A Center Transversal Theorem for Hyperplanes and Applications to Graph Drawing*

Vida Dujmović† *and* *Stefan Langerman*‡

Abstract. Motivated by an open problem from graph drawing, we study several partitioning problems for line and hyperplane arrangements. We prove a ham-sandwich cut theorem: given two sets of n lines in $\mathbb{R}^2$, there is a line ℓ such that in both line sets, for both halfplanes delimited by ℓ, there are $\sqrt{n}$ lines which pairwise intersect in that halfplane, and this bound is tight; a centerpoint theorem: for any set of n lines there is a point such that for any halfplane containing that point there are $\sqrt{n/3}$ of the lines which pairwise intersect in that halfplane. We generalize those results in higher dimension and obtain a center transversal theorem, a same-type lemma, and a positive portion Erdős-Szekeres theorem for hyperplane arrangements. This is done by formulating a generalization of the center transversal theorem which applies to set functions that are much more general than measures. Back to Graph Drawing (and in the plane), we completely solve the open problem that motivated our search: there is no set of n labelled lines that are universal for all n-vertex labelled planar graphs. As a contrast, the main result by Pach and Toth in [*J. of Graph Theory*, 2004], has, as an easy consequence, that every set of n (unlabelled) lines is universal for all n-vertex (unlabelled) planar graphs.

1 Introduction

Consider a mapping of the vertices of a graph to distinct points in the plane and represent each edge by the closed line segment between its endpoints. Such a graph representation is a *(straight-line) drawing* if the only vertices that each edge intersects are its own endpoints. A *crossing* in a drawing is a pair of edges that intersect at some point other than a common endpoint. A drawing is *crossing-free* if it has no crossings.

One main focus in graph drawing is finding methods to produce drawings or crossing-free drawings for a given graph with various restrictions on the position of the vertices of the graph in the plane. For instance, there is plethora of work where vertices are required to be placed on integer grid points or on parallel lines in 2 or 3–dimensions.

Given a set R of n regions in the plane and an n-vertex graph G, consider a class of graph drawing problems where G needs to be drawn crossing-free by placing each vertex of G in one region of R. If such a drawing exists, then R is said to *support* G. The problems studied in the literature distinguish between two scenarios: in one, each vertex of the graph is prescribed its specific region

*The preliminar version of this paper has appeared in the Proceedings of the 27th annual ACM symposium on Computational geometry, SoCG, 2011.

†School of Computer Science, Carleton University, Ottawa, Canada (vida@scs.carleton.ca). Partially supported by NSERC.

‡Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgiu (stefan.langerman@ulb.ac.be). Maître de recherches du F.R.S.-FNRS.

(that is, the vertices and the regions are labelled); in the other, each vertex is free to be assigned to any of the n regions (that is, the vertices are unlabelled).

When regions are points in the plane, Rosenstiehl and Tarjan [RT86] asked if there exists a set of n points that support all n-vertex unlabelled planar graphs. This question is answered in the negative by De Fraysseix [dFPP88, dFPP90]. On the contrary, every set of n points in general position supports all n-vertex unlabelled outerplanar graphs, as proved by Gritzmann *et al.*[GMPP91] and recapitulated in Lemma 14.7 in the text by Agarwal and Pach [AP95]. If the drawings are not restricted to be straight-line, then every set of labelled points supports every labelled planar graph, as shown by [PW01]. However $\Omega(n)$ bends per edge may be necessary in any such crossing-free drawing.

When regions are labelled lines in the plane, Estrella-Balderrama *et al.*[EBFK09] showed that for every $n \geq 6$, there is no set of n *parallel* lines in the plane that support all labelled n-vertex planar graphs. The authors moreover characterized a (sub)class of n-vertex planar graphs that are supported by every set of n-parallel lines, for every labelling of the graphs in the class. That class is mainly comprised of several special families of trees. Dujmović *et al.*[DEK$^+$10] showed that no set of n lines that *all* intersect in one common point supports all n-vertex labelled planar graphs. Moreover, they show that for every n large enough, there is a set of n lines in general position that does not support all labelled n-vertex planar graphs. They leave as the main open problem the question of whether, for every n large enough, there exists a *universal* set of n lines in the plane, that is, one that supports all labelled n-vertex planar graphs. In Section 5, as our main graph drawing result, we answer that question in the negative. The main result by Pach and Toth [PT04] on monotone drawings, has, as an easy consequence, that in the unlabelled case, every set of n-lines supports every n-vertex unlabelled planar graph. As a side note, we give an alternative and direct proof of that fact. The result illustrates the sharp contrast with the labelled case.

While the positive result is proved using little of the geometry in the arrangement, the non-existence of universal line sets required extraction of some (bad) substructure from any line arrangement. This prompted us to study several structural and partitioning problems for line and hyperplane arrangements.

Hyperplane arrangements Partitioning problems are central to our understanding of discrete and computational geometry, and while many works have focused on partitioning point sets, probability distributions or measures, much less is understood for sets of lines in $\mathbb{R}^2$ or hyperplanes in $\mathbb{R}^d$. This is partially due to the fact that a line (or a hyperplane), being infinite, can't be contained in any bounded region, or even in a halfplane (except if the boundary of the halfplane is parallel to the given line). Previous works (such as cuttings [CF90, Mat90] or equipartitions [LS03]) have focused on identifying, and bounding the number of lines/hyperplanes intersecting a set of regions. Others [CSSS89] on partitioning the vertices of the arrangements rather than the lines themselves. Those results have found numerous applications. Our graph drawing problem motivates a different approach.

An arrangement L of n lines in $\mathbb{R}^2$ is composed of vertices $V(L)$ (all pairwise intersections between lines of L), edges connecting these vertices, and half-lines. If we omit the half-lines, we are left with a finite graph which can be contained in a bounded region of the plane, in particular, it is contained in the convex hull $CH(V(L))$ of the vertices of the arrangement. Therefore, a natural way of evaluating the portion of an arrangement contained in a given convex region C is to find the largest subset L' of lines of L such that the arrangement of L' (without the half-lines) is contained in C, or equivalently, such that all pairwise intersections of lines in L' lie in C.

It is not hard to show that, in any arrangement of n lines, a line ℓ can be found such that for both closed halfplanes bounded by ℓ there are at least $\sqrt{n}$ lines which pairwise intersect in that halfplane. This provides the analogue of a bisecting line for point sets. In Section 3.1, we show that any two line arrangements can be bisected simultaneously in this manner, thus proving a ham-sandwich theorem for line arrangements. We also prove a centerpoint theorem: for any arrangement of n lines, there is a point q such that for any halfplane containing q, there are at least $\sqrt{n/3}$ lines of the arrangement that pairwise intersect in that halfplane. In Section 3.2 we generalize these notions to higher dimensions and prove a center transversal theorem: for any k and d, there is a growing function Q such that for any k sets $A_1, \ldots, A_k$ of hyperplanes in $\mathbb{R}^d$, there is a $(k-1)$-flat π such that for any halfspace h containing π there is a subset A'_i of $Q(|A_i|)$ hyperplanes from each set A_i such that any d hyperplanes of A'_i intersect in h. The bound Q we find is related to Ramsey numbers for hypergraphs.

Ham-sandwich theorems have a number of natural consequences. In Section 2 we show a same-type lemma for hyperplane arrangements: informally, for any k arrangements $A_1, \ldots, A_k$ of hyperplanes in general position (no $d+1$ share a point) and that are large enough, we can find a large subset of hyperplanes A'_i from each set A_i such that the convex hulls $CH(A'_i)$ of the vertices in the arrangements A'_i are well-separated, that is, no hyperplane hits $d+1$ of them. In the plane, we also show a positive portion Erdős-Szekeres theorem: for any integers k and c there is an integer N such that any set of N lines in general position contains k subsets $A_1, \ldots, A_k$ of c lines each such that the vertices of each arrangement A_i can be separated from those of all the others by a line.

All the results above would be relatively easy to prove if the set function we were computing – the maximum subset of hyperplanes that have all d-wise intersections in a given region – was a measure. Unfortunately it is not. However, in Section 2, we identify basic properties much weaker than those of measures which, if satisfied by a set function, guarantee a central-transversal theorem to be true.

2 Center transversal theorem

The center transversal theorem is a generalization of both the ham-sandwich cut theorem, and the centerpoint theorem discovered independently by Dol'nikov [Dol92], and Živaljević and Vrećica [ŽV90]. The version of Dol'nikov is defined for a class of set functions that is more general than measures. Let $\mathcal{H}$ be the set of all open halfspaces in $\mathbb{R}^d$ and let $\mathcal{G}$ be a family of subsets of $\mathbb{R}^d$ closed under union operations and that contains $\mathcal{H}$. A *charge* μ is a finite set function that is defined for all set $X \in \mathcal{G}$, and that is *monotone* ($\mu(X) \leq \mu(Y)$ whenever $X \subseteq Y$) and *subadditive* ($\mu(X \cup Y) \leq \mu(X) + \mu(Y)$). A charge μ is *concentrated* on a set X if for every halfspace $h \in \mathcal{H}$ s.t. $h \cap X = \emptyset$, $\mu(h) = 0$. Dol'nikov shows[1]:

Theorem 1 (Center transversal theorem [Dol92])**.** *For arbitrary k charges μ_i, $i = 1, \ldots, k$, defined on $\mathcal{G}$ and concentrated on bounded sets, there exists a $(k-1)$-flat π such that*

$$\mu_i(h) \geq \frac{\mu_i(\mathbb{R}^d)}{d-k+2}, i = 1, \ldots, k,$$

for every open halfspace $h \in \mathcal{H}$ containing π.

[1] Dol'nikov actually shows a slightly more general theorem that allows for non-concentrated charges. For the sake of simplicity we only discuss the simplified version even though our generalizations extend to the stronger original result.

A careful reading of the proof of this theorem reveals that its statement can be generalized, and the assumptions on μ_i weakened. We first notice that the subadditive property is only used in the proof for taking the union of a finite number of halfspaces from $\mathcal{H}$. Therefore, define μ to be *$\mathcal{H}$-subadditive* if

$$\mu(\cup_{h\in H}) \leq \sum_{h\in H} \mu(h)$$

for any finite set $H \subset \mathcal{H}$ of halfspaces.

Next, notice that in order for the proof to go through, the set function μ need not be real-valued. Recall [Bou07] that a *totally ordered unital magma* $(M, \oplus, \leq, e)$ is a totally ordered set M endowed with a binary operator $\oplus$ such that M is closed under $\oplus$ operations, $\oplus$ has *neutral* element e (i.e., $x \oplus e = x = e \oplus x$) and is *monotone* (i.e., $a \oplus c \leq b \oplus c$ and $c \oplus a \leq c \oplus b$ whenever $a \leq b$). Further, for all $x \in M$ and $c \in \mathbb{N}$, define the *c^{th} multiple of x* as $cx := \oplus^c x := \underbrace{x \oplus (x \oplus (\ldots \oplus x) \ldots)}_{c \text{ times}}$.

Then, it suffices that μ take values over M, and use e as the 0 used in the definition of a concentrated set function above. It is then the addition operator $\oplus$ which is to be used in the definition of the subadditive (or $\mathcal{H}$-subadditive) property and in the proof of the theorem. Thus, just by reading the proof of Dol'nikov under this new light we have:

Theorem 2. *Let μ_i, $i = 1, \ldots, k$ be k set functions defined on $\mathcal{G}$ and taking values in a totally ordered unital magma $(M, \oplus, \leq, e)$. If the functions μ_i are monotone $\mathcal{H}$-subadditive and concentrated on bounded sets, there exists a $(k-1)$-flat π such that*

$$(d - k + 2)\mu_i(h) \geq \mu_i(\mathbb{R}^d), i = 1, \ldots, k,$$

for every open halfspace $h \in \mathcal{H}$ containing π.

3 Center transversal theorem for arrangements

Let A be an arrangement of n hyperplanes in $\mathbb{R}^d$. We write $V(A)$ for the set of all vertices (intersection points between any d hyperplanes) of A and $CH(A) = CH(V(A))$ for the convex hull of those points. In the arguments that follow, by abuse of language, we will write A and mean $V(A)$ or $CH(A)$. For example, we say that the arrangement A is *above hyperplane* h when all points in $V(A)$ are above h. More generally for a region Q in $\mathbb{R}^d$, we say that the arrangement A does not intersect Q if $CH(A)$ does not intersect Q. We say that the k arrangements $A_1, A_2, \ldots, A_k$ are *disjoint* if their convex hulls do not intersect. They are *separable* if they are disjoint and no hyperplane intersects $d+1$ of them simultaneously.

Let $\mathcal{H}$ be the set of all open halfspaces in $\mathbb{R}^d$ and let $\mathcal{G}$ be a family of subsets of $\mathbb{R}^d$ closed under union operations and that contains $\mathcal{H}$. For any set $S \in \mathcal{G}$, let $\mu_A(S)$ be the maximum number of hyperplanes of A that have all their vertices inside of S, that is,

$$\mu_A(S) = \max_{A' \subseteq A, V(A') \subseteq S} |A'|.$$

In particular, $\mu_A(\mathbb{R}^d) = \mu_A(CH(A)) = n$ and $\mu_A(\emptyset) = d - 1$.

3.1 Lines in $\mathbb{R}^2$

We start with the planar case. Thus, A is a set of lines in $\mathbb{R}^2$, and $\mathcal{H}$ is the set of all open halfplanes. Recall the Erdős-Szekeres theorem [ES35]

Theorem 3 (Erdős-Szekeres)**.** *For all integers r, s, any sequence of $n > (r-1)(s-1)$ numbers contains either a non-increasing subsequence of length r or an increasing subsequence of length s.*

We show:

Lemma 4. *For any two sets $S_1 \in \mathcal{H}$ and $S_2 \in \mathcal{G}$,*

$$\mu_A(S_1 \cup S_2) \leq \mu_A(S_1)\mu_A(S_2)$$

Proof. Let ℓ be a line defining two open halfplanes ℓ^+ and ℓ^- such that $S_1 = \ell^-$ and let $S_2' = S_2 \setminus \ell^-$. Rotate and translate the plane so that ℓ is the (vertical) y axis, and ℓ^+ contains all points with positive x coordinate. Let A' be a maximum cardinality subset of A such that $V(A') \subseteq S_1 \cup S_2$. Let $l_1, \ldots, l_{|A'|}$ be the lines in A' ordered by increasing order of their slopes, and let $Y = (y_1, \ldots, y_{|A'|})$ be the y coordinates of the intersections of the lines l_i with line ℓ, in the same order. For any set $A_1 \subseteq A'$ such that the y_i values of the lines in A_1 form an increasing subsequence in Y, notice that $V(A_1) \subseteq S_1$. Likewise, for any set $A_2 \subseteq A'$ that forms a non-increasing subsequence in Y, we have $V(A_2) \subseteq S_2'$. Any such set A_1 is of size $|A_1| \leq \mu_A(S_1)$ and any such set A_2 is of size $|A_2| \leq \mu_A(S_2') \leq \mu_A(S_2)$.

Therefore, Y has no non-decreasing subsequence of length $\mu_A(S_1) + 1$ and no non-increasing subsequence of length $\mu_A(S_2)+1$, and so by Theorem 3, $\mu_A(S_1 \cup S_2) = |A'| = |Y| \leq \mu_A(S_1)\mu_A(S_2)$. □

Corollary 5. *The set function μ_A takes values in the totally ordered unital magma $(\mathbb{R}_0^+, \cdot, \leq, 1)$; it is monotone and $\mathcal{H}$-subadditive.*

We can thus apply the generalized center transversal theorem with $k = 2$ to obtain a ham-sandwich cut theorem:

Theorem 6. *For any arrangements A_1 and A_2 of lines in $\mathbb{R}^2$, there exists a line ℓ bounding closed halfplanes ℓ^+ and ℓ^- and sets A_i^σ, $i \in 1, 2$, $\sigma \in +, -$ such that $A_i^\sigma \subseteq A_i$, $|A_i^\sigma| \geq |A_i|^{1/2}$, and $V(A_i^\sigma) \in \ell^\sigma$.*

Note that this statement is similar to the result of Aronov *et al.*[AEG$^+$94] on mutually avoiding sets. Specifically, two sets A and B of points in the plane are *mutually avoiding* if no line through a pair of points in A intersects the convex hull of B, and vice versa. Note that, on the other hand, our notion of separability for lines is equivalent to the following definition in the dual. Two sets A and B of points in the plane are *separable* if there exists a point x such that all the lines through pairs of points in A are above x and all the lines through pairs of points in B are below x or vice versa. Aronov *et al.* show in Theorem 1 of [AEG$^+$94] that any two sets A_1 and A_2 of points contains two subsets $A_i' \subseteq A_i$, $|A_i'| \geq |A_i/12|^{1/2}$, $i \in \{1, 2\}$ that are mutually avoiding. That this bound is tight, up to a constant, was proved by Valtr [Val97]. In the dual, Theorem 6 states that for any two sets A_1 and A_2 of points in $\mathbb{R}^2$, there exists a point ℓ and sets A_i^σ, $i \in 1, 2$, $\sigma \in +, -$ such that $A_i^\sigma \subseteq A_i$, $|A_i^\sigma| \geq |A_i|^{1/2}$, and all lines through pairs of points in A_i^+ are above ℓ and all lines through pairs of points in A_i^- are below ℓ. While similar, neither the two results nor the two notions of mutually avoiding and separable are equivalent. It is not difficult to show that no result/notion immediately implies the other. Moreover, neither our proof of Theorem 6 nor the proof of Theorem 1 in [AEG$^+$94] give two sets that are, at the same time, mutually avoiding and separable.

Note that the bound in Theorem 6 is tight: assume n is the square of an integer. Construct the first line arrangement A_1 with $\sqrt{n}$ pencils of $\sqrt{n}$ lines each, centered at points with coordinates $(-1/2, i)$ for $i = 1, \ldots, \sqrt{n}$, and the slopes of the lines in pencil i are distinct values in $[1/2 - (i - 1)/\sqrt{n}, 1/2 - i/\sqrt{n}]$. Thus all intersections other than the pencil centers have x coordinates greater than 1/2. The line $x = 0$ delimits two halfplanes in which $\mu_{A_1}(x \leq 0) = \sqrt{n}$ since any set of more than $\sqrt{n}$ lines have lines from different pencils which intersect on the right of $x = 0$, and $\mu_{A_1}(x \geq 0) = \sqrt{n}$ since any set of more than $\sqrt{n}$ lines has two lines in the same pencil which intersect left of $x = 0$. Since μ_{A_1} is monotone, no vertical line can improve this bound on both sides. Perturb the lines so that no two intersection points have the same x coordinate. For A_2, build a copy of A_1 translated down, far enough so that no line through two vertices of A_1 intersects $CH(A_2)$ and conversely. Therefore any line not combinatorially equivalent to a vertical line (with respect to the vertices of A_1 and A_2) does not intersect one of the arrangements and so there is no better cut than $x = 0$.

Applying the generalized center transversal theorem with $k = 1$ gives a centerpoint theorem with a bound of $|A|^{1/3}$. A slightly more careful analysis improves that bound.

Theorem 7. *For any arrangement A of lines in $\mathbb{R}^2$, there exists a point q such that for every halfplane h containing q there is a set $A' \subseteq A, |A'| \geq \sqrt{|A|/3}$, such that $V(A') \in h$.*

Proof. Let H be the set of halfplanes h such that $\mu_A(h) < z = \sqrt{|A|/3}$. The *halfspace depth* $\delta(q)$ is the minimum value of $\mu_A(h)$ for any halfspace containing q. Therefore, the region of depth $\geq z$ is the intersection of the complements $\overline{h}$ of the halfplanes $h \in H$. If there is no point of depth $\geq z$ then the intersection of the complements of halfplanes in H is empty, and so (by Helly's Theorem) there must be 3 halfplanes h_1, h_2, and h_3 in H such that the intersection of their complements $\overline{h_1} \cap \overline{h_2} \cap \overline{h_3}$ is empty. But then, there is at least one point $q \in h_1 \cap h_2 \cap h_3$. Let h'_i be the translated halfplanes h_i with point q on the boundary. Since $h'_i \subseteq h_i$, $\mu_A(h'_i) \leq \mu_A(h_i) < z$. The point q and the 3 halfplanes through it are witness that there is no point of depth $\geq z$.

The 3 lines bounding those 3 halfplanes divide the plane into 6 regions. Every line misses one of the three regions $h'_1 \cap \overline{h'_2} \cap \overline{h'_3}$, $\overline{h'_1} \cap h'_2 \cap \overline{h'_3}$, and $\overline{h'_1} \cap \overline{h'_2} \cap h'_3$. Classify the lines in A depending on the first region it misses, clockwise. The largest class A' contains $\geq |A|/3$ lines. Assume without loss of generality that all lines in A' miss $h'_1 \cap \overline{h'_2} \cap \overline{h'_3}$, then all intersections between lines of A' are in $h'_2 \cup h'_3$. By Lemma 4,

$$|A|/3 \leq |A'| = \mu_{A'}(h'_2 \cup h'_3) \leq \mu_{A'}(h'_2)\mu_{A'}(h'_3) < z^2 = |A|/3,$$

a contradiction. $\square$

3.2 Hyperplanes in $\mathbb{R}^d$

We first briefly review a bichromatic version of Ramsey's theorem for hypergraphs.

Theorem 8. *For all $p, a, b \in \mathbb{N}$, there is a natural number $R = R_p(a, b)$ such that for any set S of size R and any 2-colouring $c : \binom{S}{p} \to \{1, 2\}$ of all subsets of S of size p, there is either a set A of size a*

such that all p-tuples in $\binom{A}{p}$ have colour 1 or a set B of size b such that all p-tuples in $\binom{B}{p}$ have colour 2.

Lemma 9. *For any two sets $S_1 \in \mathcal{H}$ and $S_2 \in \mathcal{G}$,*

$$\mu_A(S_1 \cup S_2) \le R_d(\mu_A(S_1) + 1, \mu_A(S_2) + 1) - 1$$

Proof. Let h be a hyperplane defining two open halfplanes h^+ and h^- such that $S_1 = h^-$ and let $S_2' = S_2 \setminus h^-$. Let A' be a maximum cardinality subset of A such that $V(A) \subseteq S_1 \cup S_2$. Colour every subset of d hyperplanes in A' with colour 1 if their intersection point is in h^- and with colour 2 otherwise.

For any set $A_1 \subseteq A'$ such that all subsets in $\binom{A_1}{d}$ have colour 1, notice that $V(A_1) \subseteq S_1$. Likewise, for any set $A_2 \subseteq A'$ such that all subsets in $\binom{A_2}{d}$ have colour 2, we have $V(A_2) \subseteq S_2'$. Any such set A_1 is of size $|A_1| \le \mu_A(S_1)$ and any such set A_2 is of size $|A_2| \le \mu_A(S_2') \le \mu_A(S_2)$.

Therefore, A' has no subset of size $\mu_A(S_1) + 1$ that has all d-tuples of colour 1, and no subset of size $\mu_A(S_2) + 1$ that has all d-tuples of colour 2, and so by Ramsey's Theorem, $\mu_A(S_1 \cup S_2) = |A'| \le R_d(\mu_A(S_1) + 1, \mu_A(S_2) + 1) - 1$. $\square$

Define the operator $\oplus$ as $a \oplus b = R_d(a + 1, b + 1) - 1$. The operator is increasing and closed on the set $\mathbb{N}_{\ge d-1}$ of naturals $\ge d - 1$. Since $R_d(d, x) = x$ for all x, $d - 1$ is a neutral element. Therefore $(\mathbb{N}_{\ge d-1}, \oplus, \le, d - 1)$ is a totally ordered unital magma. Thus we have:

Corollary 10. *The set function μ_A takes values in the totally ordered unital magma $(\mathbb{N}_{\ge d-1}, \oplus, \le, d - 1)$; it is monotone and $\mathcal{H}$-subadditive.*

Apply now the generalized center transversal theorem to obtain:

Theorem 11. *Let $A_1, \ldots, A_k$ be k sets of hyperplanes in $\mathbb{R}^d$. There exists a $(k-1)$-flat π such that for every open halfspace h that contains π,*

$$(d - k + 2)\mu_{A_i}(h) \ge |A_i|.$$

The special case when $k = d$ gives a ham-sandwich cut theorem.

Corollary 12. *Let $A_1, \ldots, A_d$ be d sets of hyperplanes in $\mathbb{R}^d$. There exists a hyperplane π bounding the two closed halfspaces π^+ and π^- and sets $A_i^\sigma \subseteq A_i$, $\sigma \in \{+, -\}$, such that $V(A_i^\sigma) \subseteq \pi^\sigma$ and $|A_i^\sigma| \oplus |A_i^\sigma| \ge |A_i|$.*

If the arrangement A has the property that no $r + 1$ hyperplanes intersect in a common point, $\mu_A(\pi) \le r$ for any hyperplane π, and so by Lemma 9, if h is an open halfspace bounded by π and $\bar{h} = \pi \cup h$ is the corresponding closed halfspace, $\mu_A(\bar{h}) \le \mu_A(h) \oplus r$.

Corollary 13. *Let $A_1, \ldots, A_d$ be d sets of hyperplanes in $\mathbb{R}^d$, no $r + 1$ of which intersect in a common point. There exists a hyperplane π bounding the two open halfspaces π^+ and π^- and sets $A_i^\sigma \subseteq A_i$, $\sigma \in \{+, -\}$, such that $V(A_i^\sigma) \subseteq \pi^\sigma$ and $(|A_i^\sigma| \oplus |A_i^\sigma|) \oplus r \ge |A_i|$.*

4 Same-type lemma for arrangements

Center transversal theorems, and especially the ham-sandwich cut theorem, are basic tools for proving many facts in discrete geometry. We show here how the same facts can be shown for hyperplane arrangements in $\mathbb{R}^d$.

A transversal of a collection of sets $X_1, \ldots, X_m$ is a m-tuple $(x_1, \ldots, x_m)$ where $x_i \in X_i$. A collection of sets $X_1, \ldots, X_m$ has *same-type transversals* if all of its transversals have the same order-type.

Note that $m \geq d+1$ sets have same-type transversals if and only if every $d+1$ of them have same-type transversals. There are several equivalent definitions for these notions.

1. The sets $X_1, \ldots, X_{d+1}$ have same-type transversals if and only if they are *well separated*, that is, if and only if for all disjoint sets of indices $I, J \subseteq \{1, \ldots, d+1\}$, there is a hyperplane separating the sets $X_i, i \in I$ from the sets $X_j, j \in J$.
2. Connected sets $C_1, \ldots, C_{d+1}$ have same-type transversals if and only if there is no hyperplane intersecting simultaneously all C_i. Sets $X_1, \ldots, X_{d+1}$ have same-type transversals if and only if there is no hyperplane intersecting simultaneously all their convex hulls $C_i = CH(X_i)$.

The same-type lemma for point sets states that there is a constant $c = c(m, d)$ such that for any collection $S_1, \ldots, S_m$ of finite point sets in $\mathbb{R}^d$, there are sets $S'_i \subseteq S_i$ such that $|S'_i| \geq c|S_i|$ and the sets $S'_1, \ldots, S'_m$ have same-type transversals. We here show a similar result for hyperplane arrangements.

A function f is *growing* if for any value y_0 there is a x_0 such that $f(x) \geq y_0$ for any $x \geq x_0$.

Lemma 14. *For any integers d, m, and r, there is a growing function $f = f_{m,d,r}$ such that for any collection of m hyperplane arrangements $A_1, \ldots, A_m$, in $\mathbb{R}^d$, where no $r+1$ hyperplanes intersect at a common point, there are sets $A'_i \subseteq A_i$ such that $|A'_i| \geq f(|A_i|)$ and the sets $CH(A'_1), \ldots, CH(A'_m)$ have same-type transversals.*

Proof. The proof will follow closely the structure of Matoušek [Mat02, Theorem 9.3.1, p.217]. First notice that the composition of two growing functions is a growing function. The proof will show how to choose successive (nested) subsets of each set A_i, c times where $c = c(m, d)$ only depends on m and d and where the size of each subset is some growing function of the previous one.

Also, it will suffice to prove the theorem for $m = d+1$, and then apply it repeatedly for each $d+1$ tuple of sets. The resulting function $f_{m,d,r}$ will be $f_{d+1,d,r}^{\binom{m}{d+1}}$, the repeated composition of $f_{d+1,d,r}$, $\binom{m}{d+1}$ times.

So, given $d+1$ sets $A_1, \ldots, A_{d+1}$ of hyperplanes in $\mathbb{R}^d$, suppose that there is an index set $I \subseteq \{1, \ldots, d+1\}$ such that $\cup_{i \in I} \mathrm{CH}(A_i)$ and $\cup_{i \notin I} \mathrm{CH}(A_i)$ are not separable by a hyperplane and assume without loss of generality that $d+1 \in I$. Let π be the ham sandwich cut hyperplane for arrangements $A_1, \ldots, A_d$ obtained by applying Corollary 13. Then for each $i \in [1, d]$, each of the two open halfspaces π^σ, $\sigma \in \{+, -\}$ bounded by π contains a subset $A_i^\sigma \subseteq A_i$ such that $V(A_i^\sigma) \subseteq \pi^\sigma$ and

$(|A_i^\sigma| \oplus |A_i^\sigma|) \oplus r \geq |A_i|$. Furthermore, because $\mu_{A_{d+1}}(\pi) \leq r$ and by Lemma 9,

$$\mu_{A_{d+1}}(\pi^+) \oplus \mu_{A_{d+1}}(\pi^-) \oplus r \geq \mu_{A_{d+1}}(\mathbb{R}^d) = |A_{d+1}|.$$

Assume without loss of generality $\mu_{A_{d+1}}(\pi^+) \geq \mu_{A_{d+1}}(\pi^-)$. Then $\mu_{A_{d+1}}(\pi^+) \oplus \mu_{A_{d+1}}(\pi^+) \oplus r \geq |A_{d+1}|$. For each $i \in I$, let $A'_i = A_i^+$ and for each $i \notin I$, let $A'_i = A_i^-$. Let $g(x) = \min\{y | y \oplus y \oplus r \geq x\}$. Then g is a growing function, and $|A'_i| \geq g(|A_i|)$.

In the worst case, we have to shrink the sets for each possible I, 2^d times. Therefore for $m = d+1$, the function f in the statement of the theorem is a composition of g, 2^d times, and is a growing function. $\square$

In the plane, the same-type lemma readily gives a positive portion Erdős-Szekeres Theorem. Recall that the Erdős-Szekeres (happy ending) theorem [ES35] states that for any k there is a number $ES(k)$ such that any set of $ES(k)$ points in general position in $\mathbb{R}^2$ contains a subset of size k which is in convex position.

Theorem 15. *For every integers k, r, and c, there is an integer N such that any arrangement A of N lines, such that no $r+1$ lines go through a common point, contains disjoint subsets $A_1, \ldots, A_k$ with $|A_i| \geq c$ and such that every transversal of $\mathrm{CH}(A_1), \ldots, \mathrm{CH}(A_k)$ is in convex position.*

Proof. Let $m = ES(k)$ and let $f = f_{m,2,r}$ be as in Lemma 14. Let N be such that $f(\lfloor N/m \rfloor) \geq c$. Partition the set A of N lines into m sets $A_1, \ldots, A_m$ of N/m lines arbitrarily. Apply Lemma 14 to obtain sets $A'_1, \ldots, A'_m$ each of size at least c. Finally, choose one transversal $(x_1, \ldots, x_m)$ from the sets $\mathrm{CH}(A'_i)$ and apply the Erdős-Szekeres theorem to obtain a subset $x_{i_1}, \ldots, x_{i_k}$ of points in convex position. Because the sets $\mathrm{CH}(A'_i)$ have the same type property, every transversal of $\mathrm{CH}(A'_{i_1}), \ldots, \mathrm{CH}(A'_{i_k})$ is in convex position. $\square$

5 Graph Drawing

Formally, a vertex labelling of a graph $G = (V, E)$ is a bijection $\pi : V \to [n]$. A set of n lines in the plane labelled from 1 to n *supports* G *with vertex labelling* π if there exists a straight-line crossing-free drawing of G where for each $i \in [n]$, the vertex labelled i in G is mapped to a point on line i. A set L of n lines labelled from 1 to n *supports* an n-vertex graph G if for every vertex labelling π of G, L supports G with vertex labelling π. In this context clearly it only makes sense to talk about planar graphs. We are interested in the existence of an n-vertex line set that supports all n-vertex planar graphs, that is, in the existence of a *universal* set of lines for planar graphs.

Theorem 16. *For some absolute constant c' and every $n \geq c'$, there exists no set of n lines in the plane that support all n-vertex planar graphs.*

The following known result will be used in the proof of this theorem.

Lemma 17. *[DEK+10] Consider the planar triangulation on 6 vertices, denoted by G_6, that is depicted on the bottom of Figure 1. G_6 has vertex labelling π such that the following holds for every set L of 6 lines labelled from 1 to 6, no two of which are parallel. For every straight-line crossing-free drawing, D, of G_6 where for each $i \in [n]$, the vertex labelled i in π is mapped to a point on line i in L, there is a point that is in an interior face of D and in* $\mathrm{CH}(L)$.

Proof of Theorem 16. Let L be any set of $n \geq c' = 5N$ lines, where N is obtained from Theorem 15 with values $k = 6$, $c = 6$, $r = 17$.

[EBFK09] proved that for every $n \geq 6$, no set of n parallel lines supports all n-vertex planar graphs. Thus if L has at least 6 lines that are pairwise parallel, then L cannot support all n-vertex planar graphs.

[DEK+10] proved that for every $n \geq 18$, no set of n lines that all go through a common point supports all n-vertex planar graphs. Thus if L has at least 18 such lines, then L cannot support all n-vertex planar graphs.

Thus assume that L has no 6 pairwise parallel lines and no 18 lines that intersect in one common point. Then L has a subset L' of $c'/5 \geq N$ lines no two of which are parallel and no 18 of which go through one common point. Then Theorem 15 implies that we can find in L' six sets $A_1, \ldots, A_6$ of six lines each, such that the set $\{\mathrm{CH}(A_1), \ldots, \mathrm{CH}(A_6)\}$ is in convex position. Assume $\mathrm{CH}(A_1), \ldots, \mathrm{CH}(A_6)$ appear in that order around their common "convex hull".

Consider an n-vertex graph H whose subgraph G is illustrated in Figure 1. $G \setminus v$ has three components, A, B, and C, each of which is a triangulation. Each of the components A, B, and C has two vertex disjoint copies of G_6 (the 6-vertex triangulation from Lemma 17). Map the vertices of the first copy of G_6 in A to A_1 and the second copy to A_4 using the mapping equivalent to π in Lemma 17. Map the vertices of the first copy of G_6 in B to A_2 and the second copy to A_5 using the mapping equivalent to π in Lemma 17. Map the vertices of the first copy of G_6 in C to A_3 and the second copy to A_6 using the mapping equivalent to π in Lemma 17. Map the remaining vertices of H arbitrarily to the remaining lines of L.

We now prove that L does not support H with such a mapping. Assume, for the sake of

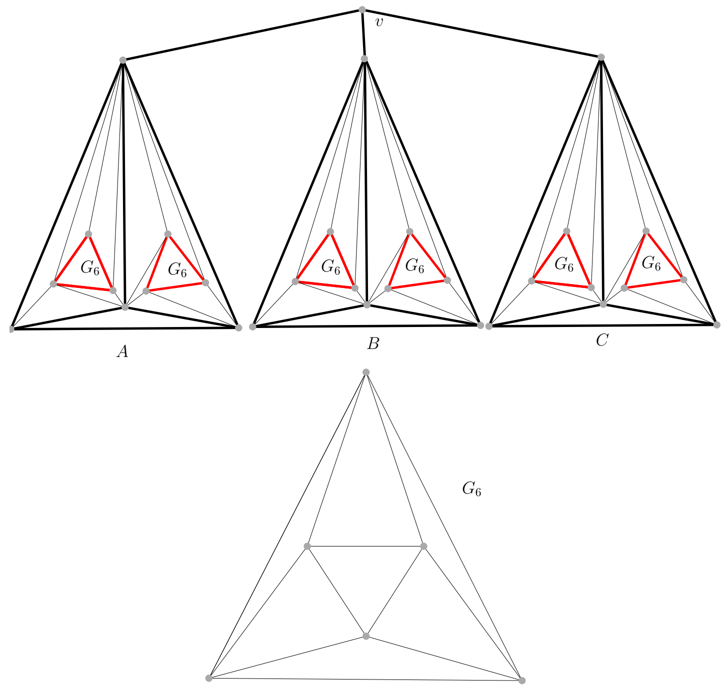

Figure 1: Illustration for the proof of Theorem 16.

contradiction, that it does and consider the resulting crossing-free drawing D of H. In D the drawing of each of A, B, and C has a triangle as an outerface. Let T_A, T_B, and T_C denote these three triangles together with their interiors in the plane.

It is simple to verify that in any crossing-free drawing of G at least two of these triangles are disjoint, meaning that there is no point p in the plane such that p is in both of these triangles. Assume, without loss of generality, that T_A and T_B are disjoint. By Lemma 17, there is a point $p_1 \in \mathrm{CH}(A_1)$ such that $p_1 \in T_A$, and a point $p_4 \in \mathrm{CH}(A_4)$ such that $p_4 \in T_A$. Thus the segment $\overline{p_1p_4}$ is in T_A. Similarly, by Lemma 17, there is a point $p_2 \in \mathrm{CH}(A_2)$ such that $p_2 \in T_B$, and a point $p_5 \in \mathrm{CH}(A_5)$ such that $p_5 \in T_B$. Thus the segment $\overline{p_2p_5}$ is in T_A.

By Theorem 15 and our ordering of $A_1, \ldots, A_6$, $\overline{p_1p_4}$ and $\overline{p_2p_5}$ intersect in some point p. That implies that $p \in T_A$ and $p \in T_B$. That provides the desired contradiction, since T_A and T_B are disjoint. □

As a sharp contrast to Theorem 16, the following theorem shows that the situation is starkly different for unlabelled planar graphs. Namely, every set of n lines supports all n-vertex unlabelled planar graphs. The proof of this theorem does not use any of the tools we introduced in the previous section and is in that sense elementary. It is not difficult to verify that the theorem also follows from the main result in [PT04] which states the following: given a drawing of a graph G in the plane where edges of G are x-monotone curves any pair of which cross even number of times, G can be redrawn as a straight-line crossing-free drawing where the x-coordinates of the vertices remain unchanged.

Theorem 18. *[PT04] Given a set L of n lines in the plane, every planar graph has a straight line crossing free drawing where each vertex of G is placed on a distinct line of L.* (*In other words, given any set L of lines, labelled from* 1 *to n, and any n-vertex planar graph G there is a vertex labelling π of G such that L supports G with vertex labelling π.*)

Proof. In this proof we will use canonical orderings introduced in [dFPP90] and a related structure called frame introduced in [BDH$^+$09]. We first recall these tools. We can assume G is an embedded edge maximal planar graph.[2] Each face of G is bounded by a 3-cycle. De Fraysseix [dFPP90] proved that G has a vertex ordering $\sigma = (v_1, v_2, v_3, \ldots, v_n)$, called a *canonical ordering*, with the following properties. Define G_i to be the embedded subgraph of G induced by $\{v_1, v_2, \ldots, v_i\}$. Let C_i be the subgraph of G induced by the edges on the boundary of the outer face of G_i. Then

- v_1, v_2 and v_n are the vertices on the outer face of G.
- For each $i \in \{3, 4, \ldots, n\}$, C_i is a cycle containing $v_1 v_2$.
- For each $i \in \{3, 4, \ldots, n\}$, G_i is biconnected and *internally* 3-*connected*; that is, removing any two interior vertices of G_i does not disconnect it.
- For each $i \in \{3, 4, \ldots, n\}$, v_i is a vertex of C_i with at least two neighbours in C_{i-1}, and these neighbours are consecutive on C_{i-1}.

For example, the ordering in Figure 2(a) is a canonical ordering of the depicted embedded graph G.

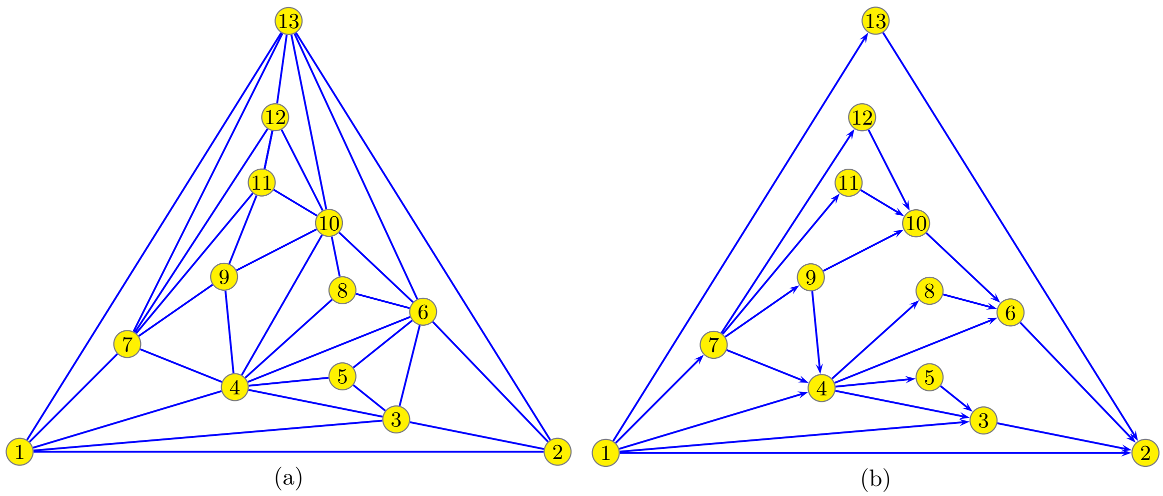

Figure 2: Illustration for the proof of Theorem 18: (a) Canonical ordering of G, (b) Frame $\mathcal{F}$ of G

A *frame* $\mathcal{F}$ of G [BDH$^+$09] is the oriented subgraph of G with vertex set $V(\mathcal{F}) := V(G)$, where:

- $v_1 v_2$ is in $E(\mathcal{F})$ and is oriented from v_1 to v_2.
- For each $i \in \{3, 4, \ldots, n\}$ in the canonical ordering σ of G, edges pv_i and $v_i p'$ are in $E(\mathcal{F})$, where p and p' are the first and the last neighbour, respectively, of v_i along the path in C_{i-1} from v_1 to v_2 not containing edge $v_1 v_2$. Edge pv_i is oriented from p to v_i, and edge $v_i p'$ is oriented from v_i to p', as illustrated in Figure 2(b).

By definition, $\mathcal{F}$ is a directed acyclic graph with one source v_1 and one sink v_2. The frame $\mathcal{F}$ defines a partial order $<_{\mathcal{F}}$ on $V(\mathcal{F})$, where $v <_{\mathcal{F}} w$ whenever there is a directed path from v to w in $\mathcal{F}$.

Translate the given set L of lines so that all vertices of the arrangement of lines have negative y coordinates, and sort the lines $\ell_i \in L$ according to the x coordinate b_i of the intersection of ℓ_i with

[2] A planar graph H is edge-maximal (also called, a *triangulation*), if for all $vw \notin E(H)$, the graph resulting from adding vw to H is not planar.

the x axis. Therefore, the lines $\ell_i \in L$ have equation $y = a_i(x - b_i)$, with $b_1 < b_2 < \ldots < b_n$. Because all intersections among lines of L have negative coordinates, all b_i are distinct, and the values $1/a_i$ are sorted. Note that the slopes a_i might be positive or negative. Let $\hat{a} = \min |a_i|$. For any segment of slope in $[-\hat{a}, \hat{a}]$ connecting two points $(x_i, y_i) \in \ell_i$ and $(x_j, y_j) \in \ell_j$ above the x axis (that is, $y_i, y_j > 0$), $x_i < x_j$ if and only if $i < j$.

Construct a linear extension $v_{\rho(1)}, v_{\rho(2)}, \ldots, v_{\rho(n)}$ of the partial order $<_{\mathcal{F}}$ and define the bijection $\pi : V \to [n]$ as $\pi(v_{\rho(i)}) = i$. That is, the vertices of G will be placed on the lines in such a way that the partial order $<_{\mathcal{F}}$ is compatible with the order determined by the values b_i of the lines.

We prove by induction that for every value $\hat{y}$ and every $i \geq 2$, it is possible to draw G_i such that v_1 and v_2 are placed on points $(b_1, 0)$, $(b_n, 0)$, and the y coordinates of all other vertices are in the horizontal slab $(0, \hat{y}]$. The base case $(i = 2)$ is obviously true.

Note that we could have formulated the induction on the slopes of the edges of G_i in the drawing. In fact those two formulations imply each other: for any value $0 < s \leq \hat{a}$, there is a $\hat{y}_s > 0$ such that any segment whose endpoints lie on distinct lines of L and have y coordinates in $[0, \hat{y}_s]$, the slope of the segment is in $[-s, s]$. This is easy to see: draw an upward cone with apex on each point $(b_i, 0)$ and bounded by the lines of slopes s and $-s$ through that point. Define $\hat{y}_s$ as the y coordinate of the lowest intersection point between any two such cones. Any segment with a slope not in $[-s, s]$ and with its lowest point inside a cone must have its highest point inside the same cone, therefore no segment connecting two different lines inside the horizontal slab $[0, \hat{y}_s]$ can have such a slope.

Assume by induction that the statement is true for G_{i-1}. We will show how to draw G_i for a specific value $\hat{y}$. The point v_i will be placed on the point on line $\pi(v_i)$ with y coordinate $\hat{y}$. Let s_1 and s_2 be the slopes of the segments $v_1 v_i$ and $v_i v_2$, and let $s = \max(|s_1|, |s_2|)/2$ or $\hat{a}$, whichever is smaller. Let y_1 be the intersection of the line of slope s through v_i and line ℓ_1 and y_2 the intersection of the line of slope $-s$ through v_i and ℓ_n. Note that y_1 and y_2 are strictly positive. Let $\hat{y}' = \min(y_1, y_2, \hat{y}_s)$. Apply the induction hypothesis to draw G_{i-1} in the horizontal slab $[0, \hat{y}']$. Thus, in the drawing of G_{i-1}, all edges have slope at most $s \leq \hat{a}$. Then by construction, the path in C_{i-1} from v_1 to v_2 not containing edge $v_1 v_2$ is x-monotone (that is, all its edges are oriented rightwards), and v_i is above the supporting line of each edge on that path. Therefore, v_i can see all vertices in C_{i-1} and all edges adjacent to v_i can be drawn. $\square$

We conclude this part with an intriguing 3D variant of this graph drawing problem. A graph is *linkless* if it has an embedding in 3D such that any two cycles of the graph are unlinked[3]. These graphs form a three-dimensional analogue of the planar graphs.

Open Problem 19. *Is there an arrangement of labelled planes in 3D such that any labelled linkless graph has a linkless straight-line embedding where each vertex is placed on the plane with the same label?*

[3]Two disjoint curves in 3D are *unlinked* if there is a continuous motion of the curves which transforms them into disjoint coplanar circles without one curve passing through the other or through itself.

Acknowledgements

We wish to thank János Pach for bringing our attention to the notion of mutually avoiding sets [AEG+94, Val97] and for pointing out that Theorem 18 was an easy consequence of [PT04]. We also wish to thank Pat Morin for helpful discussions and the anonymous referees for their comments.

References

[BDH+09] Prosenjit Bose, Vida Dujmović, Ferran Hurtado, Stefan Langerman, Pat Morin, and David Wood. A polynomial bound for untangling geometric planar graphs. *Discrete and Computational Geometry*, 42:570–585, 2009.

[Bou07] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 1 à 3.* Springer-Verlag Berlin Heidelberg, 2007.

[CF90] Bernard Chazelle and Joel Friedman. A deterministic view of random sampling and its use in geometry. *Combinatorica*, 10(3):229–249, 1990.

[CSSS89] Richard Cole, Jeffrey Salowe, William Steiger, and Endre Szemerédi. An optimal-time algorithm for slope selection. *SIAM Journal on Computing*, 18(4):792–810, 1989.

[DEK+10] Vida Dujmović, Will Evans, Stephen G. Kobourov, Giuseppe Liotta, Christophe Weibel, and Stephen Wismath. On graphs supported by line sets. In *Proceedings of the 18th Symposium on Graph Drawing (GD'10)*, pages 177–182, Volume 6502, 2011.

[dFPP88] Hubert de Fraysseix, János Pach, and Richard Pollack. Small sets supporting Fary embeddings of planar graphs. In *Proceedings 20th Symposium on Theory of Computing (STOC)*, pages 426–433, 1988.

[dFPP90] Hubert de Fraysseix, János Pach, and Richard Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.

[Dol92] Vladimir L. Dol'nikov. A generalization of the ham sandwich theorem. *Mathematical Notes*, 52:771–779, 1992.

[EBFK09] Alejandro Estrella-Balderrama, J. Joseph Fowler, and Stephen G. Kobourov. Characterization of unlabeled level planar trees. *CGTA: Computational Geometry: Theory and Applications*, 42(6-7):704–721, 2009.

[ES35] Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[GMPP91] Peter Gritzmann, Bojan Mohar, János Pach, and Richard Pollack. Embedding a planar triangulation with vertices at specified points. *The American Mathematical Monthly*, 98(2):165–166, February 1991.

[LS03] Stefan Langerman and William Steiger. Optimization in arrangements. In *Proceedings of the 20th International Symposium on Theoretical Aspects of Computer Science (STACS 2003)*, volume 2607 of *LNCS*, pages 50–61. Springer-Verlag, 2003.

[Mat90] Jiři Matoušek. Construction of ϵ-nets. *Discrete and Computational Geometry*, 5:427–448, 1990.

[Mat02] Jiři Matoušek. *Lectures on Discrete Geometry.* Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.

[PW01] János Pach and Rephael Wenger. Embedding planar graphs at fixed vertex locations. *Graphs and Combinatorics*, 17:717–728, 2001.

[RT86] Pierre Rosenstiehl and Robert E. Tarjan. Rectilinear planar layouts and bipolar orientations of planar graphs. *Discrete Computational Geometry*, 1(4):343–353, 1986.

[ŽV90] Rade T. Živaljević and Siniša T. Vrećica. An extension of the ham sandwich theorem. *Bulletin of the London Mathematical Society*, 22:183–186, 1990.

[AEG+94] Boris Aronov, Paul Erdős, Wayne Goddard, Daniel J. Kleitman, Michael Klugerman, János Pach, Leonard J. Schulman. Crossing families. *Combinatorica*, 14:127–134, 1994.

[Val97] Pavel Valtr. On mutually avoiding sets *The Mathematics of Paul Erdős II, Algorithms and Combinatorics*, 14:324–332, 1997.

[AP95] Pankaj K. Agarwal and János Pach. *Combinatorial Geometry.* Wiley-Interscience Series in Discrete Mathematics and Optimization, 1995.

[PT04] János Pach and Geza Toth. Monotone drawings of planar graphs. *Journal of Graph Theory*, 46(1):39–47, 2004.