# A New Upper Bound on 2D Online Bin Packing 

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#### Abstract

The 2D Online Bin Packing is a fundamental problem in Computer Science and the determination of its asymptotic competitive ratio has attracted great research attention. In a long series of papers, the lower bound of this ratio has been improved from $1.808,1.856$ to 1.907 and its upper bound reduced from $3.25,3.0625,2.8596,2.7834$ to 2.66013 . In this paper, we rewrite the upper bound record to 2.5545 . Our idea for the improvement is as follows.

In SODA 2002 [24], Seiden and van Stee proposed an elegant algorithm called $H \otimes B$, comprised of the Harmonic algorithm $H$ and the Improved Harmonic algorithm $B$, for the two-dimensional online bin packing problem and proved that the algorithm has an asymptotic competitive ratio of at most 2.66013. Since the best known online algorithm for one-dimensional bin packing is the Super Harmonic algorithm [25], a natural question to ask is: could a better upper bound be achieved by using the Super Harmonic algorithm instead of the Improved Harmonic algorithm? However, as mentioned in [24], the previous analysis framework does not work. In this paper, we give a positive answer for the above question. A new upper bound of 2.5545 is obtained for 2 -dimensional online bin packing. The main idea is to develop new weighting functions for the Super Harmonic algorithm and propose new techniques to bound the total weight in a rectangular bin.


## 1 Introduction

In two-dimensional bin packing, each item $\left(w_{i}, h_{i}\right)$ is a rectangle of width $w_{i} \leq 1$ and height $h_{i} \leq 1$. Given a list of such rectangular items, one is asked to pack all of them into a minimum number of square bins of side length one so that their sides are parallel to the sides of the bin. Rotation is not allowed. The problem is clearly strongly NP-hard since it is a generalization of the one-dimensional bin packing problem [7]. In this paper we will consider the online version of two-dimensional bin packing, in which the items are released one by one and we must irrevocably pack the current item into a bin without any information on the next items. Before presenting the previous results and our work, we first review the standard measure for online bin packing algorithms.

Asymptotic competitive ratio To evaluate an online algorithm for bin packing problems, we use the asymptotic competitive ratio defined as follows. Consider an online algorithm $A$. For any list $L$ of items, let $A(L)$ be the cost (number of bins used) incurred by algorithm $A$ and let $O P T(L)$ be the corresponding optimal value. Then the asymptotic competitive ratio for algorithm $A$ is

$$
R_{A}^{\infty}=\lim _{k \rightarrow \infty} \sup \max _{L}\{A(L) / O P T(L) \mid O P T(L)=k\} .
$$

Previous work Bin packing has been well-studied. For the one-dimensional case, Johnson et al. [19] showed that the First Fit algorithm (FF) has an asymptotic competitive ratio of 1.7. Yao [28] improved algorithm FF with a better upper bound of $5 / 3$. Lee and Lee [21] introduced the class of Harmonic algorithms, for which an asymptotic competitive ratio of 1.63597 was achieved. Ramanan et al. [23] further improved the upper bound to 1.61217. The best known upper bound so far is from the Super Harmonic algorithm by Seiden [25] whose asymptotic competitive ratio is at most 1.58889 . As for the negative results, Yao [28] showed that no online algorithm has asymptotic competitive ratio less than 1.5. Brown [1] and Liang [20] independently provided a better lower bound of 1.53635 . The best known lower bound to date is 1.54014 [26].

As for two-dimensional online bin packing, a lower bound of 1.6 was given by Galambos [14]. The result was gradually improved to 1.808 [15], 1.857 [27] and 1.907 [4]. Coppersmith and Raghan [9] gave the first online algorithm with asymptotic competitive ratio 3.25 . Csirik et al. [8] improved the upper bound to 3.0625 . Csirik and van Vliet [10] presented an algorithm for all $d$ dimensions, where in particular for two dimensions, they obtained a ratio of at most 2.8596 . Based on the techniques on the Improved Harmonic, Han et.al [17] improved the upper bound to 2.7834 . The best known online algorithm to date is the one called $A \otimes B$ presented by Seiden and van Stee [24], where $A$ and $B$ stand for two one-dimensional online bin packing algorithms. Basically $A$ and $B$ are applied to one dimension of the items with rounding sizes. In this seminal paper Seiden and van Stee proved that the asymptotic competitive ratio of $H \otimes B$ is at most 2.66013, where $H$ is the Harmonic algorithm [21] and $B$ is an instance of the improved Harmonic algorithm. It has been open since then to improve the upper bound. A natural idea is to use an instance of the Super Harmonic algorithm [25] instead of the improved Harmonic algorithm. However, as mentioned in paper [24], in that case, the previous analysis framework cannot be extended to Super Harmonic.

We also briefly overview the offline results on two-dimensional bin packing. Chung et al [6] showed an approximation algorithm with an asymptotic performance ratio of 2.125. Caprara [5] improved the upper bound to 1.69103 . Very recently Bansal et al. [2] derived a randomized algorithm with asymptotic performance ratio of at most 1.525 . As for the negative results, Bansal et al. [3] showed that the twodimensional bin packing problem does not admit an asymptotic polynomial time approximation scheme.

For the special case where items are squares, there is also a large number of results [9, 24, 22, 11, [12, 13, 18. Especially for bounded space online algorithms, Epstein et al. [12] gave an optimal online algorithm.
Our contributions There are two main contributions in this paper,

- we revisit 1D online bin packing algorithm: Super Harmonic, give new weighting functions for it, which are much simpler than the ones introduced in [25], and the new weighting functions have interests in its own.
- we generalize the previous analysis framework for 2D online bin packing algorithms used in [24], and show that the new analysis framework are very useful in analyzing 2D or multi-dimensional online bin packing problems.

By combining the new weighting functions with the new analysis framework, we design a new 2D online bin packing algorithm with a competitive ratio 2.5545 , which improves the previous bound of 2.66013 in SODA 2002 [24]. As mentioned in [24], the old analysis framework does not work well with the old weighting functions in [25], i.e., the old approach does not guarantee an upper bound better than 2.66013. This is testified in the following way: consider our algorithm, if we use old weighting functions with the old framework to analyze it, the competitive ratio is at least 3.04, and if we use the old weighting functions with the new framework, the competitive ratio is at least 2.79.

Organization of Paper Section 2 will review the Harmonic and Super Harmonic algorithms as preliminaries. Section 3 defines the weighting functions for Super Harmonic. Section 4 describes and analyzes the two-dimensional online bin packing algorithm $H \otimes S H+$. Section 5 concludes.

## 2 Preliminaries

We first review two online algorithms for one-dimensional bin packing, Harmonic and Super Harmonic, which are employed in designing online algorithms for two-dimensional bin packing.

### 2.1 The Harmonic algorithm

The Harmonic algorithm is a fundamental bin packing algorithm with a simple and nice structure, that was introduced by Lee and Lee [21] in 1985. The algorithm works as follows. Given a positive integer $k$, each item is immediately classified into one of $k$ types according to its size upon its arrival. In particular, if an item has a size in interval $\left(\frac{1}{i+1}, \frac{1}{i}\right]$ for some integer $i$, where $1 \leq i<k$, then it is a type- $i$ item; otherwise, it is of type- $k$. The type- $i$ item is then packed, using the simple Next Fit (NF) algorithm, into the open (not fully-packed) bin designated to contain type-i items exclusively; new bins are opened when necessary. At any time, there is at most one open bin for each type and any closed (fully-packed) bin for type- $i$ is packed exactly with $i$ items of type- $i$ for $1 \leq i<k$.

For an item of size $x$, we define a weighting function $W_{H}(x)$ for the Harmonic algorithm as follows:

$$
W_{H}(x)= \begin{cases}\frac{1}{i}, & \text { if } \frac{1}{i+1}<x \leq \frac{1}{i} \text { with } 1 \leq i<k, \\ \frac{k}{k-1} x & \text { if } 0<x \leq \frac{1}{k} .\end{cases}
$$

The following lemma is directly from [21].
Lemma 1 For any list L, we have

$$
H(L) \leq \sum_{p \in L} W_{H}(p)+O(1)
$$

where $H(L)$ is the number of bins used by the Harmonic algorithm for list $L$.

### 2.2 The Super Harmonic algorithm

The Super Harmonic algorithm [25] is a generalization of the Improved Harmonic algorithm and the Harmonic algorithm. Super Harmonic first classifies each item into one of $k+1$ types, where $k$ is a positive integer, and then assigns to the item a color of either blue or red. It allows items of up to two different types to share the same bin. In any one bin, all items of the same type have same color and items of different type have different colors. For items of type- $i(i \leq k)$, the algorithm maintains two parameters $\beta_{i}$ and $\gamma_{i}$ to bound respectively the number of blue items and the number of red items in a bin. More details are given below.

Classification into types Let $t_{1}=1>t_{2}>\ldots>t_{k}>t_{k+1}=\epsilon>t_{k+2}=0$ be real numbers. An interval $I_{i}$ is defined to be $\left(t_{i+1}, t_{i}\right]$, for $i=1, \ldots, k+1$. An item with size $x$ is of type- $i$ if $x \in I_{i}$.

Coloring red or blue Each type- $i$ item is also assigned a color, either red or blue, for $i \leq k$. The algorithm uses two sets of counters, $e_{1}, \ldots, e_{k}$ and $s_{1}, \ldots, s_{k}$, all of which are initially zero. The total number of type- $i$ items is denoted by $s_{i}$, while the number of type- $i$ red items is denoted by $e_{i}$. For $1 \leq i \leq k$, during the packing process, the fraction of type- $i$ items that are red is maintained, i.e., $e_{i}=\left\lfloor\alpha_{i} s_{i}\right\rfloor$, where $\alpha_{1}, \ldots, \alpha_{k} \in[0,1]$ are constants.

Maximal number of blue items Let $\beta_{i}=\left\lfloor\frac{1}{t_{i}}\right\rfloor$ for $1 \leq i \leq k$, which is the maximal number of blue items of type- $i$ which can be accepted in a single bin.

Space left for red items Let $\delta_{i}=1-t_{i} \beta_{i}$, which is the lower bound of the space left when a bin consists of $\beta_{i}$ blue items of type-i. If possible, we want to use the space left for small red items. Note that in the algorithm, in order to simplify the analysis, instead of using $\delta_{i}$, less space is used, namely $D=\left\{\Delta_{0}, \Delta_{1}, \ldots, \Delta_{K}\right\}$, as the spaces into which red items can be packed, where $0=\Delta_{0}<\Delta_{1}<\cdots<$ $\Delta_{K}<1 / 2$ and $K \leq k$. Let $\Delta_{\phi(i)}$ be the space to be used to accommodate red items in a bin which holds $\beta_{i}$ blue items of type- $i$, where function $\phi$ is defined as $\{1, \ldots, \mathrm{k}\} \mapsto\{0, \ldots, \mathrm{~K}\}$ such that $\phi$ satisfies $\Delta_{\phi(i)} \leq \delta_{i}$. If $\phi(i)=0$ then no red items are accepted.

For convenient use in our analysis in the next section, we introduce a function called $\varphi(i)$, which gives the index of the smallest space in $D$ into which a red item of type- $i$ can be placed:

$$
\varphi(i)=\min \left\{j \mid t_{i} \leq \Delta_{j}, 1 \leq j \leq K\right\}
$$

Maximal number of red items Now we define $\gamma_{i}$. Let $\gamma_{i}=0$ if $t_{i}>\Delta_{K}$; otherwise $\gamma_{i}=$ $\max \left\{1,\left\lfloor\Delta_{1} / t_{i}\right\rfloor\right\}$, i.e., if $\Delta_{1}<t_{i} \leq \Delta_{K}$, we set $\gamma_{i}=1$, otherwise $\gamma_{i}=\left\lfloor\Delta_{1} / t_{i}\right\rfloor$.

Naming bins It is also convenient to name the bins by groups:

$$
\begin{gathered}
\left\{(i) \mid \phi_{i}=0,1 \leq i \leq k\right\} \\
\left\{(i, ?) \mid \phi_{i} \neq 0,1 \leq i \leq k\right\} \\
\left\{(?, j) \mid \alpha_{j} \neq 0,1 \leq j \leq k\right\} \\
\left\{(i, j) \mid \phi_{i} \neq 0, \alpha_{j} \neq 0, \gamma_{j} t_{j} \leq \Delta_{\phi(i)}, 1 \leq i, j \leq k\right\} .
\end{gathered}
$$

Group $(i)$ consists of bins that hold only blue items of type-i. Group $(i, j)$ consists of bins that contain blue items of type- $i$ and red items of type- $j$. Blue group ( $i, ?$ ) and red group (?, $j$ ) are indeterminate bins currently containing only blue items of type- $i$ or red items of type- $j$ respectively. During packing, red items or blue items will be packed into indeterminate bins if necessary, i.e., indeterminate bins will be changed into $(i, j)$.

The Super Harmonic algorithm is outlined below:

## Super Harmonic

1. For each item $p: i \leftarrow$ type of $p$,
(a) if $i=k+1$ then use NF algorithm,
(b) else $s_{i} \leftarrow s_{i}+1$; if $e_{i}<\left\lfloor\alpha_{i} s_{i}\right\rfloor$ then $e_{i} \leftarrow e_{i}+1$; \{ color $p$ red $\}$
i. If there is a bin in group (?, $i$ ) with fewer than $\gamma_{i}$ type-i items, then place $p$ in it.

Else if, for any $j$, there is a bin in group $(j, i)$ with fewer than $\gamma_{i}$ type-i items then place $p$ in it.
ii. Else if there is some bin in group $(j, ?)$ such that $\Delta_{\phi(j)} \geq \gamma_{i} t_{i}$, then pack $p$ in it and change the bin into $(j, i)$.
iii. Otherwise, open a bin $(?, i)$, pack $p$ in it.
(c) else $\{$ color $p$ blue $\}$ :
i. if $\phi_{i}=0$ then if there is a bin in group $i$ with fewer than $\beta_{i}$ items then pack $p$ in it, else open a new group $i$ bin, then pack $p$ in it.
ii. Else:
A. if, for any $j$, there is a bin in group $(i, j)$ or $(i, ?)$ with fewer than $\beta_{i}$ type-i items, then pack $p$ in it.
B. Else if there is a bin in group $(?, j)$ such that $\Delta_{\phi(i)} \geq \gamma_{j} t_{j}$ then pack $p$ in it, and change the group of this bin into $(i, j)$.
C. Otherwise, open a new bin $(i, ?)$ and pack $p$ in it.

## 3 New Weighting Functions for Super Harmonic

In this section, we develop new weighting functions for Super Harmonic that are simpler than the weighting system in [25]. The weighting functions will be useful in analyzing the proposed online algorithm as we shall see in the next section.

### 3.1 Intuitions for defining weights

Weighting functions are widely used in analyzing online bin packing problems. Roughly speaking, for an item, the value by one of weight functions is the fraction of a bin occupied by the item in the online algorithm. There is a constraint in defining weights for items for an online algorithm. Let $K+1$ be the number of weighting functions. Let $W^{i}(p)$ be the weight of an item $p$, where $1 \leq i \leq K+1$. For any input $L$, the constraint is

$$
\begin{equation*}
A(L) \leq \max _{1 \leq i \leq K+1}\left\{\sum_{p \in L} W^{i}(p)\right\}+O(1) \tag{1}
\end{equation*}
$$

where $A(L)$ is the number of bins used by algorithm $A$.
Consider Super Harmonic algorithm. For $1 \leq i \leq k$, let $l_{i}$ be the number of type- $i$ pieces. For $1 \leq i, s \leq k$, let $B_{(i)}, B_{(i, s)}, B_{(i, ?)}, B_{(?, i)}$ be the number of bins in groups $(i),(i, s),(i, ?)$ and (?,i). Then we have

$$
\begin{equation*}
\sum_{i}\left\{B_{(i)}+\sum_{s} B_{(i, s)}+B_{(i, ?)}\right\}=\sum_{i} \frac{\left(1-\alpha_{i}\right) l_{i}}{\beta_{i}}+O(1) \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i}\left\{B_{(?, i)}+\sum_{s} B_{(s, i)}\right\}=\sum_{i} \frac{\alpha_{i} l_{i}}{\gamma_{i}}+O(1) \tag{3}
\end{equation*}
$$

So, for each item with size $x \in I_{i}$, where $i \leq k$, if we define its weight as below:

$$
\frac{1-\alpha_{i}}{\beta_{i}}+\frac{\alpha_{i}}{\gamma_{i}}
$$

then it is not difficult to see that the constraint (1) holds. However the above weighting function is not good enough, i.e., it always leads a competitive ratio at least 1.69103.

The main reason is that for each bin in group $(i, s)$ we account it twice, where $1 \leq i, s \leq k$. Next we give some intuitions for improving the above weighting function.

By (2) and (3), observe that

$$
\begin{equation*}
\sum_{i} \sum_{s} B_{(i, s)} \leq \sum_{i} \frac{\left(1-\alpha_{i}\right) l_{i}}{\beta_{i}}+O(1) \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i} \sum_{s} B_{(s, i)} \leq \sum_{i} \frac{\alpha_{i} l_{i}}{\gamma_{i}}+O(1) \tag{5}
\end{equation*}
$$

So, we have

$$
\begin{aligned}
\sum_{i} \sum_{s} B_{(i, s)}=\frac{\sum_{i} \sum_{s} B_{(i, s)}+\sum_{i} \sum_{s} B_{(s, i)}}{2} & \leq \sum_{i} \frac{\left(1-\alpha_{i}\right) l_{i}}{2 \beta_{i}}+\sum_{i} \frac{\alpha_{i} l_{i}}{2 \gamma_{i}}+O(1) \\
& =l_{i} \sum_{i}\left(\frac{1-\alpha_{i}}{2 \beta_{i}}+\frac{\alpha_{i}}{2 \gamma_{i}}\right)+O(1)
\end{aligned}
$$

Hence, for an item with size $x \in I_{i}$, after packing, if there is a bin in group $(i, s)$ and also a bin in group $(s, i)$, then we can define its weight as below:

$$
\frac{1-\alpha_{i}}{2 \beta_{i}}+\frac{\alpha_{i}}{2 \gamma_{i}}
$$

This is the main intuition to lead our weighting functions, which are given in the next subsection.

### 3.2 New weighting functions

Remember that in Super Harmonic, there is a set $D=\left\{\Delta_{0}, \Delta_{1}, \ldots, \Delta_{K}\right\}$ representing the "free spaces" reserved for red items. Recall the two functions $\phi(i)$ and $\varphi(i)$ are related to free spaces and have the meanings as below: $\phi(i)=j$ implies that free space $\Delta_{j}$ is reserved for red items in a bin consisting of $\beta_{i}$ blue items of type- $i$, and $\varphi(i)=j$ indicates that a red item of type- $i$ could be packed in free space $\Delta_{\geq j}$.

We are now ready to define new weighting functions. Items with size larger than $\epsilon$ will be first considered. Let $E$ be the number of indeterminate red group bins ( $?, i$ ) when the whole packing is done.

If $E=0$, i.e., every red item is placed in a bin with one or more blue items, then we define the weighting function as:

$$
\begin{equation*}
W^{1}(x)=\frac{1-\alpha_{i}}{\beta_{i}} \quad \text { if } x \in I_{i} \tag{6}
\end{equation*}
$$

Otherwise, $E>0$ implying that for some $i$, an indeterminate red group bin (?, $i$ ) exists after packing. Let $e$ be the smallest red item in indeterminate red group bins. Assume $r$ is the type of item $e$ and $j=\varphi(r)$. If $2 \leq j \leq K$ then we define the corresponding weighting functions as follows:

$$
W^{K+2-j}(x)= \begin{cases}\frac{1-\alpha_{i}}{\beta_{i}}+\frac{\alpha_{i}}{2 \gamma_{i}} & \text { if } x \in I_{i} \phi(i)<j, \text { and } \varphi(i)<j \\ \frac{1-\alpha_{i}}{\beta_{i}}+\frac{\alpha_{i}}{\gamma_{i}} & \text { if } x \in I_{i} \phi(i)<j, \text { and } \varphi(i) \geq j \\ \frac{1-\alpha_{i}}{2 \beta_{i}}+\frac{\alpha_{i}}{\gamma_{i}} & \text { if } x \in I_{i} \phi(i) \geq j, \text { and } \varphi(i) \geq j \\ \frac{1-\alpha_{i}}{2 \beta_{i}}+\frac{\alpha_{i}}{2 \gamma_{i}} & \text { if } x \in I_{i} \phi(i) \geq j, \text { and } \varphi(i)<j\end{cases}
$$

If $j=1$, we define

$$
W^{K+1}(x)= \begin{cases}\frac{1-\alpha_{i}}{\beta_{i}} & \text { if } x \in I_{i} \phi(i)=0, \text { and } \varphi(i)=0 \\ \frac{1-\alpha_{i}}{\beta_{i}}+\frac{\alpha_{i}}{\gamma_{i}} & \text { if } x \in I_{i} \phi(i)=0, \text { and } \varphi(i)>0 \\ 0 & \text { if } x \in I_{i} \phi(i)>0, \text { and } \varphi(i)=0 \\ \frac{\alpha_{i}}{\gamma_{i}} & \text { if } x \in I_{i} \phi(i)>0 \text { and } \varphi(i)>0\end{cases}
$$

Note that in the above definitions, if $\gamma_{i}=0$ then we replace $\frac{\alpha_{i}}{\gamma_{i}}$ with zero. For an item with size $x \in I_{k+1}$, we always define $W^{j}(x)=\frac{x}{1-\epsilon}$ for all $j$.

Theorem 1 For any list L, we have

$$
A(L) \leq \max _{1 \leq i \leq K+1}\left\{\sum_{p \in L} W^{i}(p)\right\}+O(1)
$$

where $A(L)$ is the number of bins used by Super Harmonic for list $L$.
Proof. Fix a list $L$. Let $D$ be the sum of the sizes of the items of type- $(k+1)$. By NEXT FIT, we know that the number of bins used for type- $(k+1)$ is at most $D /(1-\epsilon)+1$.

Again, we use $E$ to denote the number of indeterminate red group bins when all the packing is done. If $E>0$, let $e$ be the smallest red item in indeterminate red group bins. Assume $r$ is the type of item $e$ and $j=\varphi(r)$. For $1 \leq i \leq k$, let $l_{i}$ be the number of type- $i$ pieces. Let $B_{(i)}, B_{(i, s)}, B_{(i, ?)}, B_{(?, i)}$ be the number of bins in groups $(i),(i, s),(i, ?)$ and $(?, i)$.

To prove this theorem, we consider three cases.
Case 1: If $E=0$, i.e., $\sum_{i} B_{(?, i)}=0$, every red item is packed in a bin with one or more blue items. Therefore we just need to count bins containing blue items:

$$
\begin{aligned}
A(L) & \leq \frac{D}{1-\epsilon}+\sum_{i}\left\{B_{(i)}+\sum_{s} B_{(i, s)}+B_{(i, ?)}\right\}+O(1) \\
& \leq \sum_{x \in I_{k+1}} W^{1}(x)+\sum_{i} \frac{\left(1-\alpha_{i}\right) l_{i}}{\beta_{i}}+O(1) \text { by (2) } \\
& =\sum_{x \in I_{k+1}, x \in L} W^{1}(x)+\sum_{x \notin I_{k+1}, x \in L} W^{1}(x)+O(1) .
\end{aligned}
$$

Case 2: $E>0, e$ is the smallest red item in indeterminate red group bins and its type is $r$ and $\varphi(r)=j \geq 2$. Since every red item of type- $i$ is placed in a final group bin $(s, i)$, where $\varphi(i)<j$, we have

$$
\begin{equation*}
\sum_{\varphi(i)<j} B_{(?, i)}=0 . \tag{7}
\end{equation*}
$$

On the other hand, we have

$$
\begin{equation*}
\sum_{\phi(i) \geq j} B_{(i, ?)}=0 \tag{8}
\end{equation*}
$$

otherwise, $e$ would have been placed into a bin $(i, ?)$, where $\phi(i) \geq j$. According to the Super Harmonic algorithm, for any type bin $B_{(i)}$, we have

$$
\begin{equation*}
\phi(i)=0 . \tag{9}
\end{equation*}
$$

Define

$$
X=\sum_{\substack{\phi(i) \geq j \\ \varphi(s)<j}} B_{(i, s)},
$$

which is the total number of all the bins in groups $(i, s)$ such that $\phi(i) \geq j$ and $\varphi(s)<j$. Then we have

$$
\begin{align*}
A(L) & \leq \frac{D}{1-\epsilon}+\sum_{i}\left(B_{(i)}+B_{(i, ?)}+B_{(?, i)}\right)+\sum_{i} \sum_{s} B_{(i, s)}+O(1) \\
& =\frac{D}{1-\epsilon}+\sum_{i}\left(B_{(i)}+B_{(i, ?)}+B_{(?, i)}\right)+X+\sum_{\phi(i)<j} \sum_{s} B_{(i, s)}+\sum_{\varphi(i) \geq j} \sum_{s} B_{(s, i)}+O(1) \\
& =\frac{D}{1-\epsilon}+\sum_{\phi(i)<j}\left(B_{(i)}+B_{(i, ?)}+\sum_{s} B_{(i, s)}\right)+\sum_{\varphi(i) \geq j}\left(B_{(?, i)}+\sum_{s} B_{(s, i)}\right)+X+O(1) . \tag{10}
\end{align*}
$$

The last inequality follows directly from (7), (8) and (9).
Then by the definition of variable $X$, we have

$$
X \leq \sum_{j \leq \phi(i) \leq K} \sum_{s} B_{(i, s)} \text { and } X \leq \sum_{1 \leq \varphi(i) \leq j-1} \sum_{s} B_{(s, i)} .
$$

Therefore,

$$
\begin{equation*}
X \leq\left\{\sum_{j \leq \phi(i) \leq K} \sum_{s} B_{(i, s)}+\sum_{1 \leq \varphi(i) \leq j-1} \sum_{s} B_{(s, i)}\right\} / 2 . \tag{11}
\end{equation*}
$$

So, by (10) and (11), we have

$$
\begin{aligned}
A(L) \leq & \frac{D}{1-\epsilon}+\sum_{\phi(i)<j}\left(B_{(i)}+B_{(i, ?)}+\sum_{s} B_{(i, s)}\right)+\sum_{\varphi(i) \geq j}\left(B_{(?, i)}+\sum_{s} B_{(s, i)}\right) \\
& +\sum_{\phi(i) \geq j} \sum_{s} \frac{B_{(i, s)}}{2}+\sum_{\varphi(i)<j} \sum_{s} \frac{B_{(s, i)}}{2}+O(1) \\
\leq & \frac{D}{1-\epsilon}+\sum_{\phi(i)<j} \frac{\left(1-\alpha_{i}\right) l_{i}}{\beta_{i}}+\sum_{\varphi(i) \geq j} \frac{\alpha_{i} l_{i}}{\gamma_{i}}+\sum_{\phi(i) \geq j} \frac{\left(1-\alpha_{i}\right) l_{i}}{2 \beta_{i}}+\sum_{\varphi(i)<j} \frac{\alpha_{i} l_{i}}{2 \gamma_{i}}+O(1) \\
\leq & \frac{D}{1-\epsilon}+\sum_{\substack{\phi(i)<j}}\left(\frac{\left(1-\alpha_{i}\right) l_{i}}{\beta_{i}}+\frac{\alpha_{i} l_{i}}{2 \gamma_{i}}\right)+\sum_{\substack{\phi(i)<j \\
\varphi(i) \geq j}}\left(\frac{\left(1-\alpha_{i}\right) l_{i}}{\beta_{i}}+\frac{\alpha_{i} l_{i}}{\gamma_{i}}\right) \\
& +\sum_{\phi(i) \geq j}\left(\frac{\left(1-\alpha_{i}\right) l_{i}}{2 \beta_{i}}+\frac{\alpha_{i} l_{i}}{\gamma_{i}}\right)+\sum_{\phi(i) \geq j}\left(\frac{\left(1-\alpha_{i}\right) l_{i}}{2 \beta_{i}}+\frac{\alpha_{i} l_{i}}{2 \gamma_{i}}\right)+O(1) \\
= & \sum_{x \in I_{k+1}, x \in L} W^{K+2-j}(x)+\sum_{x \notin I_{k+1}, x \in L} W^{K+2-j}(x)+O(1)
\end{aligned}
$$

The second inequality follows directly from (22) and (3).
Case 3. $E>0$ and $j=1$. The arguments are analogous with Case 2. According to the Super Harmonic algorithm, for any type of bin $(i, s)$, we have $\varphi(s) \geq 1$, where $1 \leq i, s \leq k$ and $k$ is a parameter defined in Super Harmonic. So, there is no such bin $(i, s)$ with $\varphi(s)<1$. Then we have

$$
\begin{aligned}
A(L) & \leq \frac{D}{1-\epsilon}+\sum_{\phi(i)<1}\left(B_{(i)}+B_{(i, ?)}+\sum_{s} B_{(i, s)}\right)+\sum_{\varphi(i) \geq 1}\left(B_{(?, i)}+\sum_{s} B_{(s, i)}\right)+O(1) \\
& \leq \frac{D}{1-\epsilon}+\sum_{\phi(i)=0} \frac{\left(1-\alpha_{i}\right) l_{i}}{\beta_{i}}+\sum_{\varphi(i) \geq 1} \frac{\alpha_{i} l_{i}}{\gamma_{i}}+O(1) \\
& \leq \frac{D}{1-\epsilon}+\sum_{\substack{\phi(i)=0 \\
\varphi(i)=0}} \frac{\left(1-\alpha_{i}\right) l_{i}}{\beta_{i}}+\sum_{\substack{\phi(i)=0 \\
\varphi(i)>0}}\left(\frac{\left(1-\alpha_{i}\right) l_{i}}{\beta_{i}}+\frac{\alpha_{i} l_{i}}{\gamma_{i}}\right)+\sum_{\substack{\phi(i)>0 \\
\varphi(i)>0}} \frac{\alpha_{i} l_{i}}{\gamma_{i}}+O(1) \\
& =\sum_{x \in I_{k+1}, x \in L} W^{K+1}(x)+\sum_{x \notin I_{k+1}, x \in L} W^{K+1}(x)+O(1)
\end{aligned}
$$

Therefore, we have $A(L) \leq \max _{1 \leq i \leq K+1}\left\{\sum_{p \in L} W^{i}(p)\right\}+O(1)$.

## 4 Algorithm $H \otimes S H+$ and Its Analysis

In the section, we first review a class of online algorithms for two dimensional online bin packing, called $H \otimes B$ [24]. Next we introduce a new instance of algorithm $H \otimes S H+$, where $H$ is Harmonic and $S H+$ (Strange Harmonic+) is an instance of Super Harmonic. Then we propose some new techniques on how to bound the total weight in a single bin, which is crucial to obtaining a better asymptotic
competitive ratio for the $H \otimes B$ algorithm. Finally, we apply new weighting functions for $S H+$ to analyze the two-dimensional online bin packing algorithm $H \otimes S H+$ and show its competitive ratio at most 2.5545 , which implies that the new weighting functions work very well with the generalized approach of bounding the total weight in a single bin. Note that as mentioned in 24 if we apply the weighting functions of $S H+$ derived from [25] directly to analyze algorithm $H \otimes S H+$ then the upper bound cannot be improved.

### 4.1 Algorithms $H \times B$ and $H \otimes B$

Now we review two-dimensional online bin packing algorithms $H \times B$ and $H \otimes B$ [24], where $H$ is Harmonic and $B$ is Super Harmonic.

Given an item $p=(w, h), H \times B$ operates as follows:

1. Packing items into slices: If $w \geq \epsilon$ then pack $p$ into a slice of height 1 and width $t_{i}$ by $H$ (Harmonic algorithm), where $t_{i+1}<w \leq t_{i}$; else pack it into a slice of height 1 and width $\epsilon(1-\delta)^{i}$ by $H$ (Harmonic algorithm), where $\epsilon(1-\delta)^{i+1}<w \leq \epsilon(1-\delta)^{i}$ and $\delta>0$ is arbitrarily small.
2. Packing slices into bins: When a new slice is required in the above step, we allocate it from a bin using algorithm $B$.
$H \otimes B$ is a randomized algorithm, which operates as follows: before processing begins, we flip a fair coin. If the result is heads, then we run $H \times B$; otherwise we run $B \times H$, i.e., the roles of height and width are interchanged. Note that it is possible to de-randomize $H \otimes B$ without increasing its performance ratio. For details, we refer to [24].

Theorem 2 If an online $1 D$ bin packing algorithm $B$ has weighting functions $W_{B}^{i}(x)$ such that $B(L) \leq$ $\max _{i}\left\{\sum_{x \in L} W_{B}^{i}(x)\right\}+O(1)$. Then the cost by algorithm $H \otimes B$ for input $L$ is at most

$$
\frac{1}{2(1-\delta)}\left(\max _{i}\left\{\sum_{p \in L} W_{H \times B}^{i}(p)\right\}+\max _{i}\left\{\sum_{p \in L} W_{B \times H}^{i}(p)\right\}\right)+O(1),
$$

and the asymptotic competitive ratio of algorithm $H \otimes B$ is at most

$$
\frac{1}{2(1-\delta)} \max _{\forall X}\left(\max _{i}\left\{\sum_{(x, y) \in X} W_{H}(x) W_{B}^{i}(y)\right\}+\max _{i}\left\{\sum_{(x, y) \in X} W_{H}(y) W_{B}^{i}(x),\right\}\right)
$$

where $\delta$ is a parameter defined in $H \otimes B$ algorithm and $X$ is a set of items which fit in a single bin.

### 4.2 An instance of Super Harmonic $S H+$

As mentioned in [25], it is a hard problem to find appropriate parameters in designing an instance of Super Harmonic, especially setting $t_{i}$. The parameters in $S H+$ are found through a trial-and-error way
and are defined as follows:

| $i$ | $t_{i}$ | $\alpha_{i}$ | $\beta_{i}$ | $\delta_{i}$ | $\phi(i)$ | $\varphi(i)$ | $\gamma_{i}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |  |  |  |
| 2 | 0.706 | 0 | 1 | 0.294 | 1 | 0 | 0 |  |  |  |
| 3 | 0.657 | 0 | 1 | 0.343 | 2 | 0 | 0 |  |  |  |
| 4 | 0.647 | 0 | 1 | 0.353 | 3 | 0 | 0 |  |  |  |
| 5 | 0.625 | 0 | 1 | 0.375 | 4 | 0 |  |  |  |  |
| 6 | 0.6 | 0 | 1 | 0.4 | 5 | 0 | 0 |  |  |  |
| 7 | 0.58 | 0 | 1 | 0.42 | 6 | 0 | 0 |  |  |  |
| 8 | 0.5 | 0 | 2 | 0 | 0 | 0 | 0 |  |  |  |
| 9 | 0.4 | 0.162 | 2 | 0.16 | 0 | 6 | 1 |  |  |  |
| 10 | 0.4 | 0.192 | 2 | 0.2 | 0 | 5 | 1 | 1 | 0.294 | $15 . .50$ |
| 11 | 0.375 | 0.2346 | 2 | 0.25 | 0 | 4 | 1 | 2 | 0.343 | $13,15 . .50$ |
| 12 | 0.353 | 0.3004 | 2 | 0.294 | 1 | 3 | 1 | 3 | 0.353 | $12,13,15 . .50$ |
| 13 | 0.343 | 0.3077 | 2 | 0.314 | 1 | 2 | 1 | 4 | 0.375 | $11 . .13,15 . .50$ |
| 14 | $1 / 3$ | 0 | 3 | 0 | 0 | 0 | 0 | 5 | 0.4 | $10.13,15 . .50$ |
| 15 | 0.294 | 0.0816 | 3 | 0.118 | 0 | 1 | 1 | 6 | 0.42 | $9 . .13,15 . .50$ |
| 16 | $1 / 4$ | 0.186 | 4 | 0 | 0 | 1 | 1 |  |  |  |
| 17 | $1 / 5$ | 0.092 | 5 | 0 | 0 | 1 | 1 |  |  |  |
| 18 | $1 / 6$ | 0.1456 | 6 | 0 | 0 | 1 | 1 |  |  |  |
| 19 | 0.147 | 0.2162 | 6 | 0.118 | 0 | 1 | 2 |  |  |  |
| 20 | $1 / 7$ | 0.1525 | 7 | 0 | 0 | 1 | 2 |  |  |  |
| $21-49$ | $1 /(i-13)$ | $f f(i)$ | $i-13$ | 0 | 0 | 1 | $\left\lfloor\Delta_{1} / t_{i}\right\rfloor$ |  |  |  |
| 50 | $1 / 37$ | 0 | 37 | 0 | 0 | 0 | 0 |  |  |  |
| 51 | $1 / 38$ | 0 | $*$ | $*$ | $*$ | $*$ | $*$ |  |  |  |

where $f f(i)=1.35(50-i) / 37(i-12)$.
Then we have seven weighting functions for $S H+$, i.e., $W_{B}^{i}$ as defined in the last section, where $1 \leq i \leq 7$.

### 4.3 Previous framework for calculating upper bounds

In this subsection, we first introduce the previous framework for computing the upper bound of the competitive ratio of $H \otimes S H+$, then mention that the previous framework does not work well with the instance in the last subsection, i.e., the previous framework does not lead a better upper bound.

Let $p=(x, y)$ be an item. We define the following functions.

$$
W_{H \times B}^{i}(p)=W_{H}(x) W_{B}^{i}(y), W_{B \times H}^{i}(p)=W_{H}(y) W_{B}^{i}(x),
$$

and

$$
W^{i, j}(x, y)=\frac{W_{H}(x) W_{B}^{i}(y)+W_{B}^{j}(x) W_{H}(y)}{2} .
$$

Then we can obtain an upper bound on the competitive ratio $R$ of algorithm $H \otimes S H+$ as follows by Theorems 1 and 2, where $X$ is a set of items which fit in a single bin.

$$
\begin{align*}
R & \leq \frac{1}{2(1-\delta)} \max _{\forall X}\left(\max _{1 \leq i \leq 7}\left\{\sum_{p \in X} W_{H \times B}^{i}(p)\right\}+\max _{1 \leq i \leq 7}\left\{\sum_{p \in X} W_{B \times H}^{i}(p)\right\}\right) \\
& \leq \frac{1}{(1-\delta)} \max _{1 \leq i, j \leq 7, \forall X}\left\{\sum_{p \in X}\left(W_{H \times B}^{i}(p)+W_{B \times H}^{j}(p)\right) / 2\right\} \\
& =\frac{1}{(1-\delta)} \max _{1 \leq i, j \leq 7, \forall X}\left\{\sum_{p \in X} W^{i, j}(x, y)\right\} \tag{12}
\end{align*}
$$

The value of $R$ can be estimated by the following approach.
Definition 1 Let $f$ be a function mapping from $(0,1]$ to $\mathbb{R}^{+} . \mathcal{P}(f)$ is the mathematical program: maximize $\sum_{x \in X} f(x)$ subject to $\sum_{x \in X} \leq 1$, over all finite sets of real numbers $X$. We also use $\mathcal{P}(f)$ to denote the value of this mathematical program.

Lemma 2 [24] Let $f$ and $g$ be functions mapping from ( 0,1 ] to $\mathbb{R}^{+}$. Let $F=\mathcal{P}(f)$ and $G=\mathcal{P}(g)$. Then the maximum of $\sum_{p \in X} f(h(p)) g(w(p))$ over all finite multisets of items $X$ which fit in a single bin is at most $F G$, where $p$ is a rectangle and $h(p)$ and $w(p)$ are its height and width, respectively.

In [24], $f$ and $g$ are defined as below:

$$
f^{i, j}(y)=\frac{W_{H}(y)+W_{B}^{i}(y)}{2}
$$

and

$$
g^{i, j}(x)=\sup _{0<y \leq 1} \frac{W^{i, j}(x, y)}{f^{i, j}(y)}
$$

By the above definitions, we have

$$
W^{i, j}(x, y) \leq f^{i, j}(y) g^{i, j}(x),
$$

for all $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
Remarks: As mentioned in [24], the old weighting functions [25] do not work well with the calculating framework used in [25, 24], i.e., the previous framework with the old weighting function does not guarantee an upper bound better than 2.66013.

### 4.4 A new framework for calculating upper bound

In this subsection, we first generalize the previous analysis framework by introducing a new lemma and developing new functions for $f$ and $g$ in order to bound the total weight in a single bin. Then we apply our new weighting functions for Super Harmonic to algorithm $H \otimes S H+$ and obtain a new upper bound for two-dimensional online bin packing.

Lemma $3 \max _{\forall X}\left\{\sum_{p \in X} W^{i, j}(x, y)\right\}=\max _{\forall X}\left\{\sum_{p \in X} W^{j, i}(x, y)\right\}$, where $1 \leq i, j \leq 7$
Proof. By definition, observe that for any $1 \leq i, j \leq 7$,

$$
\begin{equation*}
W^{i, j}(x, y)=W^{j, i}(y, x) . \tag{13}
\end{equation*}
$$

Let $X=\left\{p_{1}, p_{2}, \ldots, p_{m}\right\}$ be a set of rectangles which fit into a single bin, where $p_{i}=\left(x_{i}, y_{i}\right)$ is the $i$-th rectangle in $X$. If we exchange roles of $x$ and $y$ of $p_{i}$ to get new rectangles $p_{i}^{\prime}=\left(y_{i}, x_{i}\right)$ for all $i$, then it
is not difficult to see that the new set $X^{\prime}=\left\{p_{1}^{\prime}, p_{2}^{\prime}, \ldots, p_{m}^{\prime}\right\}$ is also a feasible pattern, i.e., all items can fit in a single bin. On the other hand, by equation (13), we have

$$
\sum_{p \in X} W^{i, j}(p)=\sum_{p^{\prime} \in X^{\prime}} W^{j, i}\left(p^{\prime}\right),
$$

where $1 \leq i, j \leq 7$. There is a one-to-one mapping between $X$ and $X^{\prime}$ in all the feasible patterns. Therefore, we have this lemma.

New functions $f$ and $g$ : We define new functions $f$ and $g$ such that (i) Lemma 2 can be applied to bound the weight in a single bin, and (ii) the resultant bound is not too loose. The new functions $f$ and $g$ are defined as follows:

$$
f^{i, j}(y)=\lambda_{i, j} W_{H}(y)+\left(1-\lambda_{i, j}\right) W_{B}^{i}(y),
$$

where $0 \leq \lambda_{i, j} \leq 1$ and

$$
g^{i, j}(x)=\sup _{0<y \leq 1} \frac{W^{i, j}(x, y)}{f^{i, j}(y)} .
$$

Note that in [24], $\lambda_{i, j}$ are $1 / 2$ for all $i, j$. It is not difficult to see that the following inequality still holds although we have generalized the definition of the $f$ function,

$$
W^{i, j}(x, y) \leq f^{i, j}(y) g^{i, j}(x)
$$

for all $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
New approach of calculating $\mathcal{P}(f)$ : In order to use Lemma 2 to obtain the upper bound on the competitive ratio $R$ of algorithm $H \otimes B$, we need to calculate $\mathcal{P}\left(f^{i, j}\right)$ and $\mathcal{P}\left(g^{i, j}\right)$. Let $f$ be one of $f^{i, j}$ or $g^{i, j}$ for $1 \leq i, j \leq 7$. In [25], Seiden wrote a programming to enumerate all the feasible patterns to get the bounds for $\mathcal{P}(f)$. Here, we give a simple approach by calling LP solver directly to estimate $\mathcal{P}(f)$, which can be modeled as the following mixed integer program (MIP):

$$
\begin{align*}
\text { max. } \quad f= & \sum_{i=1}^{50} x_{i} w_{i}+\left(1-\sum_{i=1}^{50} x_{i}\left(t_{i+1}+\epsilon\right)\right) \times \frac{1}{1-t_{51}}  \tag{1}\\
\text { s.t. } \quad & \sum_{i=1}^{50} x_{i}\left(t_{i+1}+\epsilon\right) \leq 1, \\
& x_{i} \geq 0, \text { integer. }
\end{align*}
$$

where $x_{i}$ is the number of type- $i$ items in a feasible pattern, $w_{i}$ is the weight for an item of type- $i$, which is decided by function $f$, i.e., $w_{i}=f(p)$ if $p \in\left(t_{i+1}, t_{i}\right]$. Since $\epsilon>0$ can be arbitrarily small, we cannot find an exact value for $\epsilon$. Therefore, we set $\epsilon=0$ and re-model the above MIP as follows.

$$
\begin{align*}
& \max . \quad f=\sum_{i=1}^{50} x_{i} w_{i}+\left(1-\sum_{i=1}^{50} x_{i} t_{i+1}\right) \times \frac{1}{1-t_{51}}  \tag{2}\\
& \text { s.t. } \quad \sum_{i=1}^{50} x_{i} t_{i+1} \leq 1 \text {, } \\
& x_{i} \leq 1 \text {, for } 1 \leq i \leq 7, \quad x_{i} \leq 2 \text {, for } 8 \leq i \leq 13 \text {, } \\
& x_{i} \leq 3 \text {, for } 14 \leq i \leq 15, \quad x_{i} \leq i-12 \text {, for } 16 \leq i \leq 17 \text {, } \\
& x_{18}+x_{19} \leq 6, \quad x_{i} \leq i-13 \text {, for } 20 \leq i \leq 50, \\
& 2 x_{7}+x_{15} \leq 3.9, \quad 3 x_{7}+2 x_{13}+x_{17} \leq 5.9, \\
& 4 x_{13}+3 x_{15}+x_{24} \leq 11.9, \quad 5 x_{7}+3.53 x_{11}+1.47 x_{18} \leq 9, \\
& 12 x_{7}+8 x_{13}+3 x_{20}+x_{36} \leq 23, \quad 9 x_{7}+6 x_{13}+2 x_{21}+x_{30} \leq 17, \\
& x_{i} \geq 0 \text {, integer. }
\end{align*}
$$

Note that the new constraints do not eliminate any feasible solutions of MIP (1). For example, consider the constraint $5 x_{7}+3.53 x_{11}+1.47 x_{18} \leq 9$. Since an item of type- 7 has size larger than 0.5 , an item of type- 11 has size larger than 0.353 and an item of type- 18 has size larger than 0.147 , we have $0.5 x_{7}+0.353 x_{11}+0.147 x_{18}<1$. So, we have $5 x_{7}+3.53 x_{11}+1.47 x_{18}<10$. It is not difficult to see that the following inequality $5 x_{7}+3.53 x_{11}+1.47 x_{18} \leq 9$ is equivalent to $5 x_{7}+3.53 x_{11}+1.47 x_{18}<10$ when $x_{7}, x_{11}$ and $x_{18}$ are non-negative integers. For other constraints in MIP (2), the arguments are analogous.

To solve MIP (2), we use a tool for solving linear and integer programs called GLPK [16]. We write a program to calculate $W^{i, j}(x, y), g^{i, j}(x)$ and $f^{i, j}(y)$ for each $(i, j)$, and then call API of GLPK to calculate $\mathcal{P}\left(f^{i, j}\right)$ and $\mathcal{P}\left(g^{i, j}\right)$. The values of $\mathcal{P}\left(f^{i, j}\right)$ and $\mathcal{P}\left(g^{i, j}\right)$ are shown in the tables in Appendix.

Note that when we use Lemma 2 for the upper bound on the weight $\max _{\forall X}\left\{\sum_{p \in X} W^{i, j}(x, y)\right\}$, for all pairs $(i, j)$, the calculations are independent. For different pairs $(i, j), \lambda_{i, j}$ may be different. So, in order to get an upper bound near the true value of $\max _{\forall X}\left\{\sum_{p \in X} W^{i, j}(x, y)\right\}$, we have to select an appropriate $\lambda_{i, j}$. This can be done by a trial-and-error approach.

Theorem 3 For all $\delta>0$, the asymptotic competitive ratio of $H \otimes B$ is at most 2.5545 .
Proof. According to the tables in Appendix, by Lemma 3 and Lemma 2, we have

$$
\begin{aligned}
& \max _{\forall X}\left\{\sum_{p \in X} W^{1,2}(p)\right\}=\max _{\forall X}\left\{\sum_{p \in X} W^{2,1}(p)\right\} \leq \mathcal{P}\left(f^{1,2}\right) \mathcal{P}\left(g^{1,2}\right) \leq 2.5539 . \\
& \max _{\forall X}\left\{\sum_{p \in X} W^{1,6}(p)\right\}=\max _{\forall X}\left\{\sum_{p \in X} W^{6,1}(p)\right\} \leq \mathcal{P}\left(f^{6,1}\right) \mathcal{P}\left(g^{6,1}\right) \leq 2.5545 . \\
& \max _{\forall X}\left\{\sum_{p \in X} W^{2,5}(p)\right\}=\max _{\forall X}\left\{\sum_{p \in X} W^{5,2}(p)\right\} \leq \mathcal{P}\left(f^{5,2}\right) \mathcal{P}\left(g^{5,2}\right) \leq 2.5340 . \\
& \max _{\forall X}\left\{\sum_{p \in X} W^{2,6}(p)\right\}=\max _{\forall X}\left\{\sum_{p \in X} W^{6,2}(p)\right\} \leq \mathcal{P}\left(f^{6,2}\right) \mathcal{P}\left(g^{6,2}\right) \leq 2.5364 .
\end{aligned}
$$

For all the other $(i, j)$, by Lemma 2, we have

$$
\max _{\forall X}\left\{\sum_{p \in X} W^{i, j}(p)\right\} \leq \mathcal{P}\left(f^{i, j}\right) \mathcal{P}\left(g^{i, j}\right) \leq \mathcal{P}\left(f^{1,1}\right) \mathcal{P}\left(g^{1,1}\right) \leq 2.5545 .
$$

Remarks: If we use the weighting functions from [25] and the previous analysis framework, we find that the competitive ratio is at least 3.04. (run our programming 2DHSH.c like "./2DHSH+.exe old > yourfile") Even if we use the new weighting function, by the previous analysis framework, the competitive ratio is still at least 3.04, by running our programming 2DHSH.c like "./2DHSH+.exe new1 $>$ yourfile". We also find that if we use the old weighting function from [25] with the new analysis framework, the competitive ratio is at least 2.79. (run our programming 2DHSH.c like "./2DHSH + .exe old2 > yourfile") The reason is that: Lemma 2 does not work very well with the old weight function, i.e., the resulting value $F \cdot G$ is away from the maximum weight of items in a single bin.

## 5 Concluding Remarks

When we use the tool for solving the mixed integer programs, there are two files which are necessary: one is the model file for the linear or integer program itself (refer to Appendix), and the other is the data file where the data is stored. We write a program to generate the data and then call the tool GLPK. (Actually we call API (Application Program Interface) of GLPK. To download the source file, go to: http://sites.google.com/site/xinhan2009/Home/files/2DHSH.c).

Our framework can be applied to 3D online bin packing to result in an algorithm $H \times H \otimes S H+$ with its competitive ratio $2.5545 \times 1.69103(\approx 4.3198)$.
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## References

[1] D.J. Brown, A lower bound for on-line one-dimensional bin packing algorithms. Technical report R864, Coordinated Sci. Lab., Urbana, Illinois (1979)
[2] N. Bansal, A. Caprara and M. Sviridenko, Improved approximation algorithm for multidimensional bin packing problems, FOCS 2006: 697-708.
[3] N. Bansal, J.R. Correa, C. Kenyon and M. Sviridenko, Bin Packing in Multiple Dimensions: Inapproximability Results and Approximation Schemes, Mathematics of Operations Research, 31(1): 31-49, 2006.
[4] D. Blitz, A. van Vliet and G.J Woeginger, Lower bounds on the asymptotic worst-case ratio of online bin packing alorithms, Unpublished manuscript, 1996.
[5] A. Caprara, Packing 2-dimensional bins in harmony, FOCS 2002: 490-499.
[6] F.R.K. Chung, M.R. Garey, D.S. Johnson, On packing two-dimensional bins, SIAM J. Algebraic Discrete Methods, 3(1):66-76, 1982.
[7] E.G. Coffman, M.R. Garey and D.S. Johnson, Approximation algorithms for bin packing: a survey. In Approximation Algorithms for NP-hard Problems, D. Hochbaum, Ed. PWS, Boston, MA, 1997, chapter 2.
[8] J. Csirik, J. Frenk and M. Labbe, Two-dimensional rectangle packing: on-line methods and results, Discrete Applied Mathematics 45(3): 197-204, 1993.
[9] D. Coppersmith, P. Paghavan, Multidimensional on-line bin packing: Algorithms and worst case analysis, Oper. Res. Lett. 8:17-20, 1989.
[10] J. Csirik, A. van Vliet, An on-line algorithm for multidimensional bin packing, Operationa Research Letters 13: 149-158, 1993.
[11] L. Epstein, R. van Stee, Optimal online bounded space multidimensional packing, SODA 2004, 214-223.
[12] L. Epstein, R. van Stee, Optimal Online Algorithms for Multidimensional Packing Problems, SIAM Jouranl on Computing, 35(2), 431-448, 2005.
[13] L. Epstein, R. van Stee, Online square and cube packing, Acta Informatica 41(9), 595-606, 2005.
[14] G. Galambos, A 1.6 Lower-Bound for the Two-Dimensional On-Line Rectange Bin-Packing, Acta Cybernetica 10(1-2): 21-24, 1991.
[15] G. Galambos and A. van Vliet, Lower bounds for 1,2 and 3-dimensional online bin packing algorithms, Computing 52:281-297, 1994.
[16] http://www.gnu.org/software/glpk/.
[17] X. Han, S. Fujita and H. Guo, A Two-Dimensional Harmonic Algorithm with Performance Ratio 2.7834, IPSJ SIG Notes, No. 93 pp 43-50, 2001.
[18] X. Han, D. Ye, Y. Zhou, Improved Online Hypercube Packing, WAOA 2006: 226-239.
[19] D.S. Johnson, A.J. Demers, J.D. Ullman, M.R. Garey, R.L. Graham, Worst-Case performance bounds for simple one-dimensional packing algorithms. SIAM Journal on Computing 3(4), 299-325 (1974).
[20] F.M. Liang, A lower bound for online bin packing. Information processing letters 10,76-79 (1980).
[21] C.C. Lee and D.T. Lee, A simple on-line packing algorithm, J. ACM, 32:562-572, 1985.
[22] F.K. Miyazawa, Y. Wakabayashi, Cube packing, Theoretical Computer Sciences, 1-3(297), 355-366, 2003.
[23] P.V.Ramanan, D.J. Brown, C.C. Lee, D.T. Lee, On-line bin packing in linear time, Journal of Algorithms, 10, 305-326 (1989).
[24] S.S. Seiden and R. van Stee, New bounds for multidimensional packing, In SODA 2002, pp. 486-495. Full version in Algorithmica 36 (2003), 261-293.
[25] S.S. Seiden, On the online bin packing problem, J. ACM 49, 640-671, 2002.
[26] A. van Vliet, An improved lower bound for on-line bin packing algorithms: Information Processing Letters 43, 277-284 (1992).
[27] A. van Vliet, Lower and upper bounds for online bin packing and scheduling heuristics, Ph.D. thesis, Erasmus University, Rotterdam, 1995.
[28] A.C.-C. Yao, New Algorithms for Bin Packing. Journal of the ACM 27, 207-227, (1980).

## A Values of $f^{i, j}$ and $g^{i, j}$

| $(i, j)=$ | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | $(1,6)$ | $(1,7)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda_{i, j}$ | 0.500000 | 0.500000 | 0.540000 | 0.550000 | 0.565000 | 0.565000 | 0.600000 |
| $\mathcal{P}\left(f^{i, j}\right)$ | 1.598272 | 1.598272 | 1.605095 | 1.606845 | 1.609490 | 1.609490 | 1.615665 |
| $\mathcal{P}\left(g^{i, j}\right)$ | 1.598272 | 1.597872 | 1.574422 | 1.581742 | 1.585430 | 1.587508 | 1.575580 |
| $\mathcal{P}\left(f^{i, j}\right) \mathcal{P}\left(g^{i, j}\right)$ | 2.554474 | 2.553834 | 2.527096 | 2.541614 | 2.551734 | 2.555079 | 2.545610 |


| $(i, j)=$ | $(2,1)$ | $(2,2)$ | $(2,3)$ | $(2,4)$ | $(2,5)$ | $(2,6)$ | $(2,7)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda_{i, j}$ | 0.500000 | 0.500000 | 0.530000 | 0.550000 | 0.565000 | 0.565000 | 0.600000 |
| $\mathcal{P}\left(f^{i, j}\right)$ | 1.597328 | 1.597328 | 1.597148 | 1.597028 | 1.596938 | 1.596938 | 1.596729 |
| $\mathcal{P}\left(g^{i, j}\right)$ | 1.609235 | 1.598326 | 1.586301 | 1.595016 | 1.602278 | 1.604268 | 1.589545 |
| $\mathcal{P}\left(f^{i, j}\right) \mathcal{P}\left(g^{i, j}\right)$ | 2.570476 | 2.553051 | 2.533557 | 2.547285 | 2.558739 | 2.561917 | 2.538073 |


| $(i, j)=$ | $(3,1)$ | $(3,2)$ | $(3,3)$ | $(3,4)$ | $(3,5)$ | $(3,6)$ | $(3,7)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda_{i, j}$ | 0.500000 | 0.500000 | 0.530000 | 0.550000 | 0.565000 | 0.565000 | 0.600000 |
| $\mathcal{P}\left(f^{i, j}\right)$ | 1.573676 | 1.573676 | 1.572837 | 1.572777 | 1.572732 | 1.572732 | 1.572627 |
| $\mathcal{P}\left(g^{, i, j}\right)$ | 1.609235 | 1.598326 | 1.586301 | 1.595016 | 1.602278 | 1.604268 | 1.589545 |
| $\mathcal{P}\left(f^{i, j}\right) \mathcal{P}\left(g^{i, j}\right)$ | 2.532414 | 2.515247 | 2.494992 | 2.508604 | 2.519954 | 2.523084 | 2.499762 |


| $(i, j)=$ | $(4,1)$ | $(4,2)$ | $(4,3)$ | $(4,4)$ | $(4,5)$ | $(4,6)$ | $(4,7)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda_{i, j}$ | 0.500000 | 0.500000 | 0.535000 | 0.550000 | 0.565000 | 0.565000 | 0.600000 |
| $\mathcal{P}\left(f^{i, j}\right)$ | 1.581245 | 1.581245 | 1.577140 | 1.575380 | 1.573621 | 1.573621 | 1.569515 |
| $\mathcal{P}\left(g^{i, j}\right)$ | 1.609235 | 1.598326 | 1.586855 | 1.595016 | 1.602278 | 1.604268 | 1.589545 |
| $\mathcal{P}\left(f^{i, j}\right) \mathcal{P}\left(g^{i, j}\right)$ | 2.544594 | 2.527344 | 2.502692 | 2.512755 | 2.521378 | 2.524510 | 2.494814 |


| $(i, j)=$ | $(5,1)$ | $(5,2)$ | $(5,3)$ | $(5,4)$ | $(5,5)$ | $(5,6)$ | $(5,7)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda_{i, j}$ | 0.500000 | 0.500000 | 0.535000 | 0.550000 | 0.565000 | 0.565000 | 0.600000 |
| $\mathcal{P}\left(f^{i, j}\right)$ | 1.585370 | 1.585370 | 1.580113 | 1.577860 | 1.575607 | 1.575607 | 1.570350 |
| $\mathcal{P}\left(g^{i, j}\right)$ | 1.609542 | 1.598326 | 1.587240 | 1.595374 | 1.602747 | 1.604737 | 1.589740 |
| $\mathcal{P}\left(f^{i, j}\right) \mathcal{P}\left(g^{i, j}\right)$ | 2.551720 | 2.533939 | 2.508019 | 2.517277 | 2.525300 | 2.528436 | 2.496449 |


| $(i, j)=$ | $(6,1)$ | $(6,2)$ | $(6,3)$ | $(6,4)$ | $(6,5)$ | $(6,6)$ | $(6,7)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda_{i, j}$ | 0.500000 | 0.500000 | 0.530000 | 0.550000 | 0.565000 | 0.565000 | 0.600000 |
| $\mathcal{P}\left(f^{i, j}\right)$ | 1.586853 | 1.586853 | 1.582237 | 1.579160 | 1.576853 | 1.576853 | 1.571468 |
| $\mathcal{P}\left(g^{i, j}\right)$ | 1.609785 | 1.598326 | 1.586682 | 1.595657 | 1.603117 | 1.605107 | 1.589894 |
| $\mathcal{P}\left(f^{i, j}\right) \mathcal{P}\left(g^{i, j}\right)$ | 2.554493 | 2.536309 | 2.510507 | 2.519798 | 2.527881 | 2.531019 | 2.498468 |


| $(i, j)=$ | $(7,1)$ | $(7,2)$ | $(7,3)$ | $(7,4)$ | $(7,5)$ | $(7,6)$ | $(7,7)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda_{i, j}$ | 0.500000 | 0.515000 | 0.535000 | 0.555000 | 0.565000 | 0.570000 | 0.600000 |
| $\mathcal{P}\left(f^{i, j}\right)$ | 1.568686 | 1.560602 | 1.549821 | 1.539044 | 1.533655 | 1.530958 | 1.517143 |
| $\mathcal{P}\left(g^{i, j}\right)$ | 1.621572 | 1.609605 | 1.602462 | 1.612258 | 1.622822 | 1.638219 | 1.624359 |
| $\mathcal{P}\left(f^{i, j}\right) \mathcal{P}\left(g^{i, j}\right)$ | 2.543738 | 2.511952 | 2.483529 | 2.481335 | 2.488849 | 2.508043 | 2.464386 |

## B Model File for GLPK and Usage of Our Program 2DHSH+.c

param I:=50;

```
param c{i in 1..I}>=0;
param w{i in 1..I};
var x{i in 1..I}, integer, >=0;
maximize f: sum{i in 1..I} w[i]*x[i] + (1-sum{i in 1..I} c[i]*x[i]) * 38/37;
s.t. x0: sum{i in 1..I} c[i]*x[i] <= 1;
    x1: sum{i in 1..7} x[i] <= 1;
    x7: sum{i in 8..13} x[i] <= 2;
    x14: x[14] <= 3;
    x15: x[15] <= 3;
    x16: x[16] <= 4;
    x17: x[17] <= 5;
    x18: x[18] + x[19] <= 6;
    y715: 2*x[7] + x[15] <= 3.9;
    y71317: 3*x[7] + 2*x[13] + x[17] <= 5.9;
    y131524: 4*x[13] + 3*x[15] + x[24] <= 11.9;
    y71118: 5*x[7] + 3.53*x[11] + 1.47 *x[18] <= 9;
    y7132036: 12*x[7]+8*x[13] + 3*x[20] + x[36] <=23;
    y7132130: 9*x[7] + 6*x[13] + 2*x[21] + x[30] <=17;
    others{i in 20..50}: x[i] <= i -13;
end;
```

Whene the parameters in Super Harmonic such as $\alpha_{i}, \beta_{i}, \gamma_{i}$ and $\phi(i)$ and $\varphi(i)$ are given, we can calculate the weighting functions of Super Harmonic $W_{B}^{j}(\cdot)$. Then the weighting functions $W^{i, j}(x, y)$ for algorithm $H \otimes S H+$ can be calculated as well as $f^{i, j}(y)$ and $g^{i, j}(x)$. For each $(i, j)$, we call API of GLPK to solve $\mathcal{P}\left(f^{i, j}\right)$ and $\mathcal{P}\left(g^{i, j}\right)$.

To use our program under linux system:

- Install GLPK,
- Compile: "gcc -o 2DHSH+.exe 2DHSH+.c -lglpk"
- Run: "./2DHSH+.exe new2 $>$ yourfile"

If there is an error message like "Could not load *.so" when you compile the source, then try to set "LD_LIRARY_PATH" as follows: "LD_LIRARY_PATH=\$LD_LIRARY_PATH:/usr/local/lib", then "export LD_LIRARY_PATH".

