Approximating Low-Dimensional Coverage Problems

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Abstract

We study the complexity of the maximum coverage problem, restricted to set systems of bounded VC-dimension. Our main result is a fixed-parameter tractable approximation scheme: an algorithm that outputs a $(1-\epsilon)$-approximation to the maximum-cardinality union of k sets, in running time $O(f(\epsilon,k,d)\cdot poly(n))$ where n is the problem size, d is the VC-dimension of the set system, and $f(\epsilon,k,d)$ is exponential in $(kd/\epsilon)^c$ for some constant c. We complement this positive result by showing that the function $f(\epsilon,k,d)$ in the running-time bound cannot be replaced by a function depending only on (ϵ,d) or on (k,d), under standard complexity assumptions.

We also present an improved upper bound on the approximation ratio of the greedy algorithm in special cases of the problem, including when the sets have bounded cardinality and when they are two-dimensional halfspaces. Complementing these positive results, we show that when the sets are four-dimensional halfspaces neither the greedy algorithm nor local search is capable of improving the worst-case approximation ratio of $1-1/e$ that the greedy algorithm achieves on arbitrary instances of maximum coverage.

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1 Introduction

The Maximum Coverage problem is one of the classical NP-hard combinatorial optimization problems. An instance of Maximum Coverage is specified by a triple $(U, \mathcal{R}, k)$ where U is a finite set, $\mathcal{R}$ is a collection of subsets of U, and k is a positive integer. The objective is to output a k-tuple of elements of $\mathcal{R}$ such that their union contains as many elements as possible. (In the *weighted* version of the problem, elements $x \in U$ have non-negative weights $w(x)$ and the objective is to maximize the combined weight of elements in the union.) A very natural greedy algorithm for Maximum Coverage chooses k sets sequentially, where each new set is chosen to maximize the number (or combined weight) of elements covered by the new set but not by any of the preceding ones. It has been known for decades that this algorithm has an approximation factor of $\left(1 - \frac{1}{e}\right)$ [5]; in fact, it is a special case of the greedy algorithm for maximizing a monotone submodular function subject to a cardinality constraint [14], and the algorithm's approximation factor remains $\left(1 - \frac{1}{e}\right)$ even in this more general case. It was shown by Feige [7] that this approximation factor is the best possible, even for the unweighted Maximum Coverage problem, unless P=NP.

One of the reasons the greedy algorithm for Maximum Coverage is so widely studied is that it has innumerable applications: originally introduced by Cornuejols, Fisher, and Nemhauser [4] to model the problem of locating bank accounts in multiple cities to maximize "float," it was subsequently applied in databases [9], social networks [11], sensor placement [12], information retrieval [17, 19], and numerous other areas. A prototypical application arises in the information retrieval setting, when considering the problem of assembling a list of k documents to satisfy the information needs of as many users as possible. Equating every document with the set of users whom it satisfies, we see that this information retrieval problem is modeled by the Maximum Coverage problem.

Given the extremely broad applicability of Maximum Coverage problems, it is natural to wonder whether the approximation ratio of $1 - \frac{1}{e}$ is the strongest theoretical guarantee one can hope for. Feige's hardness result eliminates the possibility of obtaining a better worst-case approximation ratio in polynomial-time, but the problem instances arising in applications are unlikely to resemble worst-case instances of Maximum Coverage. Is it possible to identify broad classes of Maximum Coverage instances (hopefully resembling those that arise in practice) such that the greedy algorithm provably achieves an approximation factor better than $1 - \frac{1}{e}$ on these instances? If not, can one design a different polynomial-time algorithm with an improved approximation factor? These are the questions that inspired our paper.

Let us reconsider the problem of assembling a top-k list of documents, mentioned above, in light of these questions. At least two aspects of this application distinguish it from an arbitrary instance of Maximum Coverage.

(1) The value k is quite small compared to n, the input size. A typical instance might involve processing a list of thousands or millions of documents to extract a list of $k = 10$ top choices.

(2) The set system $(U, \mathcal{R})$ is likely to have a "low-dimensional" structure. For example, a natural model of users' preferences might assume that there are $d \ll n$ topics, a document contains a mix of topics described by a vector in $\mathbb{R}^d_+$, and the user's information need is satisfied if the dot product of this vector with another vector in $\mathbb{R}^d_+$ (describing the mix of topics the user seeks to read about) exceeds some threshold.

Is the approximation ratio of the greedy algorithm better than $1 - \frac{1}{e}$ under these circumstances? If not, is there some other algorithm that is significantly better?

We answer the first question negatively and the second one affirmatively. More precisely, for $d \geq 4$ we show that the greedy algorithm's approximation ratio in this special case is no better

than its worst-case approximation ratio of $1 - \left(1 - \frac{1}{k}\right)^k \approx 1 - \frac{1}{e}$, but that there is an algorithm with running time $O(f(\epsilon, k, d) \cdot poly(n))$ whose approximation factor is $(1 - \epsilon)$, for some function $f(\epsilon, k, d)$. (Of course, for *very* small values of k a trivial brute-force search over all collections of k sets in $\mathcal{R}$ finds an exactly optimal solution in $O(n^{k+1})$ time, but a fixed-parameter algorithm whose running time is exponential in k but quadratic in n is vastly faster when $k = 10$ and $n = 10^6$, for instance.)

The following subsection describes our contributions in more detail.

1.1 Our contributions

Our main contribution is a fixed-parameter approximation scheme (fpt-AS) for the Maximum Coverage problem, parameterized by the number of sets k, the approximation parameter ϵ, and the VC-dimension of the set system, d. Letting n denote the problem size — i.e. the sum of cardinalities of all the sets in $\mathcal{R}$ — the approximation scheme has running time $O(f(\epsilon, k, d) \cdot poly(n))$, where $f(\epsilon, k, d) = \exp\left(\tilde{O}(k^2 d\epsilon^{-5})\right)$[1]. The algorithm, which is presented in Section 3, is based on three ingredients. First, set systems of bounded VC-dimension have bounded-size ϵ-approximations (see Section 3 for definitions) and there is even a deterministic algorithm to find them in linear time [3]. Second, this means it is easy to design a fpt-AS for the special case of Maximum Coverage in which the set system has bounded VC-dimension *and* the optimum solution covers a constant fraction of the elements. Third, the general case can be reduced to this special case by an intricate non-deterministic algorithm, which can then be made deterministic at the cost of blowing up the running time by a factor that is exponential in $k^2 d\epsilon^{-5}$, but independent of n.

In Section 5 we show that various aspects of this result cannot be improved, under standard complexity assumptions. First, the function $f(\epsilon, k, d)$ cannot be replaced by a function depending polynomially on k unless $P = NP$. Second, it cannot be replaced by a function depending polynomially on $1/\epsilon$ unless $P = W[1]$. (The question of whether the exponential dependence on d can be eliminated is intriguing, but it is unlikely to be easily resolvable since fixed-parameter complexity theory lacks machinery analogous to the PCP Theorem for proving $W[1]$-hardness of approximation.) Furthermore, these hardness results apply even in some very simple cases: Maximum Coverage with set systems of VC-dimension 2, or with halfspaces in dimension 4, or with rectangles in dimension 2. Moreover, in all three of these special cases, the greedy algorithm fails to achieve an approximation factor better than $1 - \frac{1}{e}$.

These negative results about the greedy algorithm are counterbalanced by some positive results that we present in Section 4. We identify a parameter of the problem instance — the *covering multiplicity*, denoted by r — such that the greedy algorithm's approximation factor is never worse than $1 - (1 - \frac{1}{r})^r$. The covering multiplicity satisfies $r \leq k$, and when the inequality is strict this improves upon the worst-case approximation bound for the greedy algorithm. For problem instances whose sets have cardinality at most r, the covering multiplicity is bounded by r, and for instances in which the sets are two-dimensional halfspaces the covering multiplicity is bounded by 2, implying that the greedy algorithm is a $\frac{3}{4}$-approximation in the latter case.

1.2 Related work

As mentioned above, the Maximum Coverage problem was introduced, and the greedy algorithm analyzed, by Conuejols et al. in [4]. This work was subsequently generalized to the context of sub-

[1] $\tilde{O}$ hiding log factors

modular functions by Nemhauser et al. [14]. A matching hardness of approximation for Maximum Coverage was obtained by Feige [7], in a paper that also settled the approximation hardness of the closely related Set Cover problem, establishing that the greedy algorithm achieves the optimal approximation ratio (up to lower order terms) for both problems.

The approximability of special cases of Set Cover and Maximum Coverage was subsequently investigated in numerous papers. For example, the Maximum Vertex Coverage problem is the special case of Maximum Coverage in which U is the edge set of a graph and every set in $\mathcal{R}$ is the set of edges incident to one vertex of that graph. This special case of the problem was shown to be APX-hard by Petrank [16]. A landmark paper by Ageev and Sviridenko [1] introduced the technique of *pipage rounding* and used it to give a (non-greedy) polynomial-time $\frac{3}{4}$-approximation algorithm for Maximum Vertex Coverage; more generally, they gave a polynomial-time algorithm with approximation factor $\left(1 - \left(1 - \frac{1}{k}\right)^k\right)$ for the special case in which every element of U belongs to at most k sets in $\mathcal{R}$.

Computational geometers have intensively studied special cases of Set Cover, or the dual problem of Hitting Set, when the set system is defined geometrically, e.g. by rectangles, disks, or halfspaces. A seminal paper by Bronniman and Goodrich [2] introduced a multiplicative-weights method for approximating Hitting Set, and applied this method to design constant-factor approximation algorithms for various classes of bounded-VC-dimensional set systems, e.g. disks in the plane. The weighted case of these problems turns out to be much more challenging; see [8, 18]. A breakthrough paper by Mustafa and Ray [10] presented a new method to analyze local search algorithms for geometric hitting set problems, thereby proving that local search yields a PTAS for many interesting special cases such as three-dimensional halfspaces.

The study of fixed-parameter approximation schemes — and fixed-parameter approximation algorithms more generally — is still in its youth. An excellent survey by Marx [13] includes an fpt-AS for Maximum Vertex Coverage (also known as Partial Vertex Cover), a problem which is a special case of bounded-VC-dimensional Maximum Coverage. Thus, one consequence of our algorithm in Section 3 is an alternative fpt-AS for Partial Vertex Cover, although the techniques underlying our algorithm are very different from those in Marx's algorithm.

2 Preliminaries

An instance of the Maximum Coverage problem is specified by a finite set U, a collection of subsets $\mathcal{R}$, and a positive integer k. We will assume that the input is specified by simply listing the elements of U and those of each set in $\mathcal{R}$; thus, the problem size is $n = \sum_{R \in \mathcal{R}} |R|$. In the Weighted Maximum Coverage problem, we are also given a function $w : U \to \mathbb{R}_+$; the weight of a set $S \subseteq U$ is defined to be $w(S) = \sum_{x \in S} w(x)$ and the goal is to output a k-tuple of elements of $\mathcal{R}$ whose union has maximum weight. We will denote this maximum by $OPT(U)$.

For $A \subseteq U$, we will use the notation $\mathcal{R}|_A$ to denote the collection of all subsets $B \subseteq A$ of the form $B = A \cap R$, where $R \in \mathcal{R}$. The set A is *shattered* by $\mathcal{R}$ if $\mathcal{R}|_A$ is equal to 2^A, the collection of all subsets of A. The VC-dimension of $(U, \mathcal{R})$ is the cardinality of the largest set that is shattered by $\mathcal{R}$. If $(U, \mathcal{R})$ has VC-dimension d and $A \subseteq U$, it is known that $|\mathcal{R}|_A|$ is bounded above by $O\left(|A|^d\right)$.

Our focus will be on Maximum Coverage problems such that $(U, \mathcal{R})$ has bounded VC-dimension. Among these, two special cases of particular interest are Maximum Halfspace Coverage— in which U is a subset of $\mathbb{R}^d$ and each of the sets in $\mathcal{R}$ is the intersection of a halfspace with U — and Maximum Rectangle Coverage, in which U is again a subset of $\mathbb{R}^d$ and each of the sets in $\mathcal{R}$ is obtained by intersecting an axis-parallel rectangle with U.

3 A fixed-parameter approximation scheme

In this section, we work with the *unweighted* Maximum Coverage problem. Following Chazelle and Matoušek, we assume that $\mathcal{R}$ is represented by a *subsystem oracle of dimension d*, defined as follows.

Definition 3.1 ([3])**.** A *subsystem oracle of dimension d* for a set system $(U, \mathcal{R})$ is an algorithm which, given a subset $A \subseteq U$, returns a list of all sets in $\mathcal{R}|_A$ in time $O\left(|A|^{d+1}\right)$; the number of sets in this list must also be bounded above by $O\left(|A|^d\right)$.

The following fact is obvious but useful: a subsystem oracle of dimension d for $(U, \mathcal{R})$ also constitutes a subsystem oracle of dimension d for $(V, \mathcal{R}|_V)$, for every subset $V \subseteq U$.

We define a set $A \subseteq U$ to be an ϵ-approximation of $(U, \mathcal{R})$ if the inequality

$$\left| \frac{|A \cap R|}{|A|} - \frac{|R|}{|U|} \right| \leq \epsilon$$

holds for all $R \in \mathcal{R}$. A crucial ingredient of our approximation scheme is an algorithm, due to Chazelle and Matoušek [3], that computes an ϵ-approximation of cardinality $O\left(d\epsilon^{-2} \log(d/\epsilon)\right)$ for $(U, \mathcal{R})$ in time $O\left(d^{3d}\epsilon^{-2d} \log^d(d/\epsilon)n\right)$, given a subsystem oracle of dimension d for $(U, \mathcal{R})$.

Let $\mathcal{R}^{\cup k}$ denote the collection of all sets $R_1 \cup \cdots \cup R_k$ such that $R_1, \ldots, R_k \in \mathcal{R}$. To apply Chazelle and Matousek's algorithm, we will need a subsystem oracle for $\mathcal{R}^{\cup k}$. The existence of such an oracle is ensured by the following lemma.

Lemma 3.2. *If $(U, \mathcal{R})$ has a subsystem oracle of dimension d, then for all $k > 0$, $(U, \mathcal{R}^{\cup k})$ has a subsystem oracle of dimension kd.*

Proof. The proof is by induction on k, the base case $k = 1$ being trivial. Given subsystem oracles for $(U, \mathcal{R})$ and $(U, \mathcal{R}^{\cup k-1})$ of dimensions d and $(k-1)d$, respectively, the following simple algorithm constitutes a subsystem oracle for $(U, \mathcal{R}^{\cup k})$. First, we use the given two subsystem oracles to generate lists $\mathcal{Q}_1$ and $\mathcal{Q}_{k-1}$, consisting of all sets in $\mathcal{R}|_A$ and $\mathcal{R}^{\cup k-1}|_A$, respectively. Letting $a = |A|$, the induction hypothesis implies that $|\mathcal{Q}_1| = O(a^d)$ and $|\mathcal{Q}_{k-1}| = O(a^{(k-1)d})$, and that the two lists are generated in time $O(a^{d+1})$ and $O(a^{(k-1)d+1})$, respectively. Now, for every pair $B_1 \in \mathcal{Q}_1$ and $B_{k-1} \in \mathcal{Q}_{k-1}$, we form the set $B = B_1 \cup B_{k-1}$ and add it to $\mathcal{Q}_k$. There are $O(a^{kd})$ such pairs, and for each pair the union can be computed in $O(a)$ time, so the algorithm runs in time $O(a^{kd+1})$, as desired. □

As an easy consequence, we derive that the Maximum Coverage problem has a fpt-AS when $(U, \mathcal{R})$ has a bounded-dimensional subsystem oracle *and* the optimum is a constant fraction of $|U|$.

Lemma 3.3. *For any constants $c, \delta > 0$, consider the* Maximum Coverage *problem, restricted to set systems $(U, \mathcal{R})$ having a subsystem oracle of dimension d and satisfying $OPT(U) \geq c|U|$. This special case of the* Maximum Coverage *problem has a $\left(\frac{c-2\delta}{c}\right)$-approximation algorithm with running time bounded by $O(d^{3kd}k^{3kd}\delta^{-2kd-2} \log^{kd+1}(kd/\delta)n)$.*

Proof. The set system $(U, \mathcal{R}^{\cup k})$ has a subsystem oracle of dimension kd, so it is possible to compute a set $A \subseteq U$ which is a δ-approximation to $(U, \mathcal{R}^{\cup k})$, in time $O(d^{3kd}k^{3kd}\delta^{-2kd} \log^{kd}(kd/\delta)n)$. Furthermore, the cardinality of A is $O(kd\delta^{-2} \log(kd/\delta))$. We can solve the Maximum Coverage problem for the set system $(A, \mathcal{R}|_A)$ by brute force. First we call the subsystem oracle to obtain a list of all the sets in $\mathcal{R}|_A$; there are at most $O(k^d d^d \delta^{-2d} \log^d(kd/\delta))$ such sets. Then we enumerate all k-tuples of sets in this list, compute their union, and output the k-tuple

whose union has the largest cardinality. Computing the union of k sets requires $O(k|A|) = O(k^2 d\delta^{-2}\log(kd/\delta))$ time, and multiplying this by the number of k-tuples we obtain an overall running time of $O(k^{kd+2}d^{kd+1}\delta^{-2kd-2}\log^{kd+1}(kd/\delta))$.

Let $R_1, \ldots, R_k$ be sets in $\mathcal{R}$ whose restrictions to A constitute an optimal solution of the Maximum Coverage problem for $(A, \mathcal{R}|_A)$. Let $S_1, \ldots, S_k$ be an optimal solution of the Maximum Coverage problem for $(U, \mathcal{R})$. We have

$$\begin{aligned}
\frac{|(R_1 \cup \cdots \cup R_k) \cap A|}{|A|} &\geq \frac{|(S_1 \cup \cdots \cup S_k) \cap A|}{|A|} \\
\frac{|R_1 \cup \cdots \cup R_k|}{|U|} &\geq \frac{|S_1 \cup \cdots \cup S_k|}{|U|} - 2\delta \\
\frac{|R_1 \cup \cdots \cup R_k|}{|S_1 \cup \cdots \cup S_k|} &\geq 1 - 2\delta\left(\frac{|U|}{|S_1 \cup \cdots \cup S_k|}\right) \geq 1 - \frac{2\delta}{c} = \frac{c-2\delta}{c}
\end{aligned}$$

where the first line follows from the construction of $R_1, \ldots, R_k$, the second line follows from the fact that A is an δ-approximation for for $(U, \mathcal{R}^{\cup k})$, and the third line follows from our assumption that $|S_1 \cup \cdots \cup S_k| = OPT(U) \geq c|U|$. □

For the remainder of this section, we work on eliminating the assumption that $OPT(U) \geq c|U|$. Our plan of attack is to perform a preprocessing step that extracts a subset $V \subseteq U$ such that $OPT(V) \geq (1-\epsilon/3)OPT(U)$ and $OPT(V) \geq c|V|$, for a constant $c = c(\epsilon, k)$ depending only on ϵ and k. Then we will run the algorithm from Lemma 3.3 on $(V, \mathcal{R}|_V)$, using an appropriate choice of $\delta = \delta(\epsilon, k)$, to obtain a $(1-\epsilon)$-approximation to $OPT(U)$.

To design and analyze the preprocessing algorithm that constructs V, we must first define a new problem that we call Constrained Maximum Coverage and analyze a simple greedy algorithm for the problem.

Definition 3.4. An instance of the Constrained Maximum Coverage problem is specified by a universe U and k collections of sets $\mathcal{R}_1, \ldots, \mathcal{R}_k \subseteq 2^U$. A solution of the problem is specified by designating a k-tuple of sets $R_1, \ldots, R_k$ such that $R_i \in \mathcal{R}_i$ for $i = 1, \ldots, k$. The objective is to maximize $|R_1 \cup \cdots \cup R_k|$.

The greedy algorithm for Constrained Maximum Coverage selects $R_1, R_2, \ldots, R_k$, in that order, by choosing R_1 to be the maximum-cardinality set in $\mathcal{R}_1$ and, for $i > 2$, choosing R_i to be the set in $\mathcal{R}_i$ that maximizes $|R_i \setminus (R_1 \cup \cdots \cup R_{i-1})|$.

Note that Maximum Coverage is the special case of constrained maximum k-coverage in which $\mathcal{R}_1 = \cdots = \mathcal{R}_k$, and that the greedy algorithm specializes, in that case, to the familiar greedy algorithm for maximum k-coverage. The approximation ratio of the greedy algorithm for constrained maximum k-coverage is not equal to $1 - \frac{1}{e}$ in general; in fact it is equal to $\frac{1}{2}$. However, for our purposes the following property of the greedy algorithm will be more crucial to the analysis.

Lemma 3.5. *Given an instance of the* Constrained Maximum Coverage *problem, let $R_1, \ldots, R_k$ be the sets selected by the greedy algorithm and let $S_1, \ldots, S_k$ be any other solution. Let $\mathbf{R} = R_1 \cup \cdots \cup R_k$ and $\mathbf{S} = S_1 \cup \cdots \cup S_k$. For every $\delta > 0$, at least one of the following two alternatives holds.*

1. $|\mathbf{R}| \geq (1-\delta)|\mathbf{S}|$.
2. $|\mathbf{S} \setminus \mathbf{R}| < (1-\delta)|\mathbf{S}|$.

Proof. We will construct a one-to-one mapping from $\mathbf{S} \setminus \mathbf{R}$ into $\mathbf{R}$. This suffices to prove the lemma, since either $|\mathbf{S} \setminus \mathbf{R}| < (1-\delta)|\mathbf{S}|$ or $|\mathbf{S} \setminus \mathbf{R}| \geq (1-\delta)|\mathbf{S}|$, and in the latter case our one-to-one mapping will certify that

$$|\mathbf{R}| \geq |\mathbf{S} \setminus \mathbf{R}| \geq (1-\delta)|\mathbf{S}|.$$

To construct the one-to-one mapping, partition $\mathbf{S} \setminus \mathbf{R}$ into k sets $T_1, T_2, \ldots, T_k$, where $T_i = S_i \setminus (\mathbf{R} \cup S_1 \cup S_2 \cup \cdots \cup S_{i-1})$. Note that T_i is a subset of $S_i \setminus (R_1 \cup \cdots \cup R_{i-1})$, hence

$$|T_i| \leq |S_i \setminus (R_1 \cup \cdots \cup R_{i-1})| \leq |R_i \setminus (R_1 \cup \cdots \cup R_{i-1})|,$$

where the second inequality follows from the definition of the greedy algorithm. This means that there is a one-to-one mapping from T_i to $R_i \setminus (R_1 \cup \cdots \cup R_{i-1})$. Combining these one-to-one mappings gives us the desired one-to-one mapping from $\mathbf{S} \setminus \mathbf{R} = \coprod_{i=1}^{k} T_i$ into $\mathbf{R} = \coprod_{i=1}^{k} R_i \setminus (R_1 \cup \cdots \cup R_{i-1})$. □

We now describe and analyze a non-deterministic algorithm to solve Maximum Coverage on a set system $(U, \mathcal{R})$, given a subsystem oracle of dimension d; later we will make the algorithm deterministic. The algorithm proceeds in a sequence of phases numbered $1, \ldots, p = \left\lceil \frac{6}{\epsilon} \ln\left(\frac{3}{\epsilon}\right) \right\rceil$. In each phase q, it chooses a k-tuple of sets $R_1^q, \ldots, R_k^q$. Let

$$\mathbf{R}^q = \bigcup_{i=1}^{q} \bigcup_{j=1}^{k} R_j^i.$$

In phase q, the algorithm computes a set A^q, of cardinality $O(kd\epsilon^{-2}p^2 \log(kdp/\epsilon))$, which is an $(\epsilon/6p)$-approximation to $(\mathbf{R}^{q-1}, \mathcal{R}^{\cup k})$. It non-deterministically guesses a sequence of k subsets $B_1^q, \ldots, B_k^q \subseteq A^q$ and defines set systems $\mathcal{R}_1^q, \ldots, \mathcal{R}_k^q$ as

$$\mathcal{R}_i^q = \{R \in \mathcal{R} \mid R \cap A^q = B_i^q\}, \quad i = 1, \ldots, k.$$

It then selects the sets $R_1^q, \ldots, R_k^q$ using the greedy algorithm for Constrained Maximum Coverage, applied to the universe $U \setminus \mathbf{R}^{q-1}$ with set systems $\mathcal{R}_1^q, \ldots, \mathcal{R}_k^q$. After repeating this process for $p = \left\lceil \frac{6}{\epsilon} \ln\left(\frac{3}{\epsilon}\right) \right\rceil$ phases, it defines $V = \mathbf{R}^p = \bigcup_{i=1}^{p} \bigcup_{j=1}^{k} R_j^i$. Setting $c = 1/p$ and

$$\delta = \frac{\epsilon}{6p}\left(1 + \frac{\epsilon}{3}\right)^{-1},$$

so that $(c - 2\delta)/c \geq 1 - \epsilon/3$, it runs the algorithm of Lemma 3.3 on the set system $(V, \mathcal{R})$ to find a $(1-\epsilon/3)$-approximation to the optimum of the Maximum Coverage problem for $(V, \mathcal{R})$.

We aim to prove that there exists an execution of this non-deterministic algorithm that yields a $(1-\epsilon)$-approximation to the optimum of the Maximum Coverage problem for $(U, \mathcal{R})$. If our algorithm produces a set V satisfying $OPT(V) \geq c|V| = |V|/p$ and $OPT(V) \geq (1-\epsilon/3)OPT(U)$, then Lemma 3.3 ensures that we finish up by producing a $(1-\epsilon/3)$-approximation to $OPT(V)$, which will also be a $(1-\epsilon/3)^2 \geq (1-\epsilon)$-approximation to $OPT(U)$. Proving that $OPT(V) \geq |V|/p$ is easy: $V = \mathbf{R}^p$ is the union of p sets $\mathbf{R}^q \setminus \mathbf{R}^{q-1}$, each of which has cardinality at most $OPT(V)$ since it can be covered by the k sets $R_1^q, \ldots, R_k^q$.

To prove that there exists an execution yielding a set V such that $OPT(V) \geq (1-\epsilon/3)OPT(U)$, we use Lemma 3.5. Let $S_1, \ldots, S_k$ denote an optimal solution of the Maximum Coverage problem for $(U, \mathcal{R})$. Consider the execution in which the algorithm's choice of B_i^q is equal to $S_i \cap A^q$ for every q, i. There are two cases to consider. First, suppose that exists a phase q such that

$$|(R_1^q \cup \cdots \cup R_k^q) \setminus \mathbf{R}^{q-1}| \geq \left(1 - \frac{\epsilon}{6}\right) |(S_1 \cup \cdots \cup S_k) \setminus \mathbf{R}^{q-1}|. \tag{1}$$

Recall that $\mathcal{R}_i^q = \{R \in \mathcal{R} \mid R \cap A^q = B_i^q\}$, and that we are assuming $B_i^q = A^q \cap S_i$. Hence, we have $R_i^q \cap A^q = S_i \cap A^q$ for all i and, consequently, $(R_1^q \cup \cdots \cup R_k^q) \cap A^q = (S_1 \cup \cdots \cup S_k) \cap A^q$. Using the fact that A^q is an $(\epsilon/6p)$-approximation for $(\mathbf{R}^{q-1}, \mathcal{R}^{\cup k})$, we now obtain

$$|(R_1^q \cup \cdots \cup R_k^q) \cap \mathbf{R}^{q-1}| \geq |(S_1 \cup \cdots \cup S_k) \cap \mathbf{R}^{q-1}| - \frac{\epsilon}{6p}|\mathbf{R}^{q-1}|. \tag{2}$$

Letting $\mathbf{S}$ denote $S_1 \cup \cdots \cup S_k$, we sum (1) and (2) to obtain

$$\begin{aligned} |R_1^q \cup \cdots \cup R_k^q| &\geq \left(1 - \frac{\epsilon}{6}\right)|\mathbf{S} \setminus \mathbf{R}^{q-1}| + |\mathbf{S} \cap \mathbf{R}^{q-1}| - \frac{\epsilon}{6p}|\mathbf{R}^{q-1}| \\ &= |\mathbf{S}| - \frac{\epsilon}{6}|\mathbf{S} \setminus \mathbf{R}^{q-1}| - \frac{\epsilon}{6p}|\mathbf{R}^{q-1}|. \end{aligned} \tag{3}$$

Now, as above, $\mathbf{R}^{q-1}$ can be partitioned into sets $\mathbf{R}^i \setminus \mathbf{R}^{i-1}$, $(i = 1, \ldots, q-1)$, each having cardinality at most $OPT(U) = |\mathbf{S}|$. The number of pieces of the partition is $q - 1 < p$, so $\frac{1}{p}|\mathbf{R}^{q-1}| \leq |S|$. Substituting this back into (3), we obtain

$$OPT(V) \geq |R_1^q \cup \cdots \cup R_k^q| \geq \left(1 - \frac{\epsilon}{6} - \frac{\epsilon}{6}\right)|S| = \left(1 - \frac{\epsilon}{3}\right) OPT(U), \tag{4}$$

as desired.

Finally, there remains the case that (1) is not satisfied by any q. Then Lemma 3.5 implies that

$$|(S_1 \cup \cdots \cup S_k) \setminus \mathbf{R}^q| < \left(1 - \frac{\epsilon}{6}\right)|(S_1 \cup \cdots \cup S_k) \setminus \mathbf{R}^{q-1}| \tag{5}$$

for all q. Combining (5) for $q = 1, \ldots, p$, we get that

$$|(S_1 \cup \cdots \cup S_k) \setminus \mathbf{R}^p| < \left(1 - \frac{\epsilon}{6}\right)^p |S_1 \cup \cdots \cup S_k| \leq \frac{\epsilon}{3}|S_1 \cup \cdots \cup S_k|,$$

which implies that

$$|(S_1 \cup \cdots \cup S_k) \cap \mathbf{R}^p| > \left(1 - \frac{\epsilon}{3}\right)|S_1 \cup \cdots \cup S_k|,$$

and hence $OPT(V) \geq (1 - \epsilon/3)OPT(U)$ since $V = \mathbf{R}^p$.

To turn the non-deterministic algorithm into a deterministic one, we simply run every possible execution of the non-deterministic algorithm and output the best answer. An execution of the non-deterministic algorithm is determined by the choice of sets B_i^q, $(1 \leq q \leq p,\ 1 \leq i \leq k)$. Recall that B_i^q must be a subset of A^q and that $|A^q| = O(kd\epsilon^{-2}p^2 \log(kdp/\epsilon))$. Hence if $N(k, d, \epsilon)$ denotes the number of executions of the non-deterministic algorithm, it satisfies

$$\begin{aligned} N(k, d, \epsilon) &= \prod_{q=1}^{p} \prod_{i=1}^{k} 2^{|A_q|} \\ \log N(k, d, \epsilon) &\leq pk \cdot O(kd\epsilon^{-2}p^2 \log(kdp/\epsilon)) \\ &= O(k^2 d\epsilon^{-2}p^3 \log(kdp/\epsilon)) \\ &= \tilde{O}(k^2 d\epsilon^{-5}) \end{aligned}$$

Each iteration runs in time $O(g(k, d, \epsilon) \cdot n)$ where $\log g(k, d, \epsilon) = O(kd \log(kd/\epsilon))$. Hence, the algorithm's overall running time is $O(f(k, d, \epsilon) \cdot n)$ where $\log f(k, d, \epsilon) = \log N(k, d, \epsilon) + \log g(k, d, \epsilon) = \tilde{O}(k^2 d\epsilon^{-5})$.

In deriving this bound on the algorithm's running time, we have assumed that $(U, \mathcal{R})$ has a subsystem oracle of dimension d. If we instead assume that $(U, \mathcal{R})$ has VC dimension d and is represented in the input by simply listing all the elements of $\mathcal{R}$, the running time increases by a factor of n. This is because the trivial implementation of a subsystem oracle — computing $\mathcal{R}_A$ by enumerating each set of $\mathcal{R}$ and intersecting it with A — has running time $O(|A|^{d+1}n)$, n times slower than the bound required by the definition of a subsystem oracle.

4 Bounded Covering Multiplicity

In this section we show that the greedy algorithm gives a $1-(1-1/r)^r$-approximate solution when the *covering multiplicty* of the set system is at most r.

Definition 4.1. An instance of the maximum coverage problem $(U, \mathcal{R}, k)$ has *covering multiplicity* r if for every k-tuple of sets $a_1, \ldots, a_k \in \mathcal{R}$ there exists an optimal solution $(o_1, \ldots, o_k)$ of the maximum coverage problem, with union denoted by OPT, such that each of the sets $a_i \cap OPT$ for $1 \leq i \leq k$ is contained in the union of r elements of $\{o_1, \ldots, o_k\}$.

One of the interesting special cases which satisfies this property is when the cardinality of every set in $\mathcal{R}$ is bounded by r. In Appendix B we prove that it is also satisfied (with $r = 2$) when $U \subset \mathbb{R}^2$ and $\mathcal{R}$ consists of halfspaces in $\mathbb{R}^2$.

Let $g_1, g_2, \ldots, g_k$ be the k sets choosen by the greedy algorithm in the order that they are choosen. Let w be the coverage function and $o_1, o_2, \ldots, o_k$ be the k sets choosen by OPT.

Theorem 4.2. *Greedy algorithm is a* $1-(1-1/r)^r$ *approximation algorithm for* Maximum Coverage *with* covering multiplicity r.

Corollary 4.3. *Greedy algorithm is a* $1-(1-1/r)^r$ *approximation algorithm for* Maximum Coverage *with each set having cardinality at most* r.

4.1 Reduction to a special case

For simplying the analysis we first argue that we can consider the following special case without loss of generality. We take the problem instance on which the greedy algorithm (which we henceforth abbreviate as "greedy") has a given approximation ratio and convert it into a special instance with no better approximation ratio. Then it is enough to analyze the special case.

- The sets chosen by greedy are different from the optimal sets. This assumption can be made as we can just duplicate the sets. Note that this does not change the covering multiplicity.
- The sets chosen by greedy are disjoint. This is because if one defines a new problem instance with $\tilde{g}_i = g_i \setminus \left(\cup_{j=1}^{i-1} g_j\right)$ then the values of the optimal solution and the greedy solution are unchanged. Note that this step uses the fact that the sets chosen by greedy do not belong to the optimal solution. Also note that this does not change the covering multiplicity since we are not modifying any sets in the optimal solution.
- Let $o_1, o_2, \ldots, o_k$ be any optimal solution such that each set $g_i \cap OPT$ $(1 \leq i \leq k)$ is contained in the union of r elements of $\{o_1, \ldots, o_k\}$. We can assume that these sets o_i are pairwise disjoint. This is because we can define a new problem instance in which every point belonging to two or more of the sets in $\{o_1, \ldots, o_k\}$ is assigned to one of those sets and deleted from the others. The values of the greedy and optimal solutions are unchanged. To preserve the property that each set $g_i \cap OPT$ $(1 \leq i \leq k)$ is contained in the union of r elements of $\{o_1, \ldots, o_k\}$, we simply ensure that every element of g_i is assigned to one of those r sets, for all i. This is possible due to our previous assumption that the sets $g_1, \ldots, g_k$ are disjoint.

4.2 Simple case

Consider the simple case $k = t \cdot r$ for some integer t. We will prove the approximation for this special case to get some intuition. We will do it in steps.

- Let $x_i = \sum_{j=(i-1)\cdot t+1}^{i\cdot t} w(g_i)$.
- Let $o_1, o_2, \ldots, o_k$ be the optimal sets in decreasing order of $w(o_i)$.
- Note that $w(g_1) \geq w(o_1)$, $w(g_2) \geq w(o_{r+1})$, $w(g_3) \geq w(o_{2r+1}), \ldots, w(g_t) \geq w(o_{(t-1)\cdot r+1})$. These inequalities use the fact that the covering multiplicity is r and the sets $o_1, \ldots, o_k$ are disjoint. Now summing the t terms we get $\sum_{i=1}^{t} w(g_i) \geq \sum_{i=1}^{t} w(o_{(i-1)r+1}) \geq \frac{1}{r} w(OPT)$.
- Repeating the above step on the residual problem we get $\sum_{i=t+1}^{2t} w(g_i) \geq \frac{OPT - x_1}{r}$. Similarly we get the following series of equations.

$$\forall 1 \leq l \leq r-1, \sum_{i=l\cdot t+1}^{(l+1)t} w(g_i) \geq \frac{OPT - \sum_{i=1}^{l} x_i}{r} \tag{6}$$

- Multiplying (6) by $(1-1/r)^{r-l-1}$ and summing we get $\sum_{i=1}^{i=k} w(g_i) \geq \left(1 - \left(1 - \frac{1}{r}\right)^r\right) w(OPT)$.

4.3 General case

Let $k = t \cdot r + q$ for some $0 \leq q \leq r-1$. We will use the following lemma in the proof.

Lemma 4.4. *$\forall 1 \leq l \leq r, 0 \leq z \leq q \leq r$ we have $r \cdot \sum_{m=0}^{z} \binom{l-1}{m}\binom{r-l}{q-m-1} \geq q \cdot \sum_{m=0}^{z} \binom{l-1}{m}\binom{r-l+1}{q-m}$*

Proof. Consider $p(z) = \frac{\sum_{m=0}^{z} \binom{l-1}{m}\binom{r-l}{q-m-1}}{\sum_{m=0}^{z} \binom{l-1}{m}\binom{r-l+1}{q-m}}$. Consider a random process in which q out of r bins are chosen uniformly at random (without replacement) and a ball is added to each one of the q bins. Now $p(z)$ represents the conditional probability that a ball is in the l^{th} bin, given that at most z bins from the first $l-1$ are chosen. One can easily see that this function should be a decreasing function of z and hence $p(z) \geq p(q) = \frac{q}{r}$. □

Consider r bins and arrange the k greedy sets in the r bins with each bin having either t or $t+1$ greedy sets. Let bin 1 have the first t or $t+1$ sets, bin 2 have the second t or $t+1$ sets, and so on. Let σ be one such arrangement. We will apply inequalities similar to the simpler case. Let $x_i^{\sigma} = \sum_{g_j \in bin_i} w(g_j)$. Let $\sigma(l)$ denote the number of sets in the first l bins. Let x_{qmin}^{t} be the residual value of the q^{th} minimum set among the optimal sets after the first t greedy sets are choosen. Note that x_{qmin}^{t} is a decreasing function of t. Let $B(t)$ denote the set of bins with t sets and $B(t+1)$ denote the set of bins with $t+1$ sets.

- Consider bin l with $t+1$ items. Then $\sum_{g_i \in bin_l} w(g_i) \geq \frac{OPT - \sum_{i=1}^{l-1} x_i^{\sigma} + (r-q) x_{qmin}^{\sigma(l-1)}}{r}$. This inequality is proved similar to inequality 6.
- Consider bin l with t items. Then $\sum_{g_i \in bin_l} w(g_i) \geq \frac{OPT - \sum_{i=1}^{l-1} x_i^{\sigma} - q \cdot x_{qmin}^{\sigma(l-1)}}{r}$. This inequality is proved similar to inequality 6.

Multiplying the equation corresponding to bin l with $(1-1/r)^{r-l}$ and summing we get

$$\begin{aligned} w(greedy) \quad \geq \quad & (1 - (1-1/r)^r w(OPT) \\ & + \sum_{bin_l \in B(t+1)} \frac{r-q}{r} \cdot (1-1/r)^{r-l} x_{qmin}^{\sigma(l-1)} - \sum_{bin_l \in B(t)} \frac{q}{r} \cdot (1-1/r)^{r-l} x_{qmin}^{\sigma(l-1)} \end{aligned} \tag{7}$$

Now taking the average over all arrangements σ we get the following equation.

$$\begin{aligned}
w(greedy) &\geq (1-(1-1/r)^r w(OPT) \\
&\quad + \sum_l \frac{(1-1/r)^{r-l}}{\binom{r}{q}} \left(\frac{r-q}{r} \sum_{w=0}^{l-1} \binom{l-1}{w}\binom{r-l}{q-w-1} x_{qmin}^{(l-1)t+w} - \frac{q}{r} \sum_{w=0}^{l-1} \binom{l-1}{w}\binom{r-l}{q-w} x_{qmin}^{(l-1)t+w} \right) \\
&\geq (1-(1-1/r)^r w(OPT) \\
&\quad + \sum_l \frac{(1-1/r)^{r-l}}{r\cdot\binom{r}{q}} \left(\sum_{w=0}^{l-1} (((r-q)\binom{l-1}{w}\binom{r-l}{q-w-1} - q\binom{l-1}{w}\binom{r-l}{q-w}) x_{qmin}^{(l-1)t+w} \right) \\
&\geq (1-(1-1/r)^r w(OPT) \\
&\quad + \sum_l \frac{(1-1/r)^{r-l}}{r\cdot\binom{r}{q}} \left(\sum_{w=0}^{l-1} ((r\binom{l-1}{w}\binom{r-l}{q-w-1} - q\binom{l-1}{w}\binom{r-l+1}{q-w}) x_{qmin}^{(l-1)t+w} \right) \qquad (8)
\end{aligned}$$

Now using the fact that x_{qmin}^t is a decreasing function of t and Lemma 4.4 we get $w(greedy) \geq (1-(1-1/r)^r)w(OPT)$.

5 Lower bounds

This section considers three different low-dimensional restrictions of Maximum Coverage: set systems of VC-dimension 2, halfspaces in $\mathbb{R}^4$, and axis-parallel rectangles in $\mathbb{R}^2$. In each case, we show that the problem is APX-hard and that the greedy algorithm's approximation ratio, restricted to that special case, is no better than its worst-case approximation ratio, $1-\frac{1}{e}$.

All of these lower bounds are based on the Maximum Vertex Coverage problem, the special case of Maximum Coverage in which each element of U belongs to exactly two sets in $\mathcal{R}$. In this special case, we can identify $\mathcal{R}$ with the vertex set of a graph G, and U with its edge set, such that the endpoints of the edge corresponding to $x \in U$ are the vertices that correspond to the two sets containing x. Thus, Maximum Vertex Coverage can be defined as the problem of choosing k vertices of a graph to maximize the number of edges they cover. The problem is known to be APX-hard [16] and it is known that the approximation ratio of the greedy algorithm, specialized to Maximum Vertex Coverage, is no better than in the general case [5]. In fact, the following lemma shows that the performance of the greedy algorithm does not improve when we further specialize to bipartite instances of Maximum Vertex Coverage.

Lemma 5.1. *For any $\epsilon > 0$, there exist instances of* Maximum Vertex Coverage *in which the graph is bipartite, the instance has a vertex cover of size k, but the output of the greedy algorithm covers only $1-\frac{1}{e}+\epsilon$ fraction of the edges.*

The proof consists of taking a well-known hard example for the greedy Maximum Coverage algorithm, and encoding it in the form of a bipartite graph; the details are given in Appendix A.1.

Theorem 5.2. *Each of the following special cases of* Maximum Coverage *is APX-hard:*
(a) Set systems of VC-dimension $d \geq 2$.
(b) Halfspaces in $\mathbb{R}^d$, $d \geq 4$.
(c) Rectangular ranges in $\mathbb{R}^d$, $d \geq 2$.
Furthermore, the worst-case approximation ratio of the greedy algorithm, when restricted to any of these special cases, is $1-\frac{1}{e}$.

Proof Sketch. The full details of the proof are given in Appendix A.1. Part (a) is a restatment of the known results on Maximum Vertex Coverage. To show Part (b) we embed Maximum Vertex

COVERAGE into halfspaces in $\mathbb{R}^d$, $d \geq 4$ to reconstruct similar results. For Part (c) to show the APX-hardness, we use a reduction from BOUNDED-DEGREE VERTEX COVER, which was shown to be APX-hard by Papadimitriou and Yannakakis [15]. For the statement about the approximation ratio of the greedy algorithm, we use Lemma 5.1. □

An immediate corollary of Theorem 5.2 is the following statement, which justifies that in our fixed-parameter algorithm, the super-polynomial dependence of the running time on k and $1/\epsilon$ is unavoidable.

Corollary 5.3. *Suppose that* MAXIMUM COVERAGE*, specialized to instances with VC-dimension d, has a $(1-\epsilon)$-approximation algorithm with running time $O(f(\epsilon,k,d)\cdot \text{poly}(n))$, for every ϵ, k. If $P \neq NP$, then $f(\epsilon,k,d)$ must be super-polynomial in k. If $P \neq W[1]$, then $f(\epsilon,k,d)$ must be super-polynomial in ϵ^{-1}. In fact, both of these statements hold even if we restrict to $d=2$.*

Proof. The statement that $f(\epsilon,k,2)$ must be super-polynomial in k is a restatement of the APX-hardness of MAXIMUM COVERAGE in VC-dimension 2, which is Part (a) of Theorem 5.2. To prove that $f(\epsilon,k,2)$ must be super-polynomial in ϵ^{-1}, we observe that MAXIMUM COVERAGE, specialized to instances with VC-dimension 2, is a generalization of the $W[1]$-hard PARTIAL VERTEX COVER problem, and that approximating the optimum of PARTIAL VERTEX COVER within a factor of $(1-\epsilon)$, for $\epsilon < 1/|E|$, is equivalent to solving it exactly. □

6 Open Questions

We leave several interesting open questions.

- Improve the running time of our algorithm for sets with bounded VC-dimension.
- Give an algorithm better than $1-(1-1/r)^r$ approximation when the cardinality of each set is bounded by r. Such an algorithm could have a running time exponential in r.
- Resolve the approximability of max-coverage on 3-dimensional halfspaces. We conjecture that local search is a PTAS for the problem. Appendix B.2 presents a proof of the two-dimensional version of this conjecture.

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A Appendix

A.1 Missing Proofs from Section 5

Lemma A.1. *For any $\epsilon > 0$, there exist instances of* Maximum Vertex Coverage *in which the graph is bipartite, the instance has a vertex cover of size k, but the output of the greedy algorithm covers only $1 - \frac{1}{e} + \epsilon$ fraction of the edges.*

Proof. We construct a bipartite graph with edge set $E = \{1, \ldots, kN\}$ (for some sufficiently large N) and vertex set $U \cup W$, where $k = |U| \leq |W|$. We refer to U and W as the *left* and *right* vertex sets, respectively.

Define a sequence of positive integers $n_0, n_1, n_2, \ldots$ by the formula $n_i = \left\lceil N \cdot \left(1 - \frac{1}{k}\right)^i \right\rceil + 1$ and let $s_i = \sum_{j=0}^{i-1} n_j$ denote the sequence of partial sums, interpreting s_0 to be 0. If $r = \min\{i \mid s_i \geq kN\}$ then $W = \{w_1, \ldots, w_r\}$, while $U = \{u_1, \ldots, u_k\}$. The right endpoint of edge j is the unique w_i such that $s_{i-1} < j \leq s_i$, while the left of endpoint of j is the unique u_i such that $i \equiv j \pmod{k}$.

By construction, $|U| = k$ and U is a vertex cover. Each element of U has exactly N elements. However, the greedy algorithm instead chooses vertices $w_1, \ldots, w_k$. To prove this by induction, observe that after choosing $w_1, \ldots, w_i$, the number of remaining uncovered edges is less than $kN\left(1 - \frac{1}{k}\right)^i$, and these edges are consecutively numbered. Each element of U covers a congruence class of edges, and therefore it covers fewer than $N\left(1 - \frac{1}{k}\right)^i + 1$ of the remaining edges, whereas w_{i+1} covers n_i edges and $n_i \geq N\left(1 - \frac{1}{k}\right)^i + 1$. It follows that the greedy algorithm chooses w_{i+1} and this completes the induction step.

The number of edges covered by $w_1, \ldots, w_k$ is bounded above by $2k + N\sum_{i=0}^{k-1}\left(1 - \frac{1}{k}\right)^i = 2k + kN\left[1 - \left(1 - \frac{1}{k}\right)^k\right]$. For k, N sufficiently large, this is less than $\left(1 - \frac{1}{e} + \epsilon\right)kN$. ☐

Theorem A.2. *Each of the following special cases of* Maximum Coverage *is APX-hard:*
(a) Set systems of VC-dimension $d \geq 2$.
(b) Halfspaces in $\mathbb{R}^d$, $d \geq 4$.
(c) Rectangular ranges in $\mathbb{R}^d$, $d \geq 2$.
Furthermore, the worst-case approximation ratio of the greedy algorithm, when restricted to any of these special cases, is $1 - \frac{1}{e}$.

Proof. Recall that Maximum Vertex Coverage can be defined as the instance of Maximum Coverage in which every $x \in U$ belongs to exactly two sets in $\mathcal{R}$. Any such set system $(U, \mathcal{R})$ has VC-dimension at most 2: indeed, if $\mathcal{R}$ shatters a three-element set $\{x, y, z\}$ then there exist sets $R_1, \ldots, R_4$ in $\mathcal{R}$ whose intersections with $\{x, y, z\}$ are the sets $\{x\}$, $\{x, y\}$, $\{x, z\}$, $\{x, y, z\}$, respectively, and consequently x belongs to at least four distinct sets in $\mathcal{R}$. Thus, we see that Maximum Coverage restricted to set systems of VC-dimension d includes Maximum Vertex Coverage as a special case, as long as $d \geq 2$. Part (a) of the theorem now follows from the fact that Maximum Vertex Coverage is APX-hard [16] and from Lemma 5.1.

To prove Part (b) we again show that Maximum Vertex Coverage is a special case. To do so, consider any graph with vertex set $V = \{v_1, \ldots, v_n\}$ and associate to each vertex $v_t \in V$ the vector $b_t = (t, t^2, t^3, t^4, 0, \ldots, 0) \in \mathbb{R}^d$. Define a halfspace $\mathbf{h}_t \subset \mathbb{R}^d$ by the inequality $v_t \cdot x \geq 1$. For every edge (v_r, v_s) we construct a vector $y_{rs} \in \mathbb{R}^d$ that belongs to $\mathbf{h}_r \cap \mathbf{h}_s$ but not to $\mathbf{h}_t$ for any $t \neq i, j$. The construction is as follows. First, write the polynomial $(z - r)^2(z - s)^2$ in the form $\sum_{i=0}^{4} a_i z^i$, and then put

$$y_{rs} = -\frac{1}{a_0}(a_1, a_2, a_3, a_4, 0, \ldots, 0) \in R^d.$$

The inequality $y_{rs} \cdot v_t \geq 1$ can be rewritten as $-\sum_{i=1}^{4} a_i t^i \geq a_0$ (using the fact that $a_0 = r^2 s^2 > 0$) and it follows that the inequality is satisfied only when $(t-r)^2(t-s)^2 \leq 0$, i.e. only when $t \in \{r, s\}$. Thus, the set system defined by the vectors $\{y_{rs}\}$ and the halfspaces $\{\mathbf{h}_t\}$ is identical to the MAXIMUM VERTEX COVERAGE instance defined by G.

To prove Part (c), we specialize to rectangular ranges in $\mathbb{R}^2$. (The case of rectangular ranges in $\mathbb{R}^d$, $d > 2$ follows *a fortiori*.) To begin with, we observe that every *bipartite* instance of MAXIMUM VERTEX COVERAGE can be represented using axis-parallel rectangles in $\mathbb{R}^2$. The construction is as follows. If we label the vertices of the bipartite graph as $\{u_1, \ldots, u_p, w_1, \ldots, w_q\}$ such that every edge has one endpoint in $\{u_1, \ldots, u_p\}$ and the other endpoint in $\{w_1, \ldots, w_q\}$, then we can represent edge (u_i, w_j) using the point $(2i, 2j) \in \mathbb{R}^2$. Vertex u_i is represented by the rectangle $[2i-1, 2i+1] \times [1, 2q+1]$ and vertex w_j by the rectangle $[1, 2p+1] \times [2j-1, 2j+1]$. This construction, combined with Lemma 5.1, suffice to show that the greedy algorithm has worst-case approximation ratio $1 - \frac{1}{e}$ when specialized to rectangular ranges in $\mathbb{R}^2$. To prove APX-hardness, we need to use a different reduction that is based on bounded-degree graphs rather than bipartite graphs. We use the following theorem from [15]: there exists a constant Δ such that VERTEX COVER, restricted to graphs of maximum degree Δ, is APX-hard.

For any graph G, create an instance of MAXIMUM COVERAGE as follows. Assuming that G has vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_m\}$. For each edge e_k with endpoints v_i, v_j, the set $U \subset \mathbb{R}^2$ contains the three points $(6k-2, 2i)$, $(6k, 0)$, $(6k+2, 2j)$. These $3m$ points constitute the entire set U. The rectangles in $\mathcal{R}$ are as follows. For each edge e_k there are two rectangles $\mathbf{r}_1(e_k) = [6k-3, 6k+1] \times [-1, n+1]$ and $\mathbf{r}_2(e_k) = [6k-1, 6k+3] \times [-1, n+1]$. For each vertex v_i there is one rectangle $\mathbf{r}(v_i) = [0, 6m+3] \times [2i-1, 2i+1]$. If G has a vertex cover C of size s, then there is a set of $m+s$ rectangles in $\mathcal{R}$ that cover all the points in U: we take rectangle $\mathbf{r}(v_i)$ for each $v_i \in C$, this covers at least one of the points $(6k-2, 2i)$, $(6k+2, 2j)$ for each edge e_k and the remaining two points corresponding to that edge can be covered using either $\mathbf{r}_1(e_k)$ or $\mathbf{r}_2(e_k)$. Conversely, if U can be covered by $m+s$ elements of $\mathcal{R}$, then the covering must have a subcollection of m rectangles that contains one of the rectangles $\mathbf{r}_1(e_k), \mathbf{r}_2(e_k)$ for each k. Let T be the subset of U that is not covered by this subcollection, and let C be the set of all vertices v_i such that T contains a point whose y-coordinate is $2i$. It is easy to see that C is a vertex cover of G, and $|C| \leq s$.

Now let $\epsilon > 0, \Delta < \infty$ be chosen such that it is NP-hard to distinguish between graphs of maximum degree Δ having a vertex cover of size s (henceforth, *yes instances*) and those having no vertex cover of size less than $(1+\epsilon)s$ (*no instances*). If G is a yes instance, then the corresponding MAXIMUM COVERAGE instance with parameter $k = m+s$ has optimum value $3m$. If G is a no instance, then the corresponding *MaximumCoverage* instance with parameter $k = m+s$ has optimum value at most $3m - \epsilon s$. Indeed, if there exist $m+s$ rectangles that cover more than $3m - \epsilon s$ points, then it is trivial to find fewer than $m + (1+\epsilon)s$ rectangles that cover all $3m$ points, which is impossible if G is a no instance. If it possible for a graph with m edges and maximum degree Δ to have a vertex cover of size s then $s \geq m/\Delta$. Thus, we have shown that it a yes instance of VERTEX COVER maps to a MAXIMUM COVERAGE instance whose optimum value is $3m$ while a no instance maps to one whose optimum value is at most $(3 - \epsilon/\Delta)m$, implying the claimed APX-hardness. □

B Two-dimensional halfspaces

Theorem 5.2 rules out the possibility of designing a PTAS for MAXIMUM COVERAGE specialized to halfspaces in $\mathbb{R}^d$ for $d \geq 4$ (unless P=NP) and it likewise rules out the possibility of proving an approximation ratio better than $1 - \frac{1}{e}$ for the greedy algorithm. But in very low dimensions, the

situation is different. When $d = 1$, it is easy to see that the greedy algorithm itself always computes an optimal solution. When $d = 2$, a dynamic programming algorithm due to Har-Peled and Lee [8] computes an optimal solution in polynomial time. (The algorithm given in that paper is for SET COVER rather than MAXIMUM COVERAGE, but a trivial modification of their algorithm solves MAXIMUM COVERAGE.) Despite the existence of a polynomial-time algorithm for two-dimensional MAXIMUM HALFSPACE COVERAGE, it is interesting to investigate the approximation ratio of some other archetypical algorithms for this problem, especially since this investigation may shed light on the approximability of three-dimensional MAXIMUM HALFSPACE COVERAGE, which is NP-hard [6] and hence the two-dimensional dynamic programming algorithm is unlikely to generalize. In this section, we show that when $d = 2$ the greedy algorithm has approximation ratio 3/4, and there is a natural local search algorithm yielding a PTAS.

B.1 Analysis of the greedy algorithm

To analyze the greedy algorithm for MAXIMUM HALFSPACE COVERAGE in two dimensions we prove that the *covering multiplicity* r of the problem instance is 2. Then by Theorem 4.2 we have that greedy algorithm is a factor $\frac{3}{4}$ approximation algorithm for MAXIMUM HALFSPACE COVERAGE in two dimensions.

Lemma B.1. *The* covering multiplicity *for* MAXIMUM HALFSPACE COVERAGE *in two dimensions is* 2*.*

Proof. The proof is a series of simple observations.

(a) Without loss of generality we can assume that no set belongs to both the optimal solution and the given solution G. This is because otherwise we can duplicate the set and treat one copy as belonging to the optimal solution while the other belongs to the given solution.

(b) Consider the optimal solution which has the maximum number of sets in common with the duplicates created in the previous step.

(c) In the optimal solution O considered above, for every other set $s \notin O$, there are two sets $o_1, o_2 \in O$ such that every element of s that is covered by O belongs to $o_1 \cup o_2$. Otherwise, using the fact that this is a two-dimensional MAXIMUM HALFSPACE COVERAGE instance, we can see that one of the previous two conditions is violated.

□

Theorem B.2. *The greedy algorithm for two-dimensional* MAXIMUM HALFSPACE COVERAGE *has approximation ratio 3/4.*

Proof. The proof follows easily from Lemma B.1 and Theorem 4.2. □

Example B.3. The following example shows that the analysis of the greedy algorithm is tight. Consider the set system $s_1 = \{p_1, p_2\}$,$s_2 = \{p_3, p_4\}$ and $s_3 = \{p_1, p_3\}$ with $k = 2$. Then it should be simple to see that this can be realized an a two-dimensional instance of MAXIMUM HALFSPACE COVERAGE. One choice for optimal sets is s_1, s_2 with value of 4. One possible output for the greedy algorithm is s_3, s_1 with value 3. This gives an approximation of 3/4.

B.2 A PTAS via local search

If $(U, \mathcal{R}, k)$ is an instance of Maximum Coverage and $S = \{R_1, \ldots, R_k\}$ is a solution, define a t-swap to be the operation of transforming this solution into another solution $S' = \{R'_1, \ldots, R'_k\}$ such that there are at most t sets belonging to L but not L', and vice-versa. If $(U, \mathcal{R}, k)$ is a two-dimensional instance of Maximum Halfspace Coverage and $S = \{R_1, \ldots, R_k\}$ is a solution, define $area(S) \subseteq \mathbb{R}^2$ to be the set $\bigcup_{i=1}^{k} \mathbf{h}_i$, where $\mathbf{h}_i$ is the halfspace corresponding to R_i.

In this section we analyze the following local search algorithm. We assume an *unweighted* instance of two-dimensional Maximum Halfspace Coverage, i.e. an instance in which each element has weight 1.

1. Start with a arbitrary k-tuple of sets S.
2. While possible do a t-swap to improve the number of elements covered.
3. If there exists a 1-swap to obtain a solution S' such that $area(S) \subseteq area(S')$ and $area(S) \neq area(S')$ then perform this 1-swap and go to step 2. Otherwise terminate the algorithm.

It is simple to see that step 3 does not run for more than n times without the solution improving because once a set is deleted from S in step 3, the only event that can re-insert it is a t-swap in step 2.

We will prove that this local search algorithm achieves an approximation ratio of $2t/(2t+1)$. (This implies that we can obtain a PTAS with running time $n^{O(1/\epsilon)}$ by setting $t = 1/\epsilon$.) The proof of the approximation ratio is in two steps. We first assume the existence of a certain chain decomposition and prove that this implies a $2t/(2t+1)$ approximation ratio. Then we construct such a chain decomposition.

B.2.1 Approximation ratio assuming chain decomposition

For succinctness, we will refer to the sets in the optimal solution and in the output of the local search algorithm as *opt sets* and *local sets*, respectively. Opt sets will be denoted by o_i and local sets by ℓ_i. If L_i is a subcollection of the local sets, we will frequently use the notation $v(L_i)$ to denote the set of elements covered by L_i but not by any of the other local sets, i.e.

$$v(L_i) = \left(\bigcup_{\ell_j \in L_i} \ell_j \right) - \left(\bigcup_{\ell_j \notin L_i} \ell_j \right).$$

Let the opt sets and local sets be grouped into chains $C_1, ..., C_l$ such that the following properties are satisfied.

- the local and opt sets alternate (cyclically) in a chain C_i.
- Consider a portion of any chain (cyclically) $...o_1\ell_1o_2...\ell_to_{t+1}....$ Let $L = \{\ell_1, \ell_2, ..., \ell_t\}$, $O_1 = \{o_1, o_2, ..., o_t\}$ and $O_2 = \{o_2, o_3, ...o_{t+1}\}$ Then $v(L) \cap OPT \subseteq v(L) \cap (O_1 \cup O_2)$.

Consider L_i and O_i each having same number of sets and at most t sets. We derive some equations based on local optimality.

$$w(v(L_i) - OPT) + w(v(L_i) \cap OPT) \geq w(O_i - local) + w(O_i \cap v(L_i)) \quad (9)$$

$$\Rightarrow w(v(L_i) - OPT) + w(v(L_i) \cap (OPT - O_i)) \geq w(O_i - local) \quad (10)$$

Now we find sets used in equation (9) and then add these equations to get the desired result.

- Consider chain C_i. If number of local sets in C_i is $\leq t$, then let L_i be the collection of all local sets in C_i and let O_i be the collection of all opt sets in C_i. Make $2t$ such copies, i.e. the same equation will be used $2t$ times in the proof.
- If chain C_i chain has more than t local sets, then let L_i be any t consecutive local sets, and let O_i be the opt sets in the chain which are shifted from L_i by 1 either clockwise or counterclockwise. Note that a particular choice for L_i appears twice since there are two options for O_i.

Here are some properties of the above decomposition.

1. Each ℓ_i belongs to $2t$ of the sets L_j
2. Each o_i belongs to $2t$ of the sets O_j
3. Let L_i be associated with O_j and O_k. Then $v(L_i) \cap OPT \subseteq v(L_i) \cap (O_j \cup O_k)$. This is just a restatement of the assumed property of the chain decomposition.

Based on the above properties we derive the the final inequality. Sum the equation (9) over all L_i,O_i. Then we bound each term in the sum. Let OPT denote the set of elements covered by the opt sets, and let LOC denote the set of elements covered by the local sets.

- $\sum w(v(L_i) - OPT) \leq 2t \cdot w(LOC - OPT)$. This is due to property 1.
- $\sum w(O_i - LOC) \geq 2t \cdot w(OPT - LOC)$. This is due to property 2. Note the difference in the direction of inequalities.
- $\sum w(v(L_i) \cap (OPT - O_i)) \leq w(LOC \cap OPT)$. This is due to property 3.

From the above three equations and equation (9) we get the final necessary equation.

$$\begin{aligned}
2t \cdot w(LOC - OPT) + w(LOC \cap OPT) &\geq 2t \cdot w(OPT - LOC) \\
\Rightarrow 2t \cdot w(LOC - OPT) + (2t+1) \cdot w(LOC \cap OPT) &\geq 2t \cdot w(OPT) \\
\Rightarrow (2t+1) \cdot w(LOC - OPT) + (2t+1) \cdot w(LOC \cap OPT) &\geq 2t \cdot w(OPT) \\
\Rightarrow (2t+1) w(LOC) &\geq 2t \cdot w(OPT) \\
\Rightarrow w(LOC) &\geq \tfrac{2t}{2t+1} \cdot w(OPT) \qquad (11)
\end{aligned}$$

B.2.2 Obtaining a chain decomposition

We argue about some properties of two-dimensional Maximum Halfspace Coverage instances, based on which we get some associations. Consider the optimal solution which has the maximum number of sets in common with the output of the local search algorithm.

1. For each ℓ_i we have that $\exists o_j, o_l$ such that $\ell_i \cap OPT \subseteq o_j \cup o_l$. It is simple to see that if this is not true then we can change the optimal solution so that the number of sets in common with the local optimum increases. Now associate ℓ_i to the corresponding o_j and o_l. Let $A^{\text{init}}(o_i)$ be the local sets associated with o_i and $A^{\text{init}}(\ell_i)$ be the opt sets associated with ℓ_j.
2. For each o_i we have that $\exists \ell_j, \ell_t \in A^{\text{init}}(o_i)$ such that $o_i \cap A^{\text{init}}(o_i) \subseteq \ell_j \cup \ell_t$. This is true due to the different form of local search used. Because otherwise we can change the local optimum to increase the area. Now if $A^{\text{init}}(o_i)$ has more than two sets ℓ_j's. Then among them choose two ℓ_j, ℓ_t such that $o_i \cap A^{\text{init}}(o_i) \subseteq \ell_j \cup \ell_t$ and keep the association and remove the rest of the associations for o_i. Let the new associations be called $A^{\text{new}}(\ell_i)$ and $A^{\text{new}}(o_j)$.

3. Note that in step 2 we remove some associations. Hence it might no longer be true that $\ell_i \cap OPT \subseteq A^{\text{new}}(\ell_i)$. But also note that it is still true that $o_i \cap A^{\text{init}}(o_i) \subseteq A^{\text{new}}(o_i)$.

4. Note that due to step 1,2 we have that each local set is associated to at most two opt set and each opt set is associated to at most two local sets.

5. Now form maximal alternating pseudo chains such that a pseudo chain is a list of alternating local and optimal sets. Additionally each local set has the association to its adjacent sets and each opt set has association to its adjacent sets.

6. Now there are three kinds of pseudo chain depending on their end points. They are either $l-l$ or $o-o$ or $o-l$ (here $l-l$ means a chain starting with a local set and ending with a local set).

7. merge $l-l$ pseudo chain arbitrarily with $o-o$ to get only $o-l$ chain.

The $o-l$ chain thus got are the chain we desired in the analysis. It is left to be proven that this decomposition satisfies the properties needed.

- By construction the local and opt sets alternate (cyclically) in a chain C_i.
- Consider a portion of the chain(cyclically) $...o_1\ell_1o_2...\ell_to_{t+1}....$ Let $L = \{\ell_1, \ell_2, ..., \ell_t\}$, $O_1 = \{o_1, o_2, ..., o_t\}$ and $O_2 = \{o_2, o_3, ...o_{t+1}\}$. Then we need to prove that $v(L) \cap OPT \subseteq v(L) \cap (O_1 \cup O_2)$. The proof is by contradiction. ie. Let $x \in v(L) \cap OPT$ but $x \notin O_1 \cup O_2$. Then $x \in \ell_c \in L$. We follow through the associations.
  - Consider the initial association. Then by its property $o_j \in A^{\text{init}}(\ell_c)$ such that $x \in o_j$.
  - If $o_j \in A^{\text{new}}(\ell_c)$ then o_j is adjacent to ℓ_c in the chain and hence $o_j \in \{o_1, o_2, ..., o_{t+1}\}$ which is a contradiction to the fact that $x \notin O_1 \cup O_2$.
  - If $o_j \notin A^{\text{new}}(\ell_c)$ then in step 2 of associations $o_j \cap A^{\text{init}}(o_j) \subseteq A^{\text{new}}(o_j)$. Hence $\ell_d \in A^{\text{new}}(o_j)$ such that $x \in \ell_d$ and o_j and ℓ_d are adjacent in some chain. If $\ell_d \in \{\ell_1, \ell_2, ..., \ell_t\}$ then $o_j \in \{o_1, o_2, ..., o_{t+1}\}$ and we arrive at a contradiction that $x \notin O_1 \cup O_2$. Otherwise $x \notin v(L)$ and we still arrive at a contradiction.