Minimum Bounding Boxes for Regular Cross-Polytopes

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ABSTRACT

In recent years there has been significant use of regular cross-polytopes (regular octahedrons or hyper-diamonds) as constructs to simplify problem solving in high-dimensional database queries, collision detection algorithms and graphic rendering techniques. Many of the algorithms for these applications use minimum volume bounding boxes as approximations of the polytopes to minimize computational complexity. The standard method [1] for finding these boxes in three dimensions uses the constraint of having two edges of a polyhedron coincident with two adjacent faces of the minimum bounding box. In this paper, we show that for a uniform cross-polytope in three dimensional space, a minimum volume bounding box would have a face flush with the convex hull of the polytope for all possible orientations of the polyhedron defined. We also show that the projections of the minimum bounding box of an n-dimensional regular cross-polytope are locally optimal with respect to the projections of the enclosed cross-polytope. We use this result to provide a necessary condition for the minimum bounding boxes of such polytopes. Finally, we show that if the two dimensional planar projections of a three dimensional uniform cross-polytope are simultaneously locally optimal then the polytope itself is optimally oriented.

1. INTRODUCTION

Computing an axis-aligned bounding box for a given object is important in many applications. An axis-aligned bounding box is a box tightly fitting an object with each side of the box parallel to the axis of the coordinate system. This box can then be used as an approximation of the object for a variety of applications. For example, a range query using distance based on L1 norm (Manhattan Distance) in 2-D space is a diamond. Efficient implementation of this range query by an axis aligned bounding box is presented in [2]. In three dimensional space a range query in L1-norm is a regular octahedron and computing optimal axis aligned bounding box for this object is non-trivial. O'Rourke [1] has provided a complex method for computing an optimal axis aligned bounding box in three dimensions for a given set of points based on the necessity of having two edges of the enclosed object flush with the faces of the bounding box. Barequet et al [3] and Klowski et al [4] use an approximate minimization of volume as a metric to efficiently determine a good bounding box to reduce the complexity inherent in O'Rourke's method. In this paper we present several theorems and their proofs related to axis aligned bounding boxes for three and higher dimensional regular cross polytopes. The organization of the paper is as follows:

Section-2 presents the *two edges flush* property of optimal bounding boxes modified for the special case of a regular cross-polytope which follows from O'Rourke's work. In Section-3 we prove the existence of the *one face flush* property for regular octahedrons. In Section-4 we prove that the projections of a d-dimensional optimal bounding box in lower dimensions are optimal for the projections of the enclosed octahedron. We then generalize the two edge flush property defined earlier for three dimensions to d-dimensions. Section-5 shows that if the projections of the regular octahedron in two dimensions are simultaneously locally optimal then the bounding box of the regular octahedron is also optimal. Section-6 presents a few concluding remarks.

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2. TWO EDGES FLUSH

The problem of finding minimal volume boxes circumscribing a given set of three-dimensional points was investigated by O'Rourke in [1]. This work demonstrated that for a three dimensional polyhedron defined by such a set of points, a minimum volume bounding box would necessarily have two faces flush with two adjacent edges of the enclosed polyhedron. Given this condition we state the following theorem for the special case where the convex hull of a set of points in three dimensional space defines a regular octahedron.

Theorem 1:
Every minimal volume bounding box of a set of points describing a regular uniform octahedron in three dimensional space must have at least two faces flush with two adjacent edges of the enclosed octahedron.

Proof: The proof follows from O'Rourkes Theorem and is detailed in [1]. ■

Using this result we can develop an algorithm to determine the minimum volume bounding box for a regular octahedron. Such an algorithm would search through all possible combinations of adjacent edges in the octahedron, creating a bounding box for each pair, calculating its volume and isolating the one which has gives the smallest value. However, we can significantly improve upon this technique by leveraging one of the unique structural properties of a regular octahedron. This property is stated in the following lemma:

Lemma: *Any two adjacent edges of a uniform regular octahedron are separated by an angle of either* 90° *or* 60°

This suggests that there can only be two possible unique combinations of adjacent edges with which the faces of a bounding box can be flush. As the the octahedron is uniform, every other combination would be a reflection of these two. Utilizing this observation, we propose an additional necessary condition for the minimal volume bounding box of a regular octahedron that is significantly stricter than the one discussed above.

3. ONE FACE FLUSH

In this section we further constrain the possible orientation of a minimum volume bounding box enclosing a cross polytope, by observing that for the special case of a regular octahedron the minimum volume bounding box would have one face flush with a face of the convex hull in addition to having two faces flush with two adjacent edges of the enclosed octahedron. In the subsequent discussion we assume that without loss of generality, our cross-polytopes (regular octahedrons) are centered on the origin.

Theorem 2:
A minimal volume bounding box must have at least one face flush with a face of the enclosed regular octahedron.

Proof: We consider two cases, one in which the angle between adjacent edges is 90° and the other when the angle between the two is 90° as discussed earlier. Based on the geometry of the regular octahedron (the length of all edges being equal), all pairs of adjacent edges in the octahedron not bounding the same face of the octahedron, define three mutually orthogonal squares. Therefore, the angle between all such pairs of edges would be 90°. Conversely, in the case where two adjacent edges are at an angle of 60° , according to the geometry of the regular octahedron (the length of all edges being equal), every face of the octahedron is an equilateral triangle. Since the angle between two edges of such a triangle is always 60°, all pairs of adjacent edges bounding a face of the octahedron would have an angle of 60° between them.

Case - I: ***Angle Between Adjacent Edges is*** 90°***:***

In the first case, for simplicity we assume that the octahedron (or the 3D hyper-diamond) is rotated in such a way that one of the edges is flush with the face of the bounding box parallel to the X-Y plane and the other edge is flush with the face parallel to the Z-X plane as shown in Figure 3. Given this initial position, the projection of the octahedron on the X-Y plane takes the form shown in Figure 1. Initially, the octahedron is positioned so as to have line OA parallel to the Z-X plane. Let θ be the angle by which the figure is rotated. As is clear from the figure, the length l of the minimum volume bounding box for the octahedron is the projection of C'A' on to X-axis and its height h is equal to the projection of B'D' on to Y-axis. The breadth b of the

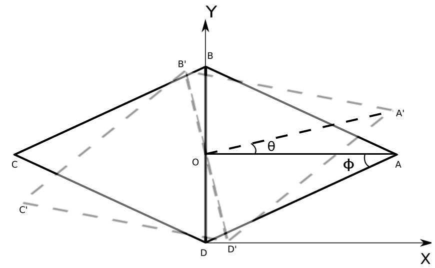

Figure 1: Projection of the Octahedron on the X-Y Plane

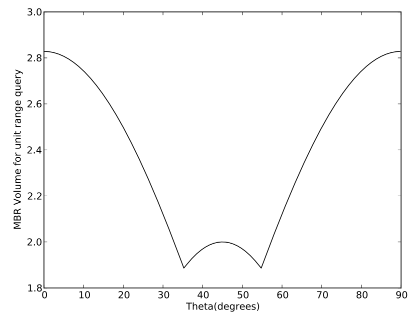

Figure 2: Volume of the MBR as a function of θ

box remains constant irrespective of θ. Hence, volume of the box can be calculated as V = lbh. As the polytope rotates Figure 2 shows volume of the minimum bounding box as a function of θ for $0 \leq \theta \leq 90$. Cases for other values of theta are similar to these with slight changes in the orientation, and are omitted hence. It can be seen that the function is discontinuous at two points which also happen to be the points of minimum volume. This is illustrated by the graph in Figure 2.

It can be shown using trigonometry that these two points correspond to the angles $\theta = \phi$ and $\theta = 90^\circ$ - ϕ, where $\phi = 35.264^\circ$. As the orientation of the cross polytope at each of these two points ensures that two parallel faces of the octahedron are flush with the two parallel faces of the MBR, it follows that if the bounding box is oriented around any two adjacent edges with an angle of 90° between them, minimal volume can only be achieved when one of its faces is flush with a face of the octahedron.

Case - II: ***Angle Between Adjacent Edges is*** 60°

The starting condition is the same in this case. An axis-aligned bounding box is constructed with two adjacent edges of a regular octahedron having an angle of 60° between them flushed with two adjacent faces as shown in Figure 4. For simplicity we assume that this bounding box is placed at the origin with the octahedron oriented as shown in the figure (the face defined by vertices $V1, V2andV3$ has the two edges with an angle of 60° between them).

Due to the constraint just mentioned (two edges need to be flushed with two adjacent faces), the only possible rotation of the cross-polytope will result in the movement of the vertice $V1$ on the Y-Axis, $V3$ on the Z-Axis and the edge $V1 - V2$ over the face of the bounding box. The distance from origin of this point is taken as a. Thus a varies over a

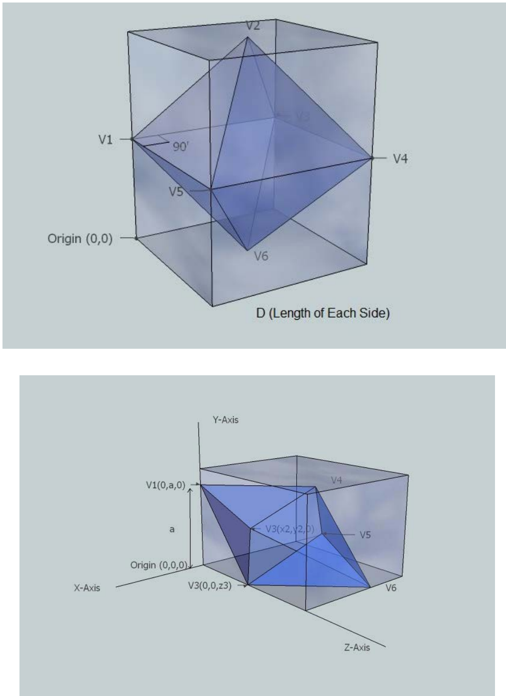

Figure 3: Position of Octahedron Against the Adjacent Faces of MBB for (a) Case-I (b) Case-II

finite range for the possible rotation.

In its initial position (as shown in Figure 4), the three vertices coincident with faces of the bounding box are given by the following coordinates.
$V_1 = (0, a, 0)$
$V_2 = (x_2, y_2, 0)$
$V_3 = (0, 0, z_3)$
Based on the fixed distance D between the vertices V_1, V_2 and V_3,we derive the following equations,

$$D^2 = (x_2)^2 + (y_2 - a)^2$$

$$D^2 = (x_2)^2 + (y_2)^2 + (z_3)^2$$

$$D^2 = a^2 + (z_3)^2$$

where D is the length of a side of the cross-polytope. Solving for x_2, y_2 and z_3, we can thus define each vertex in terms of a as given below,

$$V1 = (0, a, 0)$$

$$V2 = (\sqrt{a^2 - (\frac{2a^2 - D^2}{2a})^2}, \frac{2a^2 - D^2}{2a}, 0)$$

$$V3 = (0, 0, \sqrt{D^2 - a^2})$$

Using the fixed distances between the vertices of the cross-polytope, we can develop similar equations in terms of a and D. Solving for the variable a we obtain the following values for the coordinates of vertices $V4$,$V5$ and $V6$

$$V4 = (-(\frac{\sqrt{-a^2+d^2}(a^3-ad^2+\sqrt{2}\sqrt{-a^2(4a^4-5a^2d^2+d^4)})}{3(a^3-ad^2)}),$$
$$(\frac{\sqrt{4d^2-\frac{d^4}{a^2}}(4a^3-ad^2+\sqrt{2}\sqrt{-a^2(4a^4-5a^2d^2+d^4)})}{12a^3-3ad^2)},$$
$$\frac{(4a^3-ad^2-\sqrt{2}\sqrt{-4a^6+5a^4d^2-a^2d^4})}{3a^2})$$
$$V5 = (\frac{\sqrt{-a^2+d^2}(2a^3-2ad^2-\sqrt{2}\sqrt{-a^2(4a^4-5a^2d^2+d^4)})}{3(a^3-ad^2)},$$
$$\frac{2a^3+ad^2-2\sqrt{2}\sqrt{-4a^6+5a^4d^2-a^2d^4}}{6a^2},$$
$$\frac{\sqrt[4d^2-d^4/a^2]{(-4a^3+ad^2+2\sqrt{2}\sqrt{-a^2(4a^4-5a^2d^2+d^4)})}}{24a^3-6ad^2})$$
$$V6 = (\frac{\sqrt{-a^2+d^2}(2a^3-2ad^2-\sqrt{2}\sqrt{-a^2(4a^4-5a^2d^2+d^4)})}{3(a^3-ad^2)},$$
$$\frac{a^3-ad^2+\sqrt{2}\sqrt{-a^2(4a^4-5a^2d^2+d^4)}}{3a^2},$$
$$\frac{\sqrt{4d^2-d^4/a^2}(4a^3-ad^2+\sqrt{2}\sqrt{-a^2(4a^4-5a^2d^2+d^4]})}{12a^3-3ad^2})$$

Given that every coordinate of each vertex is a function of a, we can generalize the equation of a vertex of the cross polytope as,

$$V_i = (f_{x_i}(a), f_{y_i}(a), f_{z_i}(a)) \qquad (1)$$

where i= 1,2,...,6 and f_{x_i},f_{y_i} and f_{z_i} are the functions defining the x,y and z coordinates respectively for each vertex.

The length of each dimension of the minimum bounding box is given by the minimum of the maximum values attained by the coordinates of each vertex. From above it is clear that the orientation of the octahedron is determined by the range over which a fluctuates. Based on the fact that V_1 has to remain on Y-Axis, the edge (V_1, V_2) on the XY-plane and assuming a unit octahedron for simplicity, we find the value of a has to be between $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{3}}{2}$ Utilizing basic trigonometry and the structural properties of the octahedron, the minimum of the maximum value attained by a coordinate in each dimension is thus given by,

$$LengthofMBB_x = min_{1 \le i \le 6}(max_{\frac{1}{\sqrt{2}} \le a \le \frac{\sqrt{3}}{2}}(f_{x_i}(a)))$$

$$LengthofMBB_y = min_{1 \le i \le 6}(max_{\frac{1}{\sqrt{2}} \le a \le \frac{\sqrt{3}}{2}}(f_{y_i}(a))$$

$$LengthofMBB_Z = min_{1 \le i \le 6}(max_{\frac{1}{\sqrt{2}} \le a \le \frac{\sqrt{3}}{2}}(f_{z_i}(a)))$$

where i = *1,2,...6.*

To isolate the limits of the range of rotation of the polyhedron, we assume a unit polyhedron. Once the limits of the possible movement of the octahedron along its single degree of freedom are identified, specified in terms of the variable used as frame of reference i.e. a, we can obtain results for equations 2-4 above. This is done by finding first the local maxima and than the global minima of the coordinate equations for each vertex. After finding the local maxima for each dimension for every vertex we found that along the X-dimension only the equations for the x-coordinates of vertices $V2$ and $V6$ are differentiable and display monotone convergence as functions of a over the given range. The x-coordinate equation for $V5$ is also differentiable, it retains a lesser range of values over the given range and thus is irrelevant when considering local maxima. While the function for the x-coordinate of $V2$ is monotonically increasing, the value of the function for the x-coordinate of the other vertex is monotonically decreasing. Thus, over the given interval the global minima would be point of convergence. The equations are,

$$x_2 = \sqrt{(a^2 - ((2a^2 - d^2)/(2a))^2)}$$

$$x_6 = \frac{\sqrt{-a^2 + d^2}(2a^3 - 2ad^2 - \sqrt{2}\sqrt{-a^2(4a^4 - 5a^2d^2 + d^4)})}{3(a^3 - ad^2)}$$

However for the y-coordinate equations, no such convergence occurs. Here we determine the local maxima for the y-coordinate of $V4$, providing a measure of the height of the minimum bounding box by the following equation.

$$y_4 = \frac{\sqrt{4d^2 - \frac{d^4}{a^2}}(4a^3 - ad^2 + \sqrt{2}\sqrt{-a^2(4a^4 - 5a^2d^2 + d^4)})}{12a^3 - 3ad^2)}$$

Similarly, the z coordinate equations for vertices $V4$ and $V3$ taken as a function of a are monotonically increasing and monotonically decreasing over the given range respectively. Thus, these functions are differentiable everywhere on the given interval allowing us to determine the global minima in the next step. Also, as these two functions converge to the same point over the given interval the global minima would be the point of convergence. Hence, we find that the length of the minimum bounding box in Z-dimension is given by the following equations.

$$z_4 = \frac{(4a^3 - ad^2 - \sqrt{2}\sqrt{-4a^6 + 5a^4d^2 - a^2d^4})}{3a^2}$$

$$z_3 = \sqrt{(d^2 - a^2)}$$

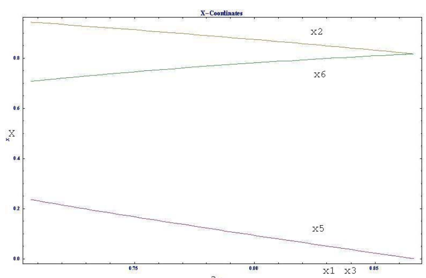

Figure 4: Local Maxima of X

The properties discussed above are easily verified by plotting the coordinate equations of each vertex as a function of a over the range of rotation identified as shown in Figures 5-7. These three extremities give the minimum volume bounding box since $Volume of MBB = Length * Breadth * Height$. Minimizing the value of the x,y and z-coordinates over the given range, we find the orientation of the octahedron (given by the value of a at point of global minima) inside the minimum bounding box. This minimization results identifying the orientation with $a = \frac{\sqrt{3}}{2}$ as the point of global minimum. As we can see from Figure 3, this orientation corresponds to the previous case where the angle $\theta = \phi$ and $\theta = 90 - \phi$, where $\phi = 35.264°$ and two parallel faces of the octahedron are flush with two parallel faces of the bounding box. Thus it follows that if the bounding box is oriented around any two adjacent edges with an angle of 60° between them, minimal volume can only be achieved when one of its faces is flush with a face of the octahedron. ■

4. BOUNDING BOX UNDER PROJECTION

We now have a trivial method to determine the bounding box of a three dimensional cross polytope. In the following section we postulate about the characteristics of a cross-polytope under projection when encapsulated within such

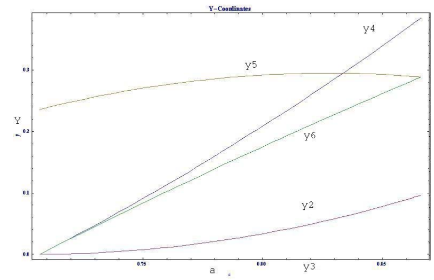

Figure 5: Local Maxima of Y

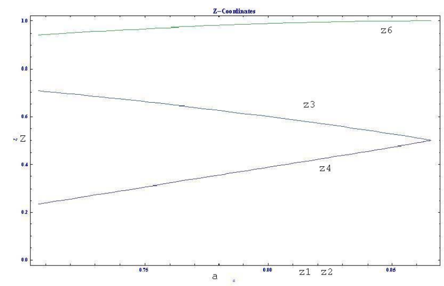

Figure 6: Local Maxima of Z

a bounding box. The result presented is applicable to d-dimensional cross-polytopes and their bounding boxes. It, therefore, also applies to the special case of the three dimensional cross-polytope and its bounding box just discussed. The discussion is made with the assumption that the cross-polytope has already been optimally oriented such that its axis aligned bounding box is also the arbitrarily oriented minimum bounding box.

4.1 Minimality of Projected Bounding Box

If the bounding box of a d-dimensional regular cross polytope is minimal, then we postulate about the nature of its projections in lower dimensions as follows,

Theorem 3:
Given an optimally oriented convex d-polytope P and its axis-aligned minimum bounding box B, any projection $p : R^d \rightarrow R^{d'}$, $d' \leq d$ *of B will also be a minimum bounding box for the corresponding projection of P.*

Proof: By way of contradiction, assume there exists a d-polytope P and rotation R such that an axis-aligned minimum bounding box B is also the minimum arbitrarily oriented minimum bounding box. Assume there exists a projection *p'* of P rotated by R such that *B'* (corresponding to the projection of B along the same dimensions)is not an arbitrarily oriented minimum bounding box of the projection *p'* of P rotated by R. Then there exists an arbitrarily oriented minimum bounding box *B"* of the projection *p'* of P rotated by R which is different from B. Let R' be the rotation that rotates B" such that it is an axis-aligned bounding box. The area of the bounding box is the product of the ranges in each dimension. Since projection discards dimensions not subject to the projection, any rotation of the projection does not change the dimensions of the axis not subject to projection for an axis aligned bounding box. However, there does now exist a rotation of the projection such that an axis aligned bounding box is smaller than B'. Hence, the composition of R and R' is a rotation that will rotate the d-polytope P such that a smaller axis-aligned minimum bounding box exists. This contradicts our earlier assumption of the optimality of the three dimensional bounding box and hence is not possible. ■

4.2 Edges Flush in d-Dimensions

Using the result just obtained we can come up with a necessary condition for the minimum bounding box of an n-dimensional cross polytope. This condition is stated by the following theorem:

Theorem 4:
Given a convex d-polytope P and an arbitrarily aligned minimum bounding box B, at least d-1 edges must be flush with d-1 orthogonal faces of B.

Proof: Without loss of generality, assume P and B are rotated such that B is an axis-aligned minimum bounding box. By the previous theorem, any 2d projection of B must also be a minimum bounding box of the 2d projection of P. By a previous theorem, at least one edge of the 2d projection of P must be flush with one edge of the 2d projection of B. Such an edge will have identical values for each the dimension corresponding to the bounding box edge and the values must be either the minimum or maximum values of that dimension. Otherwise the bounding box would not be minimum since some value in one dimension is beyond the range of the bounding box. An edge that is flush with the edge of the bounding box in a 2d projection is also flush with the corresponding face of B, since the values in that dimension are equal for both edges of the edge and are the minimum or maximum value in that dimension. Hence, for every possible pair of dimensions an edge must exist that is flush with a face of B for one of those dimensions. It is possible for an edge to serve as the flush edge for more than one 2d projection involving a given dimension *N*. The edge will always have the minimum or maximum value for dimension *N* for every 2d projection that includes that dimension. The minimum number of edges that can satisfy this constraint for all possible pairs of dimensions is d. Assume it is less than d Then there exist two dimensions for which— a flush edge does not exist. Hence, the 2d projection would not have a flush edge and we have a contradiction. ■

5. CROSS-POLYTOPE UNDER PROJECTION

The theorems just discussed confirm that if the bounding box in dimension *d* is optimal then the bounding boxes of the projections in *d-1* dimensions are also optimal. This also leads us to a definition of the local optimality of the projection of the object. This is stated as:

Local Optimality: *If the axis-aligned bounding box of a d-1 dimensional projection of a d-dimensional cross-polytope is its arbitrarily oriented minimum bounding box requiring no rotation of the projection to make it smaller, then the projection is said to be locally optimal.*

Using this definition we can further elaborate on the situations where multiple projections of the same cross-polytope exhibit local optimality at the same time. Thus:

Simultaneous Local Optimality: *If the* d-1 *dimensional projections of the* d-dimensional *cross-polytope in each coordinate plane are locally optimal then it is said to have simultaneous locally optimal projections.*

From *Theorem 3*, we see that minimum bounding boxes of each *d-1* dimensional projection of the *n-dimensional* optimally oriented object, correspond to the *d-1* dimensional projections of the minimum bounding box of the object in the corresponding planes. This, therefore, leads to the following corollary:

Corollary: *Given a convex d-polytope P, there exists at least one set of simultaneously locally optimal d-1 dimensional projections.*

5.1 Projected Polygons

Now, specifically in the case of a three dimensional cross-polytope, we can make additional statements about the nature of the projections of the polytope in two dimensions. The first of these statements is given the subsequent theorem,

Theorem 5:
Given a convex three dimensional cross-polytope P any projection $p : R^d \rightarrow R^{d_1}$, $1 < d_1 < 3$ *of P will either be a hexagon, a rhombus or a rectangle.*

Proof: The structural properties of a regular octahedron ensure that only two vertices of the polytope can be collinear at one time. Given this property, a vertex first projection onto one of the axial planes for any orientation of the octahedron would require that either one pair of vertices, two pairs of vertices or no pair of vertices be simultaneously coincident in the same planar projection.

Case - I: One Vertex Pair Coincident - The point formed from the projected pair would be bound by the convex hull of the remaining four and the planar projection obtained is rectangular.

Case - II: Two Vertex Pairs Coincident - The two pairs needs must lie on two parallel edges of the polytope thus forming two opposite points on the planar projection. The remaining two vertices are also opposite to each other and thus when projected onto the plane, form two more opposing points creating a rhombus.

Case - III: Three Vertex Pairs Coincident - Each vertice is projected as a point onto the plane forming a hexagonal envelope. ■

5.2 Unique Simultaneous Local Optimality

We can use the result of the theorem just discussed and the first theorem to define the optimal orientation of the three dimensional cross-polytope in terms of simultaneously locally optimal projections. This can be stated as:

Theorem 6:
Given a three dimensional cross-polytope P, there exists only one combination of two dimensional projections in the coordinate planes that are simultaneously locally optimal, corresponding to the optimal orientation of the polytope.

Proof: **Case - I:** ***No Two Adjacent Edges Flush(Object Orientation is Sub-Optimal) :***
For this scenario we assume that no two adjacent edges of the object P are flush with two adjacent faces of the bounding box B. This implies that the orientation of P is sub-optimal and the bounding box B of P is not minimal. From *Theorem 5* we see that the two dimensional projections in the coordinate planes are a combination of rhombuses, hexagons and rectangles. We now consider each projected shape in turn and determine the conditions under which they

become locally optimal.

Rhombus: Consider a projection *p'* of P on the *XY*-plane which forms a rhombus. For *p'* to be locally optimal, one of its edges must be coincident with one of the axes. However, we see that rotating p' to obtain the required orientation corresponds to rotating P such that one of its faces is coincident with a face of B. By Theorem 1 this corresponds to the optimal orientation of the object, contradicting our starting assumption that no two adjacent edges are flush with adjacent faces of B. Therefore, any such combination of projections would be sub-optimal

Hexagon: Consider a projection *q'* of of P on the *XZ*-plane which forms a hexagon. For *q'* to be locally optimal, at least two of its edges also need to be parallel with one of the axes and at least one must be coincident. Applying the rotation required to make the edges parallel, results in a corresponding change in the orientation of P such that the edges of P projected onto the XZ-plane to give the edges of q' being made parallel, become coincident with opposing faces of B. Due to the structure of P, any such pair of edges are joined with an adjacent edge that becomes coincident with an adjacent face of B contradicting our starting assumption. This precludes the possibility of a locally optimal hexagonal projection under the given conditions.

Rectangle: Proving the non-existence of simultaneous local optimality in all cases where the combination of projections on the coordinate planes contain a hexagon or a rhombus, leaves us with a single scenario where all of the projections are rectangles. A rectangular projection requires two pairs of adjacent edges forming a closed ring such that the angle between each pair is 90°. Due to the structure of the octahedron, at no point in time can there be more than one such ring for any orientation of the polytope. Therefore, a combination of three rectangular projections is not possible. Hence, there cannot be a simultaneously locally optimal combination of projections when two adjacent edges of the cross-polytope are not flushed with the bounding box.

Case - II: ***Two Adjacent Edges Flush: Object Orientation May Be Optimal:***

As discussed earlier, any two adjacent edges of the polytope will have an angle of either 60° or 90° between them. Assume that the polytope is oriented such that two edges with an angle of 60° between them are flush with adjacent faces of the bounding box. A projection P of the object on the XY-plane forms a hexagon, given that no two vertices of the polytope have the same coordinates in the plane. For this hexagon to be locally optimal, it needs to have at least two edges parallel to one of the axes, with one being coincident to it. Since this is not the case here, it is not locally optimal. From the orientation of the polytope and from Theorem 1, the range of rotation of the polytope is limited such that for the given range, the projection of the polytope on the XY-plane remains a non-locally optimal hexagon except at the boundary where it becomes a rhombus. At this boundary the projection is optimal. For all other orientations it is sub-optimal, meaning that the combination of projections cannot be simultaneously locally optimal.

Now, for the case where two adjacent edges with an angle of 90 between them are flush with two adjacent faces of the bounding box. As we see from Figure 2, the projection of the polytope on the XY-plane is a rhombus for the entire range of possible rotations. For the rhombus to be locally optimal, one of its edges must be coincident with one of the axes. From the discussion above, this is only possible for the optimal orientation of the polytope. This precludes the existence of a simultaneous local optimal combination of projections for all possible orientations of the polytope under the given conditions (the rhombus is sub-optimal for all these orientations).

Therefore, we see that there can only exist a single unique combination of two projections that is simultaneously locally optimal. ■

Corollary: *Given a convex three dimensional uniform regular octahedron P, any simultaneous locally optimal projections $p_i : R^d \to R^{d_1}$, $1 < d_1 < 3$, $0 < i < 4$ of P will consist of a hexagon, a rectangle and a rhombus.*

Proof: The proof follows from *Theorems 5 and 6.* ■

6. CONCLUSION

In this paper we identify face coincidence as an additional constraint for the existence of a minimum volume bounding box for a regular cross-polytope. We identify certain unique properties and characteristics of the *d-1* dimensional projections of these polytopes when encapsulated by a minimum bounding box. We also discuss the forms the convex hulls of those projections take in the projected dimensions. In future work we aim to use the results just discussed in database indexing and querying techniques with the objective of improving upon current querying methods.

Acknowledgements

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7. REFERENCES

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