Online Appendix to: Simulating Lévy Processes from Their Characteristic Functions and Financial Applications

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A. PROOF OF THEOREM 4.1

PROOF. The bias is given by

$$\begin{aligned}
\mathrm{E}[f(X)] - \mathrm{E}[f(\hat{X})] &= \int_{\mathrm{R}} f(x)p(x)dx - \int_{\mathrm{R}} f(x)\hat{p}(x)dx \\
&= \int_{\mathrm{R}} f(x)p(x)dx - \int_{x_0}^{x_K} f(x)\hat{p}(x)dx - f(x_0)\hat{F}_0 - f(x_K)(1-\hat{F}_K) \\
&= \sum_{k=1}^{K} \int_{x_{k-1}}^{x_k} f(x)(p(x)-\hat{p}(x))dx + \left(\int_{-\infty}^{x_0} + \int_{x_K}^{\infty}\right) f(x)p(x)dx \\
&\quad - f(x_0)\hat{F}_0 - f(x_K)\left(1-\hat{F}_K\right). \qquad (22)
\end{aligned}$$

If $f(x)$ is differentiable in (x_{k-1}, x_k), then for any $x \in (x_{k-1}, x_k)$, by the mean value theorem, there exists $\xi_k(x) \in (x_{k-1}, x)$ such that

$$\begin{aligned}
\left|\int_{x_{k-1}}^{x_k} f(x)(p(x)-\hat{p}(x))dx\right| &= \left|\int_{x_{k-1}}^{x_k} (f(x_{k-1}+) + f'(\xi_k(x))(x-x_{k-1}))(p(x)-\hat{p}(x))dx\right| \\
&\leq ||f||_{\mathcal{X}} \cdot |F_k - \hat{F}_k - \left(F_{k-1} - \hat{F}_{k-1}\right)| + ||f'||_{\mathcal{X}} \cdot \eta \cdot \int_{x_{k-1}}^{x_k} |p(x)-\hat{p}(x)|dx \\
&\leq 2 \cdot ||f||_{\mathcal{X}} \cdot E_{\mathcal{X}} + ||f'||_{\mathcal{X}} \cdot \eta \cdot \int_{x_{k-1}}^{x_k} |p(x)-\hat{p}(x)|dx.
\end{aligned}$$

Here $f(x_{k-1}+)$ is the right limit of f at x_{k-1}, which is finite by the assumptions. In general, if $f(x)$ is not differentiable at n_k^f points in (x_{k-1}, x_k), where $\sum_{k=1}^{K} n_k^f = n^f$, it can be shown in the same way as before that

$$\left|\int_{x_{k-1}}^{x_k} f(x)(p(x)-\hat{p}(x))dx\right| \leq 2\left(n_k^f + 1\right) \cdot ||f||_{\mathcal{X}} \cdot E_{\mathcal{X}} + ||f'||_{\mathcal{X}} \cdot \eta \cdot \int_{x_{k-1}}^{x_k} |p(x)-\hat{p}(x)|dx.$$

Note that

$$\begin{aligned}
\int_{x_{k-1}}^{x_k} |p(x)-\hat{p}(x)|dx &= \int_{x_{k-1}}^{x_k} \left|p(x) - \frac{F_k - F_{k-1}}{\eta} + \frac{F_k - F_{k-1}}{\eta} - \frac{\hat{F}_k - \hat{F}_{k-1}}{\eta}\right| dx \\
&\leq \left|F_k - \hat{F}_k - \left(F_{k-1} - \hat{F}_{k-1}\right)\right| + \int_{x_{k-1}}^{x_k} \left|p(x) - \frac{F_k - F_{k-1}}{\eta}\right| dx \\
&\leq 2E_{\mathcal{X}} + \int_{x_{k-1}}^{x_k} \left|p(x) - \frac{F_k - F_{k-1}}{\eta}\right| dx.
\end{aligned}$$

DOI 10.1145/2331140.2331142 http://doi.acm.org/10.1145/2331140.2331142

However, we have the following:

$$\int_{x_{k-1}}^{x_k} \left| p(x) - \frac{F_k - F_{k-1}}{\eta} \right| dx = \frac{1}{\eta} \int_{x_{k-1}}^{x_k} \left| \int_{x_{k-1}}^{x_k} (p(x) - p(y)) dy \right| dx.$$

Since $p(x)$ is differentiable in (x_{k-1}, x_k), using the mean value theorem again, for any $x \neq y$ in (x_{k-1}, x_k), there exists $\xi_k(x, y) \in (x, y)$ such that

$$p(y) = p(x) + p'(\xi_k(x, y))(y - x).$$

Therefore,

$$\int_{x_{k-1}}^{x_k} |p(x) - \hat{p}(x)| dx \leq 2E_{\mathcal{X}} + ||p'||_{\mathcal{X}} \cdot \eta^2.$$

Consequently,

$$\left| \sum_{k=1}^{K} \int_{x_{k-1}}^{x_k} f(x)(p(x) - \hat{p}(x)) dx \right| \leq 2E_{\mathcal{X}} \left(\left(K + n^f\right) ||f||_{\mathcal{X}} + ||f'||_{\mathcal{X}} (x_K - x_0) \right) + ||f'||_{\mathcal{X}} ||p'||_{\mathcal{X}} (x_K - x_0) \eta^2.$$

As for the last two terms in (22), we have

$$|f(x_0)\hat{F}_0| \leq |f(x_0)|(E_{\mathcal{X}} + F_0), \quad |f(x_K)\left(1 - \hat{F}_K\right)| \leq |f(x_K)|(E_{\mathcal{X}} + 1 - F_K).$$

Combining the preceding, we obtain (15). □

B. PROOF OF THEOREM 4.3

PROOF. Let F be the cdf of X. According to Theorem 4.1, the bias is bounded by (15). If $\phi \in H(\mathcal{D}_{(d_-, d_+)})$, from (7), for any $a \in (0, d_+)$, we have

$$f(x_0)F(x_0) = f(x_0) \int_{\mathrm{R}} \frac{e^{-ix_0(\xi + ia)}\phi(\xi + ia)}{-2\pi i(\xi + ia)} d\xi = f(x_0) \int_{-\infty + ia}^{+\infty + ia} \frac{e^{-ix_0 z}\phi(z)}{-2\pi i z} dz.$$

Since the integrand is analytic in $\{z \in \mathrm{C} : \Im(z) \in (0, d_+)\}$, using the condition (1) and Cauchy's integral theorem, for any $\epsilon > 0$ such that $d_+ - \epsilon > a$, we have

$$f(x_0)F(x_0) = f(x_0) \int_{-\infty + i(d_+ - \epsilon)}^{+\infty + i(d_+ - \epsilon)} \frac{e^{-ix_0 z}\phi(z)}{-2\pi i z} dz = f(x_0) e^{(d_+ - \epsilon)x_0} \int_{\mathrm{R}} \frac{e^{-ix_0\xi}\phi(\xi + i(d_+ - \epsilon))}{-2\pi i(\xi + i(d_+ - \epsilon))} d\xi.$$

Let $\epsilon \downarrow 0$, we obtain

$$|f(x_0)F(x_0)| \leq \frac{||\phi||^+}{2\pi d_+} |f(x_0)| e^{x_0 d_+}.$$

Note that the probability density $p(x)$ admits the following inverse Fourier transform representation:

$$p(x) = \frac{1}{2\pi} \int_{\mathrm{R}} e^{-izx}\phi(z) dz.$$

By Cauchy's integral theorem and the condition (1), for any $\epsilon > 0$ such that $d_+ - \epsilon > 0$, we have

$$p(x) = \frac{1}{2\pi} \int_{-\infty + i(d_+ - \epsilon)}^{+\infty + i(d_+ - \epsilon)} e^{-izx}\phi(z) dz = e^{(d_+ - \epsilon)x} \int_{\mathrm{R}} \frac{1}{2\pi} e^{-ix\xi}\phi(\xi + i(d_+ - \epsilon)) d\xi.$$

Consequently,

$$\int_{-\infty}^{x_0} |f(x)| p(x) dx \leq \frac{||\phi||^+}{2\pi} \int_{-\infty}^{x_0} |f(x)| e^{x d_+} dx.$$

Similarly, using the representation (6), we have the following:

$$|f(x_K)|(1-F(x_K)) \leq \frac{||\phi||^-}{2\pi|d_-|}|f(x_K)|e^{x_K d_-},$$

$$\int_{x_K}^{\infty}|f(x)|p(x)dx \leq \frac{||\phi||^-}{2\pi}\int_{x_K}^{\infty}|f(x)|e^{xd_-}dx.$$

Since $\xi\phi(\xi)$ is absolutely integrable on R by the assumptions on ϕ,

$$p'(x) = \frac{1}{2\pi}\frac{d}{dx}\int_{\mathrm{R}} e^{-i\xi x}\phi(\xi)d\xi = \frac{1}{2\pi i}\int_{\mathrm{R}} e^{-i\xi x}\xi\phi(\xi)d\xi,$$

where the interchange of the integration and differentiation is valid due to the dominated convergence theorem. Therefore,

$$||p'||_{\mathcal{X}} \leq \frac{1}{2\pi}\int_{\mathrm{R}}|\xi\phi(\xi)|d\xi.$$

Combining the preceding, we obtain the bound for the bias in (16). □

C. PROOF OF THEOREM 4.5

PROOF. Denote the density of $\hat{F}^{(i)}$ by $\hat{p}_i$. From Theorem 4.1, we have the following for any $1 \leq i \leq d$:

$$\int_{\mathrm{R}} g_i(x)\hat{p}_i(x)dx \leq ||p_i g_i||_1 + B_i,$$

$$\left|\int_{\mathrm{R}} f(x_1,\cdots,x_d)(p_i(x_i)-\hat{p}_i(x_i))dx_i\right| \leq B_i \prod_{j=1,\,j\neq i}^{d} g_j(x_j).$$

Following Glasserman and Liu [2010], $|\mathrm{E}[f(X_1,\cdots X_d)] - \mathrm{E}[f(\hat{X}_1,\cdots,\hat{X}_d)]|$ is bounded by the following:

$$\begin{aligned}
&\left|\mathrm{E}\left[f(X_1,X_2,\cdots,X_{d-1},X_d)\right] - \mathrm{E}\left[f\left(X_1,X_2,\cdots,X_{d-1},\hat{X}_d\right)\right]\right| \\
&+\left|\mathrm{E}\left[f\left(X_1,X_2,\cdots,X_{d-1},\hat{X}_d\right)\right] - \mathrm{E}\left[f\left(X_1,X_2,\cdots,\hat{X}_{d-1},\hat{X}_d\right)\right]\right| + \\
&\vdots \\
&+\left|\mathrm{E}\left[f\left(X_1,\hat{X}_2,\cdots,\hat{X}_{d-1},\hat{X}_d\right)\right] - \mathrm{E}\left[f\left(\hat{X}_1,\hat{X}_2,\cdots,\hat{X}_{d-1},\hat{X}_d\right)\right]\right| \\
\leq\ & B_d\prod_{j=1}^{d-1}||p_j g_j||_1 + B_{d-1}\prod_{j=1}^{d-2}||p_j g_j||_1||\hat{p}_d g_d||_1 + \cdots + B_1\prod_{j=2}^{d}||\hat{p}_j g_j||_1 \\
\leq\ & B\left(\prod_{j=1}^{d-1}||p_j g_j||_1 + \prod_{j=1}^{d-2}||p_j g_j||_1\left(||p_d g_d||_1 + B\right) + \cdots + \prod_{j=2}^{d}\left(||p_j g_j||_1 + B\right)\right),
\end{aligned}$$

where $||\hat{p}_i g_i||_1 = \int_{\mathrm{R}}\hat{p}_i(x)g_i(x)dx$, $1 \leq i \leq d$. The conclusion then follows immediately. □

D. ALGORITHM 3

ALGORITHM 3: Simulating Kou's double exponential jump diffusion

For $t > 0$, simulate $X_t = \mu t + \sigma B_t + \sum_{i=1}^{N_t} Z_i$

(1) Generate a standard normal random variable G.
(2) Generate the Poisson process N_t using the inverse transform method described on p.128 of Glasserman [2004].
(3) Generate Z_i, $1 \leq i \leq N_t$, in the following way: generate U_i that is uniform in $(0, 1)$; if $U_i < p$, generate Z_i from an $\exp(\eta_1)$ distribution; otherwise, generate Z_i from the negative of an $\exp(\eta_2)$ distribution.
(4) Let $X_t = \mu t + \sigma \sqrt{t} G + Z_1 + \cdots + Z_{N_t}$.