Undirected Connectivity of Sparse Yao Graphs

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ABSTRACT

Given a finite set S of points in the plane and a real value $d > 0$, the $d-$radius disk graph G^d contains all edges connecting pairs of points in S that are within distance d of each other. For a given graph G with vertex set S, the Yao subgraph $Y_k[G]$ with integer parameter $k > 0$ contains, for each point $p \in S$, a shortest edge $pq \in G$ (if any) in each of the k sectors defined by k equally-spaced rays with origin p. Motivated by communication issues in mobile networks with directional antennas, we study the connectivity properties of $Y_k[G^d]$, for small values of k and d. In particular, we derive lower and upper bounds on the minimum radius d that renders $Y_k[G^d]$ connected, relative to the unit radius assumed to render G^d connected. We show that $d = \sqrt{2}$ is necessary and sufficient for the connectivity of $Y_4[G^d]$. We also show that, for $d \leq 5 - \frac{2}{3}\sqrt{35}$, the graph $Y_3[G^d]$ can be disconnected, but $Y_3[G^{2/\sqrt{3}}]$ is always connected. Finally, we show that $Y_2[G^d]$ can be disconnected, for any $d \geq 1$.

1. INTRODUCTION

Let S be a finite set of points in the plane and let $G = (S, E)$ be an arbitrary (undirected) graph with node set S. The *directed Yao graph* $\overrightarrow{Y_k}[G]$ with integer parameter $k > 0$ is a subgraph of G defined as follows. At each point $p \in S$, k equally-spaced rays with origin p define k cones. In each cone, pick a shortest edge pq from G, if any, and add the directed edge $\overrightarrow{pq}$ to $\overrightarrow{Y_k}[G]$. Ties are broken arbitrarily. The *undirected* Yao graph $Y_k[G]$ ignores the directions of edges, and includes an edge pq if and only if either $\overrightarrow{pq}$ or $\overrightarrow{qp}$ is in $\overrightarrow{Y_k}[G]$.

For a fixed real value $d > 0$, let $G^d(S)$ denote the $d-$radius disk graph with node set S, in which two nodes $p, q \in S$ are adjacent if and only if $|pq| \leq d$. Most often we will refer to $G^d(S)$ simply as G^d, unless the point set S that defines G^d is unclear from the context. Under this definition, G^1 is the unit disk graph (UDG), and G^∞ is the complete Euclidean graph, in which any two points are connected by an edge. In this paper we study the connectivity of the undirected Yao graph $Y_k[G^d]$, for small values $k \in \{2, 3, 4\}$ and $d \geq 1$. Underlying our study is the assumption that G^1 is connected. (For example, G^1 can be thought of as the graph connecting all pairs of points that are within distance no greater than the length of the bottleneck edge in a minimum spanning tree for S, normalized to one.) In this context, we investigate the following problem:

> *Let S be an arbitrary set of points in the plane, and suppose that the unit radius graph G^1 defined on S is connected. What is the smallest real value $d \geq 1$ for which $Y_k[G^d]$ is connected?*

Throughout the paper, we will refer to the minimum value d that renders $Y_k[G^d]$ connected as the *connectivity radius* of $Y_k[G^d]$.

Our research is inspired by the use of wireless directional antennas in building communication networks. Unlike an omnidirectional antenna, which transmits energy in all directions, a directional antenna can concentrate its transmission energy within a narrow cone; the narrower the cone, the longer the transmission range, for a fixed transmission power level. Directional antennas are preferable over omnidirectional antennas, because they reduce interference and extend network lifetime, two criteria of utmost importance in wireless networks operating on scarce battery resources.

Directed Yao edges can be realized with narrow directional antennas (otherwise called laser-beam antennas, to imply a small cone angle, close to zero). One attractive property of Yao graphs is that they can be efficiently constructed locally, because each node can select its incident edges based on the information from nodes in its immediate neighborhood only. This enables each node to repair the communication structure quickly in the face of dynamic and kinetic changes, providing strong support for node mobility.

The limited number of antennas per node (1 to 4 in practice), raises the fundamental question of connectivity of Yao graphs Y_k, for small values of k. If the communication graph induced by antennas operating in omnidirectional mode is connected, by how much must an antenna radius increase to guarantee that k laser-beam antennas at each node, pointing in the direction of the Y_k edges, preserve connectivity? In this paper we focus our attention on small k values (2, 3 and 4) corresponding to the number of antennas commonly used in practice.

1.1 Prior Results

Yao graphs have been extensively studied in the area of computational geometry, and have been used in constructing efficient wireless communication networks [5, 8, 7, 4]. Applying the Yao structure on top of a dense communication graph, in order to obtain a sparser graph, is a very natural idea. Most existing results concern Yao graphs $Y_k[G^\infty]$ with $k \geq 6$. These graphs exhibit nice spanning properties, in the sense that the length of a shortest path between any two nodes $p, q \in Y_k[G^\infty]$ is only a constant times the Euclidean distance $|pq|$ separating p and q [2, 1, 3]. In the context of using laser-beam antennas to realize Y_k however, these results could only be applied if 6 or more antennas were available at each node, which is a rather impractical requirement. Few results exist on Yao graphs Y_k, for small values of k (below 6). It has been shown that $Y_2[G^\infty]$ and $Y_3[G^\infty]$ are not spanners [6], and that $Y_4[G^\infty]$ is a spanner [1]. However, as far as we know, no results exist on $Y_k[G^d]$, for any fixed radius $d \geq 1$.

1.2 Our Results

We develop lower and upper bounds on the connectivity radius of Y_2, Y_3 and Y_4, relative to the unit radius. (Recall that our assumption that the unit radius disk graph G^1 is connected.) We prove tight lower and upper bounds equal to $\sqrt{2} \approx 1.414$ on the connectivity radius d of $Y_4[G^d]$. Surprisingly, we prove a smaller upper bound equal to $\frac{2}{\sqrt{3}} \approx 1.155$ on the connectivity radius d of $Y_3[G^d]$. This is somewhat counterintuitive, as one would expect that fewer outgoing edges per node (3 in the case of Y_3, compared to 4 in the case of Y_4) would necessitate a higher connectivity radius, however our results show that this is not always the case. We also derive a lower bound of $5 - \frac{2}{3}\sqrt{35} \approx 1.056$ on the connectivity radius d of $Y_4[G^d]$, leaving a tiny interval $[1.056, 1.155]$ on which the connectivity of Y_4 remains uncertain. Finally, we show that $Y_2[G^d]$ can be disconnected, for any fixed value $d \geq 1$.

1.3 Definitions

Let S be a fixed set of points in the plane. At each node $p \in S$, let $r_1, r_2, \ldots, r_k$ denote the k rays originating at p, with r_1 horizontal along the $+x$ axis (see Figure 1, for $k = 3$). Let $C_i(p)$ to denote the half-open cone delimited by r_i and r_{i+1}, including r_i but excluding r_{i+1}. (Here we use r_{k+1} to mean r_1.) For any point $p \in S$, let $x(p)$ denote the $x-$coordinate of p and $y(p)$ denote the $y-$coordinate of p. For any $p, q \in S$, let $|pq|$ denote the Euclidean distance between p and q. For any point $p \in S$ and any real value $\delta > 0$, let $D(p, \delta)$ be the closed disk with center p and radius δ.

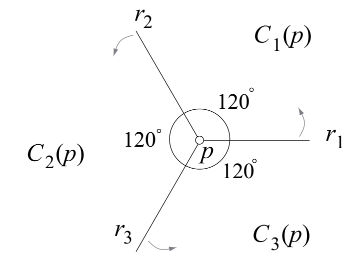

Figure 1: Rays and (half-open, half-closed) cones used in constructing Y_3.

2. CONNECTIVITY OF Y_4

In this section we derive tight lower and upper bounds on the connectivity radius d for $Y_4[G^d]$. Recall that our work relies on the assumption that G^1 is connected.

THEOREM 2.1. *There exist point sets S with the property that $G^1(S)$ is connected, but $Y_4[G^d]$ is disconnected, for any $1 \leq d < \sqrt{2}$.*

PROOF. We construct a point set S that meets the conditions of the theorem. Note that $d < \sqrt{2}$ implies that $1 - \sqrt{d^2 - 1} > 0$, meaning that there exists a real value ε such that $0 < \varepsilon < 1 - \sqrt{d^2 - 1}$, which is equivalent to

$$1 + (1 - \varepsilon)^2 > d^2$$

Let p and q be the endpoints of a vertical segment of length 1, with p below q. In Figure 2 the segment pq is shown slightly slanted to the left, merely to reinforce our convention that $pq \in C_2(p)$ and $qp \in C_4(q)$. Shoot a horizontal ray from p leftward, then slightly rotate it clockwise about p by a tiny angle α, so that the ray lies entirely in $C_2(p)$. Distribute points $a_1, a_2, \ldots, a_r$ in this order along this ray such that $|pa_1| = 1 - \varepsilon$, and $|a_i a_{i+1}| = 1$, for each i. Let b_i be the point symmetric to a_i with respect to the midpoint of pq. Let

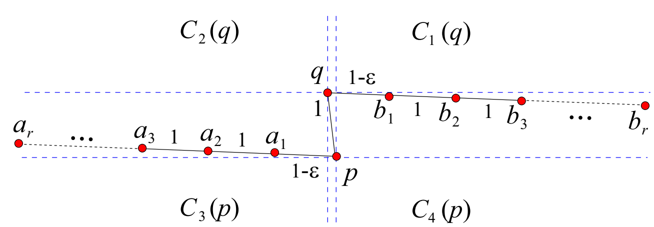

Figure 2: Point set S and $G^1(S) \equiv G^d(S)$, with $d < \sqrt{2}$; $Y_4[G^d]$ is disconnected.

$$S = \{p, q, a_i, b_i \mid 1 \leq i \leq r\}.$$

In the limit, as α approaches 0, the angle $\angle a_1pq$ approaches $\pi/2$ and $|a_1q| = \sqrt{1+(1-\varepsilon)^2} > d$. This means that a_1q is not an edge in G^d. Because $|a_ib_j| > |a_iq| \geq |a_1q| > d$ for each $i, j \geq 1$, we have that no $a-$point is directly connected to a $b-$point in G^d. It follows that the graph G^d is a path (depicted in Figure 2).

We now show that $pq \notin Y_4[G^d]$, which along with the fact that G^d is a path, yields that claim that $Y_4[G^d]$ is disconnected. First note that $\overrightarrow{pq}$ is not an edge in $Y_4[G^d]$. This is because a_1 is in the same cone $C_2(p)$ as q, and $|pa_1| = 1-\varepsilon < 1 = |pq|$. Similarly, $\overrightarrow{qp}$ is not an edge in $Y_4[G^d]$, because b_1 is in the same cone $C_4(q)$ as p, and $|qb_1| = 1-\varepsilon < 1 = |qp|$. We conclude that $Y_4[G^d]$ is disconnected. $\square$

Upper Bound $d = \sqrt{2}$.

We now show that $Y_4[G^d]$ is always connected for $d = \sqrt{2}$, matching the lower bound from Theorem 2.1. First we introduce a few definitions. For any pair of

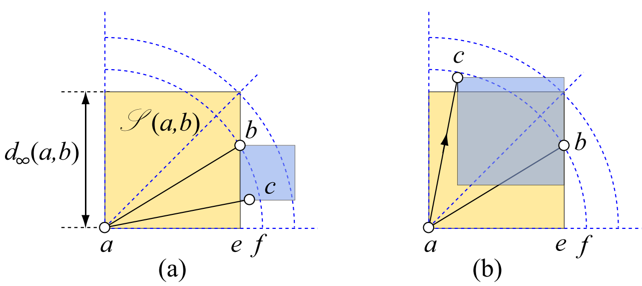

Figure 3: Theorem 2.2: $d_\infty(b,c) < d_\infty(a,b)$ **(a)** b, c **lie on the same side of the bisector (b)** b, c **lie on opposite sides of the bisector.**

points a, b, let $d_\infty(a,b)$ denote the L_∞ distance between a and b, defined as

$$d_\infty(a,b) = \max\{|x(a)-x(b)|, |y(a)-y(b)|\}$$

Let $\mathscr{S}(a,b)$ be the square with corner a whose boundary contains b, of side length $d_\infty(a,b)$ (see Figure 3a). The following inequalities follow immediately from the fact that ab is a line segment inside $\mathscr{S}(a,b)$:

$$d_\infty(a,b) \leq |ab| \leq d_\infty(a,b)\sqrt{2} \tag{1}$$

THEOREM 2.2. *For any point set S such that $G^1(S)$ is connected, $Y_4[G^{\sqrt{2}}]$ is also connected.*

PROOF. The proof is by contradiction. Assume to the contrary that G^1 is connected, but $Y_4[G^{\sqrt{2}}]$ is disconnected. Then $Y_4[G^{\sqrt{2}}]$ has at least two connected components, say J_1 and J_2. Since $G^1 \subseteq G^{\sqrt{2}}$ is connected, there is an edge $pq \in G^1$, with $p \in J_1$ and $q \in J_2$. To derive a contradiction, consider two points $a, b \in S$, with $a \in J_1$ and $b \in J_2$, that minimize $d_\infty(a,b)$. Then

$$\begin{array}{rcll} d_\infty(a,b) & \leq & d_\infty(p,q) & \text{(by choice of } ab\text{)} \\ & \leq & |pq| & \text{(by (1))} \\ & \leq & 1 & \text{(because } pq \in G^1\text{)} \end{array}$$

This along with the second inequality from (1) implies $|ab| \leq \sqrt{2}$, therefore $ab \in G^{\sqrt{2}}$. To simplify our analysis, rotate S so that b lies in the lower half of $C_1(a)$.

If $ab \in Y_4[G^{\sqrt{2}}]$, then ab connects J_1 and J_2, contradicting our assumption that J_1 and J_2 are disjoint connected components. So $ab \notin Y_4[G^{\sqrt{2}}]$. However $ab \in G^{\sqrt{2}}$ and $b \in C_1(a)$, therefore there is $\overrightarrow{ac} \in Y_4[G^{\sqrt{2}}]$, with $c \in C_1(a)$ and $|ac| \leq |ab|$. If c lies inside $\mathscr{S}(a,b)$, then $d_\infty(b,c) < d_\infty(a,b)$, because each of the horizontal and vertical distance between b and c is strictly smaller than the side length of $\mathscr{S}(a,b)$. This along with the fact that bc connects J_1 and J_2, contradicts our choice of ab. So c must lie outside of $\mathscr{S}(a,b)$ (but not outside of $D(a,|ab|)$, because $|ac| \leq |ab|$).

Let e be the lower right corner of $\mathscr{S}(a,b)$, and let f be intersection point between the boundary of $D(a,|ab|)$ and the horizontal ray through a in the direction of e (see Figure 3a). We will be using the fact that

$$|be| > |ef| \tag{2}$$

(This follows from the fact that $\angle bfe = \angle fba > \angle fbe$, and the Law of Sines applied on $\triangle bef$.)

We now derive a contradiction to our choice of ab as follows. If both b and c lie in the lower half of $C_1(a)$, as depicted in Figure 3a, then $|y(b)-y(c)| < d_\infty(a,b)$. Also $|x(b)-x(c)| < |ef|$, which by inequality (2) is no longer than $d_\infty(a,b)$. It follows that $d_\infty(b,c) < d_\infty(a,b)$, which along with the fact that bc connects J_1 and J_2, contradicts our choice of ab. If b and c lie on opposite sides of the bisector of $C_1(a)$, as depicted in Figure 3b, then the vertical distance from c to the top side of $\mathscr{S}(a,b)$ is smaller than $|ef|$, which in turn is smaller than $|be|$ (by inequality (2)). It follows that $|y(c)-y(b)| < d_\infty(a,b)$. Also, because c lies strictly to the right of a, we have that $|x(c)-x(b)| < d_\infty(a,b)$. These together show that $d_\infty(b,c) < d_\infty(a,b)$. This along with the fact that bc connects J_1 and J_2, contradicts our choice of ab. $\square$

Theorems 2.1 and 2.2 together establish matching lower and upper bounds (equal to $\sqrt{2}$) for the connectivity radius of Y_4.

3. CONNECTIVITY OF Y_3

The Yao graph Y_3 has three outgoing edges per node, compared to four outgoing edges in the case of Y_4. So one would expect that the radius necessary to maintain Y_3 connected would exceed the radius necessary to maintain Y_4 connected. However, our results show that an antenna radius equal to $\frac{2}{\sqrt{3}} < \sqrt{2}$ suffices to maintain Y_3 connected. This is a surprising result, given that a radius of $\sqrt{2}$ is necessary and sufficient to maintain Y_4 connected, as established in the previous section.

THEOREM 3.1. *There exist point sets S with the property that $G^1(S)$ is connected, but $Y_3[G^d]$ is disconnected, for any $1 \le d < 5 - \frac{2}{3}\sqrt{35}$.*

PROOF. We construct a point set S that satisfies the conditions of the theorem. Start with an isosceles trapezoid pa_1b_1q of unit altitude and bases pq and a_1b_1, with $|pq| = 1$ and $|a_1b_1| = 1 + \varepsilon$, for some small real value $0 < \varepsilon < 1$, to be determined later. Place a point x on pa_1 at distance $|pa_1|/3$ from p, and a second point y on qb_1 at distance $|qb_1|/3$ from b_1. Then simply reflect px about the vertical line through p, and qy about the vertical line through q. As we will later see, this places px and pq in the same cone of p (after a 90° counterclockwise rotation), so that px and pq compete in the edge selection process at p.

The result is the shaded polygon depicted in Figure 4a. Simple calculations show that the vertical distance between x and y is $1/3$, and the horizontal distance between x and y is

$$1 - \frac{1}{3} \cdot \frac{\varepsilon}{2} - \frac{2}{3} \cdot \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2}$$

It follows that $|xy|^2 = \left(\frac{1}{3}\right)^2 + \left(1 - \frac{\varepsilon}{2}\right)^2 = 1 + \frac{9\varepsilon^2 - 36\varepsilon + 4}{36}$. We will later require that $|xy| > d$ and $|a_1b_1| > d$, so that neither xy nor a_1b_1 is a candidate for $Y_3[G^d]$. These two inequalities reduce to

$$\begin{cases} 9\varepsilon^2 - 36\varepsilon + 40 - 36d > 0 \\ 1 + \varepsilon > d \end{cases}$$

Simple calculations yield the solution

$$1 \le d < 5 - \frac{2}{3}\sqrt{35} \tag{3}$$

$$d - 1 < \varepsilon < 2 - \frac{2}{3}\sqrt{9d - 1}. \tag{4}$$

By the triangle inequality, $|xa_1| < 2/3 + \varepsilon/2$. It can be easily verified that the above constraints on ε and d yield $|xa_1| < 1$. Similarly, each of px, qy and yb_1 has length less than 1. Also note that

$$|xq| > |xy| > d,$$

since the horizontal distance between x and q is greater than the horizontal distance between x and y, and the vertical distance is $1/3$ in both cases. Similarly, $|a_1y| > |xy| > d$.

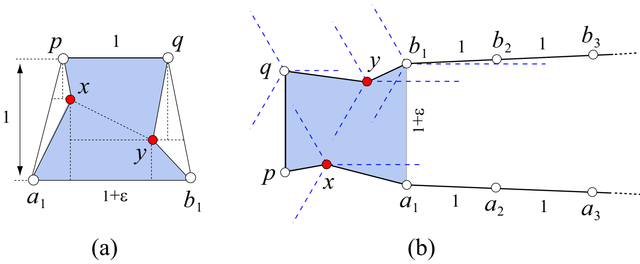

Figure 4: (a) Construction of pxa_1b_1yq (b) Point set S and $G^d(S)$, with $1 \le d < 10/9$.

We are now ready to construct S. Start by rotating the polygon pxa_1b_1yq counterclockwise by 90°, so that it lies on its side, as in Figure 4b. Shoot a horizontal ray rightward from a_1, then rotate it slightly clockwise so that it lies entirely in $C_3(a_1)$. Distribute points $a_2, a_3, \ldots, a_r$ at unit intervals along this ray. Let b_i be the reflection of a_i with respect to the horizontal through the midpoint of pq, for each $i > 1$. Our point set is

$$S = \{p, q, x, y, a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r\}.$$

The graph G^1 is a path (depicted in Figure 4b) and is therefore connected. We now show that $Y_3[G^d]$ is disconnected.

By construction, the following inequalities hold: $|a_1b_1| > d$; $|xq| > |xy| > d$; $|a_1q| > |a_1y| > |xy| > d$; and $|a_ib_{i+j}| > |a_ib_i| \ge |a_1b_1| > d$, for any $i \ge 1$ and any $j \ge 0$ (because $\angle a_ib_ib_{i+j}$ is obtuse). By symmetry, similar arguments hold for the $b-$points as well. It follows that the graph G^d is a path identical to G^1, therefore the removal of any edge from G^d disconnects it.

Next we show that $pq \notin Y_3[G^d]$, which along with the observation above implies that $Y_3[G^d]$ is disconnected. By construction, $|px| < |pq|$. This along with the fact that both x and q lie in the same cone $C_1(p)$, implies that p does not select pq for inclusion in $Y_3[G^d]$. Similarly, $|qy| < |qp|$. This along with the fact that both y and p lie in the same cone $C_3(q)$, implies that q does not select qp for inclusion in $Y_3[G^d]$. These together show that $pq \notin Y_3[G^d]$, therefore $Y_3[G^1]$ is disconnected. $\square$

***Upper Bound* $d \le 2/\sqrt{3}$.**

Next we derive an upper bound on the connectivity radius for Y_3. The approach adopted here is somewhat similar to the one employed in the proof of Theorem 2.2, but it uses a generalized distance function d_R (in place of d_∞), to measure the distance between two connected components. We define d_R as follows. For any point $a \in S$ and any point $b \in C_i(a)$, let $R(a, b)$ denote the closed rhombus with corner a and edges parallel to r_i and r_{i+1}, whose *boundary* $\partial R(a, b)$ contains b (see Figure 5). (Recall that $C_i(a)$ is the half-open cone with apex a that includes r_i and excludes r_{i+1}.) Define $d_R(a, b)$ to be the side length of $R(a, b)$. Clearly, $d_R(a, a) = 0$. Because our approach does not use the triangle inequality on d_R, we skip the proof that d_R is a distance metric, and focus instead on the symmetry property of d_R (Property (i) of Lemma 3.2 below), and the relationship between d_R and the Euclidean distance.

LEMMA 3.2. *For any pair of points $a, b \in S$ the following properties hold:*

(i) $d_R(a, b) = d_R(b, a)$.

(ii) $|ab| \le d_R(a, b)$.

(iii) $|ab| \ge d_R(a, b)\frac{\sqrt{3}}{2}$.

PROOF. To simplify our analysis, rotate S so that $b \in C_1(a)$, as depicted in Figure 6. Consider the quadrilateral $bcef$ from Figure 6a, with sides $ce \in \partial R(a, b)$ and $bf \in \partial R(b, a)$; bc and ef are parallel, since they are both parallel to r_2; and $\angle cef$ and $\angle bfe$ are each $60°$. These together show that $bcef$ is an isosceles trapezoid, meaning that $|ce| = |bf|$. Since $|ce| = d_R(a, b)$ and $|bf| = d_R(b, a)$, Property (i) holds.

Now note that $R(a, b)$ is the union of two equilateral triangles of side length $d_R(a, b)$, adjacent alongside the bisector of $C_1(a)$. Also note that ab is a segment that connects a to the opposite side in one of these triangles – call it T. It follows that ab is no longer than the side of T (thus yielding inequality (ii)), and no shorter than the height of T (thus yielding inequality (iii)). This completes the proof. $\square$

Following is an intermediate result that will help prove our main upper bound result stated in Theorem 3.4. This intermediate result will simply rule out some configurations that will occur in the analysis of the main result. To follow the logical sequence of our analysis, the reader can skip ahead to Theorem 3.4, and refer back to Lemma 3.3 only when called upon from Theorem 3.4.

LEMMA 3.3. *Let $a, b, c \in S$ be such that $b, c \in C_i(a)$, for some $i \in \{1, 2, 3\}$, and $|ac| \le |ab|$. Furthermore, assume that both b and c lie either in the half of $C_i(a)$ adjacent to r_i (excluding the bisector points), or in the half of $C_i(a)$ adjacent to r_{i+1} (including the bisector points). Then $d_R(b, c) < d_R(a, b)$.*

PROOF. To simplify our analysis, rotate S so that both b and c lie in the lower half of $C_1(a)$ (adjacent to r_1). Let $\delta = d_R(a, b)$. By Lemma 3.2(ii), $|ab| \le \delta$. This along with $|ac| \le |ab|$ implies that $c \in D(a, \delta)$. More precisely, c lies in a circular sector of angle $60°$, formed by the intersection between $D(a, \delta)$ and the lower half of $C_1(a)$.

If $c \in C_1(b)$, then $R(a, b)$ and $R(b, c)$ are similar (see Figure 7). This along with the fact that c lies in a same $60°-$sector as b implies that $d_R(b, c) < d_R(a, b)$. (The inequality is strict due to the fact that the lower half of the cone $C_1(a)$ does not include the upper bounding ray.) If $c \in C_3(b)$, then $b \in C_1(c)$ (see Figure 8a). This

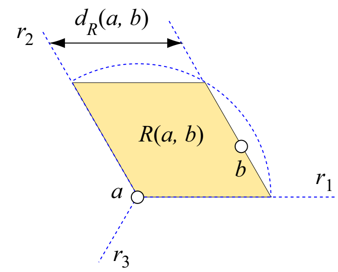

Figure 5: Rhombus $R(a, b)$ of side length $d_R(a, b)$.

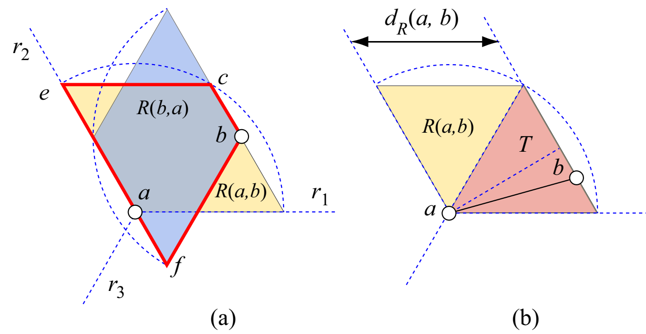

Figure 6: Lemma 3.2: (a) $d_R(a, b) = d_R(b, a)$ (b) Relationship between $d_R(a, b)$ and $|ab|$.

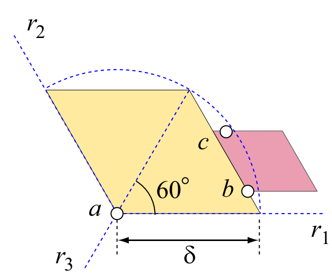

Figure 7: Lemma 3.3, case $c \in C_1(b)$: $d_R(b, c) \le d_R(a, b)$.

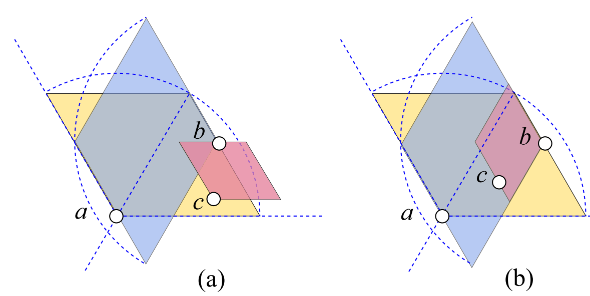

Figure 8: Lemma 3.3: $d_R(b, c) \le d_R(a, b)$ (a) $c \in C_3(b)$ (b) $c \in C_2(b)$.

case is similar to the previous one: $R(a, b)$ and $R(c, b)$ are similar, and $d_R(b, c) = d_R(c, b) < d_R(a, b)$. Finally, if $c \in C_2(b)$, then $c \in R(b, a)$ (see Figure 8b), and $R(b, c) \subset R(b, a)$. It follows that $d_R(b, c) < d_R(b, a) = d_R(a, b)$. $\square$

THEOREM 3.4. *For any point set S such that $G^1(S)$ is connected, $Y_3[G^d]$ is also connected, for $d = \frac{2}{\sqrt{3}}$.*

PROOF. The proof is by contradiction. Assume to the contrary that G^1 is connected, but $Y_3[G^d]$ is disconnected. Then $Y_3[G^d]$ has at least two connected components, say J_1 and J_2. Since $G^1 \subseteq G^d$ is connected, there is an edge $pq \in G^1$, with $p \in J_1$ and $q \in J_2$. To derive a contradiction, consider two points $a, b \in S$, with $a \in J_1$ and $b \in J_2$, that minimize $d_R(a, b)$. Then $d_R(a, b) \le d_R(p, q) \le d \cdot |pq|$. This latter inequality follows from inequality (iii) of Lemma 3.2, and the d value from the lemma statement. This along with inequality (ii) of Lemma 3.2 and the fact that $|pq| \le 1$, implies that $|ab| \le d_R(a, b) \le d$, therefore $ab \in G^d$.

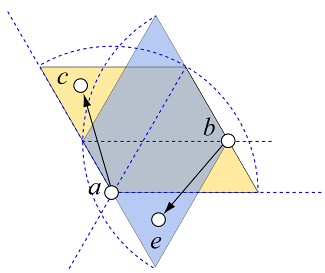

Figure 9: Theorem 3.4: case when a and d lie on a same side of the bisector of $C_2(b)$.

To simplify our analysis, rotate S so that $b \in C_1(a)$. Because J_1 and J_2 are not connected in $Y_3[G^d]$, and because $a \in J_1$ and $b \in J_2$, we have that $ab \notin Y_3[G^d]$. However $ab \in G^d$ and $b \in C_1(a)$, therefore there is $\overrightarrow{ac} \in Y_3[G^d]$, with $c \in C_1(a)$ and $|ac| \le |ab|$. If both b and c lie in the same half of $C_1(a)$ (bounded by one ray and the bisector of $C_1(a)$), then by Lemma 3.3 we have that $d_R(b, c) < d_R(a, b)$. This along with the fact that bc connects J_1 and J_2, contradicts our choice of ab. Then b and c must lie on either side of the bisector of $C_1(a)$, as depicted in Figure 9.

Assume without loss of generality that b lies in the lower half (excluding the bisector) of $C_1(a)$, and c lies in the upper half (including the bisector) of $C_1(a)$. Next we focus on $C_2(b)$. Because $a \in C_2(b)$, $ba \in G^d$, and $ba \notin Y_3[G^d]$, there must exist $\overrightarrow{be} \in Y_3[G^d]$, with $e \in C_2(b)$ and $|be| \le |ab|$. As before, if e and a lie in the same half of $C_2(b)$ (bounded by one ray and the bisector of $C_2(b)$), then by Lemma 3.3 we have that $d_R(e, a) < d_R(b, a) = d_R(a, b)$. This along with the fact that ae connects J_1 and J_2 contradicts our choice of ab. It follows that a and e lie on either side of the bisector of $C_2(b)$, as depicted in Figure 10.

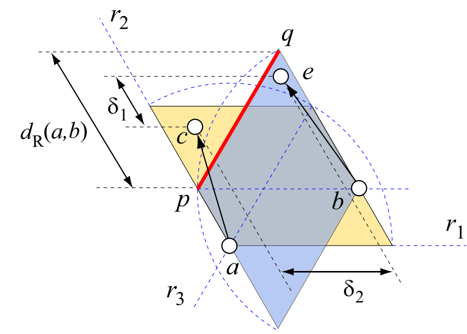

Figure 10: Theorem 3.4: case when a and d lie on opposite sides of the bisector of $C_2(b)$.

We now show that $d_R(c, e) < d_R(a, b)$. Let δ_1 be the length of the projection of ce on the ray r_2 in the (horizontal) direction of r_1. Similarly, let δ_2 be the length of the projection of ce on r_1 in the direction of r_2. (See Figure 10.) Then $d_R(c, e) = \max\{\delta_1, \delta_2\}$. We prove that $d_R(c, e) < d_R(a, b)$ by showing that each of δ_1 and δ_2 is smaller than $d_R(a, b)$.

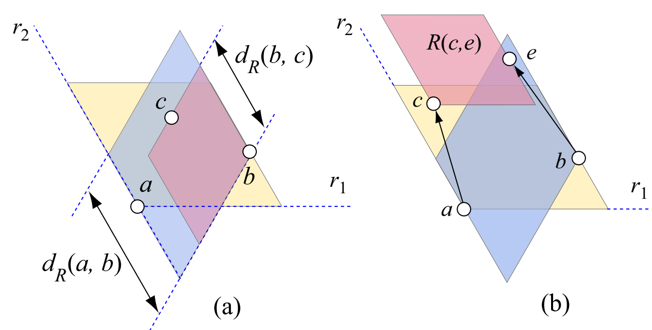

Figure 11: Theorem 3.4: (a) c inside $R(b, a)$ (b) $d_R(c, e) < d_R(a, b)$.

First note that c must lie outside of $R(b, a)$. Otherwise, if c were to lie inside $R(b, a)$, then $R(b, c) \subset R(b, a)$ (see Figure 11a). This would immediately imply that $d_R(b, c) < d_R(b, a) = d_R(a, b)$, which along with the fact that bc connects J_1 and J_2, would contradict our choice of ab. So c lies inside $D(a, |ab|)$ (because $|ac| \le |ab|$), but outside of $R(b, a)$. Similar arguments show that e lies inside $D(b, |ab|)$, but outside of $R(a, b)$. Let pq be the top left side of $R(b, a)$ (marked with a thick line in Figure 10). By the observations above, c and e cannot lie below p or above q. This implies that the horizontal projection of ce on the ray r_2 is strictly shorter than the horizontal projection of pq on r_2: $\delta_1 < d_R(a, b)$. (The claim on *strictly* shorter comes from the fact that $c \in C_1(a, b)$, and $C_1(a, b)$ does not include r_2.) Also, because c and e lie between the two lines through a and b parallel to r_2, the projection of ce on r_1 in the direction of r_2 is strictly shorter than the projection of ab on

r_1 in the direction of r_2: $\delta_2 < d_R(a, b)$.

We have established that $d_R(c, e) < d_R(a, b)$ (the rhombus $R(c, e)$ is depicted in Figure 11b). This along with the fact that ce connects J_1 and J_2, contradicts our choice of ab. We conclude that G^d is connected. $\square$

Observe that our results leave a tiny gap between the lower bound of $5 - \frac{2}{3}\sqrt{35} \approx 1.056$ from Theorem 3.1 and the upper bound of $\frac{2}{\sqrt{3}} \approx 1.155$ from Theorem 3.4 on the connectivity radius d for $Y_3[G^d]$. Nevertheless, both bounds beat the tight bound $d = \sqrt{2} \approx 1.414$ for the connectivity radius of $Y_4[G^d]$.

4. CONNECTIVITY OF Y_2

The point set S depicted in Figure 2 can be extended to show that $Y_2[G^d]$ can be disconnected, for arbitrarily large d. To see this, fix a real value $d \geq 1$, and distribute enough points a_i at unit interval along the leftward ray from p, such that the leftmost point a_r is far enough from q – in particular, we require that it satisfies the inequality $|a_r q| > d$. Similarly, we require that the rightmost point point b_r satisfies $|b_r p| > d$ (which follows immediately by symmetry). (Note that in this case $d = \Omega(|S|)$.) Recall that the leftward ray from p is *almost* horizontal, so q and all the $b-$points lie above a_r.

We now show that $Y_2[G^d]$ is disconnected. First note that a_1 is the point closest to p in $C_1(p)$, and that $C_2(p)$ is empty. Therefore, the only edge $Y_2[G^d]$ incident to p is pa_1. Also note that, for any $i < r$, a_{i+1} is the point closest to a_i in $C_1(a_i)$, and a_{i-1} is the point closest to a_i in $C_2(a_i)$ (here we use a_0 to refer to the point p). Finally, q is the point closest to a_r in $C_1(a_r)$. However, because $|a_r q| > d$, $a_r q$ is not in G^d and therefore $a_r q$ is not in $Y_2[G^d]$. The arguments are symmetric for q and the $b-$points in S. This shows that there is no edge in $Y_2[G^d]$ connecting a point in $\{q, b_i \mid 1 \leq i \leq r\}$ to a point in $\{p, a_i \mid 1 \leq i \leq r\}$. We conclude that $Y_2[G^d]$ is disconnected for connectivity radius values $d = \Omega(|S|)$.

5. CONCLUSION

In this paper we establish matching lower and upper bounds on the connectivity radius for Y_4, and a tight interval on the connectivity radius for Y_3. Reducing the gap between the lower and upper ends of this interval remains open. These results show that a small increase in the radius of a directional antenna, (compared to the unit radius of an omnidirectional antenna,) renders an efficient communication graph for mobile wireless networks, provided that each node orients its $k \in \{3, 4\}$ antennas in the direction of the Y_k edges. (Nodes are assumed to send messages in directional mode, and receive messages in omnidirectional mode). One key advantage of these graphs is that they can be quickly constructed locally, providing strong support for node mobility. We also establish that the connectivity radius for Y_2 may be arbitrarily large, which indicates that Y_2 is not a suitable communication graph for wireless networks that use narrow (laser-beam) directional antennas.

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