivity.

We now prove the size bound. The next lemma is similar to its corresponding lemma for strong connectivity in [2].

Lemma 7. *Suppose G is an undirected graph where edge e has weight w_e and standard connectivity k_e . Then, $\sum_e \frac{w_e}{k_e} \leq n - 1$.*

Proof. We use induction on the number of vertices in the graph. For a graph with a single vertex and no edge, the lemma holds vacuously. Now, suppose the lemma holds for all graphs with at most $n - 1$ vertices. Let C be a minimum cut in G , and let λ be its weight. For any edge $e \in C$, $k_e = \lambda$. Thus, $\sum_{e \in C} \frac{w_e}{k_e} = 1$. We remove all edges in C from G ; this splits G into two connected components G_1 and G_2 with n_1 and n_2 vertices respectively, where $n_1, n_2 \leq n - 1$. Further, the standard connectivity of each edge in G_1, G_2 is at most that in G . Using the inductive hypothesis, we conclude that $\sum_{e \in G_1} \frac{w_e}{k_e} \leq n_1 - 1$ and $\sum_{e \in G_2} \frac{w_e}{k_e} \leq n_2 - 1$. We conclude that

$$\sum_e \frac{w_e}{k_e} \leq n_1 - 1 + n_2 - 1 + 1 = n - 1.$$

□

Using Theorem 8, we conclude that the expected number of edges in the skeleton graph G_ε is $O(\frac{n \log^2 n}{\varepsilon^2})$ and $G_\varepsilon \in (1 \pm \varepsilon)G$ whp.

5.2 Sampling using Effective Resistances

For any edge $e = (u, v)$, set λ_e to the effective conductance of the edge, i.e. $\lambda_e = \frac{1}{R_e}$ where R_e is the effective resistance of edge e . The next two lemmas imply that the skeleton $G_\varepsilon \in (1 \pm \varepsilon)G$ whp.

Lemma 8. *Suppose that a sampling scheme (that uses the generic sampling scheme) has $\lambda_e \leq k_e$ for each edge e in graph G , where k_e is the standard connectivity of e in G . Then, the skeleton constructed is in $(1 \pm \varepsilon)G$ whp.*

Proof. We use the same definition of α , π and G_i s as in the sampling scheme with standard connectivities, and verify that π -connectivity and α -overlap continue to be satisfied. □

Lemma 9. *Suppose edge e in an undirected graph G has standard connectivity k_e and effective resistance R_e . Then, $\frac{1}{R_e} \leq k_e$.*

Proof. Consider a cut C of weight k_e separating the terminals of edge e . We contract each side of this cut into a single vertex. In other words, we reduce the resistance on each edge, other than those in C , to 0. By Rayleigh's monotonicity principle (e.g. [4]), the effective resistance of e does not increase due to this transformation. Since the effective resistance of e after the transformation is $1/k_e$, $R_e \geq 1/k_e$ in the original graph. □

The size bound follows from the following well-known fact (see e.g. [22]).⁷

Fact 1. *If R_e is the effective resistance of edge e with weight w_e in an undirected graph, then $\sum_e w_e R_e \leq n - 1$.*

It follows from Theorem 8 that the expected number of edges in skeleton G_ε is $O(\frac{n \log^2 n}{\varepsilon^2})$.

5.3 Sampling using Strong Connectivities

For any edge e , set λ_e to the strong connectivity of the edge; set $\alpha = 1$ and $\pi = 2^k$, where $k = \lceil \lg \max_{e \in E} \{\lambda_e\} \rceil$. Let G_i contain all edges in F_r for all $r \geq i$, where each edge in F_r is replicated 2^{k-r} times. We use the following property of strong connectivities that also appears in [2].

⁷There are many proofs of this fact, e.g. use linearity of expectation coupled with the fact that effective resistance of an edge is the probability that the edge is in a random spanning tree of the graph [3].

Lemma 10. *In any undirected graph G , if an edge e has strong connectivity k , then e continues to have strong connectivity k even after all edges with strong connectivity strictly less than k have been removed from G .*

Consider any cut C with an edge $e \in F_i$. Let the corresponding cut (i.e. with the same bi-partition of vertices) in G_i be C_i . We need to show that the number of edges in C_i is at least 2^k to prove π -connectivity. Let the maximum strong connectivity of an edge in C be k_C , where $2^j \leq k_C \leq 2^{j+1} - 1$ for some $j \geq i$. By the above lemma, C_i contains at least 2^j distinct edges of G , each of which is replicated at least 2^{k-j} times. Thus, C_i contains at least 2^k edges.

We now prove α -overlap. For any cut C , recall that $f_i^{(C)}$ and $e_i^{(C)}$ respectively denote the number of edges in $F_i \cap C$ and in C_i (where C_i is as defined in the previous paragraph) respectively. Then,

$$\sum_{i=0}^k \frac{e_i^{(C)} 2^{i-1}}{\pi} = \sum_{i=0}^k \sum_{r=i}^k \frac{f_r^{(C)} 2^{k-r} 2^{i-1}}{2^k} = \sum_{i=0}^k \sum_{r=i}^k \frac{f_r^{(C)}}{2^{r-i+1}} = \sum_{r=0}^k \sum_{i=0}^r \frac{f_r^{(C)}}{2^{r-i+1}} = \sum_{r=0}^k f_r^{(C)} \sum_{i=0}^r \frac{1}{2^{r-i+1}} < \sum_{r=0}^k f_r^{(C)} = c.$$

The size bound follows from the following lemma due to Benczúr and Karger.

Lemma 11 (Benczúr-Karger [2]). *If k_e is the strong connectivity of edge e with weight w_e in an undirected graph, then $\sum_e \frac{w_e}{k_e} \leq n - 1$.*

It follows from Theorem 8 that the expected number of edges in the skeleton graph G_ε is $O(\frac{n \log n}{\varepsilon^2})$ and that $G_\varepsilon \in (1 \pm \varepsilon)G$ whp.

6 Sampling in Graphs with Arbitrary Edge Weights

Unfortunately, the algorithms presented earlier for sampling in a graph with polynomial edge weights fail if the edge weights are arbitrary. In particular, we can no longer guarantee that the expected number of edges in a skeleton graph constructed by these algorithms is $\tilde{O}(n/\varepsilon^2)$, even though it continues to approximately preserve the weight of all cuts whp. Therefore, we need to modify our techniques to restore the size bounds, as described below.

We sort the edges in decreasing order of their weight, breaking ties arbitrarily. We add edges to the NI forests in this sorted order, i.e. when edge e is being added, the NI forests contain all edges of weight greater than e . To insert $e = (u, v)$, we find the NI forest with the minimum index where u and v are not connected; call this index i_e . Then, e is inserted in NI forests $T_{i_e}, T_{i_e+1}, \dots, T_{i_e+w_e-1}$. Note that this does not produce any cycle in the NI forests since Lemma 1 ensures that if u, v are disconnected in T_{i_e} , then they are not connected in T_k for any $k \geq i_e$.

For any edge $e = (u, v)$, set λ_e to the index of the first NI forest where edge e is inserted, i.e. $\lambda_e = i_e$; also set $\alpha = 2$ and $\pi = 2^{i-1}$. For any $i \geq 1$, let G_i contain all edges in NI forests $T_{2^{i-1}}, T_{2^{i-1}+1}, \dots, T_{2^i-1}$ (call this set of edges Y_i) and all edges in F_i , i.e. all edges e with $2^i \leq \lambda_e \leq 2^{i+1} - 1$. Let $G_0 = F_0$. For any $i \neq j$, $F_i \cap F_j = Y_i \cap Y_j = \emptyset$; thus, each edge appears in G_i for at most two different values of i , proving α -overlap. On the other hand, for any edge $e \in F_i$, Lemma 1 ensures that the endpoints of e are connected in each of $T_{2^{i-1}}, T_{2^{i-1}+1}, \dots, T_{2^i-1}$. It follows that e is 2^{i-1} -heavy in G_i , thereby proving π -connectivity.

We now prove the size bound on the skeleton. Partition edges into subsets S_0, S_1, \dots where S_j contains all edges e with $j < \frac{i_e}{w_e} \leq j+1$. The following lemma states that none of these subsets is large.

Lemma 12. *For any j , $|S_j| \leq n - 1$.*

Proof. We prove that the edges in any subset S_j form an acyclic graph. Suppose not; let C be a cycle formed by the edge in S_j , and $e = (u, v)$ be the edge that was inserted last in the NI forests among the edges in C . Let e' be any other edge in C . Then, $w_{e'} \geq w_e$, and hence

$$i_{e'} + w_{e'} - 1 > w_{e'}(j+1) - 1 \geq w_e(j+1) - 1 \geq i_e - 1.$$

Since both the first and last terms are integers, $i_{e'} + w_{e'} - 1 \geq i_e$. Therefore, u' and v' were connected in T_{i_e} for each $e' = (u', v')$ in C . So, u and v were connected in T_{i_e} since C is a cycle, before e was added to T_{i_e} . But, then e would not have been added to T_{i_e} , a contradiction. \square

Thus,

$$\sum_e \frac{w_e}{i_e} \leq \sum_{j: S_j \neq \emptyset} \frac{|S_j|}{j} \leq (n-1) \sum_{j: S_j \neq \emptyset} \frac{1}{j} = O(n \log n)$$

since at most $m < n^2$ of the S_j 's are non-empty. Using Theorem 8, we conclude that the skeleton G_ε has $O(\frac{n \log^2 n}{\varepsilon^2})$ edges in expectation and that $G_\varepsilon \in (1 \pm \varepsilon)G$ whp.

Finally, we need to show that the construction of NI forests where edges are added in decreasing order of weight can be done in $O(m \log^2 n)$ time. We use a data structure (call it a *partition tree*) \mathcal{P} to succinctly encode the NI forests. The leaf nodes in \mathcal{P} exactly correspond to the vertices in graph G , i.e. there is a one-one mapping between these two sets. On the other hand, each non-leaf node v of the partition tree has a number $n(v)$ associated with it that satisfies the following property: *for any two vertices x, y in the graph, if z be the least common ancestor⁸ of their corresponding leaf nodes in \mathcal{P} , then x and y are connected in exactly the first $n(z)$ NI forests.* Then, $n(z) + 1$ is the index of the first NI forest where edge (x, y) is to be inserted. Initially, all the n leaf nodes in \mathcal{P} representing the graph vertices are children of the root node r , and $n(r) = 0$. As edges are inserted in the NI forests, the partition tree evolves, but we make sure that the above property holds throughout the construction. Additionally, we also maintain the invariant that if x is a child of y in \mathcal{P} , then $n(x) > n(y)$.

We need to show that we can maintain the above properties of the partition tree as it evolves, and also retrieve the lca of any pair of vertices efficiently for this evolving partition tree. Let (x, y) be the edge being inserted, let $z = lca(x, y)$ in the partition tree, and let u and v be the children of z that are ancestors of x and y respectively. Observe that adding an edge (x, y) to trees with indices from $n_s + 1$ to $n_s + \ell$ increases the connectivity of a pair of vertices w_1, w_2 iff they were previously connected in $n_s + i$ trees for some $0 \leq i < \ell$, w_1, x were connected in $n_s + j$ trees for some $j \geq i$ and w_2, y were connected in $n_s + k$ trees for some $k \geq i$ (or vice-versa). In this case, w_1, w_2 are now connected in $n_s + \min(j, k, \ell)$ trees after adding the edge (x, y) . Further, if $n(u) - n(z) < w(x, y)$, then an edge of weight less than $w(x, y)$ must have been added to the trees according to the second invariant, which violates the fact that edges are added in decreasing order of weight. Thus, $n(u) - n(z) \geq w(x, y)$; similarly $n(v) - n(z) \geq w(x, y)$.

There are three cases:

1. $n(u) - n(z) = n(v) - n(z) = w(x, y)$. We merge u and v into a single node s that remains a child of z and $n(s) = n(u)$. The first invariant is clearly maintained. For the second invariant, observe that the only pairs of vertices w_1, w_2 whose connectivity changed were those with $lca(w_1, w_2) = z$, where w_1, w_2 are descendants of u, v respectively. Their connectivity increases to $n(u)$, which is reflected in the partition tree.

⁸The *least common ancestor* or lca of two nodes x, y in a tree is the deepest node that is an ancestor of both x and y .

2. $n(u) - n(z) = w(x, y)$ and $n(v) - n(z) > w(x, y)$ (symmetrically for $n(u) - n(z) > w(x, y)$ and $n(v) - n(z) = w(x, y)$). We make v a child of u (from being a child of z), and $n(u) = n(z) + w(x, y)$. For notational convenience in the proofs later, we replace u and v by a pair of new nodes s and t where $n(s)$ and $n(t)$ are respectively equal to the updated values of $n(u)$ and $n(v)$. The first invariant is clearly maintained. For the second invariant, observe that the only pairs of vertices w_1, w_2 whose connectivity changed were those with $\text{lca}(w_1, w_2) = z$, where w_1, w_2 are descendants of u, v respectively. Their connectivity increases to $n(z) + w(x, y)$, which is reflected in the partition tree.
3. $n(u) - n(z) > w(x, y)$ and $n(v) - n(z) > w(x, y)$. We introduce a new node r as a child of z and parent of u and v , and $n(r) = n(z) + w(x, y)$. For notational convenience in the proofs later, we replace u and v by a pair of new nodes s and t where $n(s) = n(u)$ and $n(t) = n(v)$. The first invariant is clearly maintained. For the second invariant, observe that the only pairs of vertices w_1, w_2 whose connectivity changed were those with $\text{lca}(w_1, w_2) = z$, where w_1, w_2 are descendants of u, v respectively. Their connectivity increases to $n(z) + w(x, y)$, which is reflected in the partition tree.

We use the *dynamic tree* data structure [21] for updating the partition tree. This data structure can be used to maintain a dynamically changing forest of n nodes, while supporting the following operations⁹ in $O(\log n)$ time per operation:

Cut(v) Cut the subtree under node v from the tree containing it, and make it a separate tree with root v .

Link(v, w) (w needs to be the root node of a tree not containing v .) Join the tree rooted at w and that containing v by making w a child of v .

LCA(v, w) (v and w need to be in the same tree.) Defined previously.

We maintain a dynamic tree data structure for the partition tree. Recall that the partition tree can be modified in three different ways. The last two modifications require $O(1)$ cut and link operations each. Therefore, the overall time complexity of these modifications is $O(m \log n)$. On the other hand, the first modification requires $O(d)$ cut and link operations, where d is the lesser number of children among u and v . We will prove the following lemma bounding the total number of operations due to the first type of modification.

Lemma 13. *The total number of cut and link operations due to modifications of the first type in the partition tree is $O(m \log n)$.*

Theorem 10 follows immediately.

Theorem 10. *The time complexity of constructing NI forests where edges are inserted in decreasing order of weight is $O(m \log^2 n)$ for graphs with arbitrary edge weights.*

We now prove Lemma 13.

Proof of Lemma 13. We set up a charging argument for the cut and link operations due to the first type of modification. Define a function f on the nodes of the partition tree where each node v has $f(v) = 1$ initially. In the first type of modification, we assign $f(s) = f(u) + f(v)$; in the second type of modification, $f(s) = f(u) + f(v)$ and $f(t) = 1$; in the third type of modification, $f(r) = f(u) + f(v)$ and $f(s) = f(t) = 1$. Observe that the sum of $f(\cdot)$ over all nodes in the partition tree increases by at most 2 for any of the above modifications.

⁹The dynamic tree data structure supports other operations as well; we only define the operations that we require.

Let C_u be the set of children of node u ; then, let $F_C(u) = \sum_{v \in C_u} f(v)$. We charge the cut and link operations for the first type of modification to the children of u (resp., v) if $F_C(u) \geq F_C(v)$ (resp., $F_C(v) > F_C(u)$); each child of u (resp., v) is charged $O(1)$ operations. Now, let S_u be the set of siblings of any node u in the partition tree; correspondingly, let $F_S(u) = \sum_{v \in S_u} f(v)$. Observe that whenever a node u is charged due the first type of modification, $F_S(u)$ at least doubles. Further, $F_S(u)$ never decreases for any node u due to any of the three types of modifications. Since the sum of $f(\cdot)$ over all nodes in the partition tree increases by at most 2 for any of the modifications, and there are m modifications overall, each node is charged at most $O(\log m) = O(\log n)$ times. Further, each modification introduces $O(1)$ new nodes; so the total number of operations due to modifications of the first type is $O(m \log n)$. \square

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A Proof of Theorem 9

We need the following inequality.

Lemma 14. *Let $f(x) = x - (1+x)\ln(1+x)$ and $\alpha = 1 - 2\ln 2$. Then,*

$$f(x) \leq \begin{cases} \alpha x^2 & \text{if } x \in (0, 1) \\ \alpha x & \text{if } x \geq 1. \end{cases}$$

Proof. First, consider $x \in (0, 1)$. Define

$$g(x) = \frac{f(x)}{x^2} = \frac{1}{x} - \left(\frac{1}{x} + \frac{1}{x^2} \right) \ln(1+x).$$

We can verify that $g(x)$ is an increasing function of x for $x \in (0, 1]$. Further, at $x = 1$, $g(x) = \alpha$. Thus, $f(x) < \alpha x^2$ for $x \in (0, 1)$.

Now, consider $x \geq 1$. Define

$$h(x) = \frac{f(x)}{x} = 1 - \left(1 + \frac{1}{x}\right) \ln(1+x).$$

We can verify that $h(x)$ is a decreasing function of x for $x \geq 1$. Further, at $x = 1$, $h(x) = \alpha$. Thus, $f(x) \leq \alpha x$ for $x \geq 1$. \square

We use the above inequality to prove the following lemmas.

Lemma 15. *Suppose X_1, X_2, \dots, X_n is a set of independent random variables such that each X_i , $i \in \{1, 2, \dots, n\}$, has value $1/p_i$ with probability p_i for some fixed $0 < p_i \leq 1$ and has value 0 with probability $1 - p_i$. For any $p \leq \min_i p_i$ and for any $\varepsilon > 0$,*

$$\mathbb{P} \left[\sum_i X_i > (1 + \varepsilon)n \right] < \begin{cases} e^{-0.38\varepsilon^2 pn} & \text{if } 0 < \varepsilon < 1 \\ e^{-0.38\varepsilon pn} & \text{if } \varepsilon \geq 1. \end{cases}$$

Proof. For any $t > 0$,¹⁰

$$\begin{aligned} \mathbb{P} \left[\sum_i X_i > (1 + \varepsilon)n \right] &= \mathbb{P} \left[e^{t \sum_i X_i} > e^{t(1+\varepsilon)n} \right] \\ &< \frac{\mathbb{E} \left[e^{t \sum_i X_i} \right]}{e^{t(1+\varepsilon)n}} \quad (\text{by Markov bound (see e.g. [17])}) \\ &= \prod_{i=1}^n \frac{\mathbb{E} \left[e^{t X_i} \right]}{e^{t(1+\varepsilon)}} \quad (\text{by independence of } X_1, X_2, \dots, X_n) \\ &= \prod_{i=1}^n \frac{p_i e^{t/p_i} + 1 - p_i}{e^{t(1+\varepsilon)}} \\ &= \prod_{i=1}^n \frac{1 + p_i(e^{t/p_i} - 1)}{e^{t(1+\varepsilon)}} \\ &\leq \exp \left(\sum_{i=1}^n p_i(e^{t/p_i} - 1) - t(1 + \varepsilon)n \right) \quad (\text{since } 1 + x \leq e^x, \forall x \geq 0). \end{aligned}$$

Since $p_i \geq p$ for all $i \in \{1, 2, \dots, n\}$,

$$\sum_{i=1}^n (p_i(e^{t/p_i} - 1)) \leq \sum_{i=1}^n (p(e^{t/p} - 1)) = np(e^{t/p} - 1).$$

Thus,

$$\mathbb{P} \left[\sum_i X_i > (1 + \varepsilon)n \right] < \exp(np(e^{t/p} - 1) - t(1 + \varepsilon)n).$$

Setting $t = p \ln(1 + \varepsilon)$, we get

$$\mathbb{P} \left[\sum_i X_i > (1 + \varepsilon)n \right] < \left(\frac{e^\varepsilon}{(1 + \varepsilon)^{1+\varepsilon}} \right)^{pn}.$$

¹⁰For any random variable X , $\mathbb{E}[X]$ denotes the expectation of X .

Since $1 - 2\ln 2 < -0.38$, we can use Lemma 14 to conclude that

$$\mathbb{P} \left[\sum_i X_i > (1 + \varepsilon)n \right] < \begin{cases} e^{-0.38\varepsilon^2 pn} & \text{if } 0 < \varepsilon < 1 \\ e^{-0.38\varepsilon pn} & \text{if } \varepsilon \geq 1. \end{cases} \quad \square$$

Lemma 16. Suppose X_1, X_2, \dots, X_n is a set of independent random variables such that each X_i , $i \in \{1, 2, \dots, n\}$, has value $1/p_i$ with probability p_i for some fixed $0 < p_i \leq 1$ and has value 0 with probability $1 - p_i$. For any $p \leq \min_i p_i$ and for any $\varepsilon > 0$,

$$\mathbb{P} \left[\sum_i X_i < (1 - \varepsilon)n \right] \begin{cases} < e^{-0.5\varepsilon^2 pn} & \text{if } 0 < \varepsilon < 1 \\ = 0 & \text{if } \varepsilon \geq 1. \end{cases}$$

Proof. For $\varepsilon \geq 1$,

$$\mathbb{P} \left[\sum_i X_i < (1 - \varepsilon)n \right] \leq \mathbb{P} \left[\sum_i X_i < 0 \right] = 0.$$

Now, suppose $\varepsilon \in (0, 1)$. For any $t > 0$,

$$\begin{aligned} \mathbb{P} \left[\sum_i X_i < (1 - \varepsilon)n \right] &= \mathbb{P} \left[e^{-t \sum_i X_i} > e^{-t(1 - \varepsilon)n} \right] \\ &< \frac{\mathbb{E} \left[e^{-t \sum_i X_i} \right]}{e^{-t(1 - \varepsilon)n}} \quad (\text{by Markov bound}) \\ &= \prod_{i=1}^n \frac{\mathbb{E} \left[e^{-t X_i} \right]}{e^{-t(1 - \varepsilon)n}} \quad (\text{by independence of } X_1, X_2, \dots, X_n) \\ &= \prod_{i=1}^n \frac{p_i e^{-t/p_i} + 1 - p_i}{e^{-t(1 - \varepsilon)n}} \\ &= \prod_{i=1}^n \frac{1 - p_i(1 - e^{-t/p_i})}{e^{-t(1 - \varepsilon)n}} \\ &\leq \exp \left(\sum_{i=1}^n -p_i(e^{-t/p_i} - 1) + t(1 - \varepsilon)n \right) \quad (\text{since } 1 - x \leq e^{-x}, \forall x \geq 0). \end{aligned}$$

Since $p_i \geq p$ for all $i \in \{1, 2, \dots, n\}$,

$$\sum_{i=1}^n (p_i(1 - e^{-t/p_i})) \leq \sum_{i=1}^n (p(1 - e^{-t/p})) = np(1 - e^{-t/p}).$$

Thus,

$$\mathbb{P} \left[\sum_i X_i < (1 - \varepsilon)n \right] < \exp(np(1 - e^{-t/p}) + t(1 - \varepsilon)n).$$

Setting $t = -p \ln(1 - \varepsilon)$, we get

$$\mathbb{P} \left[\sum_i X_i < (1 - \varepsilon)n \right] < \left(\frac{e^\varepsilon}{(1 - \varepsilon)^{1 - \varepsilon}} \right)^{pn} \leq e^{-0.5\varepsilon^2 pn}. \quad \square$$

We now prove Theorem 9 using the above lemmas.

Proof of Theorem 9. Let $\delta = \frac{\varepsilon N}{|C|}$. First, consider the case where $\delta \in (0, 1)$. From Lemmas 15 and 16, we conclude that

$$\begin{aligned} \mathbb{P} \left[\left| \sum_e X_e - |C| \right| > \varepsilon |C| \right] &= \mathbb{P} \left[\left| \sum_e X_e - |C| \right| > \delta |C| \right] < 2e^{-0.38\delta^2 p|C|} \\ &= 2e^{-0.38\varepsilon^2 pN(N/|C|)} \leq 2e^{-0.38\varepsilon^2 pN} \quad (\text{since } N \geq |C|). \end{aligned}$$

Now, consider the case where $\delta \geq 1$. From Lemmas 15 and 16, we conclude that

$$\mathbb{P} \left[\left| \sum_e X_e - |C| \right| > \varepsilon N \right] = \mathbb{P} \left[\left| \sum_e X_e - |C| \right| > \delta |C| \right] < e^{-0.38\delta p|C|} = e^{-0.38\varepsilon pN} \leq e^{-0.38\varepsilon^2 pN} \quad (\text{since } \varepsilon \leq 1).$$

□