# DEFECT POLYTOPES AND COUNTER-EXAMPLES WITH POLYMAKE 

MICHAEL JOSWIG AND ANDREAS PAFFENHOLZ


#### Abstract

It is demonstrated how the software system polymake can be used for computations in toric geometry. More precisely, counter-examples to conjectures related to A-determinants and defect polytopes are constructed.


## 1. Defect Polytopes

The purpose of this note is to demonstrate how polymake's features can be used to establish explicit counter-examples to some conjectures on defect polytopes in toric geometry. More precisely, we consider several invariants of lattice polytopes proposed by Di Rocco [3], show a way to compute them with polymake, and use this to give answers to questions posed in Section 6 of that paper. For basic properties of polytopes and related toric varieties we refer to Ewald [6].

We start by introducing some notation. Let $P$ be a lattice polytope, i.e., a polytope whose vertices are contained in $\mathbb{Z}^{d}$. Let $\mathrm{F}(P)$ the set of all non-empty faces of $P$, including $P$ itself. Further, we let $\mathrm{F}_{P}(k)$ be the set of $k$-dimensional faces of $P$. For $t \in \mathbb{N}$ we define

$$
\begin{equation*}
c_{t}(P):=\sum_{k=0}^{d}(-1)^{d-k} \frac{(k+t)!}{k!} \sum_{F \in \mathrm{~F}_{P}(k)} \operatorname{lvol}(F), \tag{1}
\end{equation*}
$$

where $\operatorname{lvol}(F)$ is the normalized volume of $F$ in the lattice

$$
\mathbb{Z}^{F}:=\mathbb{Z}^{d} \cap \operatorname{aff}(F) ;
$$

see [3, p. 101]. For smooth polytopes $P$ the invariant $c_{1}(P)$ is a function that records the degree of homogeneity of a certain rational function, the $A$-determinant, where $A:=P \cap \mathbb{Z}^{d}$ is the set of lattice points in $P$. Starting with the seminal monograph [8] this invariant was extensively studied; see also [2, 4, 5, 10] for recent results, and generalizations to the singular case. By $[8, \S 11$, Thm. 1.6] the number $c_{1}(P)$ is non-negative for simple lattice polytopes whose associated toric variety is quasi-smooth. (and thus, in particular, for smooth polytopes). By [4, Pro. 3.2] this also holds for arbitrary lattice simplices, while it is open for more general simple polytopes. A polytope $P$ is a defect polytope, if $P$ is a smooth lattice polytope and $c_{1}(P)=0$. In terms of the associated toric variety $P$ is a defect polytope if the dual variety is not a hypersurface. By a result of Dickenstein and Nill [3, Thm. 1.6]

[^0]any defect polytope is a smooth strict Cayley polytope. A strict Cayley polytope is a polytope that is affinely isomorphic to
$$
Q_{0} \star \cdots \star Q_{k}=\operatorname{conv}\left(Q_{0} \times\left\{\boldsymbol{e}_{0}\right\}, \ldots, Q_{k} \times\left\{\boldsymbol{e}_{k}\right\}\right)
$$
where $\boldsymbol{e}_{j}$ is a lattice basis of $\mathbb{Z}^{k+1}$ and $Q_{0}, \ldots, Q_{k}$ are strictly isomorphic lattice polytopes in $\mathbb{R}^{m}$ (i.e. having the same normal fan) such that $\operatorname{dim}\left(\operatorname{aff}\left(Q_{0}, \ldots, Q_{k}\right)\right)=$ $m$.

## 2. The Software

polymake is a software system for computations in geometric combinatorics and related areas. The project was initiated in 1995 by Gawrilow and the first author [7], and many people helped to continuously expand it since. Recently, two important additions to the system have been accomplished [9]:
(1) polymake now comes with an interactive shell similar to most computer algebra systems.
(2) polymake has been extended to allow computations specific to the class of lattice polytopes, i.e., convex polytopes with integral vertex coordinates, and their relation to combinatorial commutative algebra, toric geometry, and integer programming.
The latest release 2.9.9 of polymake was published on November 9, 2010 and can be obtained from http://www.polymake.org. It is distributed as source code and precompiled binaries for Mac OS X and several Linux distributions. It is released under the GNU GPL, version 3.
polymake's functionality is organized in various applications. Currently, these are polytope for computations with convex polyhedra, matroid and graph for purposes revealed by their names, topaz (which is short for "topology application zoo") for finite simplicial complexes, and tropical for tropical geometry. Each application centers around objects, a representation for the basic mathematical objects dealt with, e.g., Polytope<Rational> or LatticePolytope in the application polytope. Technically, objects are organized in a class hierarchy written in (a slightly extended dialect of) Perl. These extensions include the possibility to use C++-style template parameters (e.g., <Rational> to specify a coordinate domain for polytopes) and a shared-memory communication model to interface with compiled C++-code. Semantically, objects are defined by their properties with can be used to derive further properties according to an extendible set of rules. For more about the general ideas behind polymake the reader should consult [7], while [9] is a more recent account with a focus on lattice polytopes.

## 3. Computations and Code Fragments

We prepare a function, written in polymake's Perl dialect, that computes the function $c$ from (1). It takes a polytope $P$ in the first, and the parameter $t$ in the second argument.

```
sub ct_invariant {
    my ($P, $t) = @_;
    my $v = $P->VERTICES;
    my $hd = $P->HASSE_DIAGRAM;
    my $sign = 1;
    my $c = new Integer(0);
    for (my $d = $P->DIM; $d > 0; --$d) {
```

```
    foreach (@{$hd->nodes_of_dim($d)}) {
    my $F = new Polytope(VERTICES=>
                $v->minor($hd->FACES-> [$_],All));
    my $vol = $F->LATTICE_VOLUME;
    $c += $sign*fac($d+$t)/fac($d)*$vol;
}
    $sign = -$sign;
}
$c += $sign*fac($t)*$P->N_VERTICES;
return $c;
}
```

The code of all functions used in this section, together with some explanations how to use them, is also available as a tutorial in the polymake wiki at http://www. polymake.org/doku.php/tutorial/a_determinants. polymake has interfaces to several other software packages for the computation of some special properties, such as the volume with respect to a given lattice, in the example above. polymake 2.9.9 uses Normaliz [1] for this type of computation by default.

Now we can interactively apply the function ct_invariant to some examples, e.g., [3, Example 2, p. 86]. Each input line to the polymake shell is preceded with the name of the currently active application, which is polytope throughout this extended abstract:

```
polytope > $S = simplex(2);
polytope > $P = prism($S);
polytope > print $P->SMOOTH;
1
polytope > print ct_invariant($P,1);
0
```

Here we have assigned $\operatorname{conv}\{(0,0),(1,0),(0,1)\}$, the standard simplex in $\mathbb{R}^{2}$, to the variable $\$ S$ and defined $\$ P$ to be the prism over $\$ S$ (with distance 1 between base and top). We can check for non-singularity of the associated toric variety by asking for the property SMOOTH in polymake. The output " 1 " represents the boolean value "true". Calling the function ct_invariant defined above shows that, indeed, $c_{1}(P)=0$. So the prism over the standard triangle is a defect polytope. It is a Cayley polytope of 3 segments. We can repeat the same computation for the hypersimplex $\Delta(3,6)$, which is not simple and hence not smooth:

```
polytope > print ct_invariant(hypersimplex(3,6),1);
136
```

Notice that the value given in [3, Ex. 10] is not correct.
Using the connection between lattice polytopes and toric varieties Di Rocco proves that $c_{t}(P) \geq 0$ for $t \geq 1$ and any smooth lattice polytope $P$ [3, Cor. 4].

## 4. Counter-Examples

In the sequel we use polymake to give answers to three questions raised in [3].
4.1. Lattice polytopes with $c_{1}$ negative. Conjecture 4 in that paper asks if $c_{1}(P) \geq 0$ for any lattice polytope, not necessarily smooth. We introduce some more notation in order to give counter-examples. Let $P$ be a lattice polytope in $\mathbb{R}^{d}$. The lattice pyramid of $P$ is the lattice polytope

$$
\operatorname{pyr}(P):=\operatorname{conv}(P \times\{0\} \cup \boldsymbol{v} \times\{1\})
$$

where $\boldsymbol{v}$ is a vertex of $P$ (any other lattice point at distance 1 from $\mathbb{R}^{d} \times\{0\} \subseteq \mathbb{R}^{d+1}$ gives an equivalent lattice polytope). We define the r-fold lattice pyramid of $P$ inductively via

$$
\begin{array}{ll}
\operatorname{pyr}^{r}(P) & :=\operatorname{pyr}\left(\operatorname{pyr}^{r-1}(P)\right) \\
\operatorname{pyr}^{0}(P) & :=P
\end{array} \quad \text { for } r \geq 1,
$$

We define another small function to compute $r$-fold pyramids.

```
sub r_fold_pyr {
    my ($P, $r) = @_;
    for (; $r>0; --$r) {
    $P = lattice_pyramid($P);
    }
    return $P;
} \(c_{t}\) for lattice pyramids over the cube.
```

```
polytope > $C=cube(3,0);
```

polytope > $C=cube(3,0);
polytope > print ct_invariant($C,0);
polytope > print ct_invariant($C,0);
-2
-2
polytope > print ct_invariant($C,1);
polytope > print ct_invariant($C,1);
4
4
polytope > print ct_invariant(r_fold_pyr($C,1),1);
polytope > print ct_invariant(r_fold_pyr($C,1),1);
-1
-1
polytope > print ct_invariant(r_fold_pyr($C,2),2);
polytope > print ct_invariant(r_fold_pyr($C,2),2);
-2
-2
polytope > print ct_invariant(r_fold_pyr($C,3),3);
polytope > print ct_invariant(r_fold_pyr(\$C,3),3);
-6

```
-6
```

We can apply this to the standard unit cube, and compute some of the invariants

With the knowledge of this example it is not hard to see that

$$
c_{r}\left(\operatorname{pyr}^{r}(P)\right)=r!c_{0}(P)+(-1)^{d+1} r!
$$

for any $d$-dimensional lattice polytope $P$ and $r>0$. We have seen in the previous computation that $c_{0}(C)=-2$ for the standard unit cube $C$, so that

$$
c_{r}\left(\operatorname{pyr}^{r}(C)\right)=r!c_{0}(C)+r!=-r!<0
$$

4.2. Convolutions of Ehrhart polynomials. Conjecture 5 in [3] asks whether all coefficients of the polynomial

$$
\begin{aligned}
f(P, t) & :=\sum_{k=0}^{d}(-t)^{(d-k)}(k+1)!\sum_{F \in \mathrm{~F}_{P}(k)}\left|Z^{F} \cap t F\right| \\
& =\sum_{k=0}^{d} \sum_{F \in \mathrm{~F}(P)}(k+1)!\operatorname{ehr}(F, t) t^{(d-k)}
\end{aligned}
$$

in the indeterminate $t$ are non-negative. Here $\operatorname{ehr}(F, t)$ is the Ehrhart polynomial of $F$, and the coefficients of $f(P, t)$ are integral, as $k!\operatorname{ehr}(F, k)$ has integral coefficients for any $k$-dimensional lattice polytope. Again we will provide counter-examples. In polymake, we can use the following function to compute the coefficients of $f(P, t)$. sub f_poly_coeff \{
my (\$P) $=@_{-}$;
my \$d = \$P->DIM;
my \$v = \$P->VERTICES;
my \$hd = \$P->HASSE_DIAGRAM;
my $\$$ sign $=1$;

```
my $f = new Vector<Integer>($d+1);
for (my $k = $d; $k > 0; --$k) {
    foreach (@{$hd->nodes_of_dim($k)}) {
        my $q = new Polytope(VERTICES=>
                $v->minor($hd->FACES-> [$_],All));
    my $h = (zero_vector<Rational>($d-$k))
                                    |$q->EHRHART_POLYNOMIAL_COEFF;
        $f += $sign*fac($k+1)*$h;
    }
    $sign = -$sign;
}
$f += $sign*$P->N_VERTICES
                                    *unit_vector<Rational> ($d+1,$d);
    return $f;
}
```

The returned vector lists the coefficients of the polynomial with increasing degree, i.e., the last entry is the leading coefficient of the polynomial.

We can use the same example as before to show that the coefficients need not be non-negative. We start out with the 3-dimensional unit cube stored in \$C.

```
polytope > print f_poly_coeff($C);
2436244
polytope > print f_poly(lattice_pyramid($C));
120 192 114 32 -1
```

It is not hard to see that the leading coefficient of this polynomial coincides with $c_{1}(P)$. Looking at iterated pyramids, by re-using our function r_fold_pyr, we obtain:
polytope > print f_poly_coeff(r_fold_pyr(\$C,3));
$5040906055381698188-30$
polytope > print f_poly_coeff(r_fold_pyr(\$C,5));
$362880717696491304163056280861490-1500$
We are not aware of an example where another but the leading coefficient of the polynomial is negative.
4.3. Cayley polytopes. It is finally asked in [3, p. 103] whether $c_{1}(P)=0$ implies that $P$ is a strict Cayley polytope as defined above. This is not true, as the example of a two-fold pyramid over the unit square shows:
polytope > \$Q=cube $(2,0)$;
polytope > print ct_invariant(r_fold_pyr (\$Q,2),1);
0
However, in the above definition for a Cayley polytope we could drop the condition that all factors have the same normal fan. With this slightly more general notion the 2 -fold lattice pyramid over the $0 / 1$-square is a non-strict Cayley polytope of two segments and two points. On the other hand, it is known [5, Prop. 2.14] that the set of lattice points of a defect polytope is contained in two parallel hyperplanes with distance one, so any defect polytope is always a non-strict Cayley polytope over a segment.

## References

[1] W. Bruns and B. Ichim, Normaliz 2.5, 2010, http://www.mathematik.uni-osnabrueck.de/ normaliz/.
[2] Raymond Curran and Eduardo Cattani, Restriction of a-discriminants and dual defect toric varieties, J. of Symbolic Computation 42 (2007), no. 1-2, 115 - 135.
[3] S. Di Rocco, Projective duality of toric manifolds and defect polytopes., Proc. Lond. Math. Soc., III. Ser. 93 (2006), no. 1, 85-104.
[4] A. Dickenstein and B. Nill, A simple combinatorial criterion for projective toric manifolds with dual defect, Math. Res. Letters 17 (2010), no. 3, 435-448.
[5] Alexander Esterov, Newton polyhedra of discriminants of projections, Discrete Comput. Geom., Mathematical Journal 8 (2008), no. 3, 433-442.
[6] G. Ewald, Combinatorial convexity and algebraic geometry, Graduate Texts in Mathematics, vol. 168, Springer-Verlag, New York, 1996.
[7] E. Gawrilow and M. Joswig, polymake: a framework for analyzing convex polytopes, Polytopes-combinatorics and computation, DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43-73.
[8] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, Discriminants, resultants and multidimensional determinants, Birkhäuser Boston Inc., Boston, MA, 2008.
[9] M. Joswig, B. Müller, and A. Paffenholz, polymake and lattice polytopes, FPSAC 2009, Assoc. Discr. Math. Theor. Comput. Sci., Nancy, 2009, pp. 491-502.
[10] Yutaka Matsui and Kiyoshi Takeuchi, A geometric degree formula for A-discriminants and Euler obstructions of toric varieties., Adv. Math. 226 (2011), no. 2, 2040-2064.

TU Darmstadt, Dolivostr. 15,64293 Darmstadt, Germany
E-mail address: \{joswig, paffenholz\}@mathematik.tu-darmstadt.de


[^0]:    Michael Joswig is supported by DFG Research Group "Polyhedral Surfaces" (FG 565) and DFG Priority Program "Algorithmic and Experimental Methods" (SPP 1489).

    Andreas Paffenholz is supported by the DFG Priority Program "Algorithmic and Experimental Methods" (SPP 1489).

