

# Sparse Differential Resultant for Laurent Differential Polynomials\*

Wei Li, Chun-Ming Yuan, Xiao-Shan Gao<sup>†</sup>  
KLMM, Academy of Mathematics and Systems Science  
Chinese Academy of Sciences, Beijing 100190, China  
Email: {liweili,cmeyuan,xgao}@mmrc.iss.ac.cn

## Abstract

In this paper, we first introduce the concept of Laurent differentially essential systems and give a criterion for Laurent differentially essential systems in terms of their supports. Then the sparse differential resultant for a Laurent differentially essential system is defined and its basic properties are proved. In particular, order and degree bounds for the sparse differential resultant are given. Based on these bounds, an algorithm to compute the sparse differential resultant is proposed, which is single exponential in terms of the number of indeterminates, the Jacobi number of the system, and the size of the system.

**Keywords.** Sparse differential resultant, Laurent differential polynomial, differentially essential system, Jacobi number, differential toric variety, Poisson-type product formula, BKK bound, differential dimension conjecture, single exponential algorithm.

**Mathematics Subject Classification [2000].** Primary 12H05; Secondary 14M25, 14Q99, 68W30.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries</b>	<b>7</b>
2.1	Differential polynomial algebra and Kolchin topology . . . . .	7
2.2	Characteristic sets of a differential polynomial system . . . . .	9
<b>3</b>	<b>Sparse differential resultant for Laurent differential polynomials</b>	<b>10</b>
3.1	Laurent differential polynomial . . . . .	10
3.2	Definition of sparse differential resultant . . . . .	12
3.3	Necessary and sufficient condition for existence of non-polynomial solutions .	16

---

\*Partially supported by a National Key Basic Research Project of China (2011CB302400) and by grants from NSFC (60821002,11101411). Part of the results in this paper was reported in ISSAC 2011 [37] and received the Distinguished Paper Award.

<sup>†</sup>To whom correspondence should be addressed.

<b>4</b>	<b>Criterion for Laurent differentially essential system in terms of supports</b>	<b>17</b>
4.1	Sets of Laurent differential monomials in reduced and T-shape forms . . . . .	18
4.2	An algorithm to reduce Laurent differential monomials to T-shape . . . . .	22
4.3	Differential transcendence degree of generic Laurent differential polynomials .	28
<b>5</b>	<b>Basic properties of sparse differential resultant</b>	<b>32</b>
5.1	Sparse differential resultant is differentially homogeneous . . . . .	32
5.2	Order bound in terms of Jacobi number . . . . .	34
5.3	Differential toric variety and sparse differential resultant . . . . .	40
5.4	Poisson-type product formulas . . . . .	43
5.5	Structures of non-polynomial solutions . . . . .	46
5.6	Sparse differential resultant for differential polynomials with non-vanishing degree zero terms . . . . .	50
<b>6</b>	<b>A single exponential algorithm to compute the sparse differential resultant</b>	<b>53</b>
6.1	Degree of algebraic elimination ideal . . . . .	53
6.2	Degree bound for sparse differential resultant . . . . .	55
6.3	A single exponential algorithm to compute sparse differential resultant . . . .	57
6.4	Degree bound for differential resultant in terms of mixed volumes . . . . .	60
<b>7</b>	<b>Conclusion</b>	<b>66</b>

## 1 Introduction

The multivariate resultant, which gives conditions for an over-determined system of polynomial equations to have common solutions, is a basic concept in algebraic geometry [12, 19, 23, 26, 27, 41, 49]. In recent years, the multivariate resultant is emerged as one of the most powerful computational tools in elimination theory due to its ability to eliminate several variables simultaneously without introducing much extraneous solutions. Many algorithms with best complexity bounds for problems such as polynomial equation solving and first order quantifier elimination, are based on the multivariate resultant [4, 5, 14, 15, 42].

In the theory of multivariate resultants, polynomials are assumed to contain all the monomials with degrees up to a given bound. In practical problems, most polynomials are sparse in that they only contain certain fixed monomials. For such sparse polynomials, the multivariate resultant often becomes identically zero and cannot provide any useful information.

As a major advance in algebraic geometry and elimination theory, the concept of sparse resultant was introduced by Gelfand, Kapranov, Sturmfels, and Zelevinsky [19, 49]. The degree of the sparse resultant is the Bernstein-Kushnirenko-Khovanskii (BKK) bound [2] instead of the Bezout bound [19, 40, 50], which makes the computation of the sparse resultant more efficient. The concept of sparse resultant is originated from the work of Gelfand, Kapranov, and Zelevinsky on generalized hypergeometric functions, where the central concept of  $\mathcal{A}$ -discriminant is studied [17]. Kapranov, Sturmfels, and Zelevinsky introduced the concept of  $\mathcal{A}$ -resultant [28]. Sturmfels further introduced the general mixed sparse resultant and gave a single exponential algorithm to compute the sparse resultant [49, 50]. Canny and

Emiris showed that the sparse resultant is a factor of the determinant of a Macaulay style matrix and gave an efficient algorithm to compute the sparse resultant based on this matrix representation [13, 14]. D’Andrea further proved that the sparse resultant is the quotient of two Macaulay style determinants similar to the multivariate resultant [11].

Using the analogue between ordinary differential operators and univariate polynomials, the differential resultant for two linear ordinary differential operators was implicitly given by Ore [39] and then studied by Berkovich and Tsirulik [1] using Sylvester style matrices. The subresultant theory was first studied by Chardin [7] for two differential operators and then by Li [38] and Hong [24] for the more general Ore polynomials.

For nonlinear differential polynomials, the differential resultant is more difficult to define and study. The differential resultant for two nonlinear differential polynomials in one variable was defined by Ritt in [44, p.47]. In [55, p.46], Zwillinger proposed to define the differential resultant of two differential polynomials as the determinant of a matrix following the idea of algebraic multivariate resultants, but did not give details. General differential resultants were defined by Carrà-Ferro using Macaulay’s definition of algebraic resultants [6]. But, the treatment in [6] is not complete. For instance, the differential resultant for two generic differential polynomials with positive orders and degrees greater than one is always identically zero if using the definition in [6]. In [54], Yang, Zeng, and Zhang used the idea of algebraic Dixon resultant to compute the differential resultant. Although efficient, this approach is not complete, because it is not proved that the differential resultant can always be computed in this way. Differential resultants for linear ordinary differential polynomials were studied by Rueda-Sendra [46, 47]. In [16], a rigorous definition for the differential resultant of  $n + 1$  differential polynomials in  $n$  variables was first presented and its properties were proved. A generic differential polynomial with order  $o$  and degree  $d$  contains an exponential number of differential monomials in terms of  $o$  and  $d$ . Thus it is meaningful to study the sparse differential resultant which is the main focus of this paper.

Our first observation is that the sparse differential resultant is related with the non-polynomial solutions of algebraic differential equations, that is, solutions with non-vanishing derivatives to any order. As a consequence, the sparse differential resultant should be more naturally defined for Laurent differential polynomials. This is similar to the algebraic sparse resultant [19, 50], where non-zero solutions of Laurent polynomials are considered.

Consider  $n + 1$  Laurent differential polynomials in  $n$  differential indeterminates  $\mathbb{Y} = \{y_1, \dots, y_n\}$ :

$$\mathbb{P}_i = \sum_{k=0}^{l_i} u_{ik} M_{ik} \quad (i = 0, \dots, n), \quad (1)$$

where  $u_{ik} \in \mathcal{E}$  are differentially independent over  $\mathbb{Q}$  and  $M_{ik}$  are Laurent differential monomials in  $\mathbb{Y}$ . As explained later in this paper, we can assume that  $M_{ik}$  are monomials with non-negative exponent vectors  $\alpha_{ik}$ . Let  $s_i = \text{ord}(\mathbb{P}_i, \mathbb{Y})$  and denote  $M_{ik}/M_{i0} = \prod_{j=1}^n \prod_{l=0}^{s_i} (y_j^{(l)})^{t_{ikjl}} \triangleq (\mathbb{Y}^{[s_i]})^{\alpha_{ik} - \alpha_{i0}}$ , where  $y_j^{(l)}$  is the  $l$ -th derivative of  $y_j$  and  $\mathbb{Y}^{[s_i]}$  is the set  $\{y_j^{(l)} : 1 \leq j \leq n, 0 \leq l \leq s_i\}$ . Let  $\mathbf{u}_i = (u_{i0}, u_{i1}, \dots, u_{il_i})$  ( $i = 0, \dots, n$ ) be the coefficient vector of  $\mathbb{P}_i$ .

The concept of Laurent differentially essential system is introduced, which is a necessary and sufficient condition for the existence of sparse differential resultant.  $\mathbb{P}_0, \dots, \mathbb{P}_n$  are called Laurent differentially essential if  $[\mathbb{P}_0, \dots, \mathbb{P}_n] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is a prime differential ideal of

codimension one, where  $[\mathbb{P}_0, \dots, \mathbb{P}_n]$  is a differential ideal generated in  $\mathbb{Q}\{\mathbb{Y}, \mathbb{Y}^{-1}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$ . This concept is similar to (but weaker than) the concept of essential supports introduced by Sturmfels in [50], but its properties are more complicated. Precisely, we have

**Theorem 1.1** *For  $\mathbb{P}_i$  given in (1), let  $q_j = \max_{i=0}^n \text{ord}(\mathbb{P}_i, y_j)$  and  $d_{ij} = \sum_{k=0}^{l_i} u_{ik} \sum_{l=0}^{q_j} t_{ikjl} x_j^l$  ( $i = 0, \dots, n; j = 1, \dots, n$ ) where  $x_j$  are algebraic indeterminates. Denote*

$$M_{\mathbb{P}} = \begin{pmatrix} d_{01} & d_{02} & \dots & d_{0n} \\ d_{11} & d_{12} & \dots & d_{1n} \\ & & \ddots & \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{pmatrix}$$

to be the symbolic support matrix of (1). Then the following assertions hold.

- 1) *The differential transcendence degree of  $\mathbb{Q}\langle \mathbf{u}_0, \dots, \mathbf{u}_n \rangle \langle \frac{\mathbb{P}_0}{M_{00}}, \dots, \frac{\mathbb{P}_n}{M_{n0}} \rangle$  over  $\mathbb{Q}\langle \mathbf{u}_0, \dots, \mathbf{u}_n \rangle$  equals  $\text{rk}(M_{\mathbb{P}})$ .*
- 2)  *$[\mathbb{P}_0, \dots, \mathbb{P}_n] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is a prime differential ideal of codimension  $n+1 - \text{rk}(M_{\mathbb{P}})$ . So  $\mathbb{P}_0, \dots, \mathbb{P}_n$  form a Laurent differentially essential system if and only if  $\text{rk}(M_{\mathbb{P}}) = n$ .*
- 3)  *$\mathbb{P}_0, \dots, \mathbb{P}_n$  form a Laurent differentially essential system if and only if there exist  $k_i$  ( $1 \leq k_i \leq l_i$ ) such that  $\text{rk}(M_{k_0, \dots, k_n}) = n$  where  $M_{k_0, \dots, k_n}$  is the symbolic support matrix for the Laurent differential monomials  $M_{0k_0}/M_{00}, \dots, M_{nk_n}/M_{n0}$ .*

With the above theorem, computing the differential transcendence degree of certain differential polynomials is reduced to computing the rank of certain symbolic matrix. Similar to the case of linear equations, this result provides a useful tool to study generic differential polynomials. As an application of the above result, the differential dimension conjecture [45, p.178] for a class of generic differential polynomials is proved. For the  $n+1$  Laurent differential monomials  $M_{0k_0}/M_{00}, \dots, M_{nk_n}/M_{n0}$  ( $1 \leq k_i \leq l_i$ ) mentioned in 3) of Theorem 1.1, a more efficient algorithm to compute their differential transcendence degree over  $\mathbb{Q}$  is given by reducing their symbolic support matrix to a standard form called T-shape.

Before introducing the properties of the sparse differential resultant, the concept of Jacobi number is given below. Let  $\mathbb{G} = \{g_1, \dots, g_n\}$  be  $n$  differential polynomials in  $\mathbb{Y} = \{y_1, \dots, y_n\}$ . Let  $s_{ij} = \text{ord}(g_i, y_j)$  be the order of  $g_i$  in  $y_j$  if  $y_j$  occurs effectively in  $g_i$  and  $s_{ij} = -\infty$  otherwise. Then the *Jacobi bound*, or the *Jacobi number*, of  $\mathbb{G}$ , denoted as  $\text{Jac}(\mathbb{G})$ , is the maximum number of the summations of all the diagonals of  $S$ . Or equivalently,

$$\text{Jac}(\mathbb{G}) = \max \sum_{i=1}^n s_{i\sigma(i)},$$

where  $\sigma$  is a permutation of  $\{1, \dots, n\}$ . *Jacobi's Problem* conjectures that the order of the zero dimensional component of  $\mathbb{G}$  is bounded by the Jacobi number of  $\mathbb{G}$  [43].

The properties of the sparse differential resultant are summarized in the following theorem.

**Theorem 1.2** *The sparse differential resultant  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n) \in \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  of  $\mathbb{P}_0, \dots, \mathbb{P}_n$  has the following properties.*

- 1)  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  is differentially homogenous in each  $\mathbf{u}_i$  ( $i = 0, \dots, n$ ).
- 2)  $h_i = \text{ord}(\mathbf{R}, \mathbf{u}_i) \leq J_i = \text{Jac}(\mathbb{P}_i)$  where  $\mathbb{P}_i = \{\mathbb{P}_0^N, \dots, \mathbb{P}_n^N\} \setminus \{\mathbb{P}_i^N\}$ .
- 3) Let  $\mathcal{Z}_0(\mathbb{P}_0, \dots, \mathbb{P}_n)$  be the set of all specializations of the coefficients  $u_{ik}$  of  $\mathbb{P}_i$  under which  $\mathbb{P}_i = 0$  ( $i = 0, \dots, n$ ) have a common non-polynomial solution and  $\overline{\mathcal{Z}_0(\mathbb{P}_0, \dots, \mathbb{P}_n)}$  the Kolchin differential closure of  $\mathcal{Z}_0(\mathbb{P}_0, \dots, \mathbb{P}_n)$ . Then  $\mathcal{Z}_0(\mathbb{P}_0, \dots, \mathbb{P}_n) = \mathbb{V}(\text{sat}(\mathbf{R}))$ .
- 4) Assume that  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ) have the same set  $\mathcal{A}$  of monomials. The differential toric variety  $X_{\mathcal{A}}$  associated with  $\mathcal{A}$  is defined and is shown to be an irreducible projective differential variety of dimension  $n$ . Furthermore, the differential Chow form [16, 37] of  $X_{\mathcal{A}}$  is  $\mathbf{R}$ .
- 5) (Poison Type Product Formula) Let  $\mathbf{u}_0$  appear in  $\mathbf{R}$  and  $t_0 = \deg(\mathbf{R}, u_{00}^{(h_0)})$ . Then there exist  $\xi_{\tau k}$  in certain differential field  $\mathcal{F}_{\tau}$  ( $\tau = 1, \dots, t_0$ ) such that

$$\mathbf{R} = A \prod_{\tau=1}^{t_0} (u_{00} + \sum_{k=1}^{l_0} u_{0k} \xi_{\tau k})^{(h_0)},$$

where  $A$  is a polynomial in  $\mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_n \rangle [\mathbf{u}_0^{[h_0]} \setminus u_{00}^{(h_0)}]$ . Furthermore, if 1) any  $n$  of the  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ) form a differentially independent set over  $\mathbb{Q}\langle \mathbf{u}_0, \dots, \mathbf{u}_n \rangle$  and 2) for each  $j = 1, \dots, n$ ,  $\mathbf{e}_j \in \text{Span}_{\mathbb{Z}}\{\alpha_{ik} - \alpha_{i0} : k = 1, \dots, l_i; i = 0, \dots, n\}$ , then there exist  $\eta_{\tau k} \in \mathcal{F}_{\tau}$  ( $\tau = 1, \dots, t_0; k = 1, \dots, n$ ) such that

$$\mathbf{R} = A \prod_{\tau=1}^{t_0} \left[ \frac{\mathbb{P}_0(\eta_{\tau})}{M_{00}(\eta_{\tau})} \right]^{(h_0)},$$

where  $\eta_{\tau} = (\eta_{\tau 1}, \dots, \eta_{\tau n})$  and  $\mathbf{e}_i$  is the exponent vector of  $y_i$ . Moreover,  $\eta_{\tau}$  ( $\tau = 1, \dots, t_0$ ) are generic points of the prime differential ideal  $[\mathbb{P}_1, \dots, \mathbb{P}_n] : \mathfrak{m} \subset \mathcal{F}\langle \mathbf{u}_0, \dots, \mathbf{u}_n \rangle \{\mathbb{Y}\}$ , where  $\mathfrak{m}$  is the set of all differential monomials in  $\mathbb{Y}$ .

- 6)  $\deg(\mathbf{R}) \leq \prod_{i=0}^n (m_i + 1)^{h_i+1} \leq (m + 1)^{\sum_{i=0}^n (J_i+1)} \leq (m + 1)^{J+n+1}$ , where  $m_i = \deg(\mathbb{P}_i, \mathbb{Y})$ ,  $m = \max_i \{m_i\}$ , and  $J = \sum_{i=0}^n J_i$ .
- 7) Let  $s_i = \text{ord}(\mathbb{P}_i, \mathbb{Y})$ . Then  $\mathbf{R}$  has a representation

$$\prod_{i=0}^n M_{i0}^{(h_i+1)\deg(\mathbf{R})} \cdot \mathbf{R} = \sum_{i=0}^n \sum_{j=0}^{h_i} G_{ij}(\mathbb{P}_i)^{(j)}$$

where  $G_{ij} \in \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \dots, \mathbf{u}_n^{[h_n]}, \mathbb{Y}^{[h]}]$  with  $h = \max\{h_i + s_i\}$  such that  $\deg(G_{ij}(\mathbb{P}_i)^{(j)}) \leq [m + 1 + \sum_{i=0}^n (h_i + 1)\deg(M_{i0})]\deg(\mathbf{R})$ .

Although similar to the properties of algebraic sparse resultants, each property given above is an essential extension of its algebraic counterpart. For instance, it needs lots of efforts to obtain the Poisson type product formula. Property 2) is unique for the differential case and reflects the sparseness of the system in certain sense.

More properties for the sparse differential resultant are proved in this paper. For instance, the explicit condition for the equation system (1) to have a unique solution for  $\mathbb{Y}$  is given. The sparse resultant for differential polynomials with non-vanishing degree terms are also defined, which gives conditions for the existence of solutions instead of non-polynomial solutions.

Let  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ) in (1) be generic differential polynomials containing all monomials with order  $\leq s_i$  and degree  $\leq m_i$  and  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  the differential resultant of  $\mathbb{P}_0, \dots, \mathbb{P}_n$ . Then a BKK style degree bound is given:

**Theorem 1.3** *For each  $i \in \{0, 1, \dots, n\}$ ,*

$$\deg(\mathbf{R}, \mathbf{u}_i) \leq \sum_{k=0}^{s-s_i} \mathcal{M}((\mathcal{Q}_{jl})_{j \neq i, 0 \leq l \leq s-s_j}, \mathcal{Q}_{i0}, \dots, \mathcal{Q}_{i,k-1}, \mathcal{Q}_{i,k+1}, \dots, \mathcal{Q}_{i,s-s_i})$$

where  $\mathcal{Q}_{jl}$  is the Newton polytope of  $(\mathbb{P}_j)^{(l)}$  as a polynomial in  $y_1^{[s]}, \dots, y_n^{[s]}$  and  $\mathcal{M}(S)$  is the mixed volume for the polytopes in  $S$ .

In principle, the sparse differential resultant can be computed with the characteristic set method for differential polynomials via symbolic computation [45, 3, 8, 48, 53]. But in general, differential elimination procedures based on characteristic sets do not have an elementary complexity bound [20].

Based on the order and degree bounds given in 2) and 6) of Theorem 1.2, a single exponential algorithm to compute the sparse differential resultant  $\mathbf{R}$  is proposed. The idea of the algorithm is to compute  $\mathbf{R}$  with its order and degree increasing incrementally and to use linear algebra to find the coefficients of  $\mathbf{R}$  with the given order and degree. The order and degree bounds serve as the termination condition. Precisely, we have

**Theorem 1.4** *With notations introduced in Theorem 1.2, the sparse differential resultant of  $\mathbb{P}_0, \dots, \mathbb{P}_n$  can be computed with at most  $O\left(\frac{(J+n+2)^{O(l(J+1))} m^{O(l(J+1)(J+n+1))}}{n^n}\right)$   $\mathbb{Q}$ -arithmetic operations, where  $l = \sum_{i=0}^n (l_i + 1)$ ,  $m = \max_{i=0}^n m_i$ , and  $J = \sum_{i=0}^n J_i$ .*

From Theorem 1.4, the complexity of this algorithm is single exponential in terms of  $l$ , and  $J$ . The sparseness is reflected in the quantity  $l$  which is called the size of the system and the Jacobi number  $J$ . Note that even the complexity of computing the algebraic sparse resultant is single exponential [49, 14]. The algorithm seems to be the first one to eliminate several variables from nonlinear differential polynomials with a single exponential complexity.

The rest of the paper is organized as follows. In Section 2, preliminary results are introduced. In Section 3, the sparse differential resultant for Laurent differentially essential systems is defined. In Section 4, Theorem 1.1 is proved. In Section 5, properties 1) - 5) of Theorem 1.2 are proved. In Section 6, properties 6) and 7) of Theorem 1.2 and Theorems 1.3 and 1.4 are proved. In Section 7, the paper is concluded and several unsolved problems for differential sparse resultant are proposed.

## 2 Preliminaries

In this section, some basic notations and preliminary results in differential algebra will be given. For more details about differential algebra, please refer to [45, 29, 3, 48, 16].

### 2.1 Differential polynomial algebra and Kolchin topology

Let  $\mathcal{F}$  be a fixed ordinary differential field of characteristic zero, with a derivation operator  $\delta$ . An element  $c \in \mathcal{F}$  such that  $\delta c = 0$  is called a constant of  $\mathcal{F}$ . In this paper, unless otherwise indicated,  $\delta$  is kept fixed during any discussion and we use primes and exponents  $(i)$  to indicate derivatives under  $\delta$ . Let  $\Theta$  denote the free commutative semigroup with unit (written multiplicatively) generated by  $\delta$ .

A typical example of differential field is  $\mathbb{Q}(x)$  which is the field of rational functions in a variable  $x$  with  $\delta = \frac{d}{dx}$ .

Let  $S$  be a subset of a differential field  $\mathcal{G}$  which contains  $\mathcal{F}$ . We will denote respectively by  $\mathcal{F}[S]$ ,  $\mathcal{F}(S)$ ,  $\mathcal{F}\{S\}$ , and  $\mathcal{F}\langle S \rangle$  the smallest subring, the smallest subfield, the smallest differential subring, and the smallest differential subfield of  $\mathcal{G}$  containing  $\mathcal{F}$  and  $S$ . If we denote  $\Theta(S)$  to be the smallest subset of  $\mathcal{G}$  containing  $S$  and stable under  $\delta$ , we have  $\mathcal{F}\{S\} = \mathcal{F}[\Theta(S)]$  and  $\mathcal{F}\langle S \rangle = \mathcal{F}(\Theta(S))$ . A differential extension field  $\mathcal{G}$  of  $\mathcal{F}$  is said to be finitely generated if  $\mathcal{G}$  has a finite subset  $S$  such that  $\mathcal{G} = \mathcal{F}\langle S \rangle$ .

A subset  $\Sigma$  of a differential extension field  $\mathcal{G}$  of  $\mathcal{F}$  is said to be *differentially dependent* over  $\mathcal{F}$  if the set  $(\theta\alpha)_{\theta \in \Theta, \alpha \in \Sigma}$  is algebraically dependent over  $\mathcal{F}$ , and is said to be *differentially independent* over  $\mathcal{F}$ , or to be a family of *differential indeterminates* over  $\mathcal{F}$  in the contrary case. In the case  $\Sigma$  consists of one element  $\alpha$ , we say that  $\alpha$  is differentially algebraic or differentially transcendental over  $\mathcal{F}$  respectively. The maximal subset  $\Omega$  of  $\mathcal{G}$  which are differentially independent over  $\mathcal{F}$  is said to be a differential transcendence basis of  $\mathcal{G}$  over  $\mathcal{F}$ . We use  $\text{d.tr.deg } \mathcal{G}/\mathcal{F}$  (see [29, p.105-109]) to denote the *differential transcendence degree* of  $\mathcal{G}$  over  $\mathcal{F}$ , which is the cardinal number of  $\Omega$ . Considering  $\mathcal{F}$  and  $\mathcal{G}$  as ordinary algebraic fields, we denote the algebraic transcendence degree of  $\mathcal{G}$  over  $\mathcal{F}$  by  $\text{tr.deg } \mathcal{G}/\mathcal{F}$ .

A homomorphism  $\varphi$  from a differential ring  $(\mathcal{R}, \delta)$  to a differential ring  $(\mathcal{S}, \delta_1)$  is a *differential homomorphism* if  $\varphi \circ \delta = \delta_1 \circ \varphi$ . If  $\mathcal{R}_0$  is a common differential subring of  $\mathcal{R}$  and  $\mathcal{S}$  and the homomorphism  $\varphi$  leaves every element of  $\mathcal{R}_0$  invariant, it is said to be over  $\mathcal{R}_0$ . If, in addition  $\mathcal{R}$  is an integral domain and  $\mathcal{S}$  is a differential field,  $\varphi$  is called a *differential specialization* of  $\mathcal{R}$  into  $\mathcal{S}$  over  $\mathcal{R}_0$ . The following property about differential specialization will be needed in this paper, which can be proved similarly to Theorem 2.16 in [16].

**Lemma 2.1** *Let  $P_i(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\langle \mathbb{Y} \rangle\{\mathbb{U}\}$  ( $i = 1, \dots, m$ ) where  $\mathbb{U} = (u_1, \dots, u_r)$  and  $\mathbb{Y} = (y_1, \dots, y_n)$  are sets of differential indeterminates. If the set  $(P_i(\mathbb{U}, \mathbb{Y}))^{(\sigma_{ij})}$  ( $i = 1, \dots, m; j = 1, \dots, n_i$ ) are algebraically dependent over  $\mathcal{F}\langle \mathbb{U} \rangle$ , then for any differential specialization  $\mathbb{U}$  to  $\mathbb{U}^0 \subset \mathcal{F}$  over  $\mathcal{F}$ ,  $(P_i(\mathbb{U}^0, \mathbb{Y}))^{(\sigma_{ij})}$  ( $i = 1, \dots, m; j = 1, \dots, n_i$ ) are algebraically dependent over  $\mathcal{F}$ . In particular, if  $P_i(\mathbb{U}, \mathbb{Y})$  ( $i = 1, \dots, m$ ) are differentially dependent over  $\mathcal{F}\langle \mathbb{U} \rangle$ , then for any differential specialization  $\mathbb{U}$  to  $\overline{\mathbb{U}} \subset \mathcal{F}$  over  $\mathcal{F}$ ,  $P_i(\overline{\mathbb{U}}, \mathbb{Y})$  ( $i = 1, \dots, m$ ) are differentially dependent over  $\mathcal{F}$ .*

A differential extension field  $\mathcal{E}$  of  $\mathcal{F}$  is called a *universal differential extension field*, if for any finitely generated differential extension field  $\mathcal{F}_1$  of  $\mathcal{F}$  in  $\mathcal{E}$  and any finitely generated

differential extension field  $\mathcal{F}_2$  of  $\mathcal{F}_1$  not necessarily in  $\mathcal{E}$ ,  $\mathcal{F}_2$  can be embedded in  $\mathcal{E}$  over  $\mathcal{F}_1$ , i.e. there exists a differential extension field  $\mathcal{F}_3$  in  $\mathcal{E}$  that is differentially isomorphic to  $\mathcal{F}_2$  over  $\mathcal{F}_1$ . Such a differential universal extension field of  $\mathcal{F}$  always exists ([29, Theorem 2, p. 134]). By definition, any finitely generated differential extension field of  $\mathcal{F}$  can be embedded over  $\mathcal{F}$  into  $\mathcal{E}$ , and  $\mathcal{E}$  is a universal differential extension field of every finitely generated differential extension field of  $\mathcal{F}$ . In particular, for any natural number  $n$ , we can find in  $\mathcal{E}$  a subset of cardinality  $n$  whose elements are differentially independent over  $\mathcal{F}$ . Throughout the present paper,  $\mathcal{E}$  stands for a fixed universal differential extension field of  $\mathcal{F}$ .

Now suppose  $\mathbb{Y} = \{y_1, y_2, \dots, y_n\}$  is a set of differential indeterminates over  $\mathcal{E}$ . For any  $y \in \mathbb{Y}$ , denote  $\delta^k y$  by  $y^{(k)}$ . The elements of  $\mathcal{F}\{\mathbb{Y}\} = \mathcal{F}[y_j^{(k)} : j = 1, \dots, n; k \in \mathbb{N}]$  are called *differential polynomials* over  $\mathcal{F}$  in  $\mathbb{Y}$ , and  $\mathcal{F}\{\mathbb{Y}\}$  itself is called the *differential polynomial ring* over  $\mathcal{F}$  in  $\mathbb{Y}$ . A differential polynomial ideal  $\mathcal{I}$  in  $\mathcal{F}\{\mathbb{Y}\}$  is an ordinary algebraic ideal which is closed under derivation, i.e.  $\delta(\mathcal{I}) \subset \mathcal{I}$ . And a prime (resp. radical) differential ideal is a differential ideal which is prime (resp. radical) as an ordinary algebraic polynomial ideal. For convenience, a prime differential ideal is assumed not to be the unit ideal in this paper.

By a *differential affine space* we mean any one of the sets  $\mathcal{E}^n$  ( $n \in \mathbb{N}$ ). An element  $\eta = (\eta_1, \dots, \eta_n)$  of  $\mathcal{E}^n$  will be called a point. Let  $\Sigma$  be a subset of differential polynomials in  $\mathcal{F}\{\mathbb{Y}\}$ . A point  $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{E}^n$  is called a differential zero of  $\Sigma$  if  $f(\eta) = 0$  for any  $f \in \Sigma$ . The set of differential zeros of  $\Sigma$  is denoted by  $\mathbb{V}(\Sigma)$ , which is called a *differential variety* defined over  $\mathcal{F}$ . The differential varieties in  $\mathcal{E}^n$  (resp. the differential varieties in  $\mathcal{E}^n$  that are defined over  $\mathcal{F}$ ) are the closed sets in a topology called the *Kolchin topology* (resp. the Kolchin  $\mathcal{F}$ -topology).

For a differential variety  $V$  which is defined over  $\mathcal{F}$ , we denote  $\mathbb{I}(V)$  to be the set of all differential polynomials in  $\mathcal{F}\{\mathbb{Y}\}$  that vanish at every point of  $V$ . Clearly,  $\mathbb{I}(V)$  is a radical differential ideal in  $\mathcal{F}\{\mathbb{Y}\}$ . And there exists a bijective correspondence between Kolchin  $\mathcal{F}$ -closed sets and radical differential ideals in  $\mathcal{F}\{\mathbb{Y}\}$ . That is, for any differential variety  $V$  defined over  $\mathcal{F}$ ,  $\mathbb{V}(\mathbb{I}(V)) = V$  and for any radical differential ideal  $\mathcal{I}$  in  $\mathcal{F}\{\mathbb{Y}\}$ ,  $\mathbb{I}(\mathbb{V}(\mathcal{I})) = \mathcal{I}$ .

Similarly as in algebraic geometry, an  $\mathcal{F}$ -irreducible differential variety can be defined. And there is a bijective correspondence between  $\mathcal{F}$ -irreducible differential varieties and prime differential ideals in  $\mathcal{F}\{\mathbb{Y}\}$ . A point  $\eta \in \mathbb{V}(\mathcal{I})$  is called a *generic point* of a prime ideal  $\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$ , or of the irreducible variety  $\mathbb{V}(\mathcal{I})$ , if for any polynomial  $P \in \mathcal{F}\{\mathbb{Y}\}$  we have  $P(\eta) = 0 \Leftrightarrow P \in \mathcal{I}$ . It is well known that [45, p.27] a non-unit differential ideal is prime if and only if it has a generic point.

Let  $\mathcal{I}$  be a prime differential ideal in  $\mathcal{F}\{\mathbb{Y}\}$  and  $\xi = (\xi_1, \dots, \xi_n)$  a generic point of  $\mathcal{I}$  [29, p.19]. The *dimension* of  $\mathcal{I}$  or of  $\mathbb{V}(\mathcal{I})$  is defined to be the differential transcendence degree of the differential extension field  $\mathcal{F}\langle \xi_1, \dots, \xi_n \rangle$  over  $\mathcal{F}$ , that is,  $\dim(\mathcal{I}) = \text{d.tr.deg } \mathcal{F}\langle \xi_1, \dots, \xi_n \rangle / \mathcal{F}$ .

We will conclude this section by introducing some basic concepts in projective differential algebraic geometry which will be used in Section 5.3. For more details, please refer to [33, 36]. And unless otherwise stated, in the whole paper, we only consider the affine differential case.

For each  $l \in \mathbb{N}$ , consider a projective space  $\mathbf{P}(l)$  over  $\mathcal{E}$ . By a *differential projective space* we mean any one of the sets  $\mathbf{P}(l)$  ( $l \in \mathbb{N}$ ). Denote  $z_0, z_1, \dots, z_l$  to be the homogenous coordinates. Let  $\mathcal{I}$  be a differential ideal of  $\mathcal{F}\{\mathbf{z}\}$  where  $\mathbf{z} = \{z_0, z_1, \dots, z_l\}$ . Denote  $\mathcal{I} : \mathbf{z} = \{f \in \mathcal{F}\{\mathbf{z}\} \mid z_j f \in \mathcal{I} \text{ for each } j = 0, \dots, l\}$ .



**Definition 2.2** Let  $\mathcal{I}$  be a differential ideal of  $\mathcal{F}\{\mathbf{z}\}$ .  $\mathcal{I}$  is called a differentially homogenous differential ideal of  $\mathcal{F}\{\mathbf{z}\}$  if  $\mathcal{I} : \mathbf{z} = \mathcal{I}$  and for every  $P \in \mathcal{I}$  and a differential indeterminate  $\lambda$  over  $\mathcal{F}\{\mathbf{z}\}$ ,  $P(\lambda\mathbf{z}) \in \mathcal{F}\{\lambda\}\mathcal{I}$  in the differential ring  $\mathcal{F}\{\lambda, \mathbf{z}\}$ .

Consider a differential polynomial  $P \in \mathcal{E}\{\mathbf{z}\}$  and a point  $\alpha \in \mathbf{P}(l)$ . Say that  $P$  vanishes at  $\alpha$ , and that  $\alpha$  is a zero of  $P$ , if  $P$  vanishes at  $\lambda\alpha$  for every  $\lambda$  in  $\mathcal{E}$ . For a subset  $\mathcal{M}$  of  $\mathbf{P}(l)$ , let  $\mathbb{I}(\mathcal{M})$  denote the set of differential polynomials in  $\mathcal{F}\{\mathbf{z}\}$  that vanishes on  $\mathcal{M}$ . Let  $\mathbb{V}(S)$  denote the set of points of  $\mathbf{P}(l)$  that are zeros of the subset  $S$  of  $\mathcal{E}\{\mathbf{z}\}$ . And a subset  $V$  of  $\mathbf{P}(l)$  is called a *projective differential  $\mathcal{F}$ -variety* if there exists  $S \subset \mathcal{F}\{\mathbf{z}\}$  such that  $V = \mathbb{V}(S)$ . There exists a one-to-one correspondence between projective differential varieties and perfect differentially homogenous differential ideals. And a projective differential  $\mathcal{F}$ -variety  $V$  is  $\mathcal{F}$ -irreducible if and only if  $\mathbb{I}(V)$  is prime.

Let  $\mathcal{I}$  be a prime differentially homogenous ideal and  $\xi = (\xi_0, \xi_1, \dots, \xi_l)$  be a generic point of  $\mathcal{I}$  with  $\xi_0 \neq 0$ . Then the differential dimension of  $\mathbb{V}(\mathcal{I})$  is defined to be the differential transcendence degree of  $\mathcal{F}\langle(\xi_0^{-1}\xi_k)_{1 \leq k \leq l}\rangle$  over  $\mathcal{F}$ .

## 2.2 Characteristic sets of a differential polynomial system

Let  $f$  be a differential polynomial in  $\mathcal{F}\{\mathbb{Y}\}$ . We define the order of  $f$  w.r.t.  $y_i$  to be the greatest number  $k$  such that  $y_i^{(k)}$  appears effectively in  $f$ , which is denoted by  $\text{ord}(f, y_i)$ . And if  $y_i$  does not appear in  $f$ , then we set  $\text{ord}(f, y_i) = -\infty$ . The *order* of  $f$  is defined to be  $\max_i \text{ord}(f, y_i)$ , that is,  $\text{ord}(f) = \max_i \text{ord}(f, y_i)$ .

A *ranking*  $\mathcal{R}$  is a total order over  $\Theta(\mathbb{Y})$ , which is compatible with the derivations over the alphabet:

- 1)  $\delta\theta y_j > \theta y_j$  for all derivatives  $\theta y_j \in \Theta(\mathbb{Y})$ .
- 2)  $\theta_1 y_i > \theta_2 y_j \implies \delta\theta_1 y_i > \delta\theta_2 y_j$  for  $\theta_1 y_i, \theta_2 y_j \in \Theta(\mathbb{Y})$ .

By convention,  $1 < \theta y_j$  for all  $\theta y_j \in \Theta(\mathbb{Y})$ .

Two important kinds of rankings are the following:

- 1) *Elimination ranking*:  $y_i > y_j \implies \delta^k y_i > \delta^l y_j$  for any  $k, l \geq 0$ .
- 2) *Orderly ranking*:  $k > l \implies \delta^k y_i > \delta^l y_j$ , for any  $i, j \in \{1, 2, \dots, n\}$ .

Let  $p$  be a differential polynomial in  $\mathcal{F}\{\mathbb{Y}\}$  and  $\mathcal{R}$  a ranking endowed on it. The greatest derivative w.r.t.  $\mathcal{R}$  which appears effectively in  $p$  is called the *leader* of  $p$ , which will be denoted by  $u_p$  or  $\text{ld}(p)$ . The two conditions mentioned above imply that the leader of  $\theta p$  is  $\theta u_p$  for  $\theta \in \Theta$ . Let the degree of  $p$  in  $u_p$  be  $d$ . As a univariate polynomial in  $u_p$ ,  $p$  can be rewritten as

$$p = I_d u_p^d + I_{d-1} u_p^{d-1} + \dots + I_0.$$

$I_d$  is called the *initial* of  $p$  and is denoted by  $I_p$ . The partial derivative of  $p$  w.r.t.  $u_p$  is called the *separant* of  $p$ , which will be denoted by  $S_p$ . Clearly,  $S_p$  is the initial of any proper derivative of  $p$ . The *rank* of  $p$  is  $u_p^d$ , and is denoted by  $\text{rk}(p)$ .

Let  $p$  and  $q$  be two differential polynomials and  $u_p^d$  the rank of  $p$ .  $q$  is said to be *partially reduced* w.r.t.  $p$  if no proper derivatives of  $u_p$  appear in  $q$ .  $q$  is said to be *reduced* w.r.t.  $p$  if  $q$  is partially reduced w.r.t.  $p$  and  $\deg(q, u_p) < d$ . Let  $\mathcal{A}$  be a set of differential polynomials.  $\mathcal{A}$  is said to be an *auto-reduced set* if each polynomial of  $\mathcal{A}$  is reduced w.r.t. any other element of  $\mathcal{A}$ . Every auto-reduced set is finite.

Let  $\mathcal{A} = A_1, A_2, \dots, A_t$  be an auto-reduced set with  $S_i$  and  $I_i$  as the separant and initial of  $A_i$ , and  $f$  be any differential polynomial. Then there exists an algorithm, called Ritt's algorithm of reduction, which reduces  $f$  w.r.t.  $\mathcal{A}$  to a polynomial  $r$  that is reduced w.r.t.  $\mathcal{A}$ , satisfying the relation

$$\prod_{i=1}^t S_i^{d_i} I_i^{e_i} \cdot f \equiv r, \text{ mod } [\mathcal{A}],$$

where  $d_i, e_i$  ( $i = 1, 2, \dots, t$ ) are nonnegative integers. The differential polynomial  $r$  is called the *differential remainder* of  $f$  w.r.t.  $\mathcal{A}$ .

Let  $\mathcal{A}$  be an auto-reduced set. Denote  $H_{\mathcal{A}}$  to be the set of all the initials and separants of  $\mathcal{A}$  and  $H_{\mathcal{A}}^{\infty}$  to be the minimal multiplicative set containing  $H_{\mathcal{A}}$ . The *saturation ideal* of  $\mathcal{A}$  is defined to be

$$\text{sat}(\mathcal{A}) = [\mathcal{A}] : H_{\mathcal{A}}^{\infty} = \{p : \exists h \in H_{\mathcal{A}}^{\infty}, \text{ s.t. } hp \in [\mathcal{A}]\}.$$

An auto-reduced set  $\mathcal{C}$  contained in a differential polynomial set  $\mathcal{S}$  is said to be a *characteristic set* of  $\mathcal{S}$ , if  $\mathcal{S}$  does not contain any nonzero element reduced w.r.t.  $\mathcal{C}$ . A characteristic set  $\mathcal{C}$  of an ideal  $\mathcal{J}$  reduces to zero all elements of  $\mathcal{J}$ . If the ideal is prime,  $\mathcal{C}$  reduces to zero only the elements of  $\mathcal{J}$  and  $\mathcal{J} = \text{sat}(\mathcal{C})$  ([29, Lemma 2, p.167]) is valid.

In terms of the characteristic set, the cardinal number of the characteristic set of  $\mathcal{I}$  is equal to the codimension of  $\mathcal{I}$ , that is  $n - \dim(\mathcal{I})$ . When  $\mathcal{I}$  is of codimension one, it has the following property.

**Lemma 2.3** [45, p.45] *Let  $\mathcal{I}$  be a prime differential ideal of codimension one in  $\mathcal{F}\{\mathbb{Y}\}$ . Then there exists an irreducible differential polynomial  $A$  such that  $\mathcal{I} = \text{sat}(\mathcal{A})$  and  $\{A\}$  is the characteristic set of  $\mathcal{I}$  w.r.t. any ranking.*

### 3 Sparse differential resultant for Laurent differential polynomials

In this section, the concepts of Laurent differential polynomials and Laurent differentially essential systems are first introduced, and then the sparse differential resultant for Laurent differentially essential systems is defined.

#### 3.1 Laurent differential polynomial

Let  $\mathcal{F}$  be an ordinary differential field with a derivation operator  $\delta$  and  $\mathcal{F}\{\mathbb{Y}\}$  the ring of differential polynomials in the differential indeterminates  $\mathbb{Y} = \{y_1, \dots, y_n\}$ . Let  $\mathcal{E}$  be a universal differential field of  $\mathcal{F}$ . For any element  $e \in \mathcal{E}$ ,  $e^{[k]}$  is used to denote the set  $\{e^{(i)} : i = 0, \dots, k\}$ .

The sparse differential resultant is closely related with Laurent differential polynomials, which will be defined below.

**Definition 3.1** *A Laurent differential monomial of order  $s$  is a Laurent monomial in variables  $\mathbb{Y}^{[s]} = (y_i^{(k)})_{1 \leq i \leq n; 0 \leq k \leq s}$ . More precisely, it has the form  $\prod_{i=1}^n \prod_{k=0}^s (y_i^{(k)})^{d_{ik}}$  where  $d_{ik}$  are integers which can be negative. A Laurent differential polynomial is a finite linear combination of Laurent differential monomials with coefficients from  $\mathcal{E}$ .*

Clearly, the collections of all Laurent differential polynomials form a commutative differential ring under the obvious sum, product operations and the usual derivation operator  $\delta$ , where all Laurent differential monomials are invertible. We denote the differential ring of Laurent differential polynomials with coefficients in  $\mathcal{F}$  by  $\mathcal{F}\{y_1, y_1^{-1}, \dots, y_n, y_n^{-1}\}$ , or simply by  $\mathcal{F}\{\mathbb{Y}, \mathbb{Y}^{-1}\}$ .

**Remark 3.2**  $\mathcal{F}\{\mathbb{Y}, \mathbb{Y}^{-1}\} = \mathcal{F}\{y_1, y_1^{-1}, \dots, y_n, y_n^{-1}\}$  is only a notation for Laurent differential polynomial ring. It is not equal to  $\mathcal{F}[y_i^{(k)}, (y_i^{-1})^{(k)} : k \geq 0]$ .

Denote  $\mathcal{S}$  to be the set of all differential ideals in  $\mathcal{F}\{\mathbb{Y}, \mathbb{Y}^{-1}\}$ , which are finitely generated. Let  $\mathfrak{m}$  be the set of all differential monomials in  $\mathbb{Y}$  and  $\mathcal{T}$  the set of all differential ideals in  $\mathcal{F}\{\mathbb{Y}\}$ , each of which has the form

$$[f_1, \dots, f_r] : \mathfrak{m} = \{f \in \mathcal{F}\{\mathbb{Y}\} \mid \exists M \in \mathfrak{m}, \text{ s.t. } M \cdot f \in [f_1, \dots, f_r]\}$$

for arbitrary  $f_i \in \mathcal{F}\{\mathbb{Y}\}$ . Now we give a one-to-one correspondence between  $\mathcal{S}$  and  $\mathcal{T}$ .

The maps  $\phi : \mathcal{S} \rightarrow \mathcal{T}$  and  $\psi : \mathcal{T} \rightarrow \mathcal{S}$  are defined as follows:

- Given any  $\mathcal{I} = [F_1, \dots, F_s] \in \mathcal{S}$ . Since each  $F_i \in \mathcal{F}\{\mathbb{Y}, \mathbb{Y}^{-1}\}$ , we can choose a vector  $(M_1, \dots, M_s) \in \mathfrak{m}^s$  such that  $M_i F_i \in \mathcal{F}\{\mathbb{Y}\}$  ( $i = 1, \dots, s$ ). We then define  $\phi(\mathcal{I}) \triangleq [M_1 F_1, \dots, M_s F_s] : \mathfrak{m} \subset \mathcal{F}\{\mathbb{Y}\}$ .
- For any  $\mathcal{J} = [f_1, \dots, f_r] : \mathfrak{m} \in \mathcal{T}$ , define  $\psi(\mathcal{J}) = [f_1, \dots, f_r]$  in  $\mathcal{F}\{\mathbb{Y}, \mathbb{Y}^{-1}\}$ .

**Lemma 3.3** *The above maps  $\phi$  and  $\psi$  are well defined. Moreover,  $\phi \circ \psi = \text{id}_{\mathcal{T}}$  and  $\psi \circ \phi = \text{id}_{\mathcal{S}}$ .*

*Proof:*  $\psi$  is obviously well-defined. To show that  $\phi$  is well-defined, it suffices to show that given another  $(N_1, \dots, N_s) \in \mathfrak{m}^s$  with  $N_i F_i \in \mathcal{F}\{\mathbb{Y}\}$  ( $i = 1, \dots, s$ ),  $[M_1 F_1, \dots, M_s F_s] : \mathfrak{m} = [N_1 F_1, \dots, N_s F_s] : \mathfrak{m}$  follows. It follows directly from the fact that  $N_i F_i \in [M_1 F_1, \dots, M_s F_s] : \mathfrak{m}$  and  $M_i F_i \in [N_1 F_1, \dots, N_s F_s] : \mathfrak{m}$ .

For any  $\mathcal{I} = [F_1, \dots, F_s] \in \mathcal{S}$ ,  $\psi \circ \phi(\mathcal{I}) = \psi([M_1 F_1, \dots, M_s F_s] : \mathfrak{m}) = [M_1 F_1, \dots, M_s F_s] = \mathcal{I} \subset \mathcal{F}\{\mathbb{Y}, \mathbb{Y}^{-1}\}$  where  $M_i F_i \in \mathcal{F}\{\mathbb{Y}\}$ , since Laurent differential monomials are invertible. So we have  $\psi \circ \phi = \text{id}_{\mathcal{S}}$ . And for any  $\mathcal{J} = [f_1, \dots, f_r] : \mathfrak{m} \in \mathcal{T}$ ,  $\phi \circ \psi(\mathcal{J}) = \phi([f_1, \dots, f_r]) = \mathcal{J}$ . Thus,  $\phi \circ \psi = \text{id}_{\mathcal{T}}$  follows.  $\square$

From the above, for a finitely generated Laurent differential ideal  $\mathcal{I} = [F_1, \dots, F_s]$ , although  $\phi(\mathcal{I})$  is unique, different vectors  $(M_1, \dots, M_s) \in \mathfrak{m}^s$  can be chosen to give different representations for  $\phi(\mathcal{I})$ . Now the norm form for a Laurent differential polynomial is introduced to fix the choice of  $(M_1, \dots, M_s) \in \mathfrak{m}^s$  when we consider  $\phi(\mathcal{I})$ .

**Definition 3.4** *For every Laurent differential polynomial  $F \in \mathcal{E}\{\mathbb{Y}, \mathbb{Y}^{-1}\}$ , there exists a unique laurent differential monomial  $M$  such that 1)  $M \cdot F \in \mathcal{E}\{\mathbb{Y}\}$  and 2) for any Laurent differential monomial  $T$  with  $T \cdot F \in \mathcal{E}\{\mathbb{Y}\}$ ,  $T \cdot F$  is divisible by  $M \cdot F$  as differential polynomials. This  $M \cdot F$  is defined to be the norm form of  $F$ , denoted by  $F^N$ . The order of  $F^N$  is defined to be the effective order of  $F$ , denoted by  $\text{Eord}(F)$ . Clearly,  $\text{Eord}(F) \leq \text{ord}(F)$ . And the degree of  $F$  is defined to be the degree of  $F^N$ , denoted by  $\text{deg}(F)$ .*

In the following, we consider zeros for Laurent differential polynomials.

**Definition 3.5** Let  $\mathcal{E}^\wedge = \mathcal{E} \setminus \{a \in \mathcal{E} \mid \exists k \in \mathbb{N}, \text{ s.t. } a^{(k)} = 0\}$ . Let  $F$  be a Laurent differential polynomial in  $\mathcal{F}\{\mathbb{Y}, \mathbb{Y}^{-1}\}$ . A point  $(a_1, \dots, a_n) \in (\mathcal{E}^\wedge)^n$  is called a non-polynomial differential zero of  $F$  if  $F(a_1, \dots, a_n) = 0$ .

It becomes apparent why non-polynomial elements in  $\mathcal{E}^\wedge$  are considered as zeros of Laurent differential polynomials when defining the zero set of an ideal. If  $F \in \mathcal{I}$ , then  $(y_i^{(k)})^{-1}F \in \mathcal{I}$  for any positive integer  $k$ , and in order for  $(y_i^{(k)})^{-1}F$  to be meaningful, we need to assume  $y_i^{(k)} \neq 0$ . We will see later in Example 3.21, how non-polynomial solutions are naturally related with the sparse differential resultant.

### 3.2 Definition of sparse differential resultant

In this section, the definition of the sparse differential resultant will be given. Since the study of sparse differential resultants becomes more transparent if we consider not individual differential polynomials but differential polynomials with indeterminate coefficients, the sparse differential resultant for Laurent differential polynomials with differential indeterminate coefficients will be defined first. Then the sparse differential resultant for a given Laurent differential polynomial system with concrete coefficients is the value which the resultant in the generic case assumes for the given case.

Suppose  $\mathcal{A}_i = \{M_{i0}, M_{i1}, \dots, M_{il_i}\}$  ( $i = 0, 1, \dots, n$ ) where  $M_{ik} = \prod_{j=1}^n \prod_{l=0}^{s_i} (y_j^{(l)})^{d_{ikjl}} \triangleq (\mathbb{Y}^{[s_i]})^{\alpha_{ik}}$  is a Laurent differential monomial of order  $s_i$  with exponent vector  $\alpha_{ik} \in \mathbb{Z}^{n(s_i+1)}$  and for  $k_1 \neq k_2$ ,  $\alpha_{ik_1} \neq \alpha_{ik_2}$ . Consider  $n+1$  generic Laurent differential polynomials defined over  $\mathcal{A}_i$  ( $i = 0, 1, \dots, n$ ):

$$\mathbb{P}_i = \sum_{k=0}^{l_i} u_{ik} M_{ik} \quad (i = 0, \dots, n), \quad (2)$$

where all the  $u_{ik}$  are differentially independent over  $\mathbb{Q}$ . The set of exponent vectors  $\mathbb{S}_i = \{\alpha_{ik} : k = 0, \dots, l_i\}$  is called the *support* of  $\mathbb{P}_i$ . The number  $|\mathbb{S}_i| = l_i + 1$  is called the *size* of  $\mathbb{P}_i$ . Note that  $s_i$  is the order of  $\mathbb{P}_i$  and an exponent vector of  $\mathbb{P}_i$  contains  $n(s_i + 1)$  elements. Denote

$$\mathbf{u}_i = (u_{i0}, u_{i1}, \dots, u_{il_i}) \quad (i = 0, \dots, n) \text{ and } \mathbf{u} = \{u_{ik} : i = 0, \dots, n; k = 0, \dots, l_i\}. \quad (3)$$

To avoid the triviality,  $l_i \geq 1$  ( $i = 0, \dots, n$ ) are always assumed in this paper.

**Definition 3.6** A set of Laurent differential polynomials of form (2) is called a Laurent differentially essential system if there exist  $k_i$  ( $i = 0, \dots, n$ ) with  $1 \leq k_i \leq l_i$  such that  $\text{d.tr.deg } \mathbb{Q}\langle \frac{M_{0k_0}}{M_{00}}, \frac{M_{1k_1}}{M_{10}}, \dots, \frac{M_{nk_n}}{M_{n0}} \rangle / \mathbb{Q} = n$ . In this case, we also say that  $\mathcal{A}_0, \dots, \mathcal{A}_n$  or  $\mathbb{S}_0, \dots, \mathbb{S}_n$  form a Laurent differentially essential system.

Although  $M_{i0}$  are used as denominators to define differentially essential system, the following lemma shows that the definition does not depend on the choices of  $M_{i0}$ .

**Lemma 3.7** The following two conditions are equivalent.

1. There exist  $k_0, \dots, k_n$  with  $1 \leq k_i \leq l_i$  such that  $\text{d.tr.deg } \mathbb{Q}\langle \frac{M_{0k_0}}{M_{00}}, \dots, \frac{M_{nk_n}}{M_{n0}} \rangle / \mathbb{Q} = n$ .
2. There exist pairs  $(k_i, j_i)$  ( $i = 0, \dots, n$ ) with  $k_i \neq j_i \in \{0, \dots, l_i\}$  such that  $\text{d.tr.deg } \mathbb{Q}\langle \frac{M_{0k_0}}{M_{0j_0}}, \dots, \frac{M_{nk_n}}{M_{nj_n}} \rangle / \mathbb{Q} = n$ .

*Proof:* 1)  $\implies$  2) is trivial.

Now suppose 2) holds. Fix the  $n+1$  pairs  $(k_i, j_i)$ , and without loss of generality, suppose  $\frac{M_{1k_1}}{M_{1j_1}}, \dots, \frac{M_{nk_n}}{M_{nj_n}}$  are differentially independent over  $\mathbb{Q}$ . We need to show 1) holds. Suppose the contrary. Then we know that for any  $m_i \in \{1, \dots, l_i\}$ ,  $\frac{M_{1m_1}}{M_{10}}, \dots, \frac{M_{nm_n}}{M_{n0}}$  are differentially dependent over  $\mathbb{Q}$ . Since  $\frac{M_{ik_i}}{M_{ij_i}} = \frac{M_{ik_i}}{M_{i0}} / \frac{M_{ij_i}}{M_{i0}}$ , it follows that  $\frac{M_{ik_i}}{M_{ij_i}}$  ( $i = 1, \dots, n$ ) are differentially dependent over  $\mathbb{Q}$ , which is a contradiction.  $\square$

Let  $[\mathbb{P}_0, \dots, \mathbb{P}_n]$  be the differential ideal in  $\mathbb{Q}\{\mathbb{Y}, \mathbb{Y}^{-1}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$  generated by  $\mathbb{P}_i$ . By Lemma 3.3,  $[\mathbb{P}_0, \dots, \mathbb{P}_n]$  corresponds to  $[\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m} \subset \mathbb{Q}\{\mathbb{Y}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$  in a unique way. Moreover, we have the following lemma.

**Lemma 3.8**  $[\mathbb{P}_0, \dots, \mathbb{P}_n] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\} = ([\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}) \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$ .

*Proof:* Denote  $\mathbb{P}_i^N = M_i \mathbb{P}_i$  ( $i = 0, \dots, n$ ) where  $M_i$  are Laurent differential monomials. It is obvious that the right elimination ideal is contained in the left one. For the other direction, let  $G$  be any element in the left ideal. Then there exist  $H_{ij} \in \mathbb{Q}\{\mathbb{Y}, \mathbb{Y}^{-1}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$  such that  $G = \sum_{i,j} H_{ij} \mathbb{P}_i^{(j)}$ . So  $G = \sum_{i,j} H_{ij} (\frac{\mathbb{P}_i^N}{M_i})^{(j)} = \sum_{i,j} \tilde{H}_{ij} (\mathbb{P}_i^N)^{(j)}$  with  $\tilde{H}_{ij} \in \mathbb{Q}\{\mathbb{Y}, \mathbb{Y}^{-1}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$ . Thus, there exists an  $M \in \mathfrak{m}$  such that  $MG \in [\mathbb{P}_0^N, \dots, \mathbb{P}_n^N]$  and  $G \in ([\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}) \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  follows.  $\square$

In the whole paper, when talking about prime differential ideals, it is assumed that they are distinct from the unit differential ideal. The following result is the foundation for defining the sparse differential resultant.

**Theorem 3.9** Let  $\mathbb{P}_0, \dots, \mathbb{P}_n$  be Laurent differential polynomials defined in (2). Then the following assertions hold.

- $([\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m})$  is a prime differential ideal in  $\mathbb{Q}\{\mathbb{Y}, \mathbf{u}_0, \dots, \mathbf{u}_n\}$ .
- $([\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}) \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is of codimension 1 if and only if  $\mathbb{P}_0, \dots, \mathbb{P}_n$  form a Laurent differentially essential system.

*Proof:* Let  $\eta = (\eta_1, \dots, \eta_n)$  be a generic point of  $[0]$  over  $\mathbb{Q}\langle \mathbf{u} \rangle$ , where  $\mathbf{u}$  is defined in (3). Let

$$\zeta_i = - \sum_{k=1}^{l_i} u_{ik} \frac{M_{ik}(\eta)}{M_{i0}(\eta)} \quad (i = 0, 1, \dots, n). \quad (4)$$

Then we claim that  $\theta = (\eta_1, \dots, \eta_n; \zeta_0, u_{01}, \dots, u_{0l_0}; \dots; \zeta_n, u_{n1}, \dots, u_{nl_n})$  is a generic point of  $([\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m})$ , which follows that  $([\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m})$  is a prime differential ideal.

Denote  $\mathbb{P}_i^N = M_i \mathbb{P}_i$  ( $i = 0, \dots, n$ ) where  $M_i$  are Laurent differential monomials. Clearly,  $\mathbb{P}_i^N = M_i \mathbb{P}_i$  vanishes at  $\theta$  ( $i = 0, \dots, n$ ). For any  $f \in ([\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m})$ ,

there exists an  $M \in \mathfrak{m}$  such that  $Mf \in [\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N]$ . It follows that  $f(\theta) = 0$ . Conversely, let  $f$  be any differential polynomial in  $\mathbb{Q}\{\mathbb{Y}, \mathbf{u}_0, \dots, \mathbf{u}_n\}$  satisfying  $f(\theta) = 0$ . Clearly,  $\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N$  constitute an autoreduced set with  $u_{i0}$  as leaders. Let  $f_1$  be the differential remainder of  $f$  w.r.t. this autoreduced set. Then  $f_1$  is free from  $u_{i0}$  ( $i = 0, \dots, n$ ) and there exist  $k_i \geq 0$  such that  $\prod_{i=0}^n (M_i M_{i0})^{k_i} \cdot f \equiv f_1 \pmod{[\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N]}$ . Clearly,  $f_1(\theta) = 0$ . Since  $f_1 \in \mathbb{Q}\{\mathbf{u}, \mathbb{Y}\}$ ,  $f_1 = 0$ . Thus,  $f \in [\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$ . So  $[\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$  is a prime differential ideal with  $\theta$  as its generic point.

Consequently,  $([\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}) \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is a prime differential ideal with a generic point  $\zeta = (\zeta_0, u_{01}, \dots, u_{0l_0}; \dots; \zeta_n, u_{n1}, \dots, u_{nl_n})$ . From (4), it is clear that  $\text{d.tr.deg } \mathbb{Q}\langle\zeta\rangle/\mathbb{Q} \leq \sum_{i=0}^n l_i + n$ . If there exist pairs  $(i_k, j_k)$  ( $k = 1, \dots, n$ ) with  $1 \leq j_k \leq l_{i_k}$  and  $i_{k_1} \neq i_{k_2}$  ( $k_1 \neq k_2$ ) such that  $\frac{M_{i_1 j_1}}{M_{i_1 0}}, \dots, \frac{M_{i_n j_n}}{M_{i_n 0}}$  are differentially independent over  $\mathbb{Q}$ , then by Lemma 2.1,  $\zeta_{i_1}, \dots, \zeta_{i_n}$  are differentially independent over  $\mathbb{Q}\langle\mathbf{u}\rangle$ . It follows that  $\text{d.tr.deg } \mathbb{Q}\langle\zeta\rangle/\mathbb{Q} = \sum_{i=0}^n l_i + n$ . Thus,  $([\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}) \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is of codimension 1.

Conversely, assume that  $([\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}) \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is of codimension 1. That is,  $\text{d.tr.deg } \mathbb{Q}\langle\zeta\rangle/\mathbb{Q} = \sum_{i=0}^n l_i + n$ . We want to show that there exist pairs  $(i_k, j_k)$  ( $k = 1, \dots, n$ ) with  $1 \leq j_k \leq l_{i_k}$  and  $i_{k_1} \neq i_{k_2}$  ( $k_1 \neq k_2$ ) such that  $\frac{M_{i_1 j_1}}{M_{i_1 0}}, \dots, \frac{M_{i_n j_n}}{M_{i_n 0}}$  are differentially independent over  $\mathbb{Q}$ . Suppose the contrary, i.e.,  $\frac{M_{i_1 j_1}(\eta)}{M_{i_1 0}(\eta)}, \dots, \frac{M_{i_n j_n}(\eta)}{M_{i_n 0}(\eta)}$  are differentially dependent for any  $n$  different  $i_k$  and  $j_k \in \{1, \dots, l_{i_k}\}$ . Since each  $\zeta_{i_k}$  is a linear combination of  $\frac{M_{i_k j_k}(\eta)}{M_{i_k 0}(\eta)}$  ( $j_k = 1, \dots, l_{i_k}$ ), it follows that  $\zeta_{i_1}, \dots, \zeta_{i_n}$  are differentially dependent over  $\mathbb{Q}\langle\mathbf{u}\rangle$ . Thus, we have  $\text{d.tr.deg } \mathbb{Q}\langle\zeta\rangle/\mathbb{Q} < \sum_{i=0}^n l_i + n$ , a contradiction to the hypothesis.  $\square$

Combining Lemma 3.8 and Theorem 3.9, we have

**Corollary 3.10**  $[\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is a prime differential ideal of codimension one if and only if  $\{\mathbb{P}_i : i = 0, \dots, n\}$  is a Laurent differentially essential system.

Now suppose  $\{\mathbb{P}_0, \dots, \mathbb{P}_n\}$  is a Laurent differentially essential system. Denote the differential ideal  $[\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  by  $\mathcal{I}$ . Since  $\mathcal{I}$  is of codimension one, by Lemma 2.3, there exists an irreducible differential polynomial  $\mathbf{R}(\mathbf{u}; u_{00}, \dots, u_{n0}) = \mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n) \in \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  such that

$$[\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\} = \text{sat}(\mathbf{R}) \quad (5)$$

where  $\text{sat}(\mathbf{R})$  is the *saturation ideal* of  $\mathbf{R}$ . More explicitly,  $\text{sat}(\mathbf{R})$  is the whole set of differential polynomials having zero differential remainders w.r.t.  $\mathbf{R}$  under any ranking endowed on  $\mathbf{u}_0, \dots, \mathbf{u}_n$ .

Now the definition of sparse differential resultant is given as follows:

**Definition 3.11**  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n) \in \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  in (5) is defined to be the sparse differential resultant of the Laurent differentially essential system  $\mathbb{P}_0, \dots, \mathbb{P}_n$ , denoted by  $\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}$  or  $\text{Res}_{\mathbb{P}_0, \dots, \mathbb{P}_n}$ . And when all the  $\mathcal{A}_i$  are equal to the same  $\mathcal{A}$ , we simply denote it by  $\text{Res}_{\mathcal{A}}$ .

From the proof of Theorem 3.9 and equation (5),  $\mathbf{R}$  has the following useful property.

**Corollary 3.12** *Let  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  be the sparse differential resultant of  $\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n$ . Then  $\text{sat}(\mathbf{R}) \subset \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is a prime differential ideal with a generic zero  $(\mathbf{u}; \zeta_0, \dots, \zeta_n)$ , where  $\zeta_i$  are defined in (4).*

We give five examples which will be used throughout the paper.

**Example 3.13** *Let  $n = 2$  and  $\mathbb{P}_i$  has the form*

$$\mathbb{P}_i = u_{i0}y_1'' + u_{i1}y_1''' + u_{i2}y_2''' \quad (i = 0, 1, 2).$$

*It is easy to show that  $y_1'''/y_1''$  and  $y_2'''/y_1''$  are differentially independent over  $\mathbb{Q}$ . Thus,  $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$  form a Laurent differentially essential system. The sparse differential resultant is*

$$\mathbf{R} = \text{Res}_{\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2} = \begin{vmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{vmatrix}.$$

*Pay attention to the fact that  $\mathbf{R}$  does not belong to the differential ideal generated by  $\mathbb{P}_i$  in  $\mathbb{Q}\{\mathbb{Y}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$  because each  $\mathbb{P}_i$  is homogenous in  $y_1'', y_1''', y_2'''$  and  $\mathbf{R}$  does not involve  $\mathbb{Y}$ . That is why we use the ideal  $([\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2] : \mathfrak{m}) \subset \mathbb{Q}\{\mathbb{Y}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$  rather than  $[\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2] \subset \mathbb{Q}\{\mathbb{Y}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$  in Theorem 3.9. Of course,  $\mathbf{R}$  does belong to  $[\mathbb{P}_0, \dots, \mathbb{P}_n]$  when regarded as a differential ideal of the Laurent differential polynomial ring  $\mathbb{Q}\{\mathbb{Y}, \mathbb{Y}^{-1}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$ .*

The following example shows that for a Laurent differentially essential system, its sparse differential resultant may not involve the coefficients of some  $\mathbb{P}_i$ .

**Example 3.14** *Let  $n = 2$  and  $\mathbb{P}_i$  has the form*

$$\mathbb{P}_0 = u_{00} + u_{01}y_1y_1', \quad \mathbb{P}_1 = u_{10} + u_{11}y_1, \quad \mathbb{P}_2 = u_{10} + u_{11}y_2'.$$

*Clearly,  $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$  form a Laurent differentially essential system. And the sparse differential resultant of  $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$  is*

$$\mathbf{R} = u_{01}u_{10}(u_{11}u_{10}' - u_{10}u_{11}') + u_{00}u_{11}^3,$$

*which is free from the coefficients of  $\mathbb{P}_2$ .*

**Example 3.15** *Let  $\mathcal{A}_0 = \{\mathbf{1}, y_1y_2\}$ ,  $\mathcal{A}_1 = \{\mathbf{1}, y_1y_2'\}$  and  $\mathcal{A}_2 = \{\mathbf{1}, y_1'y_2'\}$ . It is easy to verify that  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$  form a Laurent differentially essential system. And  $\text{Res}_{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2} = u_{10}u_{01}u_{21}u_{11}u_{00}' - u_{10}u_{00}u_{11}u_{21}u_{01}' - u_{01}^2u_{21}u_{10}^2 - u_{01}u_{00}u_{11}^2u_{20}$ .*

**Example 3.16** *Let  $n = 1$  and  $\mathcal{A}_0 = \mathcal{A}_1 = \{y_1^2, (y_1')^2, y_1y_1'\}$ . Clearly,  $\mathcal{A}_0, \mathcal{A}_1$  form a Laurent differentially essential system and  $\text{Res}_{\mathcal{A}} = u_{11}^2u_{00}^2 - 2u_{01}u_{10}u_{11}u_{00} + u_{01}^2u_{10}^2 - u_{12}u_{02}u_{11}u_{00} - u_{12}u_{02}u_{01}u_{10} + u_{12}^2u_{01}u_{00} + u_{10}u_{11}u_{02}^2$ .*

**Example 3.17** *Let  $n = 1$  and  $\mathcal{A}_0 = \mathcal{A}_1 = \{y_1, y_1', y_1^2\}$ . Clearly,  $\mathcal{A}_0, \mathcal{A}_1$  form a Laurent differentially essential system and  $\text{Res}_{\mathcal{A}} = -u_{12}u_{01}u_{00}u_{10} - u_{12}u_{01}^2u_{10}' + u_{12}u_{01}u_{11}'u_{00} + u_{12}u_{01}u_{11}u_{00}' - u_{11}u_{02}u_{00}u_{10} + u_{11}u_{02}u_{10}'u_{01} + u_{02}u_{01}u_{10}^2 - u_{11}^2u_{02}u_{00}' + u_{11}u_{02}u_{01}'u_{10} + u_{11}u_{00}^2u_{12} + u_{11}^2u_{02}'u_{00} - u_{11}u_{02}'u_{01}u_{10} - u_{11}u_{01}u_{12}'u_{00} + u_{01}^2u_{12}'u_{10} - u_{11}u_{01}'u_{12}u_{00} - u_{11}'u_{02}u_{01}u_{10}$ .*

**Remark 3.18** When all the  $\mathcal{A}_i$  ( $i = 0, \dots, n$ ) are sets of differential monomials, unless explicitly mentioned, we always consider  $\mathbb{P}_i$  as Laurent differential polynomials. But when we regard  $\mathbb{P}_i$  as differential polynomials,  $\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}$  is also called the sparse differential resultant of the differential polynomials  $\mathbb{P}_i$ . In this paper, sometimes we regard  $\mathbb{P}_i$  as differential polynomials where we will highlight it.

We now define the sparse differential resultant for any set of specific Laurent differential polynomials over a Laurent differentially essential system. For any finite set  $\mathcal{A}$  of Laurent differential monomials, denote by  $\mathcal{L}(\mathcal{A})$  the set of Laurent differential polynomials of the form  $\sum_{M \in \mathcal{A}} a_M M$  where  $a_M \in \mathcal{E}$ . Then  $\mathcal{L}(\mathcal{A})$  can be considered as the affine space  $\mathcal{E}^l$  or the projective space  $\mathbf{P}(l-1)$  over  $\mathcal{E}$  where  $l = |\mathcal{A}|$ .

**Definition 3.19** Let  $\mathcal{A}_i = \{M_{i0}, M_{i1}, \dots, M_{il_i}\}$  ( $i = 0, 1, \dots, n$ ) be finite sets of Laurent differential monomials which form a Laurent differentially essential system. Consider  $n+1$  Laurent differential polynomials  $(F_0, F_1, \dots, F_n) \in \prod_{i=0}^n \mathcal{L}(\mathcal{A}_i)$ . The sparse differential resultant of  $F_0, F_1, \dots, F_n$ , denoted as  $\text{Res}_{F_0, \dots, F_n}$ , is obtained by replacing  $\mathbf{u}_i$  by the corresponding coefficient vector of  $F_i$  in  $\text{Res}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  which is the sparse differential resultant of the  $n+1$  generic Laurent differential polynomials in (2).

We will show in the next Section 3.3 that the sparse differential resultant  $\text{Res}_{F_0, \dots, F_n} = 0$  will approximately measure whether or not the over-determined equation system  $F_i = 0$  ( $i = 0, \dots, n$ ) have a common non-polynomial solution.

### 3.3 Necessary and sufficient condition for existence of non-polynomial solutions

In the algebraic case, the resultant gives a necessary and sufficient condition for a system of homogenous polynomials to have common solutions. We will show that this is also true for sparse differential resultants in certain sense.

To be more precise, we first introduce some notations. Let  $\mathcal{A}_0, \dots, \mathcal{A}_n$  be a Laurent differentially essential system of monomial sets. Each element  $(F_0, \dots, F_n) \in \mathcal{L}(\mathcal{A}_0) \times \dots \times \mathcal{L}(\mathcal{A}_n)$  can be represented by one and only one point  $(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathcal{E}^{l_0+1} \times \dots \times \mathcal{E}^{l_n+1}$  where  $\mathbf{v}_i = (v_{i0}, v_{i1}, \dots, v_{il_i})$  is the coefficient vector of  $F_i$ . Let  $\mathcal{Z}_0(\mathcal{A}_0, \dots, \mathcal{A}_n)$  be the subset of  $\mathcal{E}^{l_0+1} \times \dots \times \mathcal{E}^{l_n+1}$  consisting of points  $(\mathbf{v}_0, \dots, \mathbf{v}_n)$  such that the corresponding  $F_i = 0$  ( $i = 0, \dots, n$ ) have non-polynomial common solutions. That is,

$$\mathcal{Z}_0(\mathcal{A}_0, \dots, \mathcal{A}_n) = \{(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathcal{E}^{l_0+1} \times \dots \times \mathcal{E}^{l_n+1} : F_0 = \dots = F_n = 0 \text{ have a common non-polynomial solution in } (\mathcal{E}^\wedge)^n\}. \quad (6)$$

The following result shows that the vanishing of sparse differential resultant gives a necessary condition for the existence of non-polynomial solutions.

**Lemma 3.20**  $\mathcal{Z}_0(\mathcal{A}_0, \dots, \mathcal{A}_n) \subseteq \mathbb{V}(\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}))$ .

---

<sup>1</sup>Here, we can also consider the differential projective space  $\mathbf{P}(l_i)$  over  $\mathcal{E}$



*Proof:* Let  $\mathbb{P}_0, \dots, \mathbb{P}_n$  be a generic Laurent differentially essential system corresponding to  $\mathcal{A}_0, \dots, \mathcal{A}_n$  with coefficient vectors  $\mathbf{u}_0, \dots, \mathbf{u}_n$ . By (5),  $[\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\} = \text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})$ . For any point  $(\mathbf{v}_0, \dots, \mathbf{v}_n) \in Z_0(\mathcal{A}_0, \dots, \mathcal{A}_n)$ , let  $(\overline{\mathbb{P}}_0, \dots, \overline{\mathbb{P}}_n) \in \mathcal{L}(\mathcal{A}_0) \times \dots \times \mathcal{L}(\mathcal{A}_n)$  be the differential polynomial system represented by  $(\mathbf{v}_0, \dots, \mathbf{v}_n)$ . Let  $G$  be any differential polynomial in  $\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})$ . Then  $G(\mathbf{v}_0, \dots, \mathbf{v}_n) \in [\overline{\mathbb{P}}_0, \dots, \overline{\mathbb{P}}_n] \subset \mathcal{E}\{\mathbb{Y}, \mathbb{Y}^{-1}\}$ . Since  $\overline{\mathbb{P}}_0, \dots, \overline{\mathbb{P}}_n$  have a non-polynomial common zero,  $G(\mathbf{v}_0, \dots, \mathbf{v}_n)$  should be zero. Thus,  $\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})$  vanishes at  $(\mathbf{v}_0, \dots, \mathbf{v}_n)$ .  $\square$

**Example 3.21** *Continue from Example 3.13. Suppose  $\mathcal{F} = \mathbb{Q}(x)$  and  $\delta = \frac{d}{dx}$ . In this example, we have  $\text{Res}_{\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2} \neq 0$ . But  $y_1 = c_{11}x + c_{10}, y_2 = c_{22}x^2 + c_{21}x + c_{20}$  consist of a non-zero solution of  $\mathbb{P}_0 = \mathbb{P}_1 = \mathbb{P}_2 = 0$  where  $c_{ij}$  are distinct arbitrary constants. This shows that Lemma 3.20 is not correct if we do not consider non-polynomial solutions. This example also shows why we need to consider non-polynomial differential solutions, or equivalently why we consider Laurent differential polynomials instead of usual differential polynomials.*

Let  $\overline{\mathcal{Z}_0(\mathcal{A}_0, \dots, \mathcal{A}_n)}$  be the Kolchin differential closure of  $\mathcal{Z}_0(\mathcal{A}_0, \dots, \mathcal{A}_n)$  in  $\mathcal{E}^{l_0+1} \times \dots \times \mathcal{E}^{l_n+1}$ . Then we have the following theorem which gives another characterization for the sparse differential resultant.

**Theorem 3.22** *Suppose the Laurent differential monomial sets  $\mathcal{A}_i (i = 0, \dots, n)$  form a Laurent differentially essential system. Then  $\mathcal{Z}(\mathcal{A}_0, \dots, \mathcal{A}_n) = \mathbb{V}(\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}))$ .*

*Proof:* Firstly, by Lemma 3.20,  $\mathcal{Z}_0(\mathcal{A}_0, \dots, \mathcal{A}_n) \subseteq \mathbb{V}(\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}))$ . So  $\mathcal{Z}(\mathcal{A}_0, \dots, \mathcal{A}_n) = \overline{\mathcal{Z}_0(\mathcal{A}_0, \dots, \mathcal{A}_n)} \subseteq \mathbb{V}(\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}))$ .

For the other direction, follow the notations in the proof of Theorem 3.9. By Theorem 3.9,  $[\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$  is a prime differential ideal with a generic point  $(\eta, \zeta)$  where  $\eta = (\eta_1, \dots, \eta_n)$  is a generic point of  $[0]$  over  $\mathbb{Q}\langle (u_{ik})_{i=0, \dots, n; k \neq 0} \rangle$  and  $\zeta = (\zeta_0, u_{01}, \dots, u_{0l_0}; \dots; \zeta_n, u_{n1}, \dots, u_{nl_n})$ . Let  $(F_0, \dots, F_n) \in \mathcal{L}(\mathcal{A}_0) \times \dots \times \mathcal{L}(\mathcal{A}_n)$  be a set of Laurent differential polynomials represented by  $\zeta$ . Clearly,  $\eta$  is a non-polynomial solution of  $F_i = 0$ . Thus,  $\zeta \in \mathcal{Z}_0(\mathcal{A}_0, \dots, \mathcal{A}_n) \subset \mathcal{Z}(\mathcal{A}_0, \dots, \mathcal{A}_n)$ . By Corollary 3.12,  $\zeta$  is a generic point of  $\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})$ . It follows that  $\mathbb{V}(\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})) \subseteq \mathcal{Z}(\mathcal{A}_0, \dots, \mathcal{A}_n)$ . As a consequence,  $\mathbb{V}(\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})) = \mathcal{Z}(\mathcal{A}_0, \dots, \mathcal{A}_n)$ .  $\square$

The above theorem shows that the sparse differential resultant gives a sufficient and necessary condition for a differentially essential system to have non-polynomial solutions over an open set of  $\prod_{i=0}^n \mathcal{L}(\mathcal{A}_i)$  in the sense of Kolchin topology.

With Theorem 3.22, property 3) of Theorem 1.2 is proved.

## 4 Criterion for Laurent differentially essential system in terms of supports

Let  $\mathcal{A}_i (i = 0, \dots, n)$  be finite sets of Laurent differential monomials. According to Definition 3.6, in order to check whether they form a Laurent differentially essential system, we need to check whether there exist  $M_{ik_i}, M_{ij_i} \in \mathcal{A}_i (i = 0, \dots, n)$  such that  $\text{d.tr.deg } \mathbb{Q}\langle M_{0k_0}/M_{0j_0}, \dots, M_{nk_n}/M_{nj_n} \rangle / \mathbb{Q} = n$ . This can be done with the differential characteristic set method

via symbolic computation [45, 3, 48, 16]. In this section, a criterion will be given to check whether a Laurent differential system is essential in terms of their supports, which is conceptually and computationally simpler than the naive approach based on the characteristic set method.

#### 4.1 Sets of Laurent differential monomials in reduced and T-shape forms

In this section, two types of Laurent differential monomial sets are introduced, whose differential transcendence degrees are easy to compute.

Let  $B_1, B_2, \dots, B_m$  be  $m$  Laurent differential monomials, where  $B_i = \prod_{j=1}^n \prod_{k \geq 0} (y_j^{(k)})^{d_{ijk}}$ . For each  $j \in \{1, \dots, n\}$ , let  $q_j = \max_{i=1}^m \text{ord}(B_i, y_j)$ . Let  $x_1, \dots, x_n$  be new algebraic indeterminates and

$$d_{ij} = \sum_{k=0}^{q_j} d_{ijk} x_j^k \quad (i = 1, \dots, m, j = 1, \dots, n)$$

univariate polynomials in  $\mathbb{Z}[x_j]$  respectively. If  $\text{ord}(B_i, y_j) = -\infty$ , then  $d_{ij} = 0$  and we denote  $\deg(d_{ij}, x_j) = -\infty$ . The vector  $(d_{i1}, d_{i2}, \dots, d_{in})$  is called the *symbolic support vector* of  $B_i$ . The following  $m \times n$  matrix

$$M = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ & & \ddots & \\ d_{m1} & d_{m2} & \dots & d_{mn} \end{pmatrix}$$

is called the *symbolic support matrix* of  $B_1, \dots, B_m$ .

Note that there is a one-to-one correspondence between Laurent differential monomials and their symbolic support vectors, so we will not distinguish these two concepts if there is no confusion. The same is true for a set of Laurent differential monomials and its symbolic support matrix.

**Definition 4.1** *A set of Laurent differential monomials  $B_1, B_2, \dots, B_m$  or its symbolic support matrix  $M$  is called reduced if for each  $i \leq \min(m, n)$ ,  $-\infty \neq \text{ord}(B_i, y_i) > \text{ord}(B_{i+k}, y_i)$ , or equivalently  $-\infty \neq \deg(d_{ii}, x_i) > \deg(d_{i+k,i}, x_i)$ , holds for all  $k > 0$ .*

Note that a reduced symbolic support matrix is always of full rank since the term  $\prod_{i=1}^{\min(m,n)} x_i^{\text{ord}(B_i, y_i)}$  will appear effectively in the determinant of the  $\min(m, n)$ -th principal minor when expanded.

**Example 4.2** *Let  $B_1 = y_1^2 y_1'' y_2'$ ,  $B_2 = y_1^3 (y_2')^2 y_3 (y_3')^2$ ,  $B_3 = y_1' y_3'$ . Then  $q_1 = 2, q_2 = 1, q_3 = 1$ , and*

$$M = \begin{pmatrix} x_1^2 + 2 & x_2 & 0 \\ 3 & 2x_2 & 2x_3 + 1 \\ x_1 & 0 & x_3 \end{pmatrix}$$

*is reduced.*

Before giving the property of reduced symbolic support matrices, the following simple result about the differential transcendence degree will be proved.

**Lemma 4.3** For  $\eta_1, \eta_2$  in an extension field of  $\mathbb{Q}$ ,  $\text{d.tr.deg } \mathbb{Q}\langle \eta_1^{a_1}, \eta_1^{a_2} \eta_2 \rangle / \mathbb{Q} = \text{d.tr.deg } \mathbb{Q}\langle \eta_1, \eta_2 \rangle / \mathbb{Q}$ , where  $a_1, a_2$  are non-zero rational numbers.

*Proof:* For any non-zero integer  $p$ , we have

$$\begin{aligned} \text{d.tr.deg } \mathbb{Q}\langle \eta_1, \eta_2 \rangle / \mathbb{Q} &= \text{d.tr.deg } \mathbb{Q}\langle \eta_1, \eta_2 \rangle / \mathbb{Q}\langle \eta_1^p, \eta_2 \rangle + \text{d.tr.deg } \mathbb{Q}\langle \eta_1^p, \eta_2 \rangle / \mathbb{Q} \\ &= \text{d.tr.deg } \mathbb{Q}\langle \eta_1^p, \eta_2 \rangle / \mathbb{Q}. \end{aligned}$$

So for each  $a \in \mathbb{Q} \setminus \{0\}$ ,  $\text{d.tr.deg } \mathbb{Q}\langle \eta_1^a, \eta_2 \rangle / \mathbb{Q} = \text{d.tr.deg } \mathbb{Q}\langle \eta_1, \eta_2 \rangle / \mathbb{Q}$ . Let  $a_i = p_i/q_i$  ( $i = 1, 2$ ) where  $p_i, q_i$  are non-zero integers. Then,

$$\begin{aligned} \text{d.tr.deg } \mathbb{Q}\langle \eta_1^{a_1}, \eta_1^{a_2} \eta_2 \rangle / \mathbb{Q} &= \text{d.tr.deg } \mathbb{Q}\langle \eta_1^{1/q_2}, \eta_1^{p_2/q_2} \eta_2 \rangle / \mathbb{Q} \\ &= \text{d.tr.deg } \mathbb{Q}\langle \eta_1^{1/q_2}, \eta_2 \rangle / \mathbb{Q} \text{ (for } \mathbb{Q}\langle \eta_1^{1/q_2}, \eta_1^{p_2/q_2} \eta_2 \rangle = \mathbb{Q}\langle \eta_1^{1/q_2}, \eta_2 \rangle) \\ &= \text{d.tr.deg } \mathbb{Q}\langle \eta_1, \eta_2 \rangle / \mathbb{Q}. \end{aligned}$$

□

**Theorem 4.4** Let  $B_1, B_2, \dots, B_m$  be  $m$  reduced Laurent differential monomials in  $\mathbb{Y}$ . Then  $\text{d.tr.deg } \mathbb{Q}\langle B_1, B_2, \dots, B_m \rangle / \mathbb{Q} = \min(m, n)$ .

*Proof:* It suffices to prove the case  $m = n$  by the following two facts. In the case  $m > n$ , we need only to prove that  $B_1, \dots, B_n$  are differentially independent. And in the case  $m < n$ , we can treat  $y_{m+1}, \dots, y_n$  as parameters, then  $B_1, B_2, \dots, B_m$  are still reduced Laurent differential monomials. So if we have proved the result for  $m = n$ ,  $\text{d.tr.deg } \mathbb{Q}\langle B_1, B_2, \dots, B_m \rangle / \mathbb{Q} \geq \text{d.tr.deg } \mathbb{Q}\langle y_{m+1}, \dots, y_n \rangle \langle B_1, B_2, \dots, B_m \rangle / \mathbb{Q}\langle y_{m+1}, \dots, y_n \rangle = m$  follows.

Since  $B_1, B_2, \dots, B_n$  are reduced, we have  $o_i = \text{ord}(B_i, y_i) \geq 0$  for  $i \leq n$ . In this proof, a Laurent differential monomial will be treated as an algebraic Laurent monomial, or simply a monomial. Furthermore, the lex order between two monomials induced by the following variable order will be used.

$$\begin{aligned} & \boxed{y_1 > y_1' > \dots > y_1^{(o_1-1)}} \\ & > \boxed{y_2 > y_2' > \dots > y_2^{(o_2-1)}} \\ & > \dots \\ & > \boxed{y_n > y_n' > \dots > y_n^{(o_n-1)} > y_n^{(o_n)} > y_n^{(o_n+1)} > \dots} \\ & > \boxed{y_{n-1}^{(o_{n-1})} > y_{n-1}^{(o_{n-1}+1)} > \dots} \\ & > \dots \\ & > \boxed{y_1^{(o_1)} > y_1^{(o_1+1)} > \dots}. \end{aligned}$$

Under this ordering, we claim that the leading monomial of  $\delta^t B_i$  ( $1 \leq i \leq n, t \in \mathbb{N}$ ) is  $LM_{it} = \frac{B_i * y_i^{(o_i+t)}}{y_i^{(o_i)}}$ . Here by leading monomial, we mean the monomial with the highest order appearing effectively in a polynomial. Let  $B_i = N_i(y_i^{(o_i)})^{D_i}$  ( $1 \leq i \leq n$ ). If  $N_i =$

1, then the monomials of  $\delta^t B_i$  is of the form  $\prod_{k=0}^t (y_i^{(o_i+k)})^{s_k}$ , where  $s_0, \dots, s_t$  are non-negative integers such that  $\sum_{k=0}^t s_k = D_i$  and  $\sum_{k=1}^t k s_k = t$ . Among these monomials, if  $s_k > 0$  for some  $1 \leq k \leq t-1$ , then  $s_0$  is strictly less than  $D_i - 1$  and  $\prod_{k=0}^t (y_i^{(o_i+k)})^{s_k} < (y_i^{(o_i)})^{D_i-1} y_i^{(o_i+t)} = \frac{B_i * y_i^{(o_i+t)}}{y_i^{(o_i)}}$  follows. Hence, in the case  $N_i = 1$ , the claim holds. Now suppose  $N_i \neq 1$ , then it is a product of variables with lex order larger than  $y_i^{(o_i)}$ . Then  $\delta^t B_i = \sum_{k=0}^t \binom{t}{k} \delta^k N_i \delta^{t-k} (y_i^{(o_i)})^{D_i}$ . If  $k = 0$ , then similar to the case  $N_i = 1$ , we can show that the highest monomial in  $N_i \delta^t (y_i^{(o_i)})^{D_i}$  is  $N_i (y_i^{(o_i)})^{D_i-1} y_i^{(o_i+t)}$ . For each  $k > 0$ ,  $\delta^k N_i < N_i$  and  $\delta^k N_i \delta^{t-k} (y_i^{(o_i)})^{D_i} < N_i (y_i^{(o_i)})^{D_i-1} y_i^{(o_i+t)} = \frac{B_i * y_i^{(o_i+t)}}{y_i^{(o_i)}}$ . Hence, the leading monomial of  $\delta^t B_i$  is  $N_i (y_i^{(o_i)})^{D_i-1} y_i^{(o_i+t)} = \frac{B_i * y_i^{(o_i+t)}}{y_i^{(o_i)}}$ .

We claim that these leading monomials  $LM_{it} = \frac{B_i * y_i^{(o_i+t)}}{y_i^{(o_i)}}$  ( $i = 1, \dots, m; t \geq 0$ ) are algebraically independent over  $\mathbb{Q}$ . We prove this claim by showing that the algebraic transcendence degree of these monomials are the same as the number of monomials for any fixed  $t$ . Let  $Y_i = [y_i, y'_i, \dots, y_i^{(o_i-1)}]$ ,  $Y_i^* = [y_i^{(o_i+t+1)}, \dots, y_i^{(q_i+t)}]$ ,  $B_{it} = [B_i, LM_{i1}, \dots, LM_{it}]$  for  $1 \leq i \leq n$ . We denote by  $BY_i = (y_i^{(o_i)})^{D_i}$ ,  $BY_{it} = [(y_i^{(o_i)})^{D_i}, (y_i^{(o_i)})^{D_i-1} y_i^{(o_i+1)}, \dots, (y_i^{(o_i)})^{D_i-1} y_i^{(o_i+t)}]$  for  $1 \leq i \leq n$ . Then, by Lemma 4.3, we have

$$\begin{aligned} n(t+1) &\geq \text{tr.deg } \mathbb{Q}(B_{1t}, B_{2t}, \dots, B_{nt}) / \mathbb{Q} \\ &\geq \text{tr.deg } \mathbb{Q}_1(B_{1t}, B_{2t}, \dots, B_{nt}) / \mathbb{Q}_1 \\ &= \text{tr.deg } \mathbb{Q}_1(BY_{1t}, BY_{2t}, \dots, BY_{nt}) / \mathbb{Q}_1 \\ &= n(t+1) \end{aligned}$$

where  $\mathbb{Q}_1 = \mathbb{Q}(Y_1, \dots, Y_n, Y_1^*, \dots, Y_n^*)$ . Hence, this claim is proved.

Now, we prove that  $B_1, \dots, B_n$  are differentially independent over  $\mathbb{Q}$ . Suppose the contrary, then there exists a nonzero differential polynomial  $P \in \mathbb{Q}\{z_1, \dots, z_n\}$  such that  $P(B_1, \dots, B_n) = 0$ . Let  $P = \sum_k c_k P_k$ , where  $P_k$  is a monomial and  $c_k \in \mathbb{Q} \setminus \{0\}$ . Then, the leading monomial of  $P_k(B_1, \dots, B_n)$  is a product of  $LM_{it}$  ( $i = 1, \dots, n; t \geq 0$ ). We denote this product by  $LMP_k$ , then  $LMP_k \neq LMP_j$  for  $k \neq j$  since these  $LM_{it}$  are algebraically independent. But there exists one and only one product which has the highest order, which can not be eliminated by the others, which means that  $P(B_1, \dots, B_n) \neq 0$ , a contradiction.  $\square$

In general, we cannot reduce a symbolic support matrix to a reduced one. But, in the next section, we will show that any symbolic support matrix can be reduced to T-shape to be defined below.

**Definition 4.5** A set of Laurent differential monomials  $B_1, \dots, B_m$  or their symbolic support matrix  $M$  is said to be in T-shape with index  $(i, j)$ , if there exist  $1 \leq i \leq \min(m, n), 0 \leq j \leq \min(m, n) - i$  such that all elements except those in the first  $i$  rows and the  $i+1, \dots, (i+j)$ -th columns of  $M$  are zeros and the sub-matrix consisting of the first  $i+j$  columns of  $M$  is reduced. The zero sub-matrix  $(Z_1, Z_2)$  in Figure 1 is called the zero sub-matrix of  $M$ .

In Figure 1, an illustrative form of a matrix in T-shape is given, where the sub-matrices  $M_1$  and  $M_2$  of the matrix are reduced ones. It is easy to see that  $M_1$  must be an  $i \times i$  square

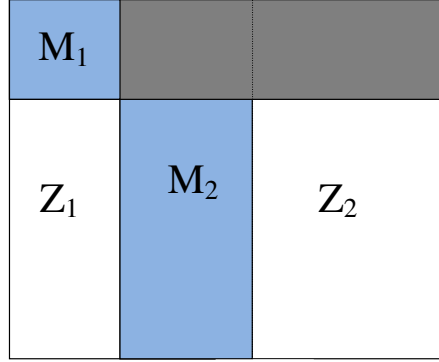


Figure 1: A T-shape Matrix

matrix. Since the first  $i + j$  columns of a T-shape matrix  $M$  is a reduced sub-matrix, we have

**Lemma 4.6** *The rank of a T-shape matrix with index  $(i, j)$  equals to  $i + j$ . Furthermore, a T-shape matrix is reduced if and only if it is of full rank, that is,  $i + j = \min(m, n)$ .*

For a zero matrix  $S$  with  $k$  rows and  $l$  columns whose elements are zeros, we define its 0-rank to be  $k + l$ . A T-shape matrix  $M$  is not of full rank if and only if  $i + j < \min(m, n)$ . As a consequence, we have

**Lemma 4.7** *A T-shape matrix of index  $(i, j)$  is not of full rank if and only if its zero sub-matrix is an  $(m - i) \times (n - j)$  zero matrix with 0-rank  $m + n - i - j \geq \max(m, n) + 1$ .*

The differential transcendence degree of  $m$  Laurent differential monomials in T-shape can be easily determined, as shown by the following result.

**Theorem 4.8** *Let  $B_1, \dots, B_m$  be  $m$  Laurent differential monomials and  $M$  their symbolic support matrix which is in T-shape with index  $(i, j)$ . Then  $\text{d.tr.deg } \mathbb{Q}\langle B_1, B_2, \dots, B_m \rangle / \mathbb{Q} = \text{rk}(M) = i + j$ .*

*Proof:* Since  $M$  is a T-shape matrix with index  $(i, j)$ , by Lemma 4.6, the rank of  $M$  is  $i + j$ .

Deleting the zero columns of the symbolic support matrix of  $B_{i+1}, \dots, B_m$ , we can get a reduced matrix. By Theorem 4.4, we have  $\text{d.tr.deg } \mathbb{Q}\langle B_{i+1}, \dots, B_m \rangle / \mathbb{Q} = j$ . Since the symbolic support matrix of  $B_1, \dots, B_i$  is also a reduced one, by Theorem 4.4, we have  $\text{d.tr.deg } \mathbb{Q}\langle B_1, \dots, B_i \rangle / \mathbb{Q} = i$ . Hence,

$$\begin{aligned}
\text{d.tr.deg } \mathbb{Q}\langle B_1, \dots, B_m \rangle / \mathbb{Q} &= \text{d.tr.deg } \mathbb{Q}\langle B_1, \dots, B_m \rangle / \mathbb{Q}\langle B_{i+1}, \dots, B_m \rangle \\
&\quad + \text{d.tr.deg } \mathbb{Q}\langle B_{i+1}, \dots, B_m \rangle / \mathbb{Q} \\
&\leq \text{d.tr.deg } \mathbb{Q}\langle B_1, \dots, B_i \rangle / \mathbb{Q} + j \\
&= i + j.
\end{aligned}$$

On the other hand, if we treat  $y_{i+1}, \dots, y_{i+j}$  and their derivatives as parameters, the symbolic support matrix of  $B_1, \dots, B_i$  is also a reduced one and the rank of this matrix

is  $i$ . By Theorem 4.4, we have  $\text{d.tr.deg } \mathbb{Q}\langle y_{i+1}, \dots, y_{i+j} \rangle \langle B_1, \dots, B_i \rangle / \mathbb{Q}\langle y_{i+1}, \dots, y_{i+j} \rangle = i$ . Since  $B_{i+1}, \dots, B_m$  are monomials in  $y_{i+1}, \dots, y_{i+j}$  (see Figure 1),  $\mathbb{Q}\langle B_{i+1}, \dots, B_m \rangle \subset \mathbb{Q}\langle y_{i+1}, \dots, y_{i+j} \rangle$ . Hence,

$$\begin{aligned} \text{d.tr.deg } \mathbb{Q}\langle B_1, \dots, B_m \rangle / \mathbb{Q} &= \text{d.tr.deg } \mathbb{Q}\langle B_1, \dots, B_m \rangle / \mathbb{Q}\langle B_{i+1}, \dots, B_m \rangle \\ &\quad + \text{d.tr.deg } \mathbb{Q}\langle B_{i+1}, \dots, B_m \rangle / \mathbb{Q} \\ &\geq \text{d.tr.deg } \mathbb{Q}\langle y_{i+1}, \dots, y_{i+j} \rangle \langle B_1, \dots, B_i \rangle / \mathbb{Q}\langle y_{i+1}, \dots, y_{i+j} \rangle + j \\ &= i + j. \end{aligned}$$

Thus,  $\text{d.tr.deg } \mathbb{Q}\langle B_1, \dots, B_m \rangle / \mathbb{Q} = \text{rk}(M) = i + j$ .  $\square$

## 4.2 An algorithm to reduce Laurent differential monomials to T-shape

In this section, an algorithm is given to reduce any set of Laurent differential monomials to a set of Laurent differential monomials in T-shape, which has the same differential transcendence degree with the original one.

First, we will define the transformations that will be used to reduce any symbolic support matrix to a T-shape one. A  $\mathbb{Q}$ -elementary transformation for a matrix  $M$  consists of two types of matrix row operations and one type of matrix column operations. To be more precise, Type 1 operations consist of interchanging two rows of  $M$ ; Type 2 operations consist of adding a rational number multiple of one row to another; and Type 3 operations consist of interchanging two columns.

Let  $B_1, \dots, B_m$  be Laurent differential monomials and  $M$  their symbolic support matrix. Then  $\mathbb{Q}$ -elementary transformations of  $M$  correspond to certain transformations of the monomials. Indeed, interchanging the  $i$ -th and the  $j$ -th rows of  $M$  means interchanging  $B_i$  and  $B_j$ , and interchanging the  $i$ -th and the  $j$ -th columns of  $M$  means interchanging  $y_i$  and  $y_j$  in  $B_1, \dots, B_m$  (or in the variable order). Multiplying the  $i$ -th row of  $M$  by a rational number  $r$  and adding the result to the  $j$ -th row means changing  $B_j$  to  $B_i^r B_j$ .

**Lemma 4.9** *Let  $B_1, \dots, B_m$  be Laurent differential monomials and  $C_1, \dots, C_m$  obtained by a series of  $\mathbb{Q}$ -elementary transformations from  $B_1, \dots, B_m$ . Then  $\text{d.tr.deg } \mathbb{Q}\langle B_1, \dots, B_m \rangle / \mathbb{Q} = \text{d.tr.deg } \mathbb{Q}\langle C_1, \dots, C_m \rangle / \mathbb{Q}$ .*

*Proof:* It is a direct consequence of Lemma 4.3.  $\square$

Now, an algorithm **RDM**( $M$ ) will be given to reduce a given symbolic support matrix to a T-shape matrix by a series of  $\mathbb{Q}$ -elementary transformations. We sketch the algorithm below. Note that we still denote by  $M$  the matrix obtained by  $\mathbb{Q}$ -elementary transformations from  $M$ . We assume that  $m \leq n$  and hence  $p = \max(m, n) = n$ . The case  $m > n$  can be shown similarly.

Let  $N$  be a sub-matrix of  $M$ . Then the *complementary matrix* of  $N$  in  $M$  is the sub-matrix of  $M$  from which all the rows and columns associated with  $N$  have been removed.

The algorithm consists of three major steps. In the first step, a procedure similar to the Gauss elimination will be used to construct a reduced square sub-matrix  $R$  of  $M$  such that the complementary matrix of  $R$  in  $M$  is a zero matrix. Precisely, choose a column of  $M$ , say the first column, which contains at least one non-zero element. Then, choose an element,

say  $d_{11}$ , of this column, which has the largest degree among all elements in the same column. If there exists a  $d_{i1}, i > 1$  such that  $\deg(d_{i1}) = \deg(d_{11})$ , then replace  $d_{ij}$  by  $d_{ij} - \frac{a_i}{a_1}d_{1j}$  for  $j = 1, \dots, n$ , where  $a_i, a_1$  are the leading coefficients of  $d_{i1}, d_{11}$  respectively. This is a  $\mathbb{Q}$ -elementary transformation of Type 2. Repeat the above procedure until the first column is in reduced form, that is  $\deg(d_{i1}) < \deg(d_{11})$  for  $i = 2, \dots, n$ . Consider the lower-right  $(m-1) \times (n-1)$  sub-matrix  $N$  of  $M$  and repeat the above procedure for  $N$ . In this way, we will obtain a reduced square matrix whose complementary matrix is a zero matrix  $Z$  in the lower-right corner of  $M$ .

In the second step, a recursive procedure is used to construct a reduced form of  $M$ . Let the zero matrix  $Z$  obtained above be an  $i \times j$  matrix. Denote  $r = i + j$  to be the 0-rank of it. If  $j = n$ , the last  $i$  rows of  $M$  are zero rows. Delete the last  $i$  rows from  $M$ , then we have a strictly smaller matrix, which can be treated recursively.

If  $r \geq n + 1$ ,  $M$  cannot be of full row rank, which will be considered in step three. Otherwise, let  $M_C$  be the lower-right  $(m+r-\max(m,n)) \times (n+r-\max(m,n)) = (m+r-n) \times r$  sub-matrix of  $M$ ,  $M_{C1}$  the lower-left  $i \times (n+i-\max(m,n)) = i \times i$  sub-matrix of  $M_C$ , and  $M_{C2}$  the upper-right  $(m+j-\max(m,n)) \times j = (m+j-n) \times j$  sub-matrix of  $M_C$ . In Figure 2(a,b),  $M_C$  is represented by the pink area. Here,  $M_C$  is chosen to be the minimal  $(m-q) \times (n-q)$  sub-matrix of  $M$  at the lower-right corner, which may have full rank.

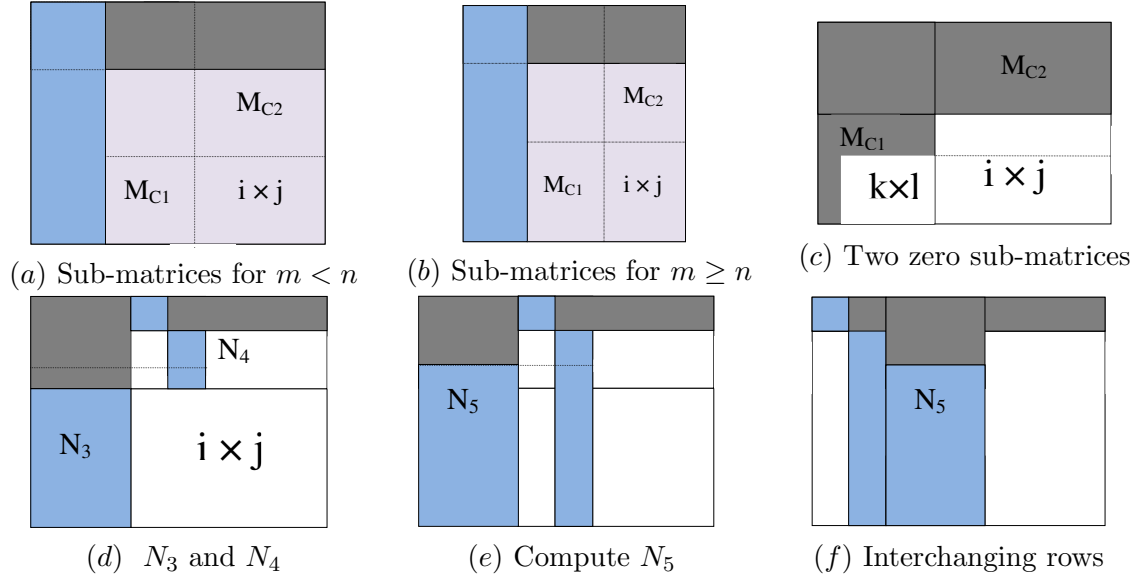


Figure 2: Matrix forms in Algorithm 1, the blue parts are reduced ones

Let  $N_1 = \mathbf{RDM}(M_{C1})$  and  $N_2 = \mathbf{RDM}(M_{C2})$ . Note that the  $\mathbb{Q}$ -elementary transformations of these sub-procedures are for the whole rows and columns of  $M$ . By doing so, the sub-matrix consisting of the first  $n - r$  columns of  $M$  remains to be a reduced one.

If  $N_1$  and  $N_2$  are reduced matrices, we can obtain a reduced matrix for  $M$  by a suitable column interchanging. Otherwise, either  $N_1$  or  $N_2$  is not of full rank. Assume  $N_1$  is not of full rank. Then merging the zero sub-matrix of  $N_1$  and  $Z$ , we obtain a zero matrix with 0-rank larger than that of  $Z$  (Figure 2(c)). Repeat the second step for  $M$  with this new zero sub-matrix.

In the third step,  $M$  contains a “large” zero sub-matrix and a T-shape matrix of  $M$  can be constructed directly as follows. Let the zero matrix  $Z$  at the lower-right corner of  $M$  be an  $i \times j$  matrix and  $r = i + j$ . Let  $M_{C3}$  be the lower-left  $i \times (n - j)$  sub-matrix of  $M$  and  $N_3 = \mathbf{RDM}(M_{C3})$ . In this case,  $M_{C3}$  has more rows than columns. We can assume that  $N_3$  is of full column rank. Otherwise, a sub-matrix of  $N_3$  can be used as  $N_3$ .

Let  $M_{C4}$  be the upper-right  $(m - i) \times j$  sub-matrix of  $M$ ,  $N_4 = \mathbf{RDM}(M_{C4})$ , and  $s = \text{rk}(N_4)$  (see Figure 2(d)). If  $N_4$  is of full row rank, then by suitable column interchangings, we can obtain a T-shape matrix. Otherwise, let the lower-left  $(m - s) \times (n - j)$  sub-matrix of  $M$  be  $M_{C5}$ , and  $N_5 = \mathbf{RDM}(M_{C5})$ , which is a reduced matrix with full column rank, see Figure 2(e). Now, by suitable column interchangings, we can obtain a T-shape matrix (see Figure 2(f)).

The idea of the algorithm is as follows. Try to use the first step to construct a reduced matrix. If the first step fails to do so, use the second step to change the matrix so that it contains a larger zero sub-matrix after each iteration. The procedure will end until either a T-shape matrix is obtained or the matrix has a zero sub-matrix with size larger than  $\max(m, n) + 1$ , in which case a T-shape matrix can be obtained directly.

We now use the following example to illustrate the first two steps of the algorithm.

**Example 4.10** Let  $B_1 = y_1 y_1' y_2''' y_3 y_3'$ ,  $B_2 = y_1^3 (y_1')^2 y_2'' (y_2''')^2 y_3^3 (y_3')^2$ ,  $B_3 = y_1^2 (y_1')^3 y_2' (y_2''')^3 y_3^3 (y_3')^3$ . Then, the symbolic support matrix is

$$M = \begin{pmatrix} x_1 + 1 & x_2^3 & x_3 + 1 \\ 2x_1 + 3 & 2x_2^3 + x_2^2 & 2x_3 + 3 \\ 3x_1 + 2 & 3x_2^3 + x_2 & 3x_3 + 3 \end{pmatrix}.$$

We will use this matrix to illustrate the algorithm.

$$M \xrightarrow{(a)} \begin{pmatrix} x_1 + 1 & x_2^3 & x_3 + 1 \\ 1 & x_2^2 & 1 \\ -1 & x_2 & 0 \end{pmatrix} \xrightarrow{(b)} \begin{pmatrix} x_1 + 1 & x_3 + 1 & x_2^3 \\ 1 & 1 & x_2^2 \\ -1 & 0 & x_2 \end{pmatrix}.$$

The matrix after  $\xrightarrow{(a)}$  is obtained with the first step of the algorithm. We first use  $d_{11} = x_1 + 1$  to reduce the degrees of  $2x_1 + 2$  and  $3x_1 + 3$  with  $\mathbb{Q}$ -elementary transformations of Type 2. Since  $x_2^2$  is of greater degree than  $x_2$ , nothing needs to do. Finally, we obtain a  $1 \times 1$  zero matrix at the lower-right corner at the end of step 1.

Now, goto the second step of the algorithm. We have  $r = 2 < \max(m, n) + 1 = 4$ .  $M_C$  is the lower-right  $2 \times 2$  sub-matrix of  $M$ ,  $M_{C1} = (x_2)$ , and  $M_{C2} = (1)$ .

Since both  $M_{C1}$  and  $M_{C2}$  are reduced, we interchange the second and third columns of  $M$  to obtain the final matrix after  $\xrightarrow{(b)}$ , which is reduced. The corresponding monomials are  $D_1 = y_1 y_1' y_2''' y_3 y_3'$ ,  $D_2 = y_1 y_2'' y_3$ , and  $D_3 = y_2' / y_1$ . It is of T-shape under the variable order  $y_1 > y_3 > y_2$ .

We use the following example to illustrate the third step of the algorithm.



**Example 4.11** Let  $B_1 = y_1'''y_2'''y_3'y_4y_5^2$ ,  $B_2 = y_1''y_2'''y_3'y_4y_5^2$ ,  $B_3 = y_1'y_3y_3'$ ,  $B_4 = y_1'$ ,  $B_5 = y_1^2$ . Then, the symbolic support matrix is  $M$  given below.

$$\begin{aligned}
M &= \begin{pmatrix} x_1^3 & x_2^3 & x_3 & 1 & 2 \\ x_1^2 & x_2^3 & x_3^2 + x_3 & 1 & 2 \\ x_1 & 0 & x_3 + 1 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} \xRightarrow{(c)} \begin{pmatrix} x_1^3 & x_2^3 & x_3 & 1 & 2 \\ -x_1^3 + x_1^2 & 0 & x_3^2 & 0 & 0 \\ x_1 & 0 & x_3 + 1 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\xRightarrow{(d)} \begin{pmatrix} x_1^3 & x_2^3 & x_3 & 1 & 2 \\ -x_1^3 + x_1^2 & 0 & x_3^2 & 0 & 0 \\ x_1 & 0 & x_3 + 1 & 0 & 0 \\ 0 & 0 & -x_3 - 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} \xRightarrow{(e)} \begin{pmatrix} x_2^3 & x_3 & x_1^3 & 1 & 2 \\ 0 & x_3^2 & -x_1^3 + x_1^2 & 0 & 0 \\ 0 & x_3 + 1 & x_1 & 0 & 0 \\ 0 & -x_3 - 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}
\end{aligned}$$

For step 1 of the algorithm, we do nothing to  $M$  and the zero matrix  $Z$  obtained at the end of this step is a  $2 \times 2$  zero sub-matrix at the lower-right corner of  $M$ . In step 2,  $M_C$  is set to be the lower-right  $4 \times 4$  sub-matrix of  $M$ ,  $M_{C1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , and  $M_{C2} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ .

Merging  $Z$  and  $M_{C1}$ , we obtain a  $2 \times 4$  zero sub-matrix at the lower-right corner of  $M$ . Up to now,  $M$  is not changed. Then, step 3 of the algorithm is applied.

In step 3, we have  $M_{C3} = \begin{pmatrix} x_1 \\ 2 \end{pmatrix}$ . Since  $M_{C3}$  is reduced and of full rank, we execute case 1 by setting  $M_{C4} = \begin{pmatrix} x_2^3 & x_3 & 1 & 2 \\ x_2^3 & x_3^2 + x_3 & 1 & 2 \\ 0 & x_3 + 1 & 0 & 0 \end{pmatrix}$  and  $N_4 = \mathbf{RDM}(M_{C4})$  which is a  $T$ -

shape matrix with index  $(1, 1)$  and is not of full rank. Now,  $M$  becomes the matrix after  $\xRightarrow{(c)}$ , which contains  $N_4$ . Since  $N_4$  is not of full rank, let  $M_{C5} = (x_1, x_1, 2)^T$  and compute  $N_5 = \mathbf{RDM}(M_{C5})$ . Now  $M$  becomes the matrix after  $\xRightarrow{(d)}$ . We interchange the first column and the 2, 3-th columns of  $M$  to obtain the final matrix which is in  $T$ -shape with index  $(1, 2)$ .

The corresponding monomials are  $D_1 = y_1'''y_2'''y_3'y_4y_5^2$ ,  $D_2 = y_1''y_3''/y_1'''$ ,  $D_3 = y_1'y_3y_3'$ ,  $D_4 = 1/(y_3y_3')$ ,  $D_5 = y_1^2$ . It is of  $T$ -shape under the variable order  $y_2 > y_3 > y_1 > y_4 > y_5$ .

---

**Algorithm 1 —  $\mathbf{RDM}(M)$** 

---

**Input:** Laurent differential monomials  $B_1, \dots, B_m$  in  $\mathbb{Y}$  or their symbolic support matrix

$$M = (d_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}.$$

**Output:** A T-shape matrix which is obtained from  $M$  by  $\mathbb{Q}$ -elementary transformations.

**Initial:** Let  $s = 1$ ,  $p = \max(m, n)$ .

1. While  $s \leq \min(m, n)$  do
  - 1.1 If for any  $j, l \geq s$ ,  $\deg(d_{jl}) = -\infty$ , let  $i = m - s + 1, j = n - s + 1$  and go to Step 2.
  - 1.2 Select  $j, l \geq s$  such that  $-\infty \neq \deg(d_{jl}) \geq \deg(d_{il})$  for any  $i \geq s$ . Interchange the  $j$ -th row and the  $s$ -th row, the  $l$ -th column and the  $s$ -th column of  $M$ . Using  $d_{ss}$  to do  $\mathbb{Q}$ -elementary transformations such that  $\deg(d_{ss}) > \deg(d_{is})$  for  $i > s$ .
  - 1.3  $s = s + 1$ .
2. Let  $r = i + j$  be the 0-rank of the  $i \times j$  zero sub-matrix in the lower-right side of  $M$ .
  - 2.1 If  $M$  is already a T-shape matrix, return  $M$ .
  - 2.2 If  $j = n$ , delete the last  $i$  rows from  $M$ , and let  $N = \mathbf{RDM}(M)$ .  
Then add  $i$  rows of zeros at the bottom of  $N$  and return this matrix.
  - 2.3 If  $r \geq p + 1$ , go to Step 3.
  - 2.4 Let  $M_C$  be the lower-right  $(m + r - p) \times (n + r - p)$  sub-matrix of  $M$ .  
Let the lower-left  $i \times (n + i - p)$  sub-matrix of  $M_C$  be  $M_{C1}$  and the upper-right  $(m + j - p) \times j$  sub-matrix of  $M_C$  be  $M_{C2}$ . (see (a, b) of Fig. 2)
  - 2.5 Let  $N_1 = \mathbf{RDM}(M_{C1})$  and  $N_2 = \mathbf{RDM}(M_{C2})$ .
  - 2.6 If  $N_1, N_2$  are reduced matrices, interchange the  $p - r + 1$  to  $n - j$  columns and the  $n - j + 1$  to  $n + m - p$  columns of  $M$ , return the obtained T-shape (reduced) matrix.
  - 2.7 If the  $k \times l$  zero sub-matrix  $Z_1$  of  $N_1$  has 0-rank  $k + l \geq \max(i, n + i - p) + 1 = i + 1$ , combine  $Z_1$  and the  $i \times j$  zero matrix to obtain a  $k \times (l + j)$  zero matrix with 0-rank  $k + l + j > i + j$  (see (c) of Fig. 2). Let  $i = k, j = l + j$ , go to Step 2.
  - 2.8 Else, the  $k \times l$  zero sub-matrix  $Z_2$  of  $N_2$  has 0-rank  $k + l \geq \max(m + j - p, j) + 1 = j + 1$ , combine  $Z_2$  and the  $i \times j$  zero matrix to obtain a  $(k + i) \times l$  zero matrix with 0-rank  $k + l + i > i + j$ . Let  $i = k + i, j = l$ , go to Step 2.
3. Let  $M_{C3}$  be the lower-left  $i \times (n - j)$  sub-matrix of  $M$  and  $N_3 = \mathbf{RDM}(M_{C3})$  with index  $(k, l)$ .
  - 3.1 If  $l = 0$ , delete the last  $i - k$  rows from  $M$ , let  $N = \mathbf{RDM}(M)$ , add  $i - k$  zero rows at the bottom of  $N$  and return this matrix.
  - 3.2 If  $N_3$  is not of full rank, put the  $k + 1, \dots, (k + l)$ -th columns as the first  $l$  columns of  $M$ . Let  $i = i - k, j = n - l$ ,  $N_3$  the lower-left  $i \times (n - j)$  sub-matrix of  $M$ .
  - 3.3 Now  $N_3$  is of full column rank. Let the upper-right  $(m - i) \times j$  sub-matrix of  $M$  be  $M_{C4}$ ,  $N_4 = \mathbf{RDM}(M_{C4})$ , and  $s = \text{rk}(N_4)$ .
  - 3.4 Let the lower-left  $(m - s) \times (n - j)$  sub-matrix of  $M$  be  $M_{C5}$  and  $N_5 = \mathbf{RDM}(M_{C5})$ .  
Interchange the first  $n - j$  columns and the  $n - j + 1$  to  $n - j + s$  columns of  $M$ , and return the obtained T-shape matrix. (See (d,e,f) of Fig. 2.)

In this algorithm, the  $\mathbb{Q}$ -elementary transformations in  $\mathbf{RDM}(M_{Ci})$  ( $i = 1, \dots, 5$ ) are also for the whole  $m \times n$  matrix. And in each step the new  $m \times n$  matrix obtained after doing  $\mathbb{Q}$ -elementary transformations is also denoted by  $M$ .

---

The following theorem proves what we claimed before.

**Theorem 4.12** *The symbolic support matrix of any Laurent differential monomials  $B_1, \dots, B_m$  can be reduced to a T-shape matrix by a finite number of  $\mathbb{Q}$ -elementary transformations.*

*Proof:* We assume that  $m \leq n$  and hence  $p = \max(m, n) = n$ . The case  $m > n$  can be proved similarly.

We prove the theorem by induction on the size of the matrix  $M$ , that is,  $m + n$ . One can easily verify that the claim is true when  $m + n = 2, 3, 4$ . Assume it holds for  $m + n \leq s - 1$ , we consider the case  $m + n = s$ .

If a T-shape matrix is obtained in Step 1, then the theorem is proved. Otherwise, let  $Z$  be the  $i \times j$  zero matrix obtained in this step. Since the complementary matrix of  $Z$  in  $M$  is a square matrix, the 0-rank of  $Z$  is larger than  $\max(m, n) - \min(m, n) + 1$ .

In Step 2.2,  $M$  contains zero rows. By deleting these zero rows, the size of  $M$  is decreased by one at least. By induction, the algorithm is valid.

In Step 2.3, from  $r \geq \max(m, n) + 1$ , we have  $r = i + j \geq n + 1$  and  $i > n - j$ . Then the  $i \times (n - j)$  left-lower sub-matrix of  $M$  has more rows than columns. As a consequence, and  $M$  cannot be of full rank.

In Step 2.4,  $M_C$  is chosen as the minimal sub-matrix of  $M$  such that it is of type  $(m - q) \times (n - q)$  which may have full row rank. This implies that  $M_{C1}$  must be an  $i \times i$  square matrix, and hence  $q = n - r$  and  $M_C$  is an  $(m + r - n) \times r$  matrix. Since the complementary matrix of  $Z$  in  $M$  is a square matrix, we have  $j \geq j - i = n - m$ . Hence  $m + r - n \geq i$  and  $M_C$  contains  $Z$  as a sub-matrix for the first loop, and this is always true since  $Z$  is from  $M_C$  and the size of  $M_C$  is increasing for each loop.

In Step 2.5, by the induction hypothesis,  $N_1 = \mathbf{RDM}(M_{C1})$  and  $N_2 = \mathbf{RDM}(M_{C2})$  can be computed. Moreover, the lower-left  $m \times (n - r)$  sub-matrix of  $M$  is always a reduced one although the  $\mathbb{Q}$ -elementary transformations are for the whole rows of  $M$ .

In Step 2.6,  $N_1$  and  $N_2$  are reduced with full rank. The algorithm terminates and returns a reduced matrix by suitable column interchanging given in the algorithm.

In Step 2.7,  $N_1$  is not of full rank. Then by Lemma 4.7, the  $k \times l$  zero sub-matrix of  $N_1$  has 0-rank  $k + l \geq \max(i, n + i - \max(m, n)) + 1 = i + 1$ . The  $i \times j$  zero sub-matrix  $Z$  and this  $k \times l$  zero sub-matrix form a  $k \times (l + j)$  zero-matrix, with 0-rank  $k + j + l \geq i + j + 1$  (Figure 2(c)). Step 2.8 can be considered similarly. Since in each loop of Step 2, the 0-rank of the zero-matrix  $Z$  of  $M$  increases strictly, this loop will terminate.

Step 3 treats the case when  $M$  is not of full rank. Note that  $M_{C3}$  has more rows than columns. Step 3.1 is correct due to the induction hypothesis.

For Step 3.2, since  $N_3$  is not of full rank and does not contain zero rows, we have  $l > 0$  and  $i > k$ . These conditions make the constructions given in the algorithm possible.

In Step 3.3,  $N_3$  is an  $i \times (n - j)$  reduced matrix with full column rank and the lower-right  $i \times j$  sub-matrix of  $M$  is a zero matrix. Due to this condition, the remaining steps are clearly valid. Also note that  $M_{C5}$  is obtained from  $N_3$  by adding several more rows. Then  $M_{C5}$  is also of full column rank and hence  $N_5$  is a reduced matrix of full column rank.  $\square$

**Theorem 4.13** *The differential transcendence degree of the Laurent differential monomials  $B_1, \dots, B_m$  over  $\mathbb{Q}$  equals to the rank of their symbolic support matrix.*

*Proof:* By Lemma 4.9,  $\mathbb{Q}$ -elementary transformations keep the differential transcendence degree. The result follows from Theorems 4.8 and 4.12.  $\square$

Theorem 4.13 can be used to check whether the Laurent polynomial system (2) is differentially essential as shown by the following result.

**Corollary 4.14** *The Laurent differential system (2) is Laurent differentially essential if and only if there exist  $M_{ij_i}$  ( $i = 0, \dots, n$ ) with  $1 \leq j_i \leq l_i$  such that the symbolic support matrix of the Laurent differential monomials  $M_{0j_0}/M_{00}, \dots, M_{nj_n}/M_{n0}$  is of rank  $n$ .*

By Corollary 3.4 of [15], the complexity to compute the determinant of a sub-matrix  $M_s$  of  $M$  with size  $k \times k$  is bounded by  $O(k^{k+2} L \gamma^{\frac{2}{k+3}} \Delta)$ , where  $L = \log \|M_s\|$ ,  $\gamma$  denotes the number of arithmetic operations required for multiplying a scalar vector by the matrix  $M_s$ , and  $\Delta$  is the degree bound of  $M_s$ . So, the complexity to compute the rank of  $M$  is single exponential at most.

**Remark 4.15** *A practical way to check whether the Laurent differential system (2) is Laurent differentially essential is given below.*

- Choose  $n + 1$  monomials  $M_{ij_i}$  ( $i = 0, \dots, n$ ) with  $1 \leq j_i \leq l_i$ .
- Use Algorithm 1 to reduce the symbolic support matrix of  $M_{0j_0}/M_{00}, \dots, M_{nj_n}/M_{n0}$  to a  $T$ -shape matrix  $M$ .
- Use Theorem 4.8 to check whether the rank of  $M$  is  $n$ .
- If the rank of  $M$  is  $n$ , then the system is essential. Otherwise, we need to choose another set of  $n + 1$  monomials and repeat the procedure.

The number of possible choices for the  $n + 1$  monomials is  $\prod_{i=0}^n l_i$ , which is very large. But, the procedure is efficient for two reasons. Firstly, Algorithm 1 is very efficient, since we are essentially doing numerical computation instead of symbolic ones. Secondly, the probability for  $n + 1$  Laurent monomials to have differential transcendence degree  $n$  is very high. As a consequence, we do not need to repeat the procedure for many choices of  $n + 1$  monomials.

By Corollary 4.14, property 3) of Theorem 1.1 is proved.

### 4.3 Differential transcendence degree of generic Laurent differential polynomials

Algorithm 1 and Theorem 4.13 show how to reduce the computation of the differential transcendence degree of a set of Laurent differential monomials to the computation of the rank of their symbolic support matrix. In this section, this result will be extended to compute the differential transcendence degree of a set of generic Laurent differential polynomials.

Consider  $m$  generic Laurent differential polynomials

$$\mathbb{P}_i = u_{i0}M_{i0} + \sum_{k=1}^{l_i} u_{ik}M_{ik} \quad (i = 1, \dots, m), \quad (7)$$

where all the  $u_{ik}$  are differentially independent over  $\mathbb{Q}$ . Let  $\beta_{ik}$  be the symbolic support vector of  $M_{ik}/M_{i0}$ . Then the vector  $w_i = \sum_{k=0}^{l_i} u_{ik}\beta_{ik}$  is called the *symbolic support vector* of  $\mathbb{P}_i$ , and the matrix  $M_{\mathbb{P}}$  with  $w_1, \dots, w_m$  as its rows is called the *symbolic support matrix* of  $\mathbb{P}_1, \dots, \mathbb{P}_m$ . Then, we have the following results.

**Lemma 4.16** *Let  $M_{k_1, \dots, k_m}$  be the symbolic support matrix of the Laurent differential monomials  $(M_{1k_1}/M_{10}, \dots, M_{mk_m}/M_{m0})$ . Then  $\text{rk}(M_{\mathbb{P}}) = \max_{1 \leq k_i \leq l_i} \text{rk}(M_{k_1, \dots, k_m})$ .*

*Proof:* Let the rank of  $M_{\mathbb{P}}$  be  $r$ . Without loss of generality, we assume that the  $r \times r$  leading principal sub-matrix of  $M_{\mathbb{P}}$ , say  $M_{\mathbb{P}, r}$ , is of full rank. By the properties of determinant,  $\det(M_{\mathbb{P}, r}) = \sum_{k_1=1}^{l_1} \cdots \sum_{k_r=1}^{l_r} \prod_{i=1}^r u_{ik_i} D(k_1, \dots, k_r)$  where  $D(k_1, \dots, k_r)$  is the determinant of the  $r \times r$  leading principal sub-matrix of  $M_{k_1, \dots, k_m}$ . So  $\det(M_{\mathbb{P}, r}) \neq 0$  if and only if there exist  $k_1, \dots, k_r$  such that  $D(k_1, \dots, k_r) \neq 0$ . Hence, the rank of  $M_{k_1, \dots, k_m}$  is no less than the rank of  $M_{\mathbb{P}}$ . On the other hand, let  $s = \max_{1 \leq k_i \leq l_i} \text{rk}(M_{k_1, \dots, k_m})$ . Without loss of generality, we assume  $D(k_1, \dots, k_s) \neq 0$ , then,  $\det(M_{\mathbb{P}, s}) \neq 0$ . Hence,  $s$  is no greater than the rank of  $M_{\mathbb{P}}$ .  $\square$

The following result is interesting in that it reduces the computation of differential transcendence degree for a set of generic differential polynomials to the computation of the rank of a matrix, which is analogue to the similar result for linear equations.

**Theorem 4.17**  $\text{d.tr.deg } \mathbb{Q}\langle \cup_{i=1}^m \mathbf{u}_i \rangle \langle \mathbb{P}_1/M_{10}, \dots, \mathbb{P}_m/M_{m0} \rangle / \mathbb{Q}\langle \cup_{i=1}^m \mathbf{u}_i \rangle = \text{rk}(M_{\mathbb{P}})$ , where  $\mathbf{u}_i = (u_{i0}, \dots, u_{il_i})$ .

*Proof:* By Lemma 2.1, the differential transcendence degree of  $\mathbb{P}_1/M_{10}, \dots, \mathbb{P}_m/M_{m0}$  is no less than the maximal differential transcendence degree of  $M_{1k_1}/M_{10}, \dots, M_{mk_m}/M_{m0}$ .

On the other hand, the differential transcendence degree will not increase by linear combinations since  $\text{d.tr.deg } \mathbb{Q}\langle \lambda \rangle \langle a_1 + \lambda \bar{a}_1, a_2, \dots, a_k \rangle / \mathbb{Q}\langle \lambda \rangle \leq \max(\text{d.tr.deg } \mathbb{Q}\langle a_1, a_2, \dots, a_k \rangle / \mathbb{Q}, \text{d.tr.deg } \mathbb{Q}\langle \bar{a}_1, a_2, \dots, a_k \rangle / \mathbb{Q})$  for any differential polynomial  $a_i$  ( $1 \leq i \leq k$ ) and  $\bar{a}_1$ . So, the differential transcendence degree of  $\mathbb{P}_1/M_{10}, \dots, \mathbb{P}_m/M_{m0}$  over  $\mathbb{Q}\langle \cup_{i=1}^m \mathbf{u}_i \rangle$  is no greater than the maximal differential transcendence degree of  $M_{1k_1}/M_{10}, \dots, M_{mk_m}/M_{m0}$ .

So, we have  $\text{d.tr.deg } \mathbb{Q}\langle \cup_{i=1}^m \mathbf{u}_i \rangle \langle \mathbb{P}_1/M_{10}, \dots, \mathbb{P}_m/M_{m0} \rangle / \mathbb{Q}\langle \cup_{i=1}^m \mathbf{u}_i \rangle = \max_{k_1, \dots, k_m} \text{d.tr.deg } \mathbb{Q}\langle M_{1k_1}/M_{10}, \dots, M_{mk_m}/M_{m0} \rangle / \mathbb{Q}$ . By Theorem 4.13 and Lemma 4.16, the differential transcendence degree of  $\mathbb{P}_1/M_{10}, \dots, \mathbb{P}_m/M_{m0}$  equals to the rank of  $M_{\mathbb{P}}$ .  $\square$

By Theorem 4.17, we have the following criterion for system (2) to be differentially essential.

**Corollary 4.18** *The Laurent differential system (2) is Laurent differentially essential if and only if  $\text{rk}(M_{\mathbb{P}}) = n$ .*

The difference between Corollary 4.14 and Corollary 4.18 is that, in the later case we need only to compute the rank of a single matrix whose elements are multivariate polynomials in  $\sum_{i=0}^n (l_i + 1) + n$  variables, while in the former case we have to compute the ranks of  $\prod_{i=0}^n l_i$  matrices whose elements are univariate polynomials in  $n$  separate variables. One also can replace  $u_{ik_i}$  by  $v_i^{k_i}$  in  $M_{\mathbb{P}}$ , where  $v_i$  is a new variable, then the elements of  $M_{\mathbb{P}}$  will be multivariate polynomials in  $2n + 1$  variables.

In the rest of this section, properties for the elimination ideal

$$\mathcal{I}_u = ([\mathbb{P}_1^N, \dots, \mathbb{P}_m^N] : \mathfrak{m}) \cap \mathbb{Q}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} \quad (8)$$

will be studied, where  $\mathbb{P}_i$  are defined in (7) and  $\mathbf{u}_i = (u_{i0}, \dots, u_{il_i})$ . These results will lead to deeper understandings for the sparse differential resultant.

**Theorem 4.19** *Let  $\mathcal{I}_u$  be defined in (8). Then  $\mathcal{I}_u$  is a differential prime ideal with codimension  $m - \text{rk}(M_{\mathbb{P}})$ .*

*Proof:* Let  $\eta = (\eta_1, \dots, \eta_n)$  be a generic point of  $[0]$  over  $\mathbb{Q}\langle \mathbf{u} \rangle$ , where  $\mathbf{u} = \{u_{ik} : i = 1, \dots, m; k = 1, \dots, l_i\}$  and

$$\zeta_i = - \sum_{k=1}^{l_i} u_{ik} \frac{M_{ik}(\eta)}{M_{i0}(\eta)} \quad (i = 1, \dots, m). \quad (9)$$

Similar to the proof of Theorem 3.9, we can show that  $\theta = (\eta_1, \dots, \eta_n; \zeta_1, u_{11}, \dots, u_{1l_1}; \dots; \zeta_m, u_{m1}, \dots, u_{ml_m})$  is a generic point of  $[\mathbb{P}_1^N, \dots, \mathbb{P}_m^N] : \mathfrak{m}$ , which follows that  $[\mathbb{P}_1^N, \dots, \mathbb{P}_m^N] : \mathfrak{m}$  is a prime differential ideal in  $\mathbb{Q}\{\mathbf{Y}, \mathbf{u}_1, \dots, \mathbf{u}_m\}$ . As a consequence,  $\mathcal{I}_u$  is a prime differential ideal. Since  $\zeta_1, \dots, \zeta_m$  are free of  $u_{i0}$  ( $i = 1, \dots, m$ ), by Theorem 4.17,

$$\begin{aligned} & \text{d.tr.deg } \mathbb{Q}\langle \mathbf{u} \rangle \langle \zeta_1, \dots, \zeta_m \rangle / \mathbb{Q}\langle \mathbf{u} \rangle \\ &= \text{d.tr.deg } \mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle \langle \zeta_1, \dots, \zeta_m \rangle / \mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle \\ &= \text{d.tr.deg } \mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle \langle \frac{\mathbb{P}_1(\eta)}{M_{10}(\eta)}, \dots, \frac{\mathbb{P}_m(\eta)}{M_{m0}(\eta)} \rangle / \mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle \\ &= \text{rk}(M_{\mathbb{P}}). \end{aligned}$$

Hence, the codimension of  $\mathcal{I}_u$  is  $m - \text{rk}(M_{\mathbb{P}})$ .  $\square$

In the following, two applications of Theorem 4.19 will be given. The first application is to identify certain  $\mathbb{P}_i$  such that their coefficients will not occur in the sparse differential resultant. This will lead to simplification in the computation of the resultant. Let  $\mathbb{T} \subset \{0, 1, \dots, n\}$ . We denote by  $\mathbb{P}_{\mathbb{T}}$  the Laurent differential polynomial set consisting of  $\mathbb{P}_i$  ( $i \in \mathbb{T}$ ), and  $M_{\mathbb{P}_{\mathbb{T}}}$  its symbolic support matrix. For a subset  $\mathbb{T} \subset \{0, 1, \dots, n\}$ , if  $\text{card}(\mathbb{T}) = \text{rk}(M_{\mathbb{P}_{\mathbb{T}}})$ , then  $\mathbb{P}_{\mathbb{T}}$ , or  $\{\mathcal{A}_i : i \in \mathbb{T}\}$ , is called a *differentially independent set*.

**Definition 4.20** *Let  $\mathbb{T} \subset \{0, 1, \dots, n\}$ . Then we say  $\mathbb{T}$  or  $\mathbb{P}_{\mathbb{T}}$  is rank essential if the following conditions hold: (1)  $\text{card}(\mathbb{T}) - \text{rk}(M_{\mathbb{P}_{\mathbb{T}}}) = 1$  and (2)  $\text{card}(\mathbb{J}) = \text{rk}(M_{\mathbb{P}_{\mathbb{J}}})$  for each proper subset  $\mathbb{J}$  of  $\mathbb{T}$ .*

Note that rank essential system is the differential analogue of essential system introduced in [50]. Using this definition, we have the following property, which is similar to Corollary 1.1 in [50].

**Theorem 4.21** *If  $\{\mathbb{P}_0, \dots, \mathbb{P}_n\}$  is a Laurent differentially essential system, then for any  $\mathbb{T} \subset \{0, 1, \dots, n\}$ ,  $\text{card}(\mathbb{T}) - \text{rk}(M_{\mathbb{P}_{\mathbb{T}}}) \leq 1$  and there exists a unique  $\mathbb{T}$  which is rank essential. In this case, the sparse differential resultant of  $\mathbb{P}_0, \dots, \mathbb{P}_n$  involves only the coefficients of  $\mathbb{P}_i$  ( $i \in \mathbb{T}$ ).*

*Proof:* Since  $n = \text{rk}(M_{\mathbb{P}}) \leq \text{rk}(M_{\mathbb{P}_{\mathbb{T}}}) + \text{card}(\mathbb{P}) - \text{card}(\mathbb{P}_{\mathbb{T}}) = n + 1 + \text{rk}(M_{\mathbb{P}_{\mathbb{T}}}) - \text{card}(\mathbb{T})$ , we have  $\text{card}(\mathbb{T}) - \text{rk}(M_{\mathbb{P}_{\mathbb{T}}}) \leq 1$ . Since  $\text{card}(\mathbb{T}) - \text{rk}(M_{\mathbb{P}_{\mathbb{T}}}) \geq 0$ , for any  $\mathbb{T}$ , either  $\text{card}(\mathbb{T}) - \text{rk}(M_{\mathbb{P}_{\mathbb{T}}}) = 0$  or  $\text{card}(\mathbb{T}) - \text{rk}(M_{\mathbb{P}_{\mathbb{T}}}) = 1$ . From this fact, it is easy to check the existence of a rank essential set  $\mathbb{T}$ . For the uniqueness, we assume that there exist two subsets  $\mathbb{T}_1, \mathbb{T}_2 \subset \{1, \dots, m\}$  which are rank essential. Then, we have

$$\begin{aligned} \text{rk}(M_{\mathbb{P}_{\mathbb{T}_1 \cup \mathbb{T}_2}}) &\leq \text{rk}(M_{\mathbb{P}_{\mathbb{T}_1}}) + \text{rk}(M_{\mathbb{P}_{\mathbb{T}_2}}) - \text{rk}(M_{\mathbb{P}_{\mathbb{T}_1 \cap \mathbb{T}_2}}) \\ &= \text{card}(\mathbb{T}_1) - 1 + \text{card}(\mathbb{T}_2) - 1 - \text{card}(\mathbb{T}_1 \cap \mathbb{T}_2) = \text{card}(\mathbb{T}_1 \cup \mathbb{T}_2) - 2, \end{aligned}$$

which means that  $M_{\mathbb{P}}$  is not of full rank, a contradiction.

Let  $\mathbb{T}$  be a rank essential set. By Theorem 4.19,  $[\mathbb{P}_i]_{i \in \mathbb{T}} \cap \mathbb{Q}\{\mathbf{u}_i\}_{i \in \mathbb{T}}$  is of codimension one, which means that the sparse differential resultant of  $\mathbb{P}_0, \dots, \mathbb{P}_n$  involves of coefficients of  $\mathbb{P}_i$  ( $i \in \mathbb{T}$ ) only.  $\square$

Using this property, one can determine which polynomial is needed for computing the sparse differential resultant, which will eventually reduce the computation complexity.

**Example 4.22** *Continue from Example 3.14.  $\{\mathbb{P}_0, \mathbb{P}_1\}$  is a rank essential sub-system since they involve  $y_1$  only.*

A more interesting example is given below.

**Example 4.23** *Let  $\mathbb{P}$  be a Laurent differential polynomial system where*

$$\begin{aligned} \mathbb{P}_0 &= u_{00}y_1y_2 + u_{01}y_3 \\ \mathbb{P}_1 &= u_{10}y_1y_2 + u_{11}y_3y_3' \\ \mathbb{P}_2 &= u_{20}y_1y_2 + u_{21}y_3' \\ \mathbb{P}_3 &= u_{30}y_1^{(o)} + u_{31}y_2^{(o)} + u_{32}y_3^{(o)} \end{aligned}$$

where  $o$  is a very large positive integer. It is easy to show that  $\mathbb{P}$  is Laurent differentially essential and  $\tilde{\mathbb{P}} = \{\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2\}$  is the rank-essential sub-system. Note that all  $y_1, y_2, y_3$  are in  $\tilde{\mathbb{P}}$ .  $\tilde{\mathbb{P}}$  is rank essential because  $y_1y_2$  can be treated as one variable.

The second application is to prove the dimension conjecture for a class of generic differential polynomials. The *differential dimension conjecture* proposed by Ritt [45, p.178] claims that the dimension of any component of  $m$  differential polynomial equations in  $n \geq m$  variables is no less than  $n - m$ . In [16], the dimension conjecture is proved for quasi-generic differential polynomials. The following theorem proves the conjecture for a larger class of differential polynomials.

**Theorem 4.24** *Let  $\mathbb{P}_i = u_{i0} + \sum_{k=1}^{l_i} u_{ik}M_{ik}$  ( $i = 1, \dots, m; m \leq n$ ) be generic differential polynomials in  $n$  differential indeterminates  $\mathbb{Y}$  and  $\mathbf{u}_i = (u_{i0}, \dots, u_{il_i})$ . Then  $[\mathbb{P}_1, \dots, \mathbb{P}_m] \subset \mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle \{\mathbb{Y}\}$  is either the unit ideal or a prime differential ideal of dimension  $n - m$ .*

*Proof:* Use the notations introduced in the proof of Theorem 4.19 with  $M_{i0} = 1$ . Let  $\mathcal{I}_0 = [\mathbb{P}_1, \dots, \mathbb{P}_m] \subset \mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m, \mathbb{Y} \rangle$  and  $\mathcal{I}_1 = [\mathbb{P}_1, \dots, \mathbb{P}_m] \subset \mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle \{\mathbb{Y}\}$ . Since  $\mathbb{P}_i$  contains a non-vanishing degree zero term  $u_{i0}$ , it is clear that  $[\mathbb{P}_1, \dots, \mathbb{P}_m] : \mathfrak{m} = \mathcal{I}_0$ .

From the proof of Theorem 4.19,  $\mathcal{I}_0$  is a prime differential ideal with  $\theta = (\eta_1, \dots, \eta_n; \zeta_1, u_{11}, \dots, u_{1l_1}; \dots; \zeta_m, u_{m1}, \dots, u_{ml_m})$  as a generic point. Note that  $\text{rk}(M_{\mathbb{P}}) \leq m$  and two cases will be considered. If  $\text{rk}(M_{\mathbb{P}}) < m$ , by Theorem 4.19,  $\mathcal{I}_u = [\mathbb{P}_1, \dots, \mathbb{P}_m] \cap \mathbb{Q}\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$  is of codimension  $m - \text{rk}(M_{\mathbb{P}}) > 0$ , which means that  $\mathcal{I}_1$  is the unit ideal in  $\mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle \{\mathbb{Y}\}$ . If  $\text{rk}(M_{\mathbb{P}}) = m$ , by the proof of Theorem 4.19,  $\text{d.tr.deg } \mathbb{Q}\langle \mathbf{u} \rangle \langle \zeta_1, \dots, \zeta_m \rangle / \mathbb{Q}\langle \mathbf{u} \rangle = m$  and  $\mathcal{I}_u = [0]$  follows. Since  $\mathcal{I}_0 = \mathcal{I}_1 \cap \mathbb{Q}\{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbb{Y}\}$  and  $\mathcal{I}_0$  is prime, it is easy to see that  $\mathcal{I}_1$  is also a differential prime ideal in  $\mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle \{\mathbb{Y}\}$ . Moreover, we have

$$\begin{aligned} n &= \text{d.tr.deg } \mathbb{Q}\langle \mathbf{u} \rangle \langle \eta_1, \dots, \eta_n, \zeta_1, \dots, \zeta_m \rangle / \mathbb{Q}\langle \mathbf{u} \rangle \\ &= \text{d.tr.deg } \mathbb{Q}\langle \mathbf{u} \rangle \langle \eta_1, \dots, \eta_n, \zeta_1, \dots, \zeta_m \rangle / \mathbb{Q}\langle \mathbf{u}, \zeta_1, \dots, \zeta_m \rangle \\ &\quad + \text{d.tr.deg } \mathbb{Q}\langle \mathbf{u}, \zeta_1, \dots, \zeta_m \rangle / \mathbb{Q}\langle \mathbf{u} \rangle \\ &= \text{d.tr.deg } \mathbb{Q}\langle \mathbf{u} \rangle \langle \eta_1, \dots, \eta_n, \zeta_1, \dots, \zeta_m \rangle / \mathbb{Q}\langle \mathbf{u}, \zeta_1, \dots, \zeta_m \rangle + m. \end{aligned}$$

Hence,  $\text{d.tr.deg } \mathbb{Q}\langle \mathbf{u}, \zeta_1, \dots, \zeta_m \rangle \langle \eta_1, \dots, \eta_n \rangle / \mathbb{Q}\langle \mathbf{u}, \zeta_1, \dots, \zeta_m \rangle = n - m$ . Without loss of generality, suppose  $\eta_1, \dots, \eta_{n-m}$  are differentially independent over  $\mathbb{Q}\langle \mathbf{u}, \zeta_1, \dots, \zeta_m \rangle$ . Since  $\mathcal{I}_0 = \mathcal{I}_1 \cap \mathbb{Q}\{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbb{Y}\}$ ,  $\{y_1, \dots, y_{n-m}\}$  is a parametric set of  $\mathcal{I}_1$ . Thus,  $[\mathbb{P}_1, \dots, \mathbb{P}_m] \subset \mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle \{\mathbb{Y}\}$  is of dimension  $n - m$ .  $\square$

By Theorem 4.17, Theorem 4.19, and Corollary 4.18, properties 1) and 2) of Theorem 1.1 are proved.

## 5 Basic properties of sparse differential resultant

In this section, we will prove some basic properties for the sparse differential resultant  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$ .

### 5.1 Sparse differential resultant is differentially homogeneous

Following Kolchin [32], we now introduce the concept of differentially homogenous polynomials.

**Definition 5.1** *A differential polynomial  $p \in \mathcal{F}\{y_0, \dots, y_n\}$  is called differentially homogenous of degree  $m$  if for a new differential indeterminate  $\lambda$ , we have  $p(\lambda y_0, \lambda y_1, \dots, \lambda y_n) = \lambda^m p(y_0, y_1, \dots, y_n)$ .*

The differential analogue of Euler's theorem related to homogenous polynomials is valid.

**Theorem 5.2** [32]  *$f \in \mathcal{F}\{y_0, y_1, \dots, y_n\}$  is differentially homogenous of degree  $m$  if and only if*

$$\sum_{j=0}^n \sum_{k \in \mathbb{N}} \binom{k+r}{r} y_j^{(k)} \frac{\partial f(y_0, \dots, y_n)}{\partial y_j^{(k+r)}} = \begin{cases} mf & r = 0 \\ 0 & r \neq 0 \end{cases}$$

Sparse differential resultants have the following property.



**Theorem 5.3** *The sparse differential resultant is differentially homogenous in each  $\mathbf{u}_i$  which is the coefficient set of  $\mathbb{P}_i$ .*

*Proof:* Suppose  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = h_i \geq 0$ . Follow the notations used in Theorem 3.9. By Corollary 3.12,  $\mathbf{R}(\mathbf{u}; \zeta_0, \dots, \zeta_n) = 0$ . Differentiating this identity w.r.t.  $u_{ij}^{(k)}$  ( $j = 1, \dots, l_i$ ) respectively, we have

$$\frac{\partial \mathbf{R}}{\partial u_{ij}} + \frac{\partial \mathbf{R}}{\partial u_{i0}} \left( -\frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right) + \frac{\partial \mathbf{R}}{\partial u'_{i0}} \left( -\left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]' \right) + \frac{\partial \mathbf{R}}{\partial u''_{i0}} \left( -\left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]'' \right) + \dots + \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} \left( -\binom{h_i}{0} \left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]^{(h_i)} \right) = 0 \quad (0*)$$

$$\frac{\partial \mathbf{R}}{\partial u'_{ij}} + 0 + \frac{\partial \mathbf{R}}{\partial u'_{i0}} \left( -\frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right) + \frac{\partial \mathbf{R}}{\partial u''_{i0}} \left( -\binom{2}{1} \left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]' \right) + \dots + \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} \left( -\binom{h_i}{1} \left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]^{(h_i-1)} \right) = 0 \quad (1*)$$

$$\frac{\partial \mathbf{R}}{\partial u''_{ij}} + 0 + 0 + \frac{\partial \mathbf{R}}{\partial u''_{i0}} \left( -\binom{2}{2} \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right) + \dots + \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} \left( -\binom{h_i}{2} \left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]^{(h_i-2)} \right) = 0 \quad (2*)$$

$$\dots \dots \dots \frac{\partial \mathbf{R}}{\partial u_{ij}^{(h_i)}} + 0 + 0 + 0 + \dots + \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} \left( -\binom{h_i}{h_i} \left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]^{(0)} \right) = 0 \quad (h_i*)$$

In the above equations,  $\frac{\partial \mathbf{R}}{\partial u_{ij}^{(k)}}$  ( $k = 0, \dots, h_i; j = 0, \dots, l_i$ ) are obtained by replacing  $u_{i0}$  by  $\zeta_i$  ( $i = 0, 1, \dots, n$ ) in each  $\frac{\partial \mathbf{R}}{\partial u_{ij}^{(k)}}$  respectively.

Now, let us consider  $\sum_{j=0}^{l_i} \sum_{k \geq 0} \binom{k+r}{r} u_{ij}^{(k)} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(k+r)}}$ . Of course, it needs only to consider  $r \leq h_i$ . For each  $r \leq h_i$  and each  $j \in \{1, \dots, l_i\}$ ,

$$\begin{aligned} 0 &= (r*) \times \binom{r}{r} u_{ij} + (r+1*) \times \binom{r+1}{r} u'_{ij} + \dots + (h_i*) \times \binom{h_i}{r} u_{ij}^{(h_i-r)} \\ &= \binom{r}{r} u_{ij} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(r)}} + \binom{r+1}{r} u'_{ij} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(r+1)}} + \dots + \binom{h_i}{r} u_{ij}^{(h_i-r)} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(h_i)}} + \frac{\partial \mathbf{R}}{\partial u_{i0}^{(r)}} \left( -u_{ij} \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right) \\ &\quad + \frac{\partial \mathbf{R}}{\partial u_{i0}^{(r+1)}} \left( -\binom{r+1}{r} u_{ij} \left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]' - \binom{r+1}{r} u'_{ij} \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right) + \dots \\ &\quad + \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i-r+1)}} \left( -\binom{h_i}{r} u_{ij} \left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]^{(h_i-r)} - \binom{r+1}{r} \binom{h_i}{r+1} u'_{ij} \left[ \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right]^{(h_i-r-1)} - \dots - \binom{h_i}{r} \binom{h_i}{h_i} u_{ij}^{(h_i-r)} \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right) \\ &= \binom{r}{r} u_{ij} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(r)}} + \binom{r+1}{r} u'_{ij} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(r+1)}} + \dots + \binom{h_i}{r} u_{ij}^{(h_i-r)} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(h_i)}} + \binom{r}{r} \frac{\partial \mathbf{R}}{\partial u_{i0}^{(r)}} \left( -u_{ij} \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right) \\ &\quad + \binom{r+1}{r} \frac{\partial \mathbf{R}}{\partial u_{i0}^{(r+1)}} \left( -u_{ij} \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right)' + \dots + \binom{h_i}{r} \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i-r+1)}} \left( -u_{ij} \frac{M_{ij}(\eta)}{M_{i0}(\eta)} \right)^{(h_i-r)} \end{aligned}$$

It follows that  $\sum_{j=1}^{l_i} \binom{r}{r} u_{ij} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(r)}} + \sum_{j=1}^{l_i} \binom{r+1}{r} u'_{ij} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(r+1)}} + \dots + \sum_{j=1}^{l_i} \binom{h_i}{r} u_{ij}^{(h_i-r)} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(h_i)}} + \binom{r}{r} \zeta_i \frac{\partial \mathbf{R}}{\partial u_{i0}^{(r)}} + \binom{r+1}{r} \zeta'_i \frac{\partial \mathbf{R}}{\partial u_{i0}^{(r+1)}} + \dots + \binom{h_i}{r} \zeta_i^{(h_i-r)} \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} = 0$ .

By Corollary 3.12,  $G = \sum_{k \geq 0} \sum_{j=0}^{l_i} \binom{r+k}{r} u_{ij}^{(k)} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(r+k)}} \in \text{sat}(\mathbf{R})$ . Since  $\text{ord}(G) \leq \text{ord}(\mathbf{R})$ ,

$G$  can be divisible by  $\mathbf{R}$ . In the case  $r = 0$ ,  $\sum_{j=0}^{l_i} \sum_{k=0}^{h_i} u_{ij}^{(k)} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(k)}} = m \cdot \mathbf{R}$  for some  $m \in \mathbb{Z}$ . While in the case  $r > 0$ , if  $G \neq 0$ , it can not be divisible by  $\mathbf{R}$ . Thus, in this case,  $G$  must be identically zero. From the above, we conclude that

$$\sum_{j=0}^{l_i} \sum_{k \geq 0} \binom{k+r}{r} u_{ij}^{(k)} \frac{\partial \mathbf{R}}{\partial u_{ij}^{(k+r)}} = \begin{cases} 0 & r \neq 0 \\ m\mathbf{R} & r = 0 \end{cases}$$

By Theorem 5.2,  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  is differentially homogenous in each  $\mathbf{u}_i$  and the theorem is obtained.  $\square$

With Theorem 5.3, property 1) of Theorem 1.2 is proved.

## 5.2 Order bound in terms of Jacobi number

In this section, we will give an order bound for the sparse differential resultant in terms of the Jacobi number of the given system.

Consider a generic Laurent differentially essential system  $\{\mathbb{P}_0, \dots, \mathbb{P}_n\}$  defined in (2) with  $\mathbf{u}_i = (u_{i0}, u_{i1}, \dots, u_{il_i})$  being the coefficient vector of  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ). Suppose  $\mathbf{R}$  is the sparse differential resultant of  $\mathbb{P}_0, \dots, \mathbb{P}_n$ . Denote  $\text{ord}(\mathbf{R}, \mathbf{u}_i)$  to be the maximal order of  $\mathbf{R}$  in  $u_{ik}$  ( $k = 0, \dots, l_i$ ), that is,  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = \max_k \text{ord}(\mathbf{R}, u_{ik})$ . If  $\mathbf{u}_i$  does not occur in  $\mathbf{R}$ , as shown in Example 3.14, then set  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = -\infty$ . Firstly, we have the following result.

**Lemma 5.4** *For each  $i$ , if  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = h_i \geq 0$ , then  $\text{ord}(\mathbf{R}, u_{ik}) = h_i$  ( $k = 0, \dots, l_i$ ).*

*Proof:* Firstly, we claim that  $\text{ord}(\mathbf{R}, u_{i0}) = h_i$ . For if not, suppose  $\text{ord}(\mathbf{R}, u_{ik}) = h_i \geq 0$  for some  $k \neq 0$ . Then by differentiating  $\mathbf{R}(\mathbf{u}; \zeta_0, \dots, \zeta_n) = 0$  w.r.t.  $u_{ik}^{(h_i)}$ , we have  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}(\mathbf{u}; \zeta_0, \dots, \zeta_n) = 0$ , where  $\zeta_i$  are defined in (4). By Corollary 3.12, we have  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} \in \text{sat}(\mathbf{R})$ , a contradiction. Thus,  $\text{ord}(\mathbf{R}, u_{i0}) = h_i$ . For each  $k \neq 0$ ,  $\text{ord}(\mathbf{R}, u_{ik}) \leq h_i$ . If  $\text{ord}(\mathbf{R}, u_{ik}) < h_i$ , differentiate  $\mathbf{R}(\mathbf{u}; \zeta_0, \dots, \zeta_n) = 0$  w.r.t.  $u_{ik}^{(h_i)}$ , we have  $\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}(\mathbf{u}; \zeta_0, \dots, \zeta_n) \cdot (-\frac{M_{ik}(\eta)}{M_{i0}(\eta)}) = 0$ . So  $\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}(\mathbf{u}; \zeta_0, \dots, \zeta_n) = 0$  and  $\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} \in \text{sat}(\mathbf{R})$ , a contradiction. Thus, for each  $k = 0, \dots, l_i$ ,  $\text{ord}(\mathbf{R}, u_{ik}) = h_i$ .  $\square$

Let  $A = (a_{ij})$  be an  $n \times n$  matrix where  $a_{ij}$  is an integer or  $-\infty$ . A *diagonal sum* of  $A$  is any sum  $a_{1i_1} + a_{2i_2} + \dots + a_{ni_n}$  with  $i_1, \dots, i_n$  a permutation of  $1, \dots, n$ . If  $A$  is an  $m \times n$  matrix with  $M = \min\{m, n\}$ , then a diagonal sum of  $A$  is a diagonal sum of any  $M \times M$  submatrix of  $A$ . The *Jacobi number* of a matrix  $A$  is the maximal diagonal sum of  $A$ , denoted by  $\text{Jac}(A)$ .

Let  $\text{ord}(\mathbb{P}_i^N, y_j) = e_{ij}$  ( $i = 0, \dots, n; j = 1, \dots, n$ ) and  $\text{Eord}(\mathbb{P}_i) = \text{ord}(\mathbb{P}_i^N) = e_i$ . We call the  $(n+1) \times n$  matrix  $A = (e_{ij})$  the *order matrix* of  $\mathbb{P}_0, \dots, \mathbb{P}_n$ . By  $A_i$ , we mean the submatrix of  $A$  obtained by deleting the  $(i+1)$ -th row from  $A$ . We use  $\mathbb{P}$  to denote the set  $\{\mathbb{P}_0^N, \dots, \mathbb{P}_n^N\}$  and by  $\mathbb{P}_i$ , we mean the set  $\mathbb{P} \setminus \{\mathbb{P}_i^N\}$ . We call  $J_i = \text{Jac}(A_i)$  the *Jacobi number* of the system  $\mathbb{P}_i$ , also denoted by  $\text{Jac}(\mathbb{P}_i)$ . Before giving an order bound for sparse differential resultant in terms of the Jacobi numbers, we first give several lemmas.

Given a vector  $\vec{K} = (k_0, k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ , we can obtain a prolongation of  $\mathbb{P}$ :

$$\mathbb{P}[\vec{K}] = \bigcup_{i=0}^n (\mathbb{P}_i^N)^{[k_i]}. \quad (10)$$

Let  $t_j = \max\{e_{0j} + k_0, e_{1j} + k_1, \dots, e_{nj} + k_n\}$ . Then  $\mathbb{P}[\vec{K}]$  is contained in  $\mathbb{Q}[\mathbf{u}^{[\vec{K}]}, \mathbb{Y}^{[\vec{K}]}]$ , where  $\mathbf{u}^{[\vec{K}]} = \bigcup_{i=0}^n \mathbf{u}_i^{[k_i]}$  and  $\mathbb{Y}^{[\vec{K}]} = \bigcup_{j=1}^n y_j^{[t_j]}$ .

Denote  $\nu(\mathbb{P}[\vec{K}])$  to be the number of  $\mathbb{Y}$  and their derivatives appearing effectively in  $\mathbb{P}[\vec{K}]$ . In order to derive a differential relation among  $\mathbf{u}_i$  ( $i = 0, \dots, n$ ) from  $\mathbb{P}[\vec{K}]$ , a sufficient condition is

$$|\mathbb{P}[\vec{K}]| \geq \nu(\mathbb{P}[\vec{K}]) + 1. \quad (11)$$

Note that  $\nu(\mathbb{P}[\vec{K}]) \leq |\mathbb{Y}[\vec{K}]| = \sum_{j=1}^n (t_j + 1) = \sum_{j=1}^n \max(e_{0j} + k_0, e_{1j} + k_1, \dots, e_{nj} + k_n) + n$ . Thus, if  $|\mathbb{P}[\vec{K}]| \geq |\mathbb{Y}[\vec{K}]| + 1$ , or equivalently,

$$k_0 + k_1 + \dots + k_n \geq \sum_{j=1}^n \max(e_{0j} + k_0, e_{1j} + k_1, \dots, e_{nj} + k_n) \quad (12)$$

is satisfied, then so is the inequality (11).

**Lemma 5.5** *Let  $\mathbb{P}$  be a Laurent differentially essential system and  $\vec{K} = (k_0, k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^{n+1}$  a vector satisfying (12). Then  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \leq k_i$  for each  $i = 0, \dots, n$ .*

*Proof:* Denote  $\mathfrak{m}^{[\vec{K}]}$  to be the set of all monomials in variables  $\mathbb{Y}^{[\vec{K}]}$ . Let  $\mathcal{I} = (\mathbb{P}^{[\vec{K}]}) : \mathfrak{m}^{[\vec{K}]}$  be an ideal in the polynomial ring  $\mathbb{Q}[\mathbb{Y}^{[\vec{K}]}, \mathbf{u}^{[\vec{K}]}]$ . Denote  $U = \mathbf{u}^{[\vec{K}]} \setminus \cup_{i=0}^n u_{i0}^{[k_i]}$ . Assume  $\mathbb{P}_i^N = \sum_{k=0}^{l_i} u_{ik} N_{ik}$  ( $i = 0, \dots, n$ ). Let  $\zeta_{il} = -(\sum_{k=1}^{l_i} u_{ik} N_{ik} / N_{i0})^{(l)}$  for  $i = 0, 1, \dots, n; l = 0, 1, \dots, k_i$ . Denote  $\zeta = (U, \zeta_{0k_0}, \dots, \zeta_{00}, \dots, \zeta_{nk_n}, \dots, \zeta_{n0})$ . It is easy to show that  $(\mathbb{Y}^{[\vec{K}]}, \zeta)$  is a generic point of  $\mathcal{I}$ . Indeed, on the one hand, each polynomial in  $\mathcal{I}$  vanishes at  $(\mathbb{Y}^{[\vec{K}]}, \zeta)$ . On the other hand, if  $f$  is an arbitrary polynomial in  $\mathbb{Q}[\mathbb{Y}^{[\vec{K}]}, \mathbf{u}^{[\vec{K}]}]$  such that  $f(\mathbb{Y}^{[\vec{K}]}, \zeta) = 0$ , substitute  $u_{i0}^{(l)} = ((\mathbb{P}_i^N - \sum_{k=1}^{l_i} u_{ik} N_{ik}) / N_{i0})^{(l)}$  into  $f$ , then we have  $\prod_{i=0}^n N_{i0}^{a_i} f \equiv f_1 \pmod{(\mathbb{P}^{[\vec{K}]})}$ , where  $f_1 \in \mathbb{Q}[\mathbb{Y}^{[\vec{K}]}], U$ . Clearly,  $f_1 = 0$  and  $f \in \mathcal{I}$  follows.

Let  $\mathcal{I}_1 = \mathcal{I} \cap \mathbb{Q}[\mathbf{u}^{[\vec{K}]}]$ . Then  $\mathcal{I}_1$  is a prime ideal with  $\zeta$  as its generic point. Since  $\mathbb{Q}(\zeta) \subset \mathbb{Q}(\mathbb{Y}^{[\vec{K}]}, U)$ ,  $\text{Codim}(\mathcal{I}_1) = |U| + \sum_{i=0}^n (k_i + 1) - \text{tr.deg } \mathbb{Q}(\zeta) / \mathbb{Q} \geq |U| + |\mathbb{P}^{[\vec{K}]}| - \text{tr.deg } \mathbb{Q}(\mathbb{Y}^{[\vec{K}]}, U) / \mathbb{Q} = |\mathbb{P}^{[\vec{K}]}| - |\mathbb{Y}^{[\vec{K}]}| \geq 1$ . Thus,  $\mathcal{I}_1 \neq (0)$ . Suppose  $f$  is any nonzero polynomial in  $\mathcal{I}_1$ . Clearly,  $\text{ord}(f, \mathbf{u}_i) \leq k_i$ . Since  $\mathcal{I}_1 \subset [\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m} \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\} = \text{sat}(\mathbf{R})$ ,  $f \in \text{sat}(\mathbf{R})$ . Note that  $\mathbf{R}$  is a characteristic set of  $\text{sat}(\mathbf{R})$  w.r.t. any ranking by Lemma 2.3. Thus,  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \leq \text{ord}(f, \mathbf{u}_i) \leq k_i$ .  $\square$

**Lemma 5.6** *Let  $\mathbb{P}$  be a Laurent differentially essential system and  $J_i \geq 0$  for each  $i = 0, \dots, n$ . Then  $k_i = J_i$  ( $i = 0, \dots, n$ ) satisfy (12) in the equality case.*

*Proof:* Let  $A = (e_{ij})$  be the  $(n+1) \times n$  order matrix of  $\mathbb{P}$ , where  $e_{ij} = \text{ord}(\mathbb{P}_i^N, y_j)$ . Without loss of generality, suppose  $J_0 = e_{11} + e_{22} + \dots + e_{nn}$ .

Firstly, we will show that for each  $k \neq 1$ ,  $e_{11} + J_1 \geq e_{k1} + J_k$ . Since  $J_k$  is the Jacobi number of  $\mathbb{P}_k$  and  $k \neq 1$ ,  $J_k$  has a summand of the form  $e_{1p_1}$ . Consider the longest sequence of summands in  $J_k$  in the following form:

$$T_0 = e_{1p_1} + e_{p_1p_2} + \dots + e_{p_{m-1}p_m}$$

and suppose  $J_k = T_0 + T_1$ . Since  $J_k$  is a diagonal sum,  $p_i \neq p_j$  for  $1 < i < j$ . For otherwise,  $J_k$  contains  $e_{p_{i-1}p_i}$  and  $e_{p_{j-1}p_i}$  as summands, a contradiction. Also note that  $p_i \neq 0$  for  $1 \leq i \leq m$ . Now we claim that  $p_m$  is either equal to 1 or equal to  $k$ . Indeed, if  $p_m = 1$  or  $p_m = k$ ,  $T_0$  cannot be any longer and these two cases may happen. But if  $p_m \neq 1$  and  $p_m \neq k$ , then we can add another summand  $e_{p_m p_{m+1}}$  to  $T_0$ , which contradicts to the fact that  $T_0$  is the longest one. Now three cases are considered.

Case 1) If  $p_1 = 1$ ,  $J_k = e_{11} + T_1$  and  $e_{k1} + J_k = e_{11} + e_{k1} + T_1$ . Since  $e_{k1} + T_1$  is a diagonal sum of  $\mathbb{P}_1$ ,  $e_{k1} + T_1 \leq J_1$ . Thus,  $e_{11} + J_1 \geq e_{k1} + J_k$ .

Case 2) If  $p_m = 1$  for  $m > 1$ ,  $T_0 = e_{1p_1} + e_{p_1p_2} + \cdots + e_{p_{m-1}1}$ . Since  $J_0 = e_{11} + \cdots + e_{nn}$ ,  $T_0 \leq e_{11} + e_{p_1p_1} + \cdots + e_{p_{m-1}p_{m-1}}$ . For otherwise, since  $p_i \neq 0$ ,  $T_0 + \sum_{k \in \{2, \dots, n\} \setminus \{p_1, \dots, p_{m-1}\}} e_{kk}$  is a diagonal sum of  $\mathbb{P}_0$  which is greater than  $J_0$ . Then  $e_{k1} + J_k = e_{k1} + T_0 + T_1 \leq e_{k1} + e_{11} + e_{p_1p_1} + \cdots + e_{p_{m-1}p_{m-1}} + T_1 \leq e_{11} + J_1$ , where the last inequality follows from the fact that  $e_{k1} + e_{p_1p_1} + \cdots + e_{p_{m-1}p_{m-1}} + T_1$  is a diagonal sum of  $\mathbb{P}_1$ .

Case 3) If  $p_m = k$ ,  $T_0 = e_{1p_1} + e_{p_1p_2} + \cdots + e_{p_{m-1}k}$ . Then, similar to case 2), we can show that  $e_{k1} + e_{1p_1} + e_{p_1p_2} + \cdots + e_{p_{m-1}k} \leq e_{11} + e_{kk} + e_{p_1p_1} + \cdots + e_{p_{m-1}p_{m-1}}$ . Thus,

$$\begin{aligned} e_{k1} + J_k &= e_{k1} + e_{1p_1} + e_{p_1p_2} + \cdots + e_{p_{m-1}k} + T_1 \\ &\leq e_{kk} + e_{11} + e_{p_1p_1} + \cdots + e_{p_{m-1}p_{m-1}} + T_1 \\ &\leq e_{11} + J_1. \end{aligned}$$

Similarly, we can prove that for each  $j$ ,  $e_{jj} + J_j \geq e_{kj} + J_k$  with  $0 \leq k \leq n$ . So

$$\begin{aligned} \sum_{j=1}^n \max(e_{0j} + J_0, \dots, e_{nj} + J_n) &= e_{11} + J_1 + e_{22} + J_2 + \cdots + e_{nn} + J_n \\ &= J_0 + J_1 + \dots + J_n. \end{aligned}$$

□

**Corollary 5.7** *Let  $\mathbb{P}$  be a Laurent differentially essential system and  $J_i \geq 0$  for each  $i = 0, \dots, n$ . Then  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \leq J_i$  ( $i = 0, \dots, n$ ).*

*Proof:* It is a direct consequence of Lemma 5.5 and Lemma 5.6. □

The above theorem shows that when all the Jacobi numbers are not less than 0, then Jacobi numbers are order bounds for the sparse differential resultant. In the following, we deal with the remaining case when some  $J_i = -\infty$ . To this end, two more lemmas are needed.

**Lemma 5.8** [9, 34] *Let  $A$  be an  $m \times n$  matrix whose entries are 0's and 1's. Let  $\text{Jac}(A) = J < \min\{m, n\}$ . Then  $A$  contains an  $a \times b$  zero sub-matrix with  $a + b = m + n - J$ .*

**Lemma 5.9** *Let  $\mathbb{P}$  be a Laurent differentially essential system with the following  $(n+1) \times n$  order matrix*

$$A = \begin{pmatrix} A_{11} & (-\infty)_{r \times t} \\ A_{21} & A_{22} \end{pmatrix},$$

where  $r + t \geq n + 1$ . Then  $r + t = n + 1$  and  $\text{Jac}(A_{22}) \geq 0$ . Moreover, when regarded as differential polynomials in  $y_1, \dots, y_{r-1}$ ,  $\{\mathbb{P}_0, \dots, \mathbb{P}_{r-1}\}$  is a Laurent differentially essential system.

*Proof:* From the structure of  $A$ , it follows that the symbolic support matrix of  $\mathbb{P}$  has the following form:

$$M_{\mathbb{P}} = \begin{pmatrix} B_{11} & 0_{r \times t} \\ B_{21} & B_{22} \end{pmatrix}.$$

Since  $\mathbb{P}$  is Laurent differentially essential, by Corollary 4.18,  $\text{rk}(M_{\mathbb{P}}) = n$ . As  $\text{rk}(M_{\mathbb{P}}) \leq \text{rk}(B_{11}) + \text{rk}(\begin{pmatrix} B_{21} & B_{22} \end{pmatrix})$ ,  $n \leq (n-t) + (n+1-r) = 2n+1-(r+t)$ . Thus,  $r+t \leq n+1$ , and  $r+t = n+1$  follows. Since the above inequality becomes equality,  $B_{11}$  has full column rank. As a consequence,  $\text{rk}(M_{\mathbb{P}}) = \text{rk}(B_{11}) + \text{rk}(B_{22})$ . Hence,  $B_{22}$  is a  $t \times t$  nonsingular matrix. Regarding  $\mathbb{P}_0, \dots, \mathbb{P}_{r-1}$  as differential polynomials in  $y_1, \dots, y_{r-1}$ , then  $B_{11}$  is the symbolic support matrix of  $\{\mathbb{P}_0, \dots, \mathbb{P}_{r-1}\}$  which is of full rank. Thus,  $\{\mathbb{P}_0, \dots, \mathbb{P}_{r-1}\}$  is a Laurent differentially essential system.

It remains to show that  $\text{Jac}(A_{22}) \geq 0$ . Suppose the contrary, i.e.  $\text{Jac}(A_{22}) = -\infty$ . Let  $\bar{A}_{22}$  be a  $t \times t$  matrix obtained from  $A_{22}$  by replacing  $-\infty$  by 0 and replacing all other elements in  $A_{22}$  by 1's. Then  $\text{Jac}(\bar{A}_{22}) < t$ , and by Lemma 5.8,  $\bar{A}_{12}$  contains an  $a \times b$  zero submatrix with  $a+b = 2t - \text{Jac}(\bar{A}_{22}) \geq t+1$ . By interchanging rows and interchanging columns when necessary, suppose such a zero submatrix is in the upper-right corner of  $\bar{A}_{22}$ . Then

$$A_{22} = \begin{pmatrix} C_{11} & (-\infty)_{a \times b} \\ C_{21} & C_{22} \end{pmatrix},$$

where  $a+b \geq t+1$ . Thus,

$$B_{22} = \begin{pmatrix} D_{11} & 0_{a \times b} \\ D_{21} & D_{22} \end{pmatrix},$$

which is singular for  $a+b \geq t+1$ , a contradiction. Thus,  $\text{Jac}(A_{22}) \geq 0$ .  $\square$

Now, we are ready to prove the main result of this section.

**Theorem 5.10** *Let  $\mathbb{P}$  be a Laurent differentially essential system and  $\mathbf{R}$  the sparse differential resultant of  $\mathbb{P}$ . Then*

$$\text{ord}(\mathbf{R}, \mathbf{u}_i) = \begin{cases} -\infty & \text{if } J_i = -\infty, \\ h_i \leq J_i & \text{if } J_i \geq 0. \end{cases}$$

*Proof:* Corollary 5.7 proves the case when  $J_i \geq 0$  for each  $i$ . Now suppose there exists at least one  $i$  such that  $J_i = -\infty$ . Without loss of generality, we assume  $J_n = -\infty$  and let  $A_n = (e_{ij})_{0 \leq i \leq n-1; 1 \leq j \leq n}$  be the order matrix of  $\mathbb{P}_{\hat{n}}$ . By Lemma 5.8 and similarly as the procedures in the proof of Lemma 5.9, we can assume that  $A_n$  is of the following form

$$A_n = \begin{pmatrix} A_{11} & (-\infty)_{r \times t} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix},$$

where  $r+t \geq n+1$ . Then the order matrix of  $\mathbb{P}$  is equal to

$$A = \begin{pmatrix} A_{11} & (-\infty)_{r \times t} \\ A_{21} & A_{22} \end{pmatrix}.$$

Since  $\mathbb{P}$  is Laurent differentially essential, by Lemma 5.9,  $r+t = n+1$  and  $\text{Jac}(A_{22}) \geq 0$ . Moreover, considered as differential polynomials in  $y_1, \dots, y_{r-1}$ ,  $\tilde{\mathbb{P}} = \{p_0, \dots, p_{r-1}\}$  is Laurent differentially essential and  $A_{11}$  is its order matrix. Let  $\tilde{J}_i = \text{Jac}((A_{11})_{\hat{i}})$ . By applying the above procedure when necessary, we can suppose that  $\tilde{J}_i \geq 0$  for each  $i = 0, \dots, r-1$ . Since  $[\mathbb{P}] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\} = [\tilde{\mathbb{P}}] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_{r-1}\} = \text{sat}(\mathbf{R})$ ,  $\mathbf{R}$  is also the sparse differential resultant of the system  $\mathbb{P}$  and  $\mathbf{u}_r, \dots, \mathbf{u}_n$  will not occur in  $\mathbf{R}$ . By Corollary 5.7,  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \leq$

$\tilde{J}_i$ . Since  $J_i = \text{Jac}(A_{22}) + \tilde{J}_i \geq \tilde{J}_i$  for  $0 \leq i \leq r-1$ ,  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \leq J_i$  for  $0 \leq i \leq r-1$  and  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = -\infty$  for  $i = r, \dots, n$ .  $\square$

**Corollary 5.11** *Let  $\mathbb{P}$  be rank essential. Then  $J_i \geq 0$  for  $i = 0, \dots, n$  and  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \leq J_i$ .*

*Proof:* From the proof of Theorem 5.10, if  $J_i = -\infty$  for some  $i$ , then  $\mathbb{P}$  contains a proper differentially essential subsystem, which contradicts to Theorem 4.21. Therefore,  $J_i \geq 0$  for  $i = 0, \dots, n$ .  $\square$

The following example shows that in spite of  $J_i \geq 0$ ,  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = -\infty$  may happen.

**Example 5.12** *Let  $\mathbb{P} = \{\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3\}$  be a Laurent differential polynomial system where*

$$\begin{aligned}\mathbb{P}_0 &= u_{00} + u_{01}y_1y_1'y_2y_2'' \\ \mathbb{P}_1 &= u_{10} + u_{11}y_1y_1'y_2y_2'' \\ \mathbb{P}_2 &= u_{20} + u_{21}y_1 + u_{22}y_2 \\ \mathbb{P}_3 &= u_{30} + u_{31}y_1' + u_{32}y_3.\end{aligned}$$

*Then, the corresponding order matrix is*

$$A = \begin{pmatrix} 1 & 2 & -\infty \\ 1 & 2 & -\infty \\ 0 & 0 & -\infty \\ 1 & -\infty & 0 \end{pmatrix}.$$

*It is easy to show that  $\mathbb{P}$  is Laurent differentially essential and  $\{\mathbb{P}_0, \mathbb{P}_1\}$  is the rank-essential sub-system. Here  $\mathbf{R} = u_{00}u_{11} - u_{01}u_{10}$ . Clearly,  $\text{ord}(\mathbf{R}, \mathbf{u}_0) = \text{ord}(\mathbf{R}, \mathbf{u}_1) = 0$  and  $\text{ord}(\mathbf{R}, \mathbf{u}_2) = \text{ord}(\mathbf{R}, \mathbf{u}_3) = -\infty$ , but  $J_0 = 2, J_1 = 2, J_3 = 3, J_4 = -\infty$ . Also note that in this example, the sub-matrix  $A_{11}$  in the proof of Theorem 5.10 corresponds to  $\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 0 \end{pmatrix}$ .*

We conclude this section by giving two improved order bounds based on the Jacobi bound given in Theorem 5.10.

For each  $j \in \{1, \dots, n\}$ , let  $\underline{\varrho}_j = \min\{k \in \mathbb{N} \mid \exists i \text{ s.t. } \deg(\mathbb{P}_i^N, y_j^{(k)}) > 0\}$ . In other words,  $\underline{\varrho}_j$  is the smallest number such that  $y_j^{(\underline{\varrho}_j)}$  occurs in  $\{\mathbb{P}_0^N, \dots, \mathbb{P}_n^N\}$ . Let  $B = (e_{ij} - \underline{\varrho}_j)$  be an  $(n+1) \times n$  matrix. We call  $\bar{J}_i = \text{Jac}(B_i)$  the *modified Jacobi number* of the system  $\mathbb{P}_i$ . Denote  $\underline{\gamma} = \sum_{j=1}^n \underline{\varrho}_j$ . Clearly,  $\bar{J}_i = J_i - \underline{\gamma}$ . Then we have the following result.

**Theorem 5.13** *Let  $\mathbb{P}$  be a Laurent differentially essential system and  $\mathbf{R}$  the sparse differential resultant of  $\mathbb{P}$ . Then*

$$\text{ord}(\mathbf{R}, \mathbf{u}_i) = \begin{cases} -\infty & \text{if } J_i = -\infty, \\ h_i \leq J_i - \underline{\gamma} & \text{if } J_i \geq 0. \end{cases}$$

*Proof:* Follow the notations given above. Let  $\hat{\mathbb{P}}_i$  be obtained from  $\mathbb{P}_i$  by replacing  $y_j^{(k)}$  by  $y_j^{(k-\underline{o}_j)}$  ( $j = 1, \dots, n; k \geq \underline{o}_j$ ) in  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ) and denote  $\hat{\mathbb{P}} = \{\hat{\mathbb{P}}_0, \dots, \hat{\mathbb{P}}_n\}$ . Since

$$M_{\mathbb{P}} = M_{\hat{\mathbb{P}}} \cdot \begin{pmatrix} x_1^{\underline{o}_1} & & 0 \\ & x_2^{\underline{o}_2} & \\ 0 & & \ddots & \\ & & & x_n^{\underline{o}_n} \end{pmatrix},$$

it follows that  $\text{rk}(M_{\hat{\mathbb{P}}}) = \text{rk}(M_{\mathbb{P}}) = n$ . Thus,  $\mathcal{I} = [\hat{\mathbb{P}}] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is a prime differential ideal of codimension 1. We claim that  $\mathcal{I} = \text{sat}(\mathbf{R})$ . Suppose  $\mathbb{P}_i = u_{i0}M_{i0} + T_i$  and  $\hat{\mathbb{P}}_i = u_{i0}\hat{M}_{i0} + \hat{T}_i$ . Let  $\zeta_i = -T_i/M_{i0}$  and  $\theta_i = -\hat{T}_i/\hat{M}_{i0}$ . Denote  $\mathbf{u} = \cup_{i=0}^n \mathbf{u}_i \setminus \{u_{i0}\}$ . Then  $\zeta = (\mathbf{u}, \zeta_0, \dots, \zeta_n)$  is a generic point of  $\text{sat}(\mathbf{R})$  and  $\theta = (\mathbf{u}, \theta_0, \dots, \theta_n)$  is a generic point of  $\mathcal{I}$ . For any differential polynomial  $G \in \text{sat}(\mathbf{R})$ ,  $G(\zeta) = 0 = (\sum \phi(\mathbb{Y})F_\phi(\mathbf{u})) / (\prod_{i=1}^n M_{i0}^{a_i})$  where  $\phi(\mathbb{Y})$  are distinct differential monomials in  $\mathbb{Y}$ . Then  $F_\phi(\mathbf{u}) \equiv 0$  for each  $\phi$ . Thus,  $G(\theta) = (\sum \hat{\phi}(\mathbb{Y})F_\phi(\mathbf{u})) / (\prod_{i=1}^n \hat{M}_{i0}^{a_i}) = 0$  and  $G \in \mathcal{I}$  follows. So  $\text{sat}(\mathbf{R}) \subseteq \mathcal{I}$ . In the similar way, we can show that  $\mathcal{I} \subseteq \text{sat}(\mathbf{R})$ . Hence,  $\mathbf{R}$  is the sparse differential resultant of  $\hat{\mathbb{P}}$ . Since  $\text{Jac}(\hat{\mathbb{P}}_i) = \text{Jac}(\mathbb{P}_i) - \underline{\gamma}$ , by Theorem 5.10, the theorem is proved.  $\square$

**Remark 5.14** Let  $\vec{K} = (e - e_0, e - e_1, \dots, e - e_n)$  where  $e = \sum_{i=0}^n e_i$ . Clearly,  $|\mathbb{P}[\vec{K}]| = ne + n + 1 = |\mathbb{Y}^{[e]}| + 1 \geq |\mathbb{Y}[\vec{K}]| + 1$ . Then by Lemma 5.5,  $\deg(\mathbf{R}, \mathbf{u}_i) \leq e - e_i \leq s - s_i$ . Here  $s_i$  is the order of  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ). If  $L_i = e - e_i - \gamma(\mathbb{P})$  where  $\gamma(\mathbb{P}) = \sum_{j=1}^n (\underline{o}_j + \bar{e}_j)$  and  $\bar{e}_j = \min_i \{e_i - \text{ord}(\mathbb{P}_i^N, y_j) | \text{ord}(\mathbb{P}_i^N, y_j) \neq -\infty\}$ . By [47],  $(L_0, \dots, L_n)$  also consists of a solution to (12). Then  $\deg(\mathbf{R}, \mathbf{u}_i) \leq L_i$ . One can easily check that  $\bar{J}_i \leq L_i \leq e - e_i$  for each  $i$ , and the modified Jacobi bound is better than the other two bounds as shown by the following example.

**Example 5.15** Let  $A = (e_{ij})_{0 \leq i \leq n, 1 \leq j \leq n}$  be the order matrix of a system  $\mathbb{P}$ :

$$A = \begin{pmatrix} 5 & -\infty & 0 \\ 5 & 0 & -\infty \\ 0 & 3 & 5 \\ 5 & 2 & -\infty \end{pmatrix}.$$

Then  $\{J_0, J_1, J_2, J_3\} = \{12, 12, 7, 10\}$ ,  $\{L_0, L_1, L_2, L_3\} = \{13, 13, 13, 13\}$ ,  $\{e - e_0, e - e_1, e - e_2, e - e_3\} = \{15, 15, 15, 15\}$ . This shows that the modified Jacobi bound could be strictly less than the other two bounds.

Now, we assume that  $\mathbb{P}$  is a Laurent differentially essential system which is not rank essential. Let  $\mathbf{R}$  be the sparse differential resultant of  $\mathbb{P}$ . We will give a better order bound for  $\mathbf{R}$ . By Theorem 4.21,  $\mathbb{P}$  contains a unique rank essential sub-system  $\mathbb{P}_{\mathbb{T}}$ . Without loss of generality, suppose  $\mathbb{T} = \{0, \dots, r\}$  with  $r < n$ . Let  $A_{\mathbb{T}}$  be the order matrix of  $\mathbb{P}_{\mathbb{T}}$  and for  $i = 0, \dots, r$ , let  $A_{\mathbb{T}\hat{i}}$  be the matrix obtained from  $A_{\mathbb{T}}$  by deleting the  $(i + 1)$ -th row. Note that  $A_{\mathbb{T}\hat{i}}$  is an  $r \times n$  matrix. Then we have the following result.

**Theorem 5.16** *With the above notations, we have*

$$\text{ord}(\mathbf{R}, \mathbf{u}_i) = \begin{cases} h_i \leq \text{Jac}(A_{\mathbb{T}_i}) & i = 0, \dots, r, \\ -\infty & i = r+1, \dots, n. \end{cases}$$

*Proof:* It suffices to show that  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \leq \text{Jac}(A_{\mathbb{T}_i})$  for  $i = 0, \dots, r$ . Let  $\mathbb{L}_i = u_{i0} + \sum_{j=1}^n u_{ij}y_j$  for  $i = r+1, \dots, n$ . Since  $\mathbb{P}_{\mathbb{T}}$  is rank essential, there exist  $\frac{M_{ik_i}}{M_{i0}}$  ( $i = 1, \dots, r$ ) such that their symbolic support matrix  $B$  is of full rank. Without loss of generality, we assume that the  $r$ -th principal submatrix of  $B$  is of full rank. Consider a new Laurent differential polynomial system  $\tilde{\mathbb{P}} = \mathbb{P}_{\mathbb{T}} \cup \{\mathbb{L}_{r+1}, \dots, \mathbb{L}_n\}$ . This system is also Laurent differentially essential since the symbolic support matrix of  $\frac{M_{1k_1}}{M_{10}}, \dots, \frac{M_{rk_r}}{M_{r0}}, y_{r+1}, \dots, y_n$  is of full rank. And  $\mathbf{R}$  is also the sparse differential resultant of  $\tilde{\mathbb{P}}$ , for  $\mathbb{P}_{\mathbb{T}}$  is the rank-essential subsystem of  $\tilde{\mathbb{P}}$ . The order vector of  $\mathbb{L}_i$  is  $(0, \dots, 0)$  for  $i = r+1, \dots, n$ . So  $\text{Jac}(\tilde{\mathbb{P}}_i) = \text{Jac}(A_{\mathbb{T}_i})$  for  $i = 0, \dots, r$ . By Theorem 5.10,  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \leq \text{Jac}(A_{\mathbb{T}_i})$  for  $i = 0, \dots, r$ .  $\square$

**Example 5.17** *Continue from Example 4.23. The corresponding order matrix is*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ o & o & o \end{pmatrix}.$$

Here  $\mathbf{R} = u_{01}u_{10}((u_{21}u_{10})'u_{20}u_{11} - u_{21}u_{10}(u_{20}u_{11})') - u_{01}u_{10}u_{20}^2u_{11}^2$ . Clearly,  $\text{ord}(\mathbf{R}, \mathbf{u}_0) = 0$ ,  $\text{ord}(\mathbf{R}, \mathbf{u}_1) = \text{ord}(\mathbf{R}, \mathbf{u}_2) = 1$ , and  $\text{ord}(\mathbf{R}, \mathbf{u}_3) = -\infty$ . But  $J_0 = J_1 = J_2 = o+1$ ,  $J_3 = 1$ , and  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \ll J_i$  for  $i = 0, 1, 2$ . If using Theorem 5.16, then  $A_{\mathbb{T}}$  consists of the first three rows of  $A$  and Jacobi numbers for  $A_{\mathbb{T}}$  are 1, 1, 1 respectively, which give much better bounds for the sparse differential resultant.

With Theorem 5.10, property 2) of Theorem 1.2 is proved.

### 5.3 Differential toric variety and sparse differential resultant

In this section, we will introduce the concept of differential toric variety and establish its relation with the sparse differential resultant.

We will deal with the special case when all the  $\mathcal{A}_i$  coincide with each other, i.e.,  $\mathcal{A}_0 = \dots = \mathcal{A}_n = \mathcal{A}$ . In this case,  $\mathcal{A}$  is said to be Laurent differentially essential when  $\mathcal{A}_0, \dots, \mathcal{A}_n$  form a Laurent differentially essential system. Let  $\mathcal{A} = \{M_0 = (\mathbb{Y}^{[o]})^{\alpha_0}, M_1 = (\mathbb{Y}^{[o]})^{\alpha_1}, \dots, M_l = (\mathbb{Y}^{[o]})^{\alpha_l}\}$  be Laurent differentially essential where  $\alpha_k \in \mathbb{Z}^{n(o+1)}$ . Then by Definition 3.6,  $l \geq n$  and there exist indices  $k_1, \dots, k_n \in \{1, \dots, l\}$  such that  $\frac{(\mathbb{Y}^{[o]})^{\alpha_{k_1}}}{(\mathbb{Y}^{[o]})^{\alpha_0}}, \dots, \frac{(\mathbb{Y}^{[o]})^{\alpha_{k_n}}}{(\mathbb{Y}^{[o]})^{\alpha_0}}$  are differentially independent over  $\mathbb{Q}$ . Let

$$\mathbb{P}_i = u_{i0}M_0 + u_{i1}M_1 + \dots + u_{il}M_l \quad (i = 0, \dots, n)$$

be  $n+1$  generic Laurent differential polynomials w.r.t  $\mathcal{A}$ .

Consider the following map

$$\phi_{\mathcal{A}} : (\mathcal{E}^{\wedge})^n \longrightarrow \mathbf{P}(l)$$



defined by

$$\phi_{\mathcal{A}}(\xi_1, \dots, \xi_n) = ((\xi^{[0]})^{\alpha_0}, (\xi^{[0]})^{\alpha_1}, \dots, (\xi^{[0]})^{\alpha_l}) \quad (13)$$

where  $\mathbf{P}(l)$  is the  $l$ -dimensional differential projective space over  $\mathcal{E}$  and  $\xi = (\xi_1, \dots, \xi_n) \in (\mathcal{E}^\wedge)^n$ . Note that  $((\xi^{[0]})^{\alpha_0}, (\xi^{[0]})^{\alpha_1}, \dots, (\xi^{[0]})^{\alpha_l})$  is never the zero vector since  $\xi_i \in \mathcal{E}^\wedge$  for all  $i$ . Thus  $\phi_{\mathcal{A}}$  is well defined on all of  $(\mathcal{E}^\wedge)^n$ , though the image of  $\phi_{\mathcal{A}}$  need not be a differential projective variety of  $\mathbf{P}(l)$ . Now we give the definition of differential toric variety.

**Definition 5.18** *The Kolchin projective differential closure of the image of  $\phi_{\mathcal{A}}$  is defined to be the differential toric variety w.r.t.  $\mathcal{A}$ , denoted by  $X_{\mathcal{A}}$ . That is,  $X_{\mathcal{A}} = \overline{\phi_{\mathcal{A}}((\mathcal{E}^\wedge)^n)}$ .*

Then we have the following theorem.

**Theorem 5.19**  *$X_{\mathcal{A}}$  is an irreducible projective differential variety over  $\mathbb{Q}$  of dimension  $n$ .*

*Proof:* Denote  $\mathbb{P}_i^N = \sum_{k=0}^l u_{ik} N_k$  ( $i = 0, \dots, n$ ) where  $N_k \in \mathfrak{m}$ . Clearly,  $M_k/M_0 = N_k/N_0$  ( $k = 1, \dots, l$ ). Let  $\mathcal{J} = [N_0 z_1 - N_1 z_0, \dots, N_0 z_l - N_l z_0] : \mathfrak{m} \in \mathbb{Q}\{\mathbb{Y}; z_0, z_1, \dots, z_l\}$ , where  $\mathfrak{m}$  denotes the set of all differential monomials in  $\mathbb{Y}$ . Let  $\eta = (\eta_1, \dots, \eta_n)$  be a generic point of  $[0]$  over  $\mathbb{Q}$  and  $v$  a differential indeterminate over  $\mathbb{Q}\langle\eta\rangle$ . Let  $\theta = (v, \frac{N_1(\eta)}{N_0(\eta)}v, \dots, \frac{N_l(\eta)}{N_0(\eta)}v)$ . We claim that  $(\eta; \theta)$  is a generic point of  $\mathcal{J}$  which follows that  $\mathcal{J}$  is a prime differential ideal. Indeed, on the one hand, since each  $N_0 z_i - N_i z_0$  ( $i = 1, \dots, l$ ) vanishes at  $(\eta; \theta)$  and  $\eta$  annihilates none of the elements of  $\mathfrak{m}$ ,  $(\eta; \theta)$  is a common zero of  $\mathcal{J}$ . On the other hand, for any  $f \in \mathbb{Q}\{\mathbb{Y}; z_0, z_1, \dots, z_l\}$  which vanishes at  $(\eta; \theta)$ , let  $f_1$  be the differential remainder of  $f$  w.r.t.  $N_0 z_i - N_i z_0$  ( $i = 1, \dots, l$ ) under the elimination ranking  $z_1 \succ \dots \succ z_l \succ z_0 \succ \mathbb{Y}$ . Then  $f_1 \in \mathbb{Q}\{\mathbb{Y}; z_0\}$  satisfies that  $N_0^a f \equiv f_1 \pmod{[N_0 z_1 - N_1 z_0, N_0 z_2 - N_2 z_0, \dots, N_0 z_l - N_l z_0]}$ . Since  $f(\eta; \theta) = 0$ ,  $f_1(\eta_1, \dots, \eta_n, v) = 0$ , and  $f_1 = 0$  follows. Thus,  $f \in \mathcal{J}$  and the claim is proved.

Let  $\mathcal{J}_1 = \mathcal{J} \cap \mathbb{Q}\{z_0, z_1, \dots, z_l\}$ . Then  $\mathcal{J}_1$  is a prime differential ideal with a generic point  $\theta$ . Denote  $\mathbf{z} = (z_0, z_1, \dots, z_l)$ . For any  $f \in \mathcal{J}_1 : \mathbf{z}$ , since  $z_0 f \in \mathcal{J}_1$ ,  $z_0 f$  vanishes at  $\theta$  and  $f(\theta) = 0$  follows. So  $f \in \mathcal{J}_1$ , and it follows that  $\mathcal{J}_1 : \mathbf{z} = \mathcal{J}_1$ . And for any  $f \in \mathcal{J}_1 \subset \mathcal{J}$  and any differential indeterminate  $\lambda$  over  $\mathbb{Q}\langle\eta, v\rangle$ , let  $f(\lambda \mathbf{z}) = \sum \phi(\lambda) f_\phi(\mathbf{z})$  where  $\phi(\lambda)$  are distinct differential monomials in  $\lambda$  and  $f_\phi(\mathbf{z}) \in \mathbb{Q}\{\mathbf{z}\}$ . Then  $f(\lambda \theta) = 0 = \sum \phi(\lambda) f_\phi(\theta)$ . So each  $f_\phi(\theta) = 0$  and  $f_\phi \in \mathcal{J}_1$  follows. Thus,  $f(\lambda \mathbf{z}) \in \mathbb{Q}\{\lambda\} \mathcal{J}_1$ . By Definition 2.2,  $\mathcal{J}_1$  is a differentially homogenous differential ideal. Then  $V = \mathbb{V}(\mathcal{J}_1)$  is an irreducible projective differential variety in  $\mathbf{P}(l)$ . Since  $\theta$  is a generic point of  $V$ ,  $\dim(V) = \text{d.tr.deg } \mathbb{Q}\langle\frac{N_1(\eta)}{N_0(\eta)}, \dots, \frac{N_l(\eta)}{N_0(\eta)}\rangle / \mathbb{Q} = n$ . If we can show  $X_{\mathcal{A}} = V$ , then it follows that  $X_{\mathcal{A}}$  is an irreducible projective differential variety of dimension  $n$ .

For any point  $\xi \in (\mathcal{E}^\wedge)^n$ , it is clear that  $(\xi; N_0(\xi), N_1(\xi), \dots, N_l(\xi))$  is a differential zero of  $\mathcal{J}$  and consequently  $(N_0(\xi), N_1(\xi), \dots, N_l(\xi)) \in \mathbb{V}(\mathcal{J}_1) = V$ . So  $\phi(\xi) = (N_0(\xi), N_1(\xi), \dots, N_l(\xi)) \in V$ . Thus  $\phi_{\mathcal{A}}((\mathcal{E}^\wedge)^n) \subseteq V$  and  $X_{\mathcal{A}} = \overline{\phi_{\mathcal{A}}((\mathcal{E}^\wedge)^n)} \subseteq V$  follows. Conversely, since  $\phi(\eta) = (1, \frac{N_1(\eta)}{N_0(\eta)}, \dots, \frac{N_l(\eta)}{N_0(\eta)}) \in X_{\mathcal{A}}$  is the generic point of  $V$ ,  $V \subseteq X_{\mathcal{A}}$ . Thus,  $V = X_{\mathcal{A}}$ .  $\square$

Now, suppose  $z_0, \dots, z_l$  are the homogenous coordinates of  $\mathbf{P}(l)$ . Let

$$\mathbb{L}_i = u_{i0} z_0 + u_{i1} z_1 + \dots + u_{il} z_l \quad (i = 0, \dots, n)$$

be generic differential hyperplanes in  $\mathbf{P}(l)$ . Then, clearly,  $\mathbb{P}_i = \mathbb{L}_i \circ \phi_{\mathcal{A}}$ . In the following, we will explore the close relation between  $\text{Res}_{\mathcal{A}}$  and the differential Chow form of  $X_{\mathcal{A}}$ . Before doing so, we first recall the concept of projective differential Chow form ([36]).

Let  $V$  be an irreducible projective differential variety of dimension  $d$  over  $\mathbb{Q}$  with a generic point  $\xi = (\xi_0, \xi_1, \dots, \xi_l)$ . Suppose  $\xi_0 \neq 0$ . Let  $\mathbb{L}_i = -\sum_{k=0}^l u_{ik} z_k$  ( $i = 0, \dots, d$ ) be  $d+1$  generic projective differential hyperplanes. Denote  $\zeta_i = -\sum_{k=1}^l u_{ik} \xi_0^{-1} \xi_k$  ( $i = 0, \dots, d$ ) and  $\mathbf{u}_i = (u_{i0}, \dots, u_{il})$ . Then we showed in [36] that the prime ideal  $\mathbb{I}(\zeta_0, \dots, \zeta_d)$  over  $\mathbb{Q}\langle \cup_i \mathbf{u}_i \setminus \{u_{i0}\} \rangle$  is of codimension one. That is, there exists an irreducible differential polynomial  $F \in \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_d\}$  such that  $\mathbb{I}(\zeta_0, u_{01}, \dots, u_{0l}; \dots; \zeta_d, u_{d1}, \dots, u_{dl}) = \text{sat}(F)$ . This  $F$  is defined to be the *differential Chow form* of  $\mathbb{V}(\mathcal{I})$  or  $\mathcal{I}$ . We list one of its properties which will be used in this section.

**Theorem 5.20** [36, Theorem 4.7] *Let  $F(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$  be the differential Chow form of  $V$  with  $\text{ord}(F) = h$  and  $S_F = \frac{\partial F}{\partial u_{00}^{(h)}}$ . Suppose that  $\mathbf{u}_i$  are differentially specialized over  $\mathbb{Q}$  to sets  $\mathbf{v}_i \subset \mathcal{E}$  and  $\bar{\mathbb{P}}_i$  are obtained by substituting  $\mathbf{u}_i$  by  $\mathbf{v}_i$  in  $\mathbb{P}_i$  ( $i = 0, \dots, d$ ). If  $\bar{\mathbb{P}}_i = 0$  ( $i = 0, \dots, d$ ) meet  $V$ , then  $\text{sat}(F)$  vanishes at  $(\mathbf{v}_0, \dots, \mathbf{v}_d)$ . Furthermore, if  $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = 0$  and  $S_F(\mathbf{v}_0, \dots, \mathbf{v}_d) \neq 0$ , then the  $d+1$  differential hyperplanes  $\bar{\mathbb{P}}_i = 0$  ( $i = 0, \dots, d$ ) meet  $V$ .*

With the above preparations, we now proceed to show that the sparse differential resultant is just the differential Chow form of  $X_{\mathcal{A}}$ .

**Theorem 5.21** *Let  $\text{Res}_{\mathcal{A}}$  be the sparse differential resultant of  $\mathbb{P}_0, \dots, \mathbb{P}_n$ . Then  $\text{Res}_{\mathcal{A}}$  is the differential Chow form of  $X_{\mathcal{A}}$  with respect to the generic hyperplanes  $\mathbb{L}_i$  ( $i = 0, \dots, n$ ).*

*Proof:* By the proof of Theorem 5.19,  $X_{\mathcal{A}}$  is an irreducible projective differential variety of dimension  $n$  with a generic point  $(1, \frac{M_1(\eta)}{M_0(\eta)}, \dots, \frac{M_l(\eta)}{M_0(\eta)})$ . Let  $\zeta_i = -\sum_{k=1}^l u_{ik} \frac{M_k(\eta)}{M_0(\eta)}$  ( $i = 0, \dots, n$ ). Then  $\text{sat}(\text{Chow}(X_{\mathcal{A}})) = \mathbb{I}((\zeta_0, u_{01}, \dots, u_{0l}; \dots; \zeta_n, u_{n1}, \dots, u_{nl}))$ . And by the definition of sparse differential resultant,  $\text{sat}(\text{Res}_{\mathcal{A}}) = \mathbb{I}((\zeta_0, u_{01}, \dots, u_{0l}; \dots; \zeta_n, u_{n1}, \dots, u_{nl}))$ . By Lemma 2.3,  $\text{Chow}(X_{\mathcal{A}})$  and  $\text{Res}_{\mathcal{A}}$  can only differ at most by a nonzero element in  $\mathbb{Q}$ . Thus,  $\text{Res}_{\mathcal{A}}$  is just the differential Chow form of  $X_{\mathcal{A}}$ .  $\square$

Based on Theorem 5.20, we give another characterization of the vanishing of sparse differential resultants below, where the zeros are taken from  $\mathcal{E}$  instead of  $\mathcal{E}^\wedge$ .

**Corollary 5.22** *Let  $\bar{\mathbb{L}}_i = v_{i0}z_0 + v_{i1}z_1 + \dots + v_{il}z_l = 0$  ( $i = 0, \dots, n$ ) be projective differential hyperplanes over  $\mathcal{E}$  with  $\mathbf{v}_i = (v_{i0}, \dots, v_{il})$ . Denote  $\text{ord}(\text{Res}_{\mathcal{A}}) = h$  and  $S_{\mathbf{R}} = \frac{\partial \text{Res}_{\mathcal{A}}}{\partial u_{00}^{(h)}}$ . If  $X_{\mathcal{A}}$  meets  $\bar{\mathbb{L}}_i = 0$  ( $i = 0, \dots, n$ ), then  $\text{Res}_{\mathcal{A}}(\mathbf{v}_0, \dots, \mathbf{v}_n) = 0$ . And if  $\text{Res}_{\mathcal{A}}(\mathbf{v}_0, \dots, \mathbf{v}_n) = 0$  and  $S_{\mathbf{R}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$ , then  $X_{\mathcal{A}}$  meets  $\bar{\mathbb{L}}_i = 0$  ( $i = 0, \dots, n$ ).*

*Proof:* It follows directly from Theorems 5.21 and 5.20.  $\square$

**Example 5.23** *Continue from Example 3.17. Following the proof of Theorem 5.19, consider  $\mathcal{J} = [y_1z_1 - y_1'z_0, y_1z_2 - y_1'^2z_0] : \mathfrak{m}$ . It is easy to show that  $X_{\mathcal{A}}$  is the general component of  $z_1z_2 - (z_0z_2' - z_0'z_2)$ , that is,  $X_{\mathcal{A}} = \mathbb{V}(\text{sat}(z_1z_2 - (z_0z_2' - z_0'z_2)))$ . And  $\text{Res}_{\mathcal{A}}$  is equal to the differential Chow form of  $X_{\mathcal{A}}$ .*

By Theorems 5.19 and 5.21, property 4) of Theorem 1.2 is proved.

## 5.4 Poisson-type product formulas

In this section, we prove formulas for sparse differential resultants, which are similar to the Poisson-type product formulas for multivariate resultants [40].

Denote  $\text{ord}(\mathbf{R}, \mathbf{u}_i)$  by  $h_i$  ( $i = 0, \dots, n$ ), and suppose  $h_0 \geq 0$ . Let  $\tilde{\mathbf{u}} = \cup_{i=0}^n \mathbf{u}_i \setminus \{u_{00}\}$  and  $\mathbb{Q}_0 = \mathbb{Q}\langle \tilde{\mathbf{u}} \rangle(u_{00}^{(0)}, \dots, u_{00}^{(h_0-1)})$ . Consider  $\mathbf{R}$  as an irreducible algebraic polynomial  $r(u_{00}^{(h_0)})$  in  $\mathbb{Q}_0[u_{00}^{(h_0)}]$ . In a suitable algebraic extension field of  $\mathbb{Q}_0$ ,  $r(u_{00}^{(h_0)}) = 0$  has  $t_0 = \deg(r, u_{00}^{(h_0)}) = \deg(\mathbf{R}, u_{00}^{(h_0)})$  roots  $\gamma_1, \dots, \gamma_{t_0}$ . Thus

$$\mathbf{R}(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n) = A \prod_{\tau=1}^{t_0} (u_{00}^{(h_0)} - \gamma_\tau) \quad (14)$$

where  $A \in \mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_n \rangle[\mathbf{u}_0^{[h_0]} \setminus u_{00}^{(h_0)}]$ . For each  $\tau$  such that  $1 \leq \tau \leq t_0$ , let

$$\mathbb{Q}_\tau = \mathbb{Q}_0(\gamma_\tau) = \mathbb{Q}\langle \tilde{\mathbf{u}} \rangle(u_{00}^{(0)}, \dots, u_{00}^{(h_0-1)}, \gamma_\tau) \quad (15)$$

be an algebraic extension field of  $\mathbb{Q}_0$  defined by  $r(u_{00}^{(h_0)}) = 0$ . We will define a derivation operator  $\delta_\tau$  on  $\mathbb{Q}_\tau$  so that  $\mathbb{Q}_\tau$  becomes a  $\delta_\tau$ -field. This can be done in a very natural way. For  $e \in \mathbb{Q}\langle \tilde{\mathbf{u}} \rangle$ , define  $\delta_\tau e = \delta e = e'$ . Define  $\delta_\tau^i u_{00} = u_{00}^{(i)}$  for  $i = 0, \dots, h_0 - 1$  and

$$\delta_\tau^{h_0} u_{00} = \gamma_\tau.$$

Since  $\mathbf{R}$ , regarded as an algebraic polynomial  $r$  in  $u_{00}^{(h_0)}$ , is a minimal polynomial of  $\gamma_\tau$ ,  $\mathbf{S}_\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u_{00}^{(h_0)}}$  does not vanish at  $u_{00}^{(h_0)} = \gamma_\tau$ . Now, we define the derivatives of  $\delta_\tau^i u_{00}$  for  $i > h_0$  by induction. Firstly, since  $r(\gamma_\tau) = 0$ ,  $\delta_\tau(r(\gamma_\tau)) = \mathbf{S}_\mathbf{R}|_{u_{00}^{(h_0)}=\gamma_\tau} \delta_\tau(\gamma_\tau) + T|_{u_{00}^{(h_0)}=\gamma_\tau} = 0$ , where  $T = \mathbf{R}' - \mathbf{S}_\mathbf{R} u_{00}^{(h_0+1)}$ . We define  $\delta_\tau^{h_0+1} u_{00}$  to be  $\delta_\tau(\gamma_\tau) = -\frac{T}{\mathbf{S}_\mathbf{R}}|_{u_{00}^{(h_0)}=\gamma_\tau}$ . Supposing the derivatives of  $\delta_\tau^{h_0+j} u_{00}$  with order less than  $j < i$  have been defined, we now define  $\delta_\tau^{h_0+i} u_{00}$ . Since  $\mathbf{R}^{(i)} = \mathbf{S}_\mathbf{R} u_{00}^{(h_0+i)} + T_i$  is linear in  $u_{00}^{(h_0+i)}$ , we define  $\delta_\tau^{h_0+i} u_{00}$  to be  $-\frac{T_i}{\mathbf{S}_\mathbf{R}}|_{u_{00}^{(h_0+j)}=\delta_\tau^{h_0+j} u_{00}, j < i}$ .

In this way,  $(\mathbb{Q}_\tau, \delta_\tau)$  is a differential field which can be considered as a finitely generated differential extension of  $\mathbb{Q}\langle \tilde{\mathbf{u}} \rangle$ . Recall that  $\mathbb{Q}\langle \tilde{\mathbf{u}} \rangle$  is a finitely generated differential extension field of  $\mathbb{Q}$  contained in  $\mathcal{E}$ . By the definition of universal differential extension field, there exists a differential extension field  $\mathcal{G} \subset \mathcal{E}$  of  $\mathbb{Q}\langle \tilde{\mathbf{u}} \rangle$  and a differential isomorphism  $\varphi_\tau$  over  $\mathbb{Q}\langle \tilde{\mathbf{u}} \rangle$  from  $(\mathbb{Q}_\tau, \delta_\tau)$  to  $(\mathcal{G}, \delta)$ . Summing up the above results, we have

**Lemma 5.24**  *$(\mathbb{Q}_\tau, \delta_\tau)$  defined above is a finitely generated differential extension field of  $\mathbb{Q}\langle \tilde{\mathbf{u}} \rangle$ , which is differentially  $\mathbb{Q}\langle \tilde{\mathbf{u}} \rangle$ -isomorphic to a subfield of  $\mathcal{E}$ .*

Let  $p$  be a differential polynomial in  $\mathcal{F}\{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n\} = \mathcal{F}\{\tilde{\mathbf{u}}, u_{00}\}$ . For convenience, by the symbol  $p|_{u_{00}^{(h_0)}=\gamma_\tau}$ , we mean substituting  $u_{00}^{(h_0+i)}$  by  $\delta_\tau^i \gamma_\tau$  ( $i \geq 0$ ) in  $p$ . Similarly, by saying  $p$  vanishes at  $u_{00}^{(h_0)} = \gamma_\tau$ , we mean  $p|_{u_{00}^{(h_0)}=\gamma_\tau} = 0$ . It is easy to prove the following lemma.

**Lemma 5.25** *Let  $p$  be a differential polynomial in  $\mathcal{F}\{\tilde{\mathbf{u}}, u_{00}\}$ . Then  $p \in \text{sat}(\mathbf{R})$  if and only if  $p$  vanishes at  $u_{00}^{(h_0)} = \gamma_\tau$ .*

When a differential polynomial  $p \in \mathbb{Q}\langle \tilde{\mathbf{u}} \rangle \{Y\}$  vanishes at a point  $\eta \in \mathbb{Q}_\tau^n$ , it is easy to see that  $p$  vanishes at  $\varphi_\tau(\eta) \in \mathcal{E}^n$ . For convenience, by saying  $\eta$  is in a differential variety  $V$  over  $\mathbb{Q}\langle \tilde{\mathbf{u}} \rangle$ , we mean  $\varphi_\tau(\eta) \in V$ .

With these preparations, we now give the following theorem.

**Theorem 5.26** *Let  $\mathbf{R}(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n)$  be the sparse differential resultant of  $\mathbb{P}_0, \dots, \mathbb{P}_n$  with  $\text{ord}(\mathbf{R}, \mathbf{u}_0) = h_0 \geq 0$ . Let  $\deg(\mathbf{R}, u_{00}^{(h_0)}) = t_0$ . Then there exist  $\xi_{\tau k}$  in an extension field  $(\mathbb{Q}_\tau, \delta_\tau)$  of  $(\mathbb{Q}\langle \tilde{\mathbf{u}} \rangle, \delta)$  for  $\tau = 1, \dots, t_0$  and  $k = 1, \dots, l_0$  such that*

$$\mathbf{R} = A \prod_{\tau=1}^{t_0} (u_{00} + \sum_{k=1}^{l_0} u_{0k} \xi_{\tau k})^{(h_0)}, \quad (16)$$

where  $A$  is a polynomial in  $\mathbb{Q}\langle \mathbf{u}_1, \dots, \mathbf{u}_n \rangle [\mathbf{u}_0^{[h_0]} \setminus u_{00}^{(h_0)}]$ . Note that equation (16) is formal and should be understood in the following precise meaning:  $(u_{00} + \sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})^{(h_0)} \triangleq \delta^{h_0} u_{00} + \delta_\tau^{h_0} (\sum_{\rho=1}^n u_{0\rho} \xi_{\tau\rho})$ .

*Proof:* We will follow the notations introduced in the proof of Lemma 5.24. Since  $\mathbf{R}$  is irreducible, we have  $\mathbf{R}_{\tau 0} = \frac{\partial \mathbf{R}}{\partial u_{00}^{(h_0)}} \Big|_{u_{00}^{(h_0)} = \gamma_\tau} \neq 0$ . Let  $\xi_{\tau\rho} = \mathbf{R}_{\tau\rho} / \mathbf{R}_{\tau 0}$  ( $\rho = 1, \dots, l_0$ ), where  $\mathbf{R}_{\tau\rho} = \frac{\partial \mathbf{R}}{\partial u_{0\rho}^{(h_0)}} \Big|_{u_{00}^{(h_0)} = \gamma_\tau}$ . Note that  $\mathbf{R}_{\tau\rho}$  and  $\xi_{\tau\rho}$  are in  $\mathcal{F}_\tau$ . We will prove

$$\gamma_\tau = -\delta_\tau^{h_0} (u_{01} \xi_{\tau 1} + u_{02} \xi_{\tau 2} + \dots + u_{0l_0} \xi_{\tau l_0}).$$

Differentiating the equality  $\mathbf{R}(\mathbf{u}; \zeta_0, \zeta_1, \dots, \zeta_n) = 0$  w.r.t.  $u_{0\rho}^{(h_0)}$ , we have

$$\frac{\overline{\partial \mathbf{R}}}{\partial u_{0\rho}^{(h_0)}} + \frac{\overline{\partial \mathbf{R}}}{\partial u_{00}^{(h_0)}} \left( -\frac{M_{0\rho}(\eta)}{M_{00}(\eta)} \right) = 0, \quad (17)$$

where  $\frac{\overline{\partial \mathbf{R}}}{\partial u_{0\rho}^{(h_0)}}$  are obtained by substituting  $\zeta_i$  to  $u_{i0}$  ( $i = 0, 1, \dots, n$ ) in  $\frac{\partial \mathbf{R}}{\partial u_{0\rho}^{(h_0)}}$ . Multiplying  $u_{0\rho}$  to the above equation and for  $\rho$  from 1 to  $l_0$ , adding them together, we have

$$\sum_{\rho=1}^{l_0} u_{0\rho} \frac{\overline{\partial \mathbf{R}}}{\partial u_{0\rho}^{(h_0)}} + \frac{\overline{\partial \mathbf{R}}}{\partial u_{00}^{(h_0)}} \left( -\sum_{\rho=1}^{l_0} u_{0\rho} \frac{M_{0\rho}(\eta)}{M_{00}(\eta)} \right) = \sum_{\rho=1}^{l_0} u_{0\rho} \frac{\overline{\partial \mathbf{R}}}{\partial u_{0\rho}^{(h_0)}} + \zeta_0 \frac{\overline{\partial \mathbf{R}}}{\partial u_{00}^{(h_0)}} = 0.$$

Thus,  $q = \sum_{\rho=1}^{l_0} u_{0\rho} \frac{\partial \mathbf{R}}{\partial u_{0\rho}^{(h_0)}} + u_{00} \frac{\partial \mathbf{R}}{\partial u_{00}^{(h_0)}} \in \text{sat}(\mathbf{R})$ . Since  $q$  is of order not greater than  $\mathbf{R}$ , it must be divisible by  $\mathbf{R}$ . Since  $q$  and  $\mathbf{R}$  have the same degree, there exists an  $a \in \mathbb{Q}$  such that  $q = a\mathbf{R}$ . Setting  $u_{00}^{(h_0)} = \gamma_\tau$  in both sides of  $q = a\mathbf{R}$ , we have  $\sum_{\rho=1}^{l_0} u_{0\rho} \mathbf{R}_{\tau\rho} + u_{00} \mathbf{R}_{\tau 0} = 0$ . Hence, as an algebraic equation, we have

$$u_{00} + \sum_{\rho=1}^{l_0} u_{0\rho} \xi_{\tau\rho} = 0 \quad (18)$$

under the constraint  $u_{00}^{(h_0)} = \gamma_\tau$ . Equivalently, the above equation is valid in  $(\mathcal{F}_\tau, \delta_\tau)$ . As a consequence,  $\gamma_\tau = -\delta_\tau^{h_0}(\sum_{\rho=1}^{l_0} u_{0\rho} \xi_{\tau\rho})$ . Substituting them into equation (14), the theorem is proved.  $\square$

Note that the quantities  $\xi_{\tau\rho}$  are not expressions in terms of  $y_i$ . In the following theorem, we will show that if  $\{\mathcal{A}_i (i = 0, \dots, n)\}$  satisfies certain conditions, Theorem 5.26 can be strengthened to make  $\xi_{\tau\rho}$  as productions of certain values of  $y_i$  and its derivatives. Following the notations introduced before Lemma 5.31, we have

**Theorem 5.27** *Assume that 1) any  $n$  of the  $\mathcal{A}_i (i = 0, \dots, n)$  form a differentially independent set and 2) for each  $j = 1, \dots, n$ ,  $\mathbf{e}_j \in \text{Span}_{\mathbb{Z}}\{\alpha_{ik} - \alpha_{i0} : k = 1, \dots, l_i; i = 0, \dots, n\}$ . Then there exist  $\eta_{\tau k}$  ( $\tau = 1, \dots, t_0; k = 1, \dots, n$ ) such that*

$$\begin{aligned} \mathbf{R} &= A \prod_{\tau=1}^{t_0} \left( u_{00} + \sum_{k=1}^{l_0} u_{0k} \frac{M_{0k}(\eta_\tau)}{M_{00}(\eta_\tau)} \right)^{(h_0)} \\ &= A \prod_{\tau=1}^{t_0} \left[ \frac{\mathbb{P}_0(\eta_\tau)}{M_{00}(\eta_\tau)} \right]^{(h_0)}, \quad \text{where } \eta_\tau = (\eta_{\tau 1}, \dots, \eta_{\tau n}). \end{aligned} \quad (19)$$

Moreover,  $\eta_\tau$  ( $\tau = 1, \dots, t_0$ ) lies on  $\mathbb{P}_1, \dots, \mathbb{P}_n$ .

*Proof:* Follow the notations in this section and those introduced before Lemma 5.31. By condition 1), each  $h_i \geq 0$ . Denote  $\mathbb{P}_i^N = M_i \mathbb{P}_i (i = 0, \dots, n)$  where  $M_i$  are Laurent differential polynomials. Then  $(\eta; \zeta_0, u_{01}, \dots, u_{0l_0}; \dots; \zeta_n, u_{n1}, \dots, u_{nl_n})$  is a generic point of  $[\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$  where  $\eta = (\eta_1, \dots, \eta_n)$  is a generic point of  $[0]$  over  $\mathbb{Q}\langle \mathbf{u} \rangle$  and  $\zeta_i = -\sum_{k=1}^{l_i} u_{ik} \frac{M_{ik}(\eta)}{M_{i0}(\eta)}$ . By the proof of Lemma 5.31, there exist  $S_j$  and  $T_j$  which are products of nonnegative powers of  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  such that  $S_j y_j - T_j \in [\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$ . That is,  $\eta_j = \overline{T_j} / \overline{S_j}$  for  $j = 1, \dots, n$ , where  $\overline{T_j}, \overline{S_j}$  are obtained by substituting  $(u_{00}, \dots, u_{n0}) = (\zeta_0, \dots, \zeta_n)$  in  $T_j, S_j$  respectively. Since  $\mathbf{R}$  is an irreducible polynomial, every  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  does not vanishes at  $u_{00}^{(h_0)} = \gamma_\tau$ . Let  $\eta_{\tau j} = \frac{T_j}{S_j} \big|_{u_{00}^{(h_0)} = \gamma_\tau}$  and  $\eta_\tau = (\eta_{\tau 1}, \dots, \eta_{\tau n})$ . By (17),  $\frac{M_{0k}(\eta)}{M_{00}(\eta)} = \prod_{j=1}^n \prod_{k=1}^{s_0} (\eta_j^{(k)})^{(\alpha_{0k} - \alpha_{00})_{jk}} = \frac{\overline{\partial \mathbf{R}}}{\partial u_{0k}^{(h_0)}} / \frac{\overline{\partial \mathbf{R}}}{\partial u_{00}^{(h_0)}}$ . So  $\prod_{j=1}^n \prod_{k=1}^{s_0} \left[ \left( \frac{\overline{T_j}}{\overline{S_j}} \right)^{(k)} \right]^{(\alpha_{0k} - \alpha_{00})_{jk}} = \frac{\overline{\partial \mathbf{R}}}{\partial u_{0k}^{(h_0)}} / \frac{\overline{\partial \mathbf{R}}}{\partial u_{00}^{(h_0)}}$ . Let  $\mathcal{S}$  be the differential polynomial set consisting of  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  and  $(S_j)^{m+1} \left( \frac{T_j}{S_j} \right)^{(m)}$  for all  $i = 0, \dots, n; k = 0, \dots, l_i; j = 1, \dots, n$  and  $m \in \mathbb{N}$ . By Corollary 3.12, there exists a finite set  $\mathcal{S}_1$  of  $\mathcal{S}$  and  $a \in \mathbb{N}$  such that  $H = \left( \prod_{S \in \mathcal{S}_1} S \right)^a \left( \prod_{j=1}^n \prod_{k=1}^{s_0} \left[ (T_j / S_j)^{(k)} \right]^{(\alpha_{0k} - \alpha_{00})_{jk}} - \frac{\partial \mathbf{R}}{\partial u_{0k}^{(h_0)}} / \frac{\partial \mathbf{R}}{\partial u_{00}^{(h_0)}}$   $\right) \in \text{sat}(\mathbf{R})$ . By Lemma 5.25,  $H$  vanishes at  $u_{00}^{(h_0)} = \gamma_\tau$ . And by the proof of Theorem 5.29,  $\mathcal{S} \cap \text{sat}(\mathbf{R}) = \emptyset$ . So  $\xi_{\tau k} = \frac{M_{0k}(\eta_\tau)}{M_{00}(\eta_\tau)}$ . By Theorem 5.26,  $\mathbf{R} = A \prod_{\tau=1}^{t_0} (u_{00} + \sum_{k=1}^{l_0} u_{0k} \xi_{\tau k})^{(h_0)}$ . Thus, (19) follows.

To prove the second part of this theorem, we need first to show that  $\delta_\tau^k \eta_{\tau j} \neq 0$  for each  $k \geq 0$ . Suppose the contrary, that is, there exists some  $k$  such that  $\delta_\tau^k \eta_{\tau j} = 0$ . From

$\eta_{\tau j} = \frac{T_j}{S_j} \Big|_{u_{00}^{(h_0)} = \gamma_\tau}$ ,  $\delta_\tau^k \eta_{\tau j} = \left(\frac{T_j}{S_j}\right)^{(k)} \Big|_{u_{00}^{(h_0)} = \gamma_\tau} = 0$ . Thus,  $S_j^{k+1} \left(\frac{T_j}{S_j}\right)^{(k)} \in \text{sat}(\mathbf{R})$ . It follows that  $\eta_j^{(k)} = \left(\frac{T_j}{S_j}\right)^{(k)} = 0$ , a contradiction to the fact that  $\eta_j$  is a differential indeterminate.

Follow the above procedure, we can show that  $\frac{M_{ik}(\eta_\tau)}{M_{i0}(\eta_\tau)} = \frac{\widehat{\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}}}{\widehat{\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}}}$  where  $\widehat{\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}} = \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} \Big|_{u_{00}^{(h_0)} = \gamma_\tau}$ . From (20), it is easy to show that  $\sum_{k=0}^{l_i} u_{ik} \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} = b \mathbf{R}$  for some  $b$  in  $\mathbb{Q}$ . So, for each  $i \neq 0$ ,  $\sum_{k=0}^{l_i} u_{ik} \widehat{\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}} = 0$ . It follows that for each  $i \neq 0$ ,  $\mathbb{P}_i(\eta_\tau) = \sum_{k=0}^{l_i} u_{ik} M_{ik}(\eta_\tau) = \frac{M_{i0}(\eta_\tau)}{\widehat{\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}}} \left( \sum_{k=0}^{l_i} u_{ik} \widehat{\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}} \right) = 0$ . So  $\eta_\tau$  lies on  $\mathbb{P}_1, \dots, \mathbb{P}_n$ .  $\square$

Under the conditions of Theorem 5.27, we further have the following theorem.

**Theorem 5.28** *The elements  $\eta_\tau$  ( $\tau = 1, \dots, t_0$ ) defined in Theorem 5.27 are generic points of the prime ideal  $[\mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m} \subset \mathbb{Q}\langle \hat{\mathbf{u}} \rangle \{ \mathbb{Y} \}$ , where  $\hat{\mathbf{u}} = \cup_{i=1}^n \mathbf{u}_i$ .*

*Proof:* Follow the notations in this section. Let  $\mathcal{J}_0 = [\mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m} \subset \mathbb{Q}\{ \mathbb{Y}, \hat{\mathbf{u}} \}$  and  $\mathcal{J} = [\mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m} \subset \mathbb{Q}\langle \hat{\mathbf{u}} \rangle \{ \mathbb{Y} \}$ . Similar to the proof of Theorem 3.9, it is easy to show that  $\mathcal{J}_0$  is a prime differential ideal. And by condition 1),  $\mathcal{J}_0 \cap \mathbb{Q}\{ \hat{\mathbf{u}} \} = [0]$ . Thus,  $\mathcal{J} = [\mathcal{J}_0]$  is a prime differential ideal and  $\mathcal{J} \cap \mathbb{Q}\{ \mathbb{Y}, \hat{\mathbf{u}} \} = \mathcal{J}_0$ . Let  $\xi = (\xi_1, \dots, \xi_n)$  be a generic point of  $\mathcal{J}$ . Then  $(\xi; \hat{\mathbf{u}})$  is a generic point of  $\mathcal{J}_0$ . Let  $\beta = -\sum_{k=1}^{l_0} u_{0k} M_{0k}(\xi) / M_{00}(\xi)$ . Then  $(\xi; \beta, u_{01}, \dots, u_{0l_0}; \hat{\mathbf{u}})$  is a generic point of  $\mathcal{I} = [\mathbb{P}_0^N, \mathbb{P}_1^N, \dots, \mathbb{P}_n^N] : \mathfrak{m} \subset \mathbb{Q}\{ \mathbb{Y}; \mathbf{u}_0, \hat{\mathbf{u}} \}$ . Since  $S_j y_j - T_j \in \mathcal{I}$  ( $j = 1, \dots, n$ ),  $\xi_j = \frac{T_j}{S_j}(\beta, u_{01}, \dots, u_{0l_0}; \hat{\mathbf{u}})$ .

By Theorem 5.27,  $\eta_\tau$  is a common non-polynomial solution of  $\mathbb{P}_i^N = 0$  ( $i = 1, \dots, n$ ), thus also a differential zero of  $\mathcal{J}$ . Recall  $\eta_{\tau j} = \frac{T_j}{S_j} \Big|_{u_{00}^{(h_0)} = \gamma_\tau}$ . If  $f$  is any differential polynomial in  $\mathbb{Q}\langle \hat{\mathbf{u}} \rangle \{ \mathbb{Y} \}$  such that  $f(\eta_\tau) = 0$ , then  $f\left(\frac{T_1}{S_1}, \dots, \frac{T_n}{S_n}\right) \Big|_{u_{00}^{(h_0)} = \gamma_\tau} = 0$ . There exist  $a_j \in \mathbb{N}$  such that  $g = \prod_j S_j^{a_j} f\left(\frac{T_1}{S_1}, \dots, \frac{T_n}{S_n}\right) \in \mathbb{Q}\{ \mathbb{Y}; \mathbf{u}_0, \hat{\mathbf{u}} \}$ . Then  $g \Big|_{u_{00}^{(h_0)} = \gamma_\tau} = 0$ . By Lemma 5.25,  $g \in \text{sat}(\mathbf{R})$  while  $S_j \notin \text{sat}(\mathbf{R})$ . As a consequence,  $g(\beta, u_{01}, \dots, u_{0l_0}; \hat{\mathbf{u}}) = 0$  and  $S_j(\beta, u_{01}, \dots, u_{0l_0}; \hat{\mathbf{u}}) \neq 0$ . It follows that  $f(\xi_1, \dots, \xi_n) = 0$  and  $f \in \mathcal{J}$ . Thus,  $\eta_\tau$  is a generic point of  $\mathcal{J}$ .  $\square$

With Theorems 5.26, 5.27, and 5.28, property 4) of Theorem 1.2 is proved.

## 5.5 Structures of non-polynomial solutions

In this section, we will analyze the structures of the non-polynomial solutions. Firstly, we will give the following theorem which shows the relation between the original differential system and their sparse differential resultant.

Let  $\mathcal{A}_i$  ( $i = 0, \dots, n$ ) be a Laurent differentially essential system of monomial sets. Then by Theorem 4.21,  $\mathcal{A}_i$  ( $i = 0, \dots, n$ ) can be divided into two disjoint sets  $\{\mathcal{A}_i : i \in \mathbb{T}\}$  and  $\{\mathcal{A}_i : i \in \{0, 1, \dots, n\} \setminus \mathbb{T}\}$ , where  $\mathbb{T} \subseteq \{0, 1, \dots, n\}$  is rank essential. In this section, we will assume that  $\{0, 1, \dots, n\}$  is rank essential, that is, any  $n$  of the  $\mathcal{A}_i$  ( $i = 0, \dots, n$ ) form a differentially independent set, which is equivalent to the fact that each  $\mathbf{u}_i$  occurs in  $\mathbf{R}$  effectively.

**Theorem 5.29** Let  $\mathbb{P}_i = \sum_{k=0}^{l_i} u_{ik} M_{ik}$  ( $i = 0, \dots, n$ ) be a system of differential polynomials such that any  $n$  of the  $\mathbb{P}_i$  form a differentially independent set. Let  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  be the sparse differential resultant of  $\mathbb{P}_i$  with  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = h_i$ . Denote  $Q_{ik} = \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} M_{ik} - \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} M_{i0}$  and  $S$  to be the set consisting of  $\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}$  ( $i = 0, \dots, n$ ) and  $(y_i^{(k)})_{1 \leq i \leq n; k \geq 0}$ . Then

$$[\mathbb{P}_0, \dots, \mathbb{P}_n] : \mathfrak{m} = [\mathbf{R}, (Q_{ik})_{0 \leq i \leq n; 1 \leq k \leq l_i}] : S^\infty$$

in  $\mathbb{Q}\{\mathbb{Y}, \mathbf{u}_0, \dots, \mathbf{u}_n\}$ .

*Proof:* Let  $\mathcal{I} = [\mathbb{P}_0, \dots, \mathbb{P}_n] : \mathfrak{m}$ . Following the notations in the proof of Theorem 3.9,  $\mathcal{I}$  is a prime differential ideal with a generic point  $(\eta; \zeta_0, u_{01}, \dots, u_{0l_0}; \dots; \zeta_n, u_{n1}, \dots, u_{nl_n})$  where  $\zeta_i = -\sum_{k=1}^{l_i} u_{ik} \frac{M_{ik}(\eta)}{M_{i0}(\eta)}$ . By the definition of sparse differential resultant,  $\mathcal{I} \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\} = \text{sat}(\mathbf{R})$ . Differentiating the equality  $\mathbf{R}(\mathbf{u}; \zeta_0, \zeta_1, \dots, \zeta_n) = 0$  w.r.t.  $u_{ik}^{(h_i)}$ , we have

$$\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} + \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} \left( -\frac{M_{ik}(\eta)}{M_{i0}(\eta)} \right) = 0 \quad (20)$$

where  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  are obtained by substituting  $\zeta_i$  to  $u_{i0}$  ( $i = 0, 1, \dots, n$ ) in  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$ . Let  $Q_{ik} = \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} M_{ik} - \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} M_{i0}$ . Clearly,  $Q_{ik} \in \mathcal{I}$ .

Since any  $n$  of the  $\mathbb{P}_i$  form a differentially independent set,  $\text{ord}(\mathbf{R}, \mathbf{u}_i) \geq 0$ . Substituting  $M_{ik}$  by  $(Q_{ik} + M_{i0} \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}) / \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}$  in each  $\mathbb{P}_i$ , we have  $\mathbb{P}_i = \sum_{k=0}^{l_i} u_{ik} M_{ik} = u_{i0} M_{i0} + \sum_{k=1}^{l_i} u_{ik} (Q_{ik} + M_{i0} \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}) / \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}$ . So  $\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} \mathbb{P}_i = \sum_{k=1}^{l_i} u_{ik} Q_{ik} + (\sum_{k=0}^{l_i} u_{ik} \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}) M_{i0}$ . Since  $Q_{ik} \in \mathcal{I}$ ,  $\sum_{k=0}^{l_i} u_{ik} \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} \in \mathcal{I}$ . Thus, there exists some  $a \in \mathbb{Q}$  such that  $\sum_{k=0}^{l_i} u_{ik} \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} = a \mathbf{R}$ . It follows that  $\mathbb{P}_i \in [\mathbf{R}, (Q_{ik})_{0 \leq i \leq n; 1 \leq k \leq l_i}] : S^\infty$ . For any differential polynomial  $f \in \mathcal{I}$ , there exists a differential monomial  $M \in \mathfrak{m}$  such that  $Mf \in [\mathbb{P}_0, \dots, \mathbb{P}_n] \subset ([\mathbf{R}, (Q_{ik})_{0 \leq i \leq n; 1 \leq k \leq l_i}] : S^\infty)$ . Thus,  $f \in [\mathbf{R}, (Q_{ik})_{0 \leq i \leq n; 1 \leq k \leq l_i}] : S^\infty$  and  $\mathcal{I} \subseteq [\mathbf{R}, (Q_{ik})_{0 \leq i \leq n; 1 \leq k \leq l_i}] : S^\infty$  follows. Conversely, for any differential polynomial  $g \in [\mathbf{R}, (Q_{ik})_{0 \leq i \leq n; 1 \leq k \leq l_i}] : S^\infty$ , there exist some differential monomial  $M$  and some  $b \in \mathbb{N}$  such that  $M(\prod_i \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}})^b g \in [\mathbf{R}, Q_{ik}] \subset \mathcal{I}$ . Since  $\mathcal{I}$  is a prime differential ideal,  $g \in \mathcal{I}$ . Hence,  $\mathcal{I} = [\mathbf{R}, (Q_{ik})_{0 \leq i \leq n; 1 \leq k \leq l_i}] : S^\infty$ .  $\square$

Theorem 5.29 shows that under the condition  $\mathbf{R} = 0$ , the non-polynomial solutions of  $\mathbb{P}_i = 0$  ( $i = 0, \dots, n$ ) are the solutions of some differential polynomials of two terms.

**Corollary 5.30** Let  $\mathbb{P}_0, \dots, \mathbb{P}_n$  be a Laurent differentially essential system of the form (2) and  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  its sparse differential resultant. Suppose  $\text{ord}(\mathbf{R}, \mathbf{u}_0) = h_0 \geq 0$  and denote  $S_{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial u_{00}^{(h_0)}}$ . Suppose that when  $\mathbf{u}_i$  ( $i = 0, \dots, n$ ) are specialized to sets  $\mathbf{v}_i$  which are elements in an extension field of  $\mathcal{F}$ ,  $\mathbb{P}_i$  are specialized to  $\overline{\mathbb{P}}_i$  ( $i = 0, \dots, n$ ). Suppose  $S_{\mathbf{R}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$ . If  $\overline{\mathbb{P}}_i = 0$  ( $i = 0, \dots, n$ ) have a common non-polynomial differential solution  $\xi$ , then for each  $k$ , we have

$$\frac{M_{0k}(\xi)}{M_{00}(\xi)} = \frac{\partial \mathbf{R}}{\partial u_{0k}^{(h_0)}}(\mathbf{v}_0, \dots, \mathbf{v}_n) / S_{\mathbf{R}}(\mathbf{v}_0, \dots, \mathbf{v}_n). \quad (21)$$

*Proof:* Denote  $\mathbb{P}_i^N = M_i \mathbb{P}_i$  ( $i = 0, \dots, n$ ) where  $M_i$  are Laurent differential monomials. By the proof of Theorem 5.29, for each  $k = 1, \dots, l_0$ , the polynomial  $S_{\mathbf{R}} M_0 M_{0k} - \frac{\partial \mathbf{R}}{\partial u_{0k}^{(h_0)}} M_0 M_{00} \in [\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$ . Thus, if  $\xi$  is a common non-polynomial differential solution of  $\bar{\mathbb{P}}_i = 0$ , then  $S_{\mathbf{R}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot M_{0k}(\xi) - \frac{\partial \mathbf{R}}{\partial u_{0k}^{(h_0)}}(\mathbf{v}_0, \dots, \mathbf{v}_n) M_{00}(\xi) = 0$ . Since  $S_{\mathbf{R}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$ , (21) follows.  $\square$

We conclude this section by giving a sufficient condition for a differentially essential system to have a unique non-polynomial solution.

Follow the notations in section 3.2, that is,  $\mathcal{A}_i = \{M_{i0}, M_{i1}, \dots, M_{il_i}\}$  are finite sets of Laurent differential monomials where  $M_{ik} = (\mathbb{Y}^{[s_i]})^{\alpha_{ik}}$ , and  $\mathbb{P}_i = \sum_{k=0}^{l_i} u_{ik} M_{ik}$  ( $i = 0, \dots, n$ ).  $\alpha_{ik} \in \mathbb{Z}^{n(s_i+1)}$  is an exponent vector written in terms of the degrees of  $y_1, \dots, y_n, y'_1, \dots, y'_n, \dots, y_1^{(s_i)}, \dots, y_n^{(s_i)}$ . Let  $o = \max_i \{s_i\}$ . Of course, every vector in  $\mathbb{Z}^{n(s_i+1)}$  can be embedded in  $\mathbb{Z}^{n(o+1)}$ . Let  $\mathbf{e}_i$  be the exponent vector for  $y_i$  in  $\mathbb{Z}^{n(o+1)}$  whose  $i$ -th coordinate is 1 and other coordinates are equal to zero.

**Lemma 5.31** *Assume that 1) any  $n$  of the  $\mathcal{A}_i$  ( $i = 0, \dots, n$ ) form a differentially independent set and 2) for each  $j = 1, \dots, n$ ,  $\mathbf{e}_j \in \text{Span}_{\mathbb{Z}}\{\alpha_{ik} - \alpha_{i0} : k = 1, \dots, l_i; i = 0, \dots, n\}$ . Denote  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = h_i$  where  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n) = \text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}$ . Let  $\bar{\mathbb{P}}_i$  be a specialization of  $\mathbb{P}_i$  with coefficient vector  $\mathbf{v}_i$  ( $i = 0, \dots, n$ ). If  $\mathbf{R}(\mathbf{v}_0, \dots, \mathbf{v}_n) = 0$  and  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$  for each  $i$  and  $k$ , then  $\bar{\mathbb{P}}_i = 0$  ( $i = 0, \dots, n$ ) have at most one common non-polynomial solution.*

*Proof:* By hypothesis 1), each  $h_i \geq 0$ . Denote  $\mathbb{P}_i^N = M_i \mathbb{P}_i$  where  $M_i$  are Laurent differential monomials. Similar to procedures to derive equation (20), we have  $\frac{M_{ik}(\eta)}{M_{i0}(\eta)} = \frac{\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}}{\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}}$ , where  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  are obtained by substituting  $\zeta_i$  to  $u_{i0}$  ( $i = 0, 1, \dots, n$ ) in  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$ . By hypothesis 2), there exist  $t_{jik} \in \mathbb{Z}$  such that  $\sum_{i,k} t_{jik}(\alpha_{ik} - \alpha_{i0}) = \mathbf{e}_j$  for  $j = 1, \dots, n$ . So  $\prod_{i,k} \left(\frac{M_{ik}}{M_{i0}}\right)^{t_{jik}} = y_j$ . Thus,  $\prod_{i,k} \left(\frac{M_{ik}(\eta)}{M_{i0}(\eta)}\right)^{t_{jik}} = \eta_j = \prod_{i,k} \left(\frac{\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}}{\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}}\right)^{t_{jik}}$ . It follows that  $S_j y_j - T_j \in [\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$  where  $S_j$  and  $T_j$  are products of nonnegative powers of  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  obtained from the above identity. Since  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$  for each  $i$  and  $k$ ,  $T_j(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot S_j(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$ . Let  $\bar{y}_j = T_j(\mathbf{v}_0, \dots, \mathbf{v}_n)/S_j(\mathbf{v}_0, \dots, \mathbf{v}_n)$ . If  $\xi = (\xi_1, \dots, \xi_n)$  is any common non-polynomial solution of  $\bar{\mathbb{P}}_i = 0$  ( $i = 0, \dots, n$ ), then  $\xi_j = T_j(\mathbf{v}_0, \dots, \mathbf{v}_n)/S_j(\mathbf{v}_0, \dots, \mathbf{v}_n) = \bar{y}_j$ . Thus  $\xi = (\bar{y}_1, \dots, \bar{y}_n)$  and  $\bar{\mathbb{P}}_i = 0$  ( $i = 0, \dots, n$ ) have at most one common non-polynomial solution.  $\square$

**Theorem 5.32** *Assume that 1) any  $n$  of the  $\mathcal{A}_i$  ( $i = 0, \dots, n$ ) form a differentially independent set and 2) for each  $j = 1, \dots, n$ ,  $\mathbf{e}_j \in \text{Span}_{\mathbb{Z}}\{\alpha_{ik} - \alpha_{i0} : k = 1, \dots, l_i; i = 0, \dots, n\}$ . Denote  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = h_i$  where  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n) = \text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}$ . Let  $\bar{\mathbb{P}}_i$  be a specialization of  $\mathbb{P}_i$  with coefficient vector  $\mathbf{v}_i$  ( $i = 0, \dots, n$ ). Then there exists a differential polynomial set  $\mathcal{S} \subset \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  such that  $\mathbb{V}(\mathbf{R}) \setminus \bigcup_{S \in \mathcal{S}} \mathbb{V}(S) \neq \emptyset$  and whenever  $(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathbb{V}(\mathbf{R}) \setminus \bigcup_{S \in \mathcal{S}} \mathbb{V}(S)$ ,  $\bar{\mathbb{P}}_i = 0$  ( $i = 0, \dots, n$ ) have a unique common non-polynomial solution.*



*Proof:* Follow the notations in the proof of Lemma 5.31. Denote  $\mathbb{P}_i^N = M_i \mathbb{P}_i$  ( $i = 0, \dots, n$ ) where  $M_i$  are Laurent differential monomials. Then by the proof of Lemma 5.31, there exist  $S_j$  and  $T_j$  which are products of nonnegative powers of  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  such that  $S_j y_j - T_j \in [\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$ . Thus,  $\mathbf{R}, S_1 y_1 - T_1, \dots, S_n y_n - T_n$  is a characteristic set of  $[\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$  w.r.t. any elimination ranking  $u_{ik} \prec y_1 \prec \dots \prec y_n$ .

Let  $\mathcal{S}$  be the differential polynomial set consisting of  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  and  $(S_j)^{m+1} \left(\frac{T_j}{S_j}\right)^{(m)}$  for all  $i = 0, \dots, n; k = 0, \dots, l_i; j = 1, \dots, n$  and  $m \in \mathbb{N}$ . Firstly, we show that  $\mathbb{V}(\mathbf{R}) / \bigcup_{S \in \mathcal{S}} \mathbb{V}(S) \neq \emptyset$ . Suppose the contrary, viz.  $\mathbb{V}(\mathbf{R}) \subset \bigcup_{S \in \mathcal{S}} \mathbb{V}(S)$ . In particular, there exists one  $S \in \mathcal{S}$  such that  $S$  vanishes at the generic point  $\zeta$  of  $\text{sat}(\mathbf{R})$ . It is obvious that  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  does not vanish at  $\zeta$ . If  $(S_j)^{m+1} \left(\frac{T_j}{S_j}\right)^{(m)}$  vanishes at  $\zeta$  for some  $m$ ,  $(S_j)^{m+1} \left(\frac{T_j}{S_j}\right)^{(m)} \in \text{sat}(\mathbf{R})$ . Since  $S_j^{m+1} y_j^{(m)} - (S_j)^{m+1} \left(\frac{T_j}{S_j}\right)^{(m)} \in [\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$ , it follows that  $S_j^{m+1} \in [\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$ , a contradiction.

Suppose  $(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathbb{V}(\mathbf{R}) / \bigcup_{S \in \mathcal{S}} \mathbb{V}(S)$ . Since  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$  for each  $i$  and  $k$ ,  $T_j(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot S_j(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$ . Let  $\bar{y}_j = \frac{T_j}{S_j}(\mathbf{v}_0, \dots, \mathbf{v}_n)$  and denote  $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n)$ . And for each  $m \in \mathbb{N}$ ,  $\bar{y}_j^{(m)} = \left(\frac{T_j}{S_j}\right)^{(m)}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$ . Thus,  $\bar{y} \in (\mathcal{E}^\wedge)^n$ . Since  $\mathbf{R}, S_1 y_1 - T_1, \dots, S_n y_n - T_n$  is a characteristic set of  $[\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$ ,  $H \cdot M_i \mathbb{P}_i \equiv 0, \text{ mod } [\mathbf{R}, S_1 y_1 - T_1, \dots, S_n y_n - T_n]$  where  $H$  is a product of powers of  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$ . Hence,  $M_i(\bar{y}) \cdot \bar{\mathbb{P}}_i(\bar{y}) = 0$ , which follows that  $\bar{\mathbb{P}}_i(\bar{y}) = 0$ . Thus,  $\bar{y}$  is a non-polynomial common solution of  $\bar{\mathbb{P}}_i$ . On the other hand, for every non-polynomial common solution of  $\bar{\mathbb{P}}_i$ ,  $S_j(\mathbf{v}_0, \dots, \mathbf{v}_n) y_j - T_j(\mathbf{v}_0, \dots, \mathbf{v}_n)$  vanishes at it. Thus, it must be equal to the point  $\bar{y}$ . As a consequence, we have proved that  $\bar{\mathbb{P}}_i = 0$  have a unique common non-polynomial solution.  $\square$

Theorem 5.32 can be rephrased as the following geometric form.

**Corollary 5.33** *Let  $\mathcal{Z}_1(\mathcal{A}_0, \dots, \mathcal{A}_n)$  be a subset of  $\mathcal{E}^{l_0+1} \times \dots \times \mathcal{E}^{l_n+1}$  consisting of points  $(\mathbf{v}_0, \dots, \mathbf{v}_n)$  for which the corresponding Laurent differential polynomials  $F_i = 0$  ( $i = 0, \dots, n$ ) have a unique non-polynomial common solution and  $\overline{\mathcal{Z}_1(\mathcal{A}_0, \dots, \mathcal{A}_n)}$  the Kolchin closure of  $\mathcal{Z}_1(\mathcal{A}_0, \dots, \mathcal{A}_n)$ . Then under the condition of Theorem 5.32, we have  $\overline{\mathcal{Z}_1(\mathcal{A}_0, \dots, \mathcal{A}_n)} = \mathbb{V}(\text{sat}(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}))$ .*

**Example 5.34** *Continue from Example 3.14. In this example, the sparse differential resultant  $\mathbf{R}$  of  $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$  is free from the coefficients of  $\mathbb{P}_2$ . The system can be solved as follows:  $y_1$  can be solved from  $\mathbb{P}_0 = \mathbb{P}_1 = 0$  and  $\mathbb{P}_2 = u_{10} + u_{11} y_2'$  is of order one in  $y_2$  which will lead to an infinite number of solutions. Thus the system can not have a unique solution This shows the importance of the first condition in Theorem 5.32.*

**Example 5.35** *Continue from Example 3.15. In this example, the characteristic set of  $[\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2]$  w.r.t. the elimination ranking  $u_{ik} \prec y_2 \prec y_1$  is  $\mathbf{R}, u_{11} u_{00} y_2' - u_{01} u_{10} y_2, u_{01} y_2 y_1 + u_{00}$ . Here  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$  do not satisfy condition 2) and the system  $\{\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2\}$  does not have a unique solution under the condition  $\mathbf{R} = 0$ .*

## 5.6 Sparse differential resultant for differential polynomials with non-vanishing degree zero terms

As pointed out in the preceding sections, for Laurent differential polynomials, non-polynomial zeros are considered. But, for certain differential polynomials, this condition seems to be too demanding. In this section, we restrict to consider the sparse differential resultant for differential polynomials with non-vanishing degree zero terms. To be more precise, consider  $n + 1$  differential polynomials of the form

$$\mathbb{P}_i = u_{i0} + \sum_{k=1}^{l_i} u_{ik} M_{ik} \quad (i = 0, \dots, n) \quad (22)$$

where  $M_{ik} = (\mathbb{Y}^{[s_i]})^{\alpha_{ik}}$  is a monomial in  $\{y_1, \dots, y_n, \dots, y_1^{(s_i)}, \dots, y_n^{(s_i)}\}$  whose exponent vector  $\alpha_{ik} \in \mathbb{Z}_{\geq 0}^{n(s_i+1)}$  with  $|\alpha_{ik}| \geq 1$ , and all the  $u_{ik}$  are differentially independent over  $\mathbb{Q}$ . Denote  $\mathcal{B}_i = \{1, M_{i1}, \dots, M_{il_i}\}$  ( $i = 0, \dots, n$ ). The set of exponent vectors  $\mathbb{S}_i = \{\mathbf{0}, \alpha_{ik} : k = 1, \dots, l_i\}$  is called the *support* of  $\mathbb{P}_i$ , where  $\mathbf{0}$  is the exponent vector for the constant term. Denote  $\mathbf{u}_i = (u_{i0}, \dots, u_{il_i})$  ( $i = 0, \dots, n$ ) and  $\mathbf{u} = \cup_i \mathbf{u}_i \setminus \{u_{i0}\}$ .

**Definition 5.36** Let  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ) be a differential polynomial system of the form (22).  $\{\mathbb{P}_0, \dots, \mathbb{P}_n\}$  is called a differentially essential system if they form a Laurent differentially essential system when considered as Laurent differential polynomials. In this case, we also call  $\mathcal{B}_0, \dots, \mathcal{B}_n$  a differentially essential system.

All results for sparse differential resultants proved in the previous sections can be naturally rephrased in this case by just setting  $M_{i0}$  in (2) to 1. The main difference is that we do not need to consider non-polynomial solutions and hence results in section 5.5 could be modified.

First, we show that Theorem 3.9 can be modified as follows.

**Theorem 5.37** Let  $\mathbb{P}_0, \dots, \mathbb{P}_n$  be differential polynomials as defined in (22). Then  $[\mathbb{P}_0, \dots, \mathbb{P}_n]$  is a prime differential ideal in  $\mathbb{Q}\{\mathbb{Y}, \mathbf{u}_0, \dots, \mathbf{u}_n\}$ . And  $([\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n]) \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  is of codimension 1 if and only if  $\{\mathbb{P}_i, i = 0, \dots, n\}$  is a differentially essential system.

*Proof:* Let  $\eta = (\eta_1, \dots, \eta_n)$  be a generic point of  $[0]$  over  $\mathbb{Q}\langle \mathbf{u} \rangle$ . Denote  $\zeta_i = -\sum_{k=1}^{l_i} u_{ik} M_{ik}(\eta)$ . It is easy to show that  $(\eta; \zeta)$  is a generic point of  $[\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n] \subset \mathbb{Q}\{\mathbb{Y}, \mathbf{u}_0, \dots, \mathbf{u}_n\}$  where  $\zeta = (\zeta_0, u_{01}, \dots, u_{0l_0}; \dots; \zeta_n, u_{n1}, \dots, u_{nl_n})$ , and it follows that  $[\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n]$  is a prime differential ideal. So  $[\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n] : \mathfrak{m} = [\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n]$ . By Theorem 3.9, the second part follows.  $\square$

Then, for a differentially essential system  $\{\mathbb{P}_0, \dots, \mathbb{P}_n\}$  of form (22), its sparse differential resultant  $\mathbf{R}$  can be defined as

$$[\mathbb{P}_0, \dots, \mathbb{P}_n] \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\} = \text{sat}(\mathbf{R}). \quad (23)$$

This equation is different from equation (5) in that the differential ideal  $[\mathbb{P}_0, \dots, \mathbb{P}_n]$  here is a differential ideal in  $\mathbb{Q}\{\mathbb{Y}, \mathbf{u}_0, \dots, \mathbf{u}_n\}$  while the other one is generated in the Laurent differential polynomial ring.

Now we first introduce some notations similar to Section 5.5. Let  $\mathcal{B}_0, \dots, \mathcal{B}_n$  be a differentially essential system of monomial sets. For every specific differential polynomial set  $(F_0, \dots, F_n)$  with  $F_i = v_{i0} + \sum_{k=1}^{l_i} v_{ik} M_{ik} \in \mathcal{E}\{\mathbb{Y}\}$ , we also represent it by  $(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathcal{E}^{l_0+1} \times \dots \times \mathcal{E}^{l_n+1}$  where  $\mathbf{v}_i = (v_{i0}, v_{i1}, \dots, v_{il_i})$ . Let

$$\mathcal{Z}_0(\mathcal{B}_0, \dots, \mathcal{B}_n) = \{(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathcal{E}^{l_0+1} \times \dots \times \mathcal{E}^{l_n+1} : F_0 = \dots = F_n = 0 \text{ have a common solution in } \mathcal{E}^n\} \quad (24)$$

Let  $\overline{\mathcal{Z}(\mathcal{B}_0, \dots, \mathcal{B}_n)}$  be the Kolchin differential closure of  $\mathcal{Z}_0(\mathcal{B}_0, \dots, \mathcal{B}_n)$  in  $\mathcal{E}^{l_0+1} \times \dots \times \mathcal{E}^{l_n+1}$ . Note that zeros from  $\mathcal{E}$  are considered, instead of  $\mathcal{E}^\wedge$  as in (6).

The following result shows that the vanishing of sparse differential resultant gives a necessary condition for the existence of solutions, and as well as gives a sufficient condition in some sense.

**Theorem 5.38** *Suppose  $\mathcal{B}_i$  ( $i = 0, \dots, n$ ) form a differentially essential system. Then we have  $\mathcal{Z}(\mathcal{B}_0, \dots, \mathcal{B}_n) = \mathbb{V}(\text{sat}(\text{Res}_{\mathcal{B}_0, \dots, \mathcal{B}_n}))$ .*

*Proof:* Since  $\text{sat}(\text{Res}_{\mathcal{B}_0, \dots, \mathcal{B}_n}) \subset [\mathbb{P}_0, \dots, \mathbb{P}_n] \subset \mathbb{Q}\{\mathbb{Y}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$ , it follows directly that  $\mathcal{Z}_0(\mathcal{B}_0, \dots, \mathcal{B}_n) \subseteq \mathbb{V}(\text{sat}(\text{Res}_{\mathcal{B}_0, \dots, \mathcal{B}_n}))$ . Consequently,  $\mathcal{Z}(\mathcal{B}_0, \dots, \mathcal{B}_n) \subseteq \mathbb{V}(\text{sat}(\text{Res}_{\mathcal{B}_0, \dots, \mathcal{B}_n}))$ .

For the other direction, follow the notations in the proof of Theorem 5.37. By Theorem 5.37,  $[\mathbb{P}_0, \dots, \mathbb{P}_n]$  is a prime differential ideal with a generic point  $(\eta, \zeta)$  where  $\eta = (\eta_1, \dots, \eta_n)$  is a generic point of  $[0]$  over  $\mathbb{Q}\langle \mathbf{u} \rangle$  and  $\zeta = (\zeta_0, u_{01}, \dots, u_{0l_0}; \dots; \zeta_n, u_{n1}, \dots, u_{nl_n})$ . Let  $(F_0, \dots, F_n)$  be a set of differential polynomials represented by  $\zeta$ . Clearly,  $\eta$  is a solution of  $F_i = 0$ . Thus,  $\zeta \in \mathcal{Z}_0(\mathcal{B}_0, \dots, \mathcal{B}_n) \subset \mathcal{Z}(\mathcal{B}_0, \dots, \mathcal{B}_n)$ . Since  $\zeta$  is a generic point of  $\text{sat}(\text{Res}_{\mathcal{B}_0, \dots, \mathcal{B}_n})$ , it follows that  $\mathbb{V}(\text{sat}(\text{Res}_{\mathcal{B}_0, \dots, \mathcal{B}_n})) \subseteq \mathcal{Z}(\mathcal{B}_0, \dots, \mathcal{B}_n)$ . As a consequence,  $\mathbb{V}(\text{sat}(\text{Res}_{\mathcal{B}_0, \dots, \mathcal{B}_n})) = \mathcal{Z}(\mathcal{B}_0, \dots, \mathcal{B}_n)$ .  $\square$

In the following, we will analyze the properties of the solutions as we did in section 5.5. The following lemma shows the relation between the original differential system and their sparse differential resultant, which is a direct consequence of Theorem 5.29.

**Lemma 5.39** *Let  $\mathbb{P}_i = u_{i0} + \sum_{k=1}^{l_i} u_{ik} M_{ik}$  ( $i = 0, \dots, n$ ) be a system of differential polynomials satisfying that any  $n$  of the  $\mathbb{P}_i$  form a differentially independent set. Let  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  be the sparse differential resultant of  $\mathbb{P}_i$  with  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = h_i$ . Denote  $Q_{ik} = \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} M_{ik} - \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  and  $S$  to be the set consisting of  $\frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}$  ( $i = 0, \dots, n$ ). Then*

$$[\mathbb{P}_0, \dots, \mathbb{P}_n] = [\mathbf{R}, (Q_{ik})_{0 \leq i \leq n; 1 \leq k \leq l_i}] : S^\infty$$

in  $\mathbb{Q}\{\mathbb{Y}, \mathbf{u}_0, \dots, \mathbf{u}_n\}$ .

*Proof:* It is a direct consequence of Theorem 5.29 by setting  $M_{i0} = 1$  and from the fact that  $[\mathbb{P}_0, \dots, \mathbb{P}_n] : \mathbf{m} = [\mathbb{P}_0, \dots, \mathbb{P}_n]$  as differential ideals in  $\mathbb{Q}\{\mathbb{Y}, \mathbf{u}_0, \dots, \mathbf{u}_n\}$ .  $\square$

Lemma 5.39 shows that under the condition  $\mathbf{R} = 0$ , the solutions of  $\mathbb{P}_i = 0$  ( $i = 0, \dots, n$ ) generally are the solutions of some differential polynomials of two terms.

**Corollary 5.40** *Let  $\mathbb{P}_0, \dots, \mathbb{P}_n$  be a differentially essential system of the form (22) and  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  their sparse differential resultant. Suppose  $\text{ord}(\mathbf{R}, \mathbf{u}_0) = h_0 \geq 0$  and denote  $S_{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial u_{00}^{(h_0)}}$ . Suppose that when  $\mathbf{u}_i$  ( $i = 0, \dots, n$ ) are specialized to sets  $\mathbf{v}_i$  over  $\mathbb{Q}$  which are elements in an extension field of  $\mathcal{F}$ ,  $\mathbb{P}_i$  are specialized to  $\overline{\mathbb{P}}_i$  ( $i = 0, \dots, n$ ). If  $S_{\mathbf{R}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$ , in the case that  $\overline{\mathbb{P}}_i = 0$  ( $i = 0, \dots, n$ ) have a common differential solution  $\xi$ , then for each  $k$ , we have*

$$M_{0k}(\xi) = \frac{\partial \mathbf{R}}{\partial u_{0k}^{(h_0)}}(\mathbf{v}_0, \dots, \mathbf{v}_n) / S_{\mathbf{R}}(\mathbf{v}_0, \dots, \mathbf{v}_n). \quad (25)$$

*Proof:* By Lemma 5.39, for each  $k = 1, \dots, l_0$ , the polynomial  $S_{\mathbf{R}} M_{0k} - \frac{\partial \mathbf{R}}{\partial u_{0k}^{(h_0)}} \in [\mathbb{P}_0, \dots, \mathbb{P}_n]$ . Thus, if  $\xi$  is a common differential solution of  $\overline{\mathbb{P}}_i = 0$ , then  $S_{\mathbf{R}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot M_{0k}(\xi) - \frac{\partial \mathbf{R}}{\partial u_{0k}^{(h_0)}}(\mathbf{v}_0, \dots, \mathbf{v}_n) = 0$ . Since  $S_{\mathbf{R}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$ , (25) follows.  $\square$

In the rest of this section, we will give a sufficient condition for a differentially essential system to have a unique solution.

**Theorem 5.41** *Assume that 1) any  $n$  of the  $\mathcal{B}_i$  ( $i = 0, \dots, n$ ) form a differentially independent set and 2) for each  $j = 1, \dots, n$ ,  $\mathbf{e}_j \in \text{Span}_{\mathbb{Z}}\{\alpha_{ik} : k = 1, \dots, l_i; i = 0, \dots, n\}$ . Denote  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = h_i$  where  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n) = \text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}$ . Let  $\overline{\mathbb{P}}_i$  be a specialization of  $\mathbb{P}_i$  over  $\mathbb{Q}$  with coefficient vector  $\mathbf{v}_i$  ( $i = 0, \dots, n$ ). If  $\mathbf{R}(\mathbf{v}_0, \dots, \mathbf{v}_n) = 0$  and  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$  for each  $i$  and  $k$ , then  $\overline{\mathbb{P}}_i = 0$  ( $i = 0, \dots, n$ ) have a unique common solution.*

*Proof:* By hypothesis 1), each  $h_i \geq 0$ . By Lemma 5.39,  $Q_{ik} = \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} M_{ik} - \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} \in [\mathbb{P}_0, \dots, \mathbb{P}_n] \subset \mathbb{Q}\{\mathbf{Y}; \mathbf{u}_0, \dots, \mathbf{u}_n\}$ . Since  $(\eta; \zeta)$  is a generic point of  $[\mathbb{P}_0, \dots, \mathbb{P}_n]$ ,  $M_{ik}(\eta) = \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} / \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}}$ , where  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  are obtained by substituting  $\zeta_i$  to  $u_{i0}$  ( $i = 0, 1, \dots, n$ ) in  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$ . By hypothesis 2), there exist  $t_{jik} \in \mathbb{Z}$  such that  $\sum_{i,k} t_{jik} \alpha_{ik} = \mathbf{e}_j$  for  $j = 1, \dots, n$ . So  $\prod_{i,k} (M_{ik})^{t_{jik}} = y_j$ . Thus,  $\prod_{i,k} (M_{ik}(\eta))^{t_{jik}} = \eta_j = \prod_{i,k} \left( \frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}} / \frac{\partial \mathbf{R}}{\partial u_{i0}^{(h_i)}} \right)^{t_{jik}}$ . It follows that  $S_j y_j - T_j \in [\mathbb{P}_0, \dots, \mathbb{P}_n]$  where  $S_j$  and  $T_j$  are products of nonnegative powers of  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  obtained from the above identity. Thus,  $R, S_1 y_1 - T_1, \dots, S_n y_n - T_n$  is a characteristic set of  $[\mathbb{P}_0, \dots, \mathbb{P}_n]$  w.r.t. any elimination ranking  $u_{ik} \prec y_1 \prec \dots \prec y_n$ . For each  $\mathbb{P}_i$ , there exists a product  $A_i$  of nonnegative powers of  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}$  such that  $A_i \mathbb{P}_i \in [R, S_1 y_1 - T_1, \dots, S_n y_n - T_n]$ . Now specialize  $\mathbf{u}_i$  to  $\mathbf{v}_i$  over  $\mathbb{Q}$  ( $i = 0, \dots, n$ ). Since  $\frac{\partial \mathbf{R}}{\partial u_{ik}^{(h_i)}}(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$  for each  $i$  and  $k$ ,  $T_j(\mathbf{v}_0, \dots, \mathbf{v}_n) \cdot S_j(\mathbf{v}_0, \dots, \mathbf{v}_n) \neq 0$ . Let  $\bar{y}_j = T_j(\mathbf{v}_0, \dots, \mathbf{v}_n) / S_j(\mathbf{v}_0, \dots, \mathbf{v}_n)$  and  $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n)$ . Clearly,  $\overline{\mathbb{P}}_i(\bar{y}) = 0$ . Thus,  $\bar{y}$  is a common solution of  $\overline{\mathbb{P}}_i$  ( $i = 0, \dots, n$ ).

On the other hand, for any solution  $\xi$  of  $\overline{\mathbb{P}}_i$ ,  $S_j(\mathbf{v}_0, \dots, \mathbf{v}_n) y_j - T_j(\mathbf{v}_0, \dots, \mathbf{v}_n)$  ( $j = 1, \dots, n$ ) vanishes at it. So  $\xi = \bar{y}$ . As a consequence, we have proved that in this case,  $\overline{\mathbb{P}}_i = 0$  ( $i = 0, \dots, n$ ) have a unique solution.  $\square$

## 6 A single exponential algorithm to compute the sparse differential resultant

In this section, we give an algorithm to compute the sparse differential resultant for a Laurent differentially essential system with single exponential complexity. The idea is to estimate the degree bounds for the resultant and then to use linear algebra to find the coefficients of the resultant.

### 6.1 Degree of algebraic elimination ideal

In this section, we will prove several properties about the degrees of elimination ideals in the algebraic case, which will be used to estimate the degree bound for sparse differential resultants.

Let  $P$  be a polynomial in  $K[\mathbb{X}]$  where  $K$  is an algebraic field and  $\mathbb{X} = \{x_1, \dots, x_n\}$  a set of algebraic indeterminates. We use  $\deg(P)$  to denote the total degree of  $P$ . Let  $\mathcal{I}$  be a prime algebraic ideal in  $K[\mathbb{X}]$  with dimension  $d$ . We use  $\deg(\mathcal{I})$  to denote the *degree* of  $\mathcal{I}$ , which is defined to be the number of solutions of the zero dimensional prime ideal  $(\mathcal{I}, \mathbb{L}_1, \dots, \mathbb{L}_d)$  in  $K(U)[\mathbb{X}]$ , where  $\mathbb{L}_i = u_{i0} + \sum_{j=1}^n u_{ij}x_j$  ( $i = 1, \dots, d$ ) are  $d$  generic hyperplanes [23] and  $U = \{u_{ij} \mid (i = 1, \dots, d, j = 0, \dots, n)\}$ . That is,

$$\deg(\mathcal{I}) = |\mathbb{V}(\mathcal{I}, \mathbb{L}_1, \dots, \mathbb{L}_d)|. \quad (26)$$

Clearly,  $\deg(\mathcal{I}) = \deg(\mathcal{I}, \mathbb{L}_1, \dots, \mathbb{L}_i)$  for  $i = 1, \dots, d$ .  $\deg(\mathcal{I})$  is also equal to the maximal number of intersection points of  $\mathbb{V}(\mathcal{I})$  with  $d$  hyperplanes under the condition that the number of these points is finite [25]. That is,

$$\deg(\mathcal{I}) = \max\{|\mathbb{V}(\mathcal{I}) \cap H_1 \cap \dots \cap H_d| : H_i \text{ are affine hyperplanes} \\ \text{such that } |\mathbb{V}(\mathcal{I}) \cap H_1 \cap \dots \cap H_d| < \infty\} \quad (27)$$

We investigate the relation between the degree of an ideal and that of its elimination ideal by proving Theorem 6.2.

**Lemma 6.1** *Let  $\mathcal{I}$  be a prime ideal of dimension zero in  $K[\mathbb{X}]$  and  $\mathcal{I}_k = \mathcal{I} \cap K[x_1, \dots, x_k]$  the elimination ideal of  $\mathcal{I}$  with respect to  $x_1, \dots, x_k$ . Then  $\deg(\mathcal{I}_k) \leq \deg(\mathcal{I})$ .*

*Proof:* Since both  $\mathcal{I}$  and  $\mathcal{I}_k$  are prime ideals of dimension zero,  $\deg(\mathcal{I}_k) = |\mathbb{V}(\mathcal{I}_k)|$  and  $\deg(\mathcal{I}) = |\mathbb{V}(\mathcal{I})|$ . To show  $\deg(\mathcal{I}_k) \leq \deg(\mathcal{I})$ , it suffices to prove that every point of  $\mathbb{V}(\mathcal{I}_k)$  can be extended to a point of  $\mathbb{V}(\mathcal{I})$ . Let  $(\xi_1, \dots, \xi_k) \in \mathbb{V}(\mathcal{I}_k)$ . For any point  $(\eta_1, \dots, \eta_n) \in \mathbb{V}(\mathcal{I})$ ,  $(\eta_1, \dots, \eta_k)$  is a zero point of  $\mathcal{I}_k$ . So we have  $K(\xi_1, \dots, \xi_k) \cong K(\eta_1, \dots, \eta_k)$ . By [52, Proposition 9, Chapter 1, §3], there exist  $\xi_{k+1}, \dots, \xi_n$  such that  $K(\xi_1, \dots, \xi_n) \cong K(\eta_1, \dots, \eta_n)$ . Thus,  $(\xi_1, \dots, \xi_n)$  is a zero of  $\mathcal{I}$ , which completes the proof.  $\square$

**Theorem 6.2** *Let  $\mathcal{I}$  be a prime ideal in  $K[\mathbb{X}]$  and  $\mathcal{I}_k = \mathcal{I} \cap K[x_1, \dots, x_k]$  for any  $1 \leq k \leq n$ . Then  $\deg(\mathcal{I}_k) \leq \deg(\mathcal{I})$ .*

*Proof:* Suppose  $\dim(\mathcal{I}) = d$  and  $\dim(\mathcal{I}_k) = d_1$ . Two cases are considered:

Case (a):  $d_1 = d$ . Let  $\mathbb{P}_i = u_{i0} + u_{i1}x_1 + \cdots + u_{ik}x_k$  ( $i = 1, \dots, d$ ). Denote  $\mathbf{u} = \{u_{ij} : i = 1, \dots, d; j = 0, \dots, k\}$ . Then by [23, Theorem 1, p. 54],  $\mathcal{J} = (\mathcal{I}_k, \mathbb{P}_1, \dots, \mathbb{P}_d)$  is a prime ideal of dimension zero in  $K(\mathbf{u})[x_1, \dots, x_k]$  and has the same degree as  $\mathcal{I}_k$ . We claim that

i)  $(\mathcal{I}, \mathbb{P}_1, \dots, \mathbb{P}_d) \cap K(\mathbf{u})[x_1, \dots, x_k] = \mathcal{J}$ .

ii)  $(\mathcal{I}, \mathbb{P}_1, \dots, \mathbb{P}_d)$  is a 0-dimensional prime ideal over  $K(\mathbf{u})$ .

To prove i), it suffices to show that whenever  $f$  is in the left ideal,  $f$  belongs to  $\mathcal{J}$ . Without loss of generality, suppose  $f \in K[\mathbf{u}][x_1, \dots, x_k]$ . Then there exist  $h_l, q_i \in K[\mathbf{u}][\mathbb{X}]$  and  $g_l \in \mathcal{I}$  such that  $f = \sum_l h_l g_l + \sum_{i=1}^d q_i \mathbb{P}_i$ . Substituting  $u_{i0} = -u_{i1}x_1 - \cdots - u_{ik}x_k$  into the above equality, we get  $\bar{f} = \sum_l \bar{h}_l g_l \in \mathcal{I}$ . Thus,  $\bar{f} \in \mathcal{I}_k$ . But  $f \equiv \bar{f} \pmod{(\mathbb{P}_1, \dots, \mathbb{P}_d)}$ , so  $f \in \mathcal{J}$ , and i) follows.

To prove ii), let  $(\xi_1, \dots, \xi_n)$  be a generic point of  $\mathcal{I}$ . Denote  $U_0 = \{u_{10}, \dots, u_{d0}\}$ . Then  $\mathcal{J}_0 = (\mathcal{I}, \mathbb{P}_1, \dots, \mathbb{P}_d) \subseteq K(\mathbf{u} \setminus U_0)[\mathbb{X}, U_0]$  is a prime ideal of dimension  $d$  with a generic point  $(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_d)$ , where  $\zeta_i = -\sum_{j=1}^k u_{ij}\xi_j$  ( $i = 1, \dots, d$ ). Since  $d_1 = d$ , there exist  $d$  elements in  $\{\xi_1, \dots, \xi_k\}$  algebraically independent over  $K$ . So by [22, p.168-169],  $\zeta_1, \dots, \zeta_d$  are algebraically independent over  $K(\mathbf{u} \setminus U_0)$ . Thus,  $\mathcal{J}_0 \cap K(\mathbf{u} \setminus U_0)[U_0] = (0)$  and ii) follows.

By Lemma 6.1,  $\deg(\mathcal{J}) \leq \deg(\mathcal{I}, \mathbb{P}_1, \dots, \mathbb{P}_d)$ . So by (27),  $\deg(\mathcal{I}) \geq |\mathbb{V}(\mathcal{I}, \mathbb{P}_1, \dots, \mathbb{P}_d)| \geq \deg(\mathcal{J}) = \deg(\mathcal{I}_k)$ .

Case (b):  $d_1 < d$ . Let  $\mathbb{L}_i = u_{i0} + u_{i1}x_1 + \cdots + u_{in}x_n$  ( $i = 1, \dots, d - d_1$ ). By [23, Theorem 1, p. 54],  $\mathcal{J} = (\mathcal{I}, \mathbb{L}_1, \dots, \mathbb{L}_{d-d_1}) \subseteq K(\mathbf{u})[\mathbb{X}]$  is a prime ideal of dimension  $d_1$  and  $\deg(\mathcal{J}) = \deg(\mathcal{I})$ , where  $\mathbf{u} = \{u_{ij} : i = 1, \dots, d - d_1; j = 0, \dots, n\}$ . Let  $\mathcal{J}_k = \mathcal{J} \cap K(\mathbf{u})[x_1, \dots, x_k]$ . We claim that  $\mathcal{J}_k = (\mathcal{I}_k)$  in  $K(\mathbf{u})[x_1, \dots, x_k]$ . Of course,  $\mathcal{J}_k \supseteq (\mathcal{I}_k)$ . Since both  $\mathcal{J}_k$  and  $(\mathcal{I}_k)$  are prime ideals and  $\dim((\mathcal{I}_k)) = d_1$ , it suffices to prove that  $\dim(\mathcal{J}_k) = d_1$ .

Suppose  $(\xi_1, \dots, \xi_n)$  is a generic point of  $\mathcal{I}$ , then  $(\xi_1, \dots, \xi_k)$  is that of  $\mathcal{I}_k$ . Let  $\mathcal{J}_0 = (\mathcal{I}, \mathbb{L}_1, \dots, \mathbb{L}_{d-d_1}) \subseteq K(\mathbf{u} \setminus U_0)[\mathbb{X}, U_0]$ , then  $(\xi_1, \dots, \xi_n, -\sum_{j=1}^n u_{1j}\xi_j, \dots, -\sum_{j=1}^n u_{d-d_1,j}\xi_j)$  is a generic point of it, where  $U_0 = \{u_{10}, \dots, u_{d-d_1,0}\}$ . Since  $\dim(\mathcal{I}_k) = d_1$ , without loss of generality, suppose  $\xi_1, \dots, \xi_{d_1}$  is a transcendence basis of  $K(\xi_1, \dots, \xi_k)/K$  and  $\xi_1, \dots, \xi_{d_1}, \xi_{k+1}, \dots, \xi_{k+(d-d_1)}$  is that of  $K(\xi_1, \dots, \xi_n)/K$ . Then by [22, p.168-169], it is easy to show that  $\mathcal{J}_0 \cap K(\mathbf{u} \setminus U_0)[x_1, \dots, x_{d_1}, U_0] = (0)$ , and  $\mathcal{J}_k \cap K(\mathbf{u})[x_1, \dots, x_{d_1}] = 0$  follows. So  $\dim(\mathcal{J}_k) = d_1$  and  $\mathcal{J}_k = (\mathcal{I}_k)$ .

Since  $\dim(\mathcal{J}_k) = \dim(\mathcal{J})$ , by case (a), we have  $\deg(\mathcal{J}_k) \leq \deg(\mathcal{J}) = \deg(\mathcal{I})$ . Due to the fact that  $\deg(\mathcal{J}_k) = \deg((\mathcal{I}_k)) = \deg(\mathcal{I}_k)$ ,  $\deg(\mathcal{I}_k) \leq \deg(\mathcal{I})$  follows.  $\square$

In this article, we will use the following two results.

**Lemma 6.3** [51, Corollary 2.28] *Let  $V_1, \dots, V_r \subset \mathbf{P}^n$  ( $r \geq 2$ ) be pure dimensional projective varieties in  $\mathbf{P}^n$ . Then*

$$\prod_{i=1}^r \deg(V_i) \geq \sum_C \deg(C)$$

where  $C$  runs through all irreducible components of  $V_1 \cap \cdots \cap V_r$ .

**Lemma 6.4** [35, Proposition 1, p.151] *Let  $F_1, \dots, F_m \in K[\mathbb{X}]$  be polynomials generating an ideal  $\mathcal{I}$  of dimension  $r$ . Suppose  $\deg(F_1) \geq \cdots \geq \deg(F_m)$  and let  $D := \prod_{i=1}^{n-r} \deg(F_i)$ . Then  $\deg(\mathcal{I}) \leq D$ .*

## 6.2 Degree bound for sparse differential resultant

In this section, we give an upper bound for the degree of the sparse differential resultant, which will be crucial to our algorithm to compute the sparse resultant.

**Theorem 6.5** *Let  $\mathbb{P}_0, \dots, \mathbb{P}_n$  be a Laurent differentially essential system of form (2) with  $\text{Eord}(\mathbb{P}_i) = e_i$  and  $\deg(\mathbb{P}_i^N, \mathbb{Y}) = m_i$ . Suppose  $\mathbb{P}_i^N = \sum_{k=0}^{l_i} u_{ik} N_{ik}$  and  $J_i$  is the modified Jacobi number of  $\{\mathbb{P}_0^N, \dots, \mathbb{P}_n^N\} \setminus \{\mathbb{P}_i^N\}$ . Denote  $m = \max_i \{m_i\}$ . Let  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  be the sparse differential resultant of  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ). Suppose  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = h_i$  for each  $i$ . Then the following assertions hold:*

1)  $\deg(\mathbf{R}) \leq \prod_{i=0}^n (m_i + 1)^{h_i+1} \leq (m + 1)^{\sum_{i=0}^n (J_i+1)}$ , where  $m = \max_i \{m_i\}$ .

2)  $\mathbf{R}$  has a representation

$$\prod_{i=0}^n N_{i0}^{(h_i+1)\deg(\mathbf{R})} \cdot \mathbf{R} = \sum_{i=0}^n \sum_{j=0}^{h_i} G_{ij}(\mathbb{P}_i^N)^{(j)} \quad (28)$$

where  $G_{ij} \in \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \dots, \mathbf{u}_n^{[h_n]}, \mathbb{Y}^{[h]}]$  and  $h = \max\{h_i + e_i\}$  such that  $\deg(G_{ij}(\mathbb{P}_i^N)^{(j)}) \leq [m + 1 + \sum_{i=0}^n (h_i + 1)\deg(N_{i0})]\deg(\mathbf{R})$ .

*Proof:* 1) By the definition of sparse differential resultant,  $[\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m} \cap \mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\} = \text{sat}(\mathbf{R})$ . Let  $\eta = (\eta_1, \dots, \eta_n)$  be a generic point of  $[0]$ . Denote  $\zeta_i = -\sum_{k=1}^{l_i} u_{ik} \frac{M_{ik}(\eta)}{M_{i0}(\eta)}$  ( $i = 0, \dots, n$ ). Then  $(\eta; \zeta_0, u_{01}, \dots, u_{0l_0}; \dots; \zeta_n, u_{n1}, \dots, u_{nl_n})$  is a generic point of  $[\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$ . Clearly,  $\mathbb{P}_0^N, \dots, \mathbb{P}_n^N$  is a characteristic set of  $[\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$  w.r.t. the elimination ranking  $u_{n0} \succ \dots \succ u_{10} \succ u_{00} \succ \mathbf{u} \succ \mathbb{Y}$ . Taking the differential remainder of  $\mathbf{R}$  w.r.t. this characteristic set, we have

$$\prod_{i=0}^n N_{i0}^{a_i} \mathbf{R} = \sum_{i=0}^n \sum_{k=0}^{h_i} G_{ik}(\mathbb{P}_i^N)^{(k)}$$

for  $a_i \in \mathbb{N}$ . Denote  $h = \max_i \{h_i + e_i\}$  and by  $\mathfrak{m}^{[h]}$  we mean the set of all monomials in  $\mathbb{Y}^{[h]}$ . Let  $\mathcal{J} = ((\mathbb{P}_0^N)^{[h_0]}, \dots, (\mathbb{P}_n^N)^{[h_n]}) : \mathfrak{m}^{[h]}$  be an ideal in  $\mathcal{R} = \mathbb{Q}[\mathbb{Y}^{[h]}, \mathbf{u}_0^{[h_0]}, \dots, \mathbf{u}_n^{[h_n]}]$ . Then  $\mathbf{R} \in \mathcal{J}$ . Furthermore, it is easy to show that  $\mathcal{J}$  is a prime ideal in  $\mathcal{R}$  with a generic point  $(\eta^{[h]}; \tilde{\mathbf{u}}, \zeta_0^{[h_0]}, \dots, \zeta_n^{[h_n]})$  and  $\mathcal{J} \cap \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \dots, \mathbf{u}_n^{[h_n]}] = (\mathbf{R})$ , where  $\tilde{\mathbf{u}} = \cup_i \mathbf{u}_i^{[h_i]} \setminus \{u_{i0}^{[h_i]}\}$ . Let  $H_{ik}$  be the homogenous polynomial corresponding to  $(\mathbb{P}_i^N)^{(k)}$  with  $x_0$  the variable of homogeneity. Then  $\mathcal{J}^0 = ((H_{ik})_{1 \leq i \leq n; 0 \leq k \leq h_i}) : \tilde{\mathfrak{m}}$  is a prime ideal in  $\mathbb{Q}[x_0, \mathbb{Y}^{[h]}, \mathbf{u}_0^{[h_0]}, \dots, \mathbf{u}_n^{[h_n]}]$  with a generic point  $(v, v\eta^{[h]}; v\tilde{\mathbf{u}}, v\zeta_0^{[h_0]}, \dots, v\zeta_n^{[h_n]})$  where  $\tilde{\mathfrak{m}}$  is the whole set of monomials in  $\mathbb{Y}^{[h]}$  and  $x_0$ . Then  $\deg(\mathcal{J}^0) = \deg(\mathcal{J})$ .

Since  $\mathbb{V}((H_{ik})_{1 \leq i \leq n; 0 \leq k \leq h_i}) = \mathbb{V}(\mathcal{J}^0) \cup \mathbb{V}(H_{ik}, x_0) \cup \cup_{j,l} \mathbb{V}(H_{ik}, y_j^{(l)})$ ,  $\mathbb{V}(\mathcal{J}^0)$  is an irreducible component of  $\mathbb{V}((H_{ik})_{1 \leq i \leq n; 0 \leq k \leq h_i})$ . By Lemma 6.3,  $\deg(\mathcal{J}^0) \leq \prod_{i=0}^n \prod_{k=0}^{h_i} (m_i + 1) = \prod_{i=0}^n (m_i + 1)^{h_i+1}$ . Thus,  $\deg(\mathcal{J}) \leq \prod_{i=0}^n (m_i + 1)^{h_i+1}$ . Since  $\mathcal{J} \cap \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \dots, \mathbf{u}_n^{[h_n]}] = (\mathbf{R})$ , by Theorem 6.2,  $\deg(\mathbf{R}) \leq \deg(\mathcal{J}) \leq \prod_{i=0}^n (m_i + 1)^{h_i+1} \leq (m + 1)^{\sum_{i=0}^n (J_i+1)}$  follows. The last inequality holds because  $h_i \leq J_i$  by Theorem 5.13.

2) To obtain the degree bounds for this representation for  $\mathbf{R}$ , we first substitute  $u_{i0}$  by  $(\mathbb{P}_i^N - \sum_{k=1}^{l_i} u_{ik} N_{ik})/N_{i0}$  into  $\mathbf{R}$  and then expand it. To be more precise, we take one monomial  $M(\mathbf{u}; u_{00}, \dots, u_{n0})$  in  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  for an example. Denote  $M = \mathbf{u}^\gamma \prod_{i=0}^n \prod_{k=0}^{h_i} (u_{i0}^{(k)})^{d_{ik}}$  with  $|\gamma| + \sum_{i=0}^n \sum_{k=0}^{h_i} d_{ik} = \deg(\mathbf{R})$ , where  $\mathbf{u}^\gamma$  represents a monomial in  $\mathbf{u}$  and their derivatives with exponent vector  $\gamma$ . Substitute  $u_{i0}$  by  $(\mathbb{P}_i^N - \sum_{k=1}^{l_i} u_{ik} N_{ik})/N_{i0}$  into  $M$ , we have

$$M(\mathbf{u}; u_{00}, \dots, u_{n0}) = \mathbf{u}^\gamma \prod_{i=0}^n \prod_{k=0}^{h_i} \left( \left( (\mathbb{P}_i^N - \sum_{k=1}^{l_i} u_{ik} N_{ik}) / N_{i0} \right)^{(k)} \right)^{d_{ik}}.$$

When expanded, the denominator is of the form  $\prod_{i=0}^n N_{i0}^{\sum_k (k+1)d_{ik}}$  and every term of the numerator has total degree  $|\gamma| + \sum_{i=0}^n \sum_{k=0}^{h_i} [(k+1)\deg(N_{i0}) + (m_i + 1 - \deg(N_{i0}))]d_{ik}$  in  $\mathbf{u}_0^{[h_0]}, \dots, \mathbf{u}_n^{[h_n]}$  and  $\mathbb{Y}^{[h]}$  with  $h = \max\{h_i + e_i\}$ . For every monomial  $M$  in  $\mathbf{R}$ ,  $\sum_{k=0}^{h_i} (k+1)d_{ik} \leq (h_i + 1)\deg(\mathbf{R})$ . Thus,  $\prod_{i=0}^n N_{i0}^{(h_i+1)\deg(\mathbf{R})} \cdot \mathbf{R} = \sum_{i=0}^n \sum_{j=0}^{h_i} G_{ij} (\mathbb{P}_i^N)^{(j)} + T$  where  $G_{ij}, T \in \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \dots, \mathbf{u}_n^{[h_n]}, \mathbb{Y}^{[h]}]$  and  $T$  is free from  $u_{i0}$  for  $i = 0, \dots, n$ . It is easy to see that  $T \in [\mathbb{P}_0^N, \dots, \mathbb{P}_n^N] : \mathfrak{m}$  and  $T = 0$  follows. By the above substitution for every monomial in  $\mathbf{R}$ , we can see that

$$\begin{aligned} & \deg(G_{ij} (\mathbb{P}_i^N)^{(j)}) \\ \leq & \max_{(\gamma, d_{ik})} \{ |\gamma| + \sum_{i=0}^n \sum_{k=0}^{h_i} [(k+1)\deg(N_{i0}) + (m_i + 1 - \deg(N_{i0}))]d_{ik} \\ & + \sum_{i=0}^n (h_i + 1)\deg(\mathbf{R})\deg(N_{i0}) - \sum_{i=0}^n (\sum_k (k+1)d_{ik})\deg(N_{i0}) \} \\ \leq & \deg(\mathbf{R}) + \sum_{i,k} (m_i - \deg(N_{i0}))d_{ik} + \deg(\mathbf{R}) \sum_{i=0}^n (h_i + 1)\deg(N_{i0}) \\ \leq & (m+1)\deg(\mathbf{R}) + \deg(\mathbf{R}) \sum_{i=0}^n (h_i + 1)\deg(N_{i0}) \\ = & [m+1 + \sum_{i=0}^n (h_i + 1)\deg(N_{i0})]\deg(\mathbf{R}). \end{aligned}$$

□

**Example 6.6** Continue from Example 3.15. In this example,  $J_0 = 2$ ,  $J_1 = J_2 = 1$  and  $m_0 = m_1 = m_2 = 2$ . The expression of  $\mathbf{R}$  shows that  $h_0 = \text{ord}(\mathbf{R}, \mathbf{u}_0) = 1 < J_0$ ,  $h_i = \text{ord}(\mathbf{R}, \mathbf{u}_i) = 0 < J_i$  ( $i = 1, 2$ ) and  $\deg(\mathbf{R}) = 5 < 3^4 = \prod_{i=0}^2 (m_i + 1)^{h_i+1}$ .

For a differentially essential system of form (22), the second part of Theorem 6.5 can be improved as follows.

**Theorem 6.7** Let  $\mathbb{P}_0, \dots, \mathbb{P}_n$  be a differentially essential system of form (22) with  $m = \max_i \{\deg(\mathbb{P}_i^N, \mathbb{Y})\}$  and  $J_i$  the modified Jacobi number of  $\{\mathbb{P}_0^N, \dots, \mathbb{P}_n^N\} \setminus \{\mathbb{P}_i^N\}$ . Let  $\mathbf{R}(\mathbf{u}_0, \dots,$



$\mathbf{u}_n$ ) be the sparse differential resultant of  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ). Suppose  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = h_i$  for each  $i$  and  $h = \max\{h_i + s_i\}$ . Then  $\mathbf{R}$  has a representation

$$\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n) = \sum_{i=0}^n \sum_{j=0}^{h_i} G_{ij} \mathbb{P}_i^{(j)}$$

where  $G_{ij} \in \mathbb{Q}[\mathbf{u}_0^{[h_0]}, \dots, \mathbf{u}_n^{[h_n]}, \mathbb{Y}^{[h]}]$  such that  $\deg(G_{ij} \mathbb{P}_i^{(j)}) \leq (m+1)\deg(\mathbf{R}) \leq (m+1) \sum_{i=0}^n (J_i+1)+1$ .

*Proof:* Regarding  $\mathbb{P}_i$  as Laurent differential polynomials,  $\mathbb{P}_i^N = \mathbb{P}_i$  and  $N_{i0} = 1$ . By setting  $N_{i0} = 1$  in Theorem 6.5 these three assertions directly follow.  $\square$

The following result gives an effective differential Nullstellensatz under certain conditions.

**Corollary 6.8** *Let  $f_0, \dots, f_n \in \mathcal{F}\{y_1, \dots, y_n\}$  have no common solutions with  $\deg(f_i) \leq m$ . Let  $\text{Jac}(\{f_0, \dots, f_n\} \setminus \{f_i\}) = J_i$ . If the sparse differential resultant of  $f_0, \dots, f_n$  is nonzero, then there exist  $H_{ij} \in \mathcal{F}\{y_1, \dots, y_n\}$  s.t.  $\sum_{i=0}^n \sum_{j=0}^{J_i} H_{ij} f_i^{(j)} = 1$  and  $\deg(H_{ij} f_i^{(j)}) \leq (m+1) \sum_{i=0}^n (J_i+1)+1$ .*

*Proof:* The hypothesis implies that  $\mathbb{P}(f_i)$  form a differentially essential system. Clearly,  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  has the property stated in Theorem 6.7, where  $\mathbf{u}_i$  are coefficients of  $\mathbb{P}(f_i)$ . The result follows directly from Theorem 6.7 by specializing  $\mathbf{u}_i$  to the coefficients of  $f_i$ .  $\square$

With Theorem 6.5, properties 6) and 7) of Theorem 1.2 are proved.

### 6.3 A single exponential algorithm to compute sparse differential resultant

If a polynomial  $R$  is the linear combination of some known polynomials  $F_i$  ( $i = 1, \dots, s$ ), that is  $R = \sum_{i=1}^s H_i F_i$ , and we know the upper bounds of the degrees of  $R$  and  $H_i F_i$ , then a general idea to estimate the computational complexity of  $R$  is to use linear algebra to find the coefficients of  $R$ .

For sparse differential resultant, we already gave its degree bound and the degrees of the expressions in the linear combination in Theorem 6.5.

Now, we give the algorithm **SDResultant** to compute sparse differential resultants based on the linear algebra techniques. The algorithm works adaptively by searching for  $\mathbf{R}$  with an order vector  $(h_0, \dots, h_n) \in \mathbb{N}^{n+1}$  with  $h_i \leq J_i$  by Theorem 6.5. Denote  $o = \sum_{i=0}^n h_i$ . We start with  $o = 0$ . And for this  $o$ , choose one vector  $(h_0, \dots, h_n)$  at a time. For this  $(h_0, \dots, h_n)$ , we search for  $\mathbf{R}$  from degree  $d = 1$ . If we cannot find an  $\mathbf{R}$  with such a degree, then we repeat the procedure with degree  $d+1$  until  $d > \prod_{i=0}^n (m_i+1)^{h_i+1}$ . In that case, we choose another  $(h_0, \dots, h_n)$  with  $\sum_{i=0}^n h_i = o$ . But if for all  $(h_0, \dots, h_n)$  with  $h_i \leq J_i$  and  $\sum_{i=0}^n h_i = o$ ,  $\mathbf{R}$  cannot be found, then we repeat the procedure with  $o+1$ . In this way, we will find an  $\mathbf{R}$  with the smallest order satisfying equation (28), which is the sparse resultant.

**Theorem 6.9** *Let  $\mathbb{P}_0, \dots, \mathbb{P}_n$  be a Laurent differentially essential system of form (2). Denote  $\mathbb{P} = \{\mathbb{P}_0^N, \dots, \mathbb{P}_n^N\}$ ,  $J_i = \text{Jac}(\mathbb{P}_i)$ ,  $J = \sum_{i=0}^n J_i$  and  $m = \max_{i=0}^n \deg(\mathbb{P}_i, \mathbb{Y})$ . Algorithm **SDResultant** computes sparse differential resultant  $\mathbf{R}$  of  $\mathbb{P}_0, \dots, \mathbb{P}_n$  with the following complexities:*

---

**Algorithm 2 — SDResultant**( $\mathbb{P}_0, \dots, \mathbb{P}_n$ )

---

**Input:** A generic Laurent differentially essential system  $\mathbb{P}_0, \dots, \mathbb{P}_n$ .

**Output:** The sparse differential resultant  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  of  $\mathbb{P}_0, \dots, \mathbb{P}_n$ .

1. For  $i = 0, \dots, n$ , set  $P_i^N = \sum_{k=0}^{l_i} u_{ik} N_{ik}$  with  $\deg(N_{i0}) \leq \deg(N_{ik})$ .  
Set  $e_{ij} = \text{ord}(\mathbb{P}_i^N, y_j)$ ,  $m_i = \deg(\mathbb{P}_i^N)$ ,  $m_{i0} = \deg(N_{i0})$ ,  $\mathbf{u}_i = \text{coeff}(\mathbb{P}_i)$  and  $|\mathbf{u}_i| = l_i + 1$ .  
Set  $A = (e_{ij})$  and compute  $J_i = \text{Jac}(A_i)$ .
2. Set  $\mathbf{R} = 0$ ,  $o = 0$ ,  $m = \max_i \{m_i\}$ .
3. While  $\mathbf{R} = 0$  do
  - 3.1. For each vector  $(h_0, \dots, h_n) \in \mathbb{N}^{n+1}$  with  $\sum_{i=0}^n h_i = o$  and  $h_i \leq J_i$  do
    - 3.1.1.  $U = \cup_{i=0}^n \mathbf{u}_i^{[h_i]}$ ,  $h = \max_i \{h_i + e_i\}$ ,  $d = 1$ .
    - 3.1.2. While  $\mathbf{R} = 0$  and  $d \leq \prod_{i=0}^n (m_i + 1)^{h_i+1}$  do
      - 3.1.2.1. Set  $\mathbf{R}_0$  to be a homogenous GPol of degree  $d$  in  $U$ .
      - 3.1.2.2. Set  $\mathbf{c}_0 = \text{coeff}(\mathbf{R}_0, U)$ .
      - 3.1.2.3. Set  $H_{ij}$  ( $i = 0, \dots, n; j = 0, \dots, h_i$ ) to be GPols of degree  $[m + 1 + \sum_{i=0}^n (h_i + 1)m_{i0}]d - m_i - 1$  in  $\mathbb{Y}^{[h]}, U$ .
      - 3.1.2.4. Set  $\mathbf{c}_{ij} = \text{coeff}(H_{ij}, \mathbb{Y}^{[h]} \cup U)$ .
      - 3.1.2.5. Set  $\mathcal{P}$  to be the set of coefficients of  $\prod_{i=0}^n N_{i0}^{(h_i+1)d} \mathbf{R}_0(\mathbf{u}_0, \dots, \mathbf{u}_n) - \sum_{i=0}^n \sum_{j=0}^{h_i} H_{ij}(\mathbb{P}_i^N)^{(j)}$  as a polynomial in  $\mathbb{Y}^{[h]}, U$ .
      - 3.1.2.6. Solve the linear equation  $\mathcal{P} = 0$  in variables  $\mathbf{c}_0$  and  $\mathbf{c}_{ij}$ .
      - 3.1.2.7. If  $\mathbf{c}_0$  has a nonzero solution, then substitute it into  $\mathbf{R}_0$  to get  $\mathbf{R}$  and go to Step 4, else  $\mathbf{R} = 0$ .
      - 3.1.2.8.  $d := d + 1$ .
    - 3.2.  $o := o + 1$ .
  4. Return  $\mathbf{R}$ .

/\*/ GPol stands for generic algebraic polynomial.

/\*/  $\text{coeff}(P, V)$  returns the set of coefficients of  $P$  as an ordinary polynomial in variables  $V$ .

---

1) In terms of the degree bound  $D$  of  $\mathbf{R}$ , the algorithm needs at most  $O\left(\frac{(mD(J+n+2))^{O(l(J+1))}}{n^n}\right)$   $\mathbb{Q}$ -arithmetic operations, where  $l = \sum_{i=0}^n (l_i + 1)$  is the size of all  $\mathbb{P}_i$ .

2) The algorithm needs at most  $O\left(\frac{(J+n+2)^{O(l(J+1))} m^{O(l(J+1)(J+n+1))}}{n^n}\right)$   $\mathbb{Q}$ -arithmetic operations.

*Proof:* The algorithm finds a differential polynomial  $P$  in  $\mathbb{Q}\{\mathbf{u}_0, \dots, \mathbf{u}_n\}$  satisfying equation (28), which has the smallest order and the smallest degree in those with the same order. Existence for such a differential polynomial is guaranteed by Theorem 6.5. Such a  $P$  must be in  $\text{sat}(\mathbf{R})$  by equation (5). Since each differential polynomial in  $\text{sat}(\mathbf{R})$  not equal to  $\mathbf{R}$  either has greater order than  $\mathbf{R}$  or has the same order but greater degree than  $\mathbf{R}$ ,  $P$  must be  $\mathbf{R}$ .

We will estimate the complexity of the algorithm below. Denote  $D$  to be the degree bound of  $\mathbf{R}$ . By Theorem 6.5,  $D \leq (m+1)^{\sum_{i=0}^n (J_i+1)} = (m+1)^{J+n+1}$ , where  $J = \sum_{i=0}^n J_i$ .

In each loop of Step 3, the complexity of the algorithm is clearly dominated by Step 3.1.2, where we need to solve a system of linear equations  $\mathcal{P} = 0$  over  $\mathbb{Q}$  in  $\mathbf{c}_0$  and  $\mathbf{c}_{ij}$ . It is easy to show that  $|\mathbf{c}_0| = \binom{d+L-1}{L-1}$  and  $|\mathbf{c}_{ij}| = \binom{d_1-m_i-1+L+n(h+1)}{L+n(h+1)}$ , where  $L = \sum_{i=0}^n (h_i + 1)(l_i + 1)$  and  $d_1 = [m + 1 + \sum_{i=0}^n (h_i + 1)m_{i0}]d$ . Then  $\mathcal{P} = 0$  is a linear equation system with  $N = \binom{d+L-1}{L-1} + \sum_{i=0}^n (h_i + 1) \binom{d_1-m_i-1+L+n(h+1)}{L+n(h+1)}$  variables and  $M = \binom{d_1+L+n(h+1)}{L+n(h+1)}$  equations. To solve it, we need at most  $(\max\{M, N\})^\omega$  arithmetic operations over  $\mathbb{Q}$ , where  $\omega$  is the matrix multiplication exponent and the currently best known  $\omega$  is 2.376.

The iteration in Step 3.1.2 may go through 1 to  $\prod_{i=0}^n (m_i + 1)^{h_i+1} \leq (m + 1)^{\sum_{i=0}^n (J_i+1)}$ , and the iteration in Step 3.1 at most will repeat  $\prod_{i=0}^n (J_i + 1)$  times. And by Theorem 6.5, Step 3 may loop from  $o = 0$  to  $\sum_{i=0}^n (J_i + 1)$ . The whole algorithm needs at most

$$\begin{aligned}
& \sum_{o=0}^{\sum_{i=0}^n J_i} \sum_{\substack{h_i \leq J_i \\ \sum_i h_i = o}} \prod_{i=0}^n (m_i + 1)^{h_i+1} \sum_{d=1} (\max\{M, N\})^{2.376} \\
& \leq (J + 1) \left( \prod_{i=0}^n (J_i + 1) \right) \cdot D \left[ (J + n + 2) \binom{[m + 1 + \sum_{i=0}^n (J_i + 1)m_{i0}]D + L + n(h + 1)}{L + n(h + 1)} \right]^{2.376} \\
& \leq (J + n + 2)^{3.376} \left( \frac{\sum_{i=0}^n (J_i + 1)}{n + 1} \right)^{n+1} \cdot D \cdot [(m(J + n + 2)D)^{2.376(L+n(h+1))}] \\
& \leq (J + n + 2)^{3.376} \frac{(J + n + 1)^{n+1}}{n^n} \cdot D \cdot (m(J + n + 2)D)^{2.376((l+n)(J+1)+n)}
\end{aligned}$$

arithmetic operations over  $\mathbb{Q}$ . The above inequalities follow from the fact that  $h \leq J$ ,  $L = \sum_{i=0}^n (h_i + 1)(l_i + 1) \leq lJ + l$ ,  $L + n(h + 1) \leq (l + n)J + l + n = (l + n)(J + 1)$  and the assumption  $[m + 1 + \sum_{i=0}^n (J_i + 1)m]D \geq L + n(h + 1)$ .

Since  $l \geq 2(n + 1)$ , the complexity bound is  $O\left(\frac{(mD(J+n+2))^{O(l(J+1))}}{n^n}\right)$ . Our complexity assumes an  $O(1)$ -complexity cost for all field operations over  $\mathbb{Q}$ . Thus, the complexity follows. Now 1) is proved. To prove 2), we just need to replace  $D$  by the degree bound for  $\mathbf{R}$  in Theorem 6.5 in the complexity bound in 1).  $\square$

**Remark 6.10** *As we indicated at the end of Section 3.3, if we first use Algorithm 1 to compute the rank-essential set  $\mathbb{T}$ , then the algorithm can be improved by only considering the Laurent differential polynomials  $\mathbb{P}_i$  ( $i \in \mathbb{T}$ ) in the linear combination of the sparse resultant.*

**Remark 6.11** *Algorithm **SDResultant** can be improved by using a better search strategy. If  $d$  is not big enough, instead of checking  $d + 1$ , we can check  $2d$ . Repeating this procedure, we may find a  $k$  such that  $2^k \leq \deg(\mathbf{R}) \leq 2^{k+1}$ . We then bisecting the interval  $[2^k, 2^{k+1}]$  again to find the proper degree for  $\mathbf{R}$ . This will lead to a better complexity, which is still single exponential.*

**Remark 6.12** *If the given system is algebraic, that is  $J = 0$ , then the complexity bound given in 1) of Theorem 6.9 is essentially the same as that given in [49][p. 288] since  $D \gg m$  and  $D \gg n$ .*

For differential polynomials with non-vanishing degree terms given in (22), a better degree bound is given in Theorem 6.7. Based on this bound, we can simplify the Algorithm **SDResultant** to compute the sparse differential resultant by removing the computation for  $P_i^N$  and  $N_{i0}$  in the first step where  $N_{i0}$  is exactly equal to 1.

**Theorem 6.13** *Algorithm **SDResultant** computes sparse differential resultants for a differentially essential system of form (22) with at most  $O\left(\frac{(J+n+1)^{O(n)} m^{O(l(J+1)(J+n+1))}}{n^n}\right)$   $\mathbb{Q}$ -arithmetic operations.*

*Proof:* Follow the proof process of Theorem 6.9, it can be shown that the complexity is one mentioned in the theorem.  $\square$

With Theorem 6.9, Theorem 1.4 is proved.

## 6.4 Degree bound for differential resultant in terms of mixed volumes

The degree bound given in Theorem 6.5 is essentially a Bézout type bound. In this section, a BKK style degree bound for differential resultant will be given, which is the sum of the mixed volumes of certain polytopes generated by the supports of certain differential polynomials and their derivatives.

We first recall results about the degree of algebraic sparse resultant given by Sturmfels ([50]). Let  $\mathcal{K}[\mathbb{X}] = \mathcal{K}[x_1, \dots, x_n]$  be the polynomial ring defined over a field  $\mathcal{K}$ . For any vector  $\alpha = (a_1, \dots, a_n) \in \mathbb{Z}^n$ , denote the Laurent monomial  $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$  by  $\mathbb{X}^\alpha$ . Let  $\mathcal{B}_0, \dots, \mathcal{B}_n \subset \mathbb{Z}^n$  be subsets which jointly span the affine lattice  $\mathbb{Z}^n$ . Suppose  $\mathbf{0} = (0, \dots, 0) \in \mathcal{B}_i$  for each  $i$  and  $|\mathcal{B}_i| = l_i + 1 \geq 2$ . Let

$$\mathbb{F}_i(x_1, \dots, x_n) = c_{i0} + \sum_{\alpha \in \mathcal{B}_i \setminus \{\mathbf{0}\}} c_{i,\alpha} \mathbb{X}^\alpha \quad (i = 0, 1, \dots, n) \quad (29)$$

be generic sparse Laurent polynomials defined w.r.t  $\mathcal{B}_i$  ( $i = 0, 1, \dots, n$ ).  $\mathcal{B}_i$  or  $\{\mathbb{X}^\alpha : \alpha \in \mathcal{B}_i\}$  are called the support of  $\mathbb{F}_i$ . Denote  $\mathbf{c}_i = (c_{i,\alpha})_{\alpha \in \mathcal{B}_i}$  and  $\mathbf{c} = \cup_i (\mathbf{c}_i \setminus \{c_{i0}\})$ . Let  $\mathcal{Q}_i$  be the convex hull of  $\mathcal{B}_i$  in  $\mathbb{R}^n$ , which is the smallest convex set containing  $\mathcal{B}_i$ .  $\mathcal{Q}_i$  is also called the *Newton polytope* of  $\mathbb{F}_i$ , denoted by  $\text{NP}(\mathbb{F}_i)$ . In [50], Sturmfels gave the definition of algebraic essential set and proved that a necessary and sufficient condition for the existence of sparse resultant is that there exists a unique subset  $\{\mathcal{B}_i\}_{i \in I}$  which is essential. Now, we restate the definition of essential sets in our words for the sake of later use.

**Definition 6.14** *Follow the notations introduced above.*

- A collection of  $\{\mathcal{B}_i\}_{i \in J}$ , or  $\{\mathbb{F}_i\}_{i \in J}$  of the form (29), is said to be algebraically independent if  $\text{tr.deg } \mathbb{Q}(\mathbf{c})(\mathbb{F}_i - c_{i0} : i \in J) / \mathbb{Q}(\mathbf{c}) = |J|$ . Otherwise, they are said to be algebraically dependent.
- A collection of  $\{\mathcal{B}_i\}_{i \in I}$  is said to be essential if  $\{\mathcal{B}_i\}_{i \in I}$  is algebraically dependent and for each proper subset  $J$  of  $I$ ,  $\{\mathcal{B}_i\}_{i \in J}$  are algebraically independent.

In the case that  $\{\mathcal{B}_0, \dots, \mathcal{B}_n\}$  is essential, the degree of the sparse resultant can be described by mixed volumes.

**Theorem 6.15 ([50])** *Suppose that  $\{\mathcal{B}_0, \dots, \mathcal{B}_n\}$  is essential. For each  $i \in \{0, 1, \dots, n\}$ , the degree of the sparse resultant in  $\mathbf{c}_i$  is a positive integer, equal to the mixed volume*

$$\mathcal{M}(\mathcal{Q}_0, \dots, \mathcal{Q}_{i-1}, \mathcal{Q}_{i+1}, \dots, \mathcal{Q}_n) = \sum_{J \subset \{0, \dots, i-1, i+1, \dots, n\}} (-1)^{n-|J|} \text{vol}\left(\sum_{j \in J} \mathcal{Q}_j\right)$$

where  $\text{vol}(\mathcal{Q})$  means the  $n$ -dimensional volume of  $\mathcal{Q} \subset \mathbb{R}^n$  and  $\mathcal{Q}_1 + \mathcal{Q}_2$  means the Minkowski sum of  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$ .

The mixed volume of the Newton polytopes of a polynomial system is important in that it relates to the number of solutions of these polynomial equations contained in  $(\mathbb{C}^*)^n$ , which is the famous BKK bound. The following theorem explains it.

**Theorem 6.16 (Bernstein's Theorem)** ([2]) *Given polynomials  $f_1, \dots, f_n$  over  $\mathbb{C}$  with finitely many common zeroes in  $(\mathbb{C})^n$ , let  $\mathcal{Q}_i$  be the Newton polytope of  $f_i$  in  $\mathbb{R}^n$ . Then the number of common zeroes of the  $f_i$  in  $(\mathbb{C}^*)^n$  is bounded by the mixed volume  $\mathcal{M}(\mathcal{Q}_1, \dots, \mathcal{Q}_n)$ . Moreover, for generic choices of the coefficients in the  $f_i$ , the number of common solutions in  $(\mathbb{C}^*)^n$  is exactly  $\mathcal{M}(\mathcal{Q}_1, \dots, \mathcal{Q}_n)$ .*

It is well known that for a given polynomial system over  $\mathbb{C}$ , the Bézout bound gives a bound for the number of isolated solutions in  $(\mathbb{C})^n$ . Comparing the BKK bound with Bézout bound, we have the following lemma.

**Lemma 6.17** *Follow the notations in Theorem 6.16. Then  $\mathcal{M}(\mathcal{Q}_1, \dots, \mathcal{Q}_n) \leq \prod_{i=1}^n \deg(f_i)$ .*

*Proof:* Suppose  $f_i$  ( $i = 1, \dots, m$ ) are a system with generic coefficients. Then by Theorem 6.16, the number of common zeroes of the  $f_i$  in  $(\mathbb{C}^*)^n$  is equal to the mixed volume  $\mathcal{M}(\mathcal{Q}_1, \dots, \mathcal{Q}_n)$ . And by Lemma 6.4, the number of common zeroes of the  $f_i$  in  $(\mathbb{C})^n$  is bounded by  $\prod_{i=1}^n \deg(f_i)$ . Thus,  $\mathcal{M}(\mathcal{Q}_1, \dots, \mathcal{Q}_n) \leq \prod_{i=1}^n \deg(f_i)$  follows.  $\square$

In the rest of this section, the degree of sparse resultant will be used to give a degree bound for differential resultant in terms of mixed volumes. A system of  $n + 1$  generic differential polynomials with degrees  $m_0, \dots, m_n$  and orders  $s_0, \dots, s_n$  respectively of the form

$$\mathbb{P}_i = u_{i0} + \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n(s_i+1)} \\ 1 \leq |\alpha| \leq m_i}} u_{i\alpha} (\mathbb{Y}^{[s_i]})^\alpha \quad (i = 0, \dots, n), \quad (30)$$

of course forms a differentially essential system and their sparse differential resultant is exactly equal to their differential resultant defined in [16]. So Theorem 6.7 also gives a degree bound for differential resultant. But when we use Theorem 6.7 to estimate the degree of  $\mathbf{R}$ , not only Bézout bound is used, but also the degrees of  $\mathbb{P}_i$  in both  $\mathbb{Y}$  and  $\mathbf{u}_i$  are considered.

The following theorem gives a better upper bound for degrees of differential resultants, the proof of which is not valid for sparse differential resultants. Precisely, in the following result, when estimate the degree of  $\mathbf{R}$ , the BKK bound is used rather than the Bézout bound as did in Theorem 6.7.

**Theorem 6.18** Let  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ) be generic differential polynomials in  $\mathbb{Y} = \{y_1, \dots, y_n\}$  with order  $s_i$ , degree  $m_i$ , and coefficients  $\mathbf{u}_i$  respectively. Let  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  be the differential resultant of  $\mathbb{P}_0, \dots, \mathbb{P}_n$ . Denote  $s = \sum_{i=0}^n s_i$ . Then for each  $i \in \{0, 1, \dots, n\}$ ,

$$\deg(\mathbf{R}, \mathbf{u}_i) \leq \sum_{k=0}^{s-s_i} \mathcal{M}((\mathcal{Q}_{jl})_{j \neq i, 0 \leq l \leq s-s_j}, \mathcal{Q}_{i0}, \dots, \mathcal{Q}_{i,k-1}, \mathcal{Q}_{i,k+1}, \dots, \mathcal{Q}_{i,s-s_i}) \quad (31)$$

where  $\mathcal{Q}_{jl}$  is the Newton polytope of  $\mathbb{P}_j^{(l)}$  as a polynomial in  $y_1^{[s]}, \dots, y_n^{[s]}$ .

*Proof:* By [16, Theorem 6.8],  $\text{ord}(\mathbf{R}, \mathbf{u}_i) = s - s_i$  ( $i = 0, \dots, n$ ) and  $(\mathbf{R}) = (\mathbb{P}_0^{[s-s_0]}, \dots, \mathbb{P}_n^{[s-s_n]}) \cap \mathbb{Q}[\mathbf{u}_0^{[s-s_0]}, \dots, \mathbf{u}_n^{[s-s_n]}]$ . Regard  $\mathbb{P}_i^{(k)}$  ( $i = 0, \dots, n, k = 0, \dots, s - s_i$ ) as polynomials in the  $n(s+1)$  variables  $\mathbb{Y}^{[s]} = \{y_1, \dots, y_n, y'_1, \dots, y'_n, \dots, y_1^{(s)}, \dots, y_n^{(s)}\}$ , and we denote its support by  $\mathcal{B}_{ik}$ . Let  $\mathbb{F}_{ik}$  be the generic sparse polynomial with support  $\mathcal{B}_{ik}$ . Denote  $\mathbf{v}_{ik}$  to be the set of coefficients of  $\mathbb{F}_{ik}$  and in particular, suppose  $v_{ik0}$  is the coefficient of the monomial 1 in  $\mathbb{F}_{ik}$ . Now we claim that

C1)  $\overline{\mathcal{B}} = \{\mathcal{B}_{ik} : 0 \leq i \leq n; 0 \leq k \leq s - s_i\}$  is an essential set.

C2)  $\overline{\mathcal{B}} = \{\mathcal{B}_{ik} : 0 \leq i \leq n; 0 \leq k \leq s - s_i\}$  jointly span the affine lattice  $\mathbb{Z}^{n(s+1)}$ .

Note that  $|\overline{\mathcal{B}}| = n(s+1) + 1$ . To prove C1), it suffices to show that any  $n(s+1)$  of distinct  $\mathbb{F}_{ik}$  are algebraically independent. Without loss of generality, we prove that for a fixed  $k \in \{0, \dots, s - s_0\}$ ,

$$S_k = \{(\mathbb{F}_{jl})_{1 \leq j \leq n; 0 \leq l \leq s-s_j}, \mathbb{F}_{00}, \dots, \mathbb{F}_{0,k-1}, \mathbb{F}_{0,k+1}, \dots, \mathbb{F}_{0,s-s_0}\}$$

is an algebraically independent set. Remember that  $\{y_1, \dots, y_n, y'_1, \dots, y'_n, \dots, y_1^{(s_1+l)}, \dots, y_n^{(s_n+l)}\}$  is a subset of the support of  $\mathbb{F}_{il}$ . Now we choose a monomial from each  $\mathbb{F}_{jl}$  and denote it by  $m(\mathbb{F}_{jl})$ . For each  $j \in \{1, \dots, n\}$  and  $l \in \{0, \dots, s - s_j\}$ , let  $m(\mathbb{F}_{jl}) = y_j^{(s_j+l)}$  which belongs to the support of  $\mathbb{F}_{jl}$ . For the fixed  $k$ , there exists a  $\tau \in \{0, 1, \dots, n-1\}$  such that either  $\sum_{i=1}^{\tau} s_i \leq k \leq \sum_{i=1}^{\tau+1} s_i - 1$  for some  $\tau \in \{0, 1, \dots, n-2\}$  or  $\sum_{i=1}^{\tau} s_i \leq k \leq \sum_{i=1}^{\tau+1} s_i$  for  $\tau = n-1$ . Here when  $\tau = 0$ , it means  $0 \leq k \leq s_1 - 1$ . Then for  $l \neq k$ , let

$$m(\mathbb{F}_{0l}) = \begin{cases} y_1^{(l)} & 0 \leq l \leq s_1 - 1 \\ y_2^{(l-s_1)} & s_1 \leq l \leq s_1 + s_2 - 1 \\ \vdots & \vdots \\ y_{\tau+1}^{(l-\sum_{i=1}^{\tau} s_i)} & \sum_{i=1}^{\tau} s_i \leq l \leq k-1 \\ y_{\tau+1}^{(l-\sum_{i=1}^{\tau} s_i-1)} & k+1 \leq l \leq \sum_{i=1}^{\tau+1} s_i \\ y_{\tau+2}^{(l-\sum_{i=1}^{\tau+1} s_i-1)} & \sum_{i=1}^{\tau+1} s_i + 1 \leq l \leq \sum_{i=1}^{\tau+2} s_i \\ \vdots & \vdots \\ y_n^{(l-\sum_{i=1}^{n-1} s_i-1)} & \sum_{i=1}^{n-1} s_i + 1 \leq l \leq \sum_{i=1}^n s_i = s - s_0 \end{cases}$$

So  $m(S_k)$  is equal to  $\{y_j^{[s]} : 1 \leq j \leq n\}$ , which are algebraically independent over  $\mathbb{Q}$ . Thus, the  $n(s+1)$  members of  $S_k$  are algebraically independent over  $\mathbb{Q}$ . For if not,

$\mathbb{F}_{jl} - v_{jl0}$  are algebraically dependent over  $\mathbb{Q}(\mathbf{v})$  where  $\mathbf{v} = \cup_{i=0}^n \sum_{k=0}^{s-s_i} \mathbf{v}_{ik} \setminus \{v_{ik0}\}$ . Now specialize the coefficient of  $m(\mathbb{F}_{jl})$  in  $\mathbb{F}_{jl}$  to 1, and all the other coefficients of  $\mathbb{F}_{jl} - v_{jl0}$  to 0, by the algebraic version of Lemma 2.1,  $\{m(\mathbb{F}_{jl}) : \mathbb{F}_{jl} \in S_k\}$  are algebraically dependent, which is a contradiction. Thus, claim C1) is proved. Claim C2) follows from the fact that  $\{1, y_j^{[s]} : 1 \leq j \leq n\}$  is contained in the support of  $\mathbb{F}_{0,s-s_0}$ .

From the claims C1) and C2), the sparse resultant of  $(\mathbb{F}_{ik})_{0 \leq i \leq n; 0 \leq k \leq s-s_i}$  exists and we denote it by  $G$ . Then  $(G) = ((\mathbb{F}_{ik})_{0 \leq i \leq n; 0 \leq k \leq s-s_i}) \cap \mathbb{Q}[(\mathbf{v}_{ik})_{0 \leq i \leq n; 0 \leq k \leq s-s_i}]$ , and by Theorem 6.15,  $\deg(G, \mathbf{v}_{ik}) = \mathcal{M}((\mathcal{Q}_{jl})_{j \neq i, 0 \leq l \leq s-s_j}, \mathcal{Q}_{i0}, \dots, \mathcal{Q}_{i,k-1}, \mathcal{Q}_{i,k+1}, \dots, \mathcal{Q}_{i,s-s_i})$ .

Now suppose  $\xi$  is a generic point of the zero ideal (0) in  $\mathbb{Q}(\mathbf{v})[\mathbb{Y}^{[s]}]$ . Let  $\zeta_{ik} = -\mathbb{F}_{ik}(\xi) + v_{ik0}$  and  $\bar{\zeta}_{ik} = -\mathbb{P}_i^{(k)}(\xi) + u_{i0}^{(k)}$  ( $i = 0, \dots, n; k = 0, \dots, s-s_i$ ). Clearly,  $\zeta_i$  and  $\bar{\zeta}_i$  are free of  $v_{ik0}$  and  $u_{i0}^{(k)}$  respectively. It is easy to see that  $(\xi; \mathbf{v}, \zeta_{00}, \dots, \zeta_{0,s-s_0}, \dots, \zeta_{n0}, \dots, \zeta_{n,s-s_n})$  is a generic point of the algebraic prime ideal  $((\mathbb{F}_{ik})_{0 \leq i \leq n; 0 \leq k \leq s-s_i}) \subset \mathbb{Q}[\mathbb{Y}^{[s]}, (\mathbf{v}_{ik})_{0 \leq i \leq n; 0 \leq k \leq s-s_i}]$ , while  $(\xi; \cup_{i=0}^n (\mathbf{u}_i \setminus \{u_{i0}\})^{[s-s_i]}, \bar{\zeta}_{00}, \dots, \bar{\zeta}_{0,s-s_0}, \dots, \bar{\zeta}_{n0}, \dots, \bar{\zeta}_{n,s-s_n})$  is a generic point of the algebraic prime ideal  $((\mathbb{P}_i^{(k)})_{0 \leq i \leq n; 0 \leq k \leq s-s_i}) \subset \mathbb{Q}[\mathbb{Y}^{[s]}, \mathbf{u}_0^{[s-s_0]}, \dots, \mathbf{u}_n^{[s-s_n]}]$ . If we regard  $G$  as a polynomial in  $v_{ik0}$  over  $\mathbb{Q}(\mathbf{v})$ , then  $G$  is the vanishing polynomial of  $(\zeta_{00}, \dots, \zeta_{0,s-s_0}, \dots, \zeta_{n0}, \dots, \zeta_{n,s-s_n})$  over  $\mathbb{Q}(\mathbf{v})$ . Now specialize the coefficients  $\mathbf{v}_{ik}$  of  $\mathbb{F}_{ik}$  to the corresponding coefficients of  $\mathbb{P}_i^{(k)}$ . Then  $\zeta_i$  are specialized to  $\bar{\zeta}_i$ . In particular,  $v_{ik0}$  are specialized to  $u_{i0}^{(k)}$  which are algebraically independent over the field  $\mathbb{Q}(\xi, \cup_{i=0}^n \mathbf{u}_i^{[s-s_i]} \setminus u_{i0}^{[s-s_i]})$ . We claim that there exists a nonzero polynomial  $H(\cup_{i=0}^n \mathbf{u}_i^{[s-s_i]} \setminus u_{i0}^{[s-s_i]}; u_{00}, \dots, u_{00}^{(s-s_0)}, \dots, u_{n0}, \dots, u_{n0}^{(s-s_n)}) \in \mathbb{Q}[\mathbf{u}_0^{[s-s_0]}, \dots, \mathbf{u}_n^{[s-s_n]}]$  such that

$$\text{C3)} \quad H(\cup_{i=0}^n \mathbf{u}_i^{[s-s_i]} \setminus u_{i0}^{[s-s_i]}; \bar{\zeta}_{00}, \dots, \bar{\zeta}_{0,s-s_0}, \dots, \bar{\zeta}_{n0}, \dots, \bar{\zeta}_{n,s-s_n}) = 0 \text{ and}$$

$$\text{C4)} \quad \deg(H, \mathbf{u}_i^{[s-s_i]}) \leq \deg(G, \cup_{k=0}^{s-s_i} \mathbf{v}_{ik}).$$

We obtain  $H$  by specializing  $\mathbf{v}$  one by one in  $G$ . For each  $v \in \mathbf{v}$ , denote  $u$  to be its corresponding coefficient in  $\mathbb{P}_i^{(k)}$ . Now we first specialize  $v$  to  $u$  and suppose  $\zeta_{ik}$  is specialized to  $\tilde{\zeta}_{ik}$  correspondingly. Clearly,  $G(\mathbf{v} \setminus \{v\}, u; \tilde{\zeta}_{00}, \dots, \tilde{\zeta}_{0,s-s_0}, \tilde{\zeta}_{n0}, \dots, \tilde{\zeta}_{n,s-s_n}) = 0$ . If  $\bar{G} = G(\mathbf{v} \setminus \{v\}, u; v_{000}, v_{010}, \dots, v_{0,s-s_0,0}, \dots, v_{n00}, v_{n10}, \dots, v_{n,s-s_n,0}) \neq 0$ , denote  $\bar{G}$  by  $H_1$ . Otherwise, there exists some  $a \in \mathbb{N}$  such that  $G = (v - u)^a G_1$  with  $G_1|_{v=u} \neq 0$ . But  $G(\mathbf{v} \setminus \{v\}, u; \tilde{\zeta}_{00}, \dots, \tilde{\zeta}_{0,s-s_0}, \tilde{\zeta}_{n0}, \dots, \tilde{\zeta}_{n,s-s_n}) = 0 = (v - u)^a G_1(\mathbf{v} \setminus \{v\}, u; \tilde{\zeta}_{00}, \dots, \tilde{\zeta}_{0,s-s_0}, \tilde{\zeta}_{n0}, \dots, \tilde{\zeta}_{n,s-s_n})$ , so  $G_1(\mathbf{v} \setminus \{v\}, u; \tilde{\zeta}_{00}, \dots, \tilde{\zeta}_{0,s-s_0}, \tilde{\zeta}_{n0}, \dots, \tilde{\zeta}_{n,s-s_n}) = 0$ . Denote  $G_1|_{v=u}$  by  $H_1$ . Clearly,  $\deg(H_1, \mathbf{u}_i^{[s-s_i]} \cup \cup_k \mathbf{v}_{ik}) \leq \deg(G, \cup_k \mathbf{v}_{ik})$  for each  $i$ . Continuing this process for  $|\mathbf{v}|$  times till each  $v \in \mathbf{v}$  is specialized to its corresponding element  $u$ , we will obtain a nonzero polynomial  $H_{|\mathbf{v}|}(\cup_{i=0}^n (\mathbf{u}_i \setminus \{u_{i0}\})^{[s-s_i]}; v_{000}, v_{010}, \dots, v_{0,s-s_0,0}, \dots, v_{n00}, v_{n10}, \dots, v_{n,s-s_n,0})$  satisfying  $H_{|\mathbf{v}|}(\cup_{i=0}^n (\mathbf{u}_i \setminus \{u_{i0}\})^{[s-s_i]}; \bar{\zeta}_{00}, \dots, \bar{\zeta}_{0,s-s_0}, \bar{\zeta}_{n0}, \dots, \bar{\zeta}_{n,s-s_n}) = 0$  and moreover, for each  $i$ ,  $\deg(H_{|\mathbf{v}|}, \mathbf{u}_i^{[s-s_i]} \cup \cup_k \{v_{ik0}\}) \leq \deg(G, \cup_k \mathbf{v}_{ik})$ . Since  $u_{i0}^{(k)}$  are algebraically independent over the field  $\mathbb{Q}(\xi, \cup_{i=0}^n (\mathbf{u}_i \setminus \{u_{i0}\})^{[s-s_i]})$ ,  $H = H_{|\mathbf{v}|}(\cup_{i=0}^n (\mathbf{u}_i \setminus \{u_{i0}\})^{[s-s_i]}; u_{00}, \dots, u_{00}^{(s-s_0)}, \dots, u_{n0}, \dots, u_{n0}^{(s-s_n)}) \in \mathbb{Q}[\mathbf{u}_0^{[s-s_0]}, \dots, \mathbf{u}_n^{[s-s_n]}]$  is a polynomial satisfying C3) and C4).

From C3),  $H \in (\mathbb{P}_0^{[s-s_0]}, \dots, \mathbb{P}_n^{[s-s_n]})$ . Since  $(\mathbb{P}_0^{[s-s_0]}, \dots, \mathbb{P}_n^{[s-s_n]}) \cap \mathbb{Q}[\mathbf{u}_0^{[s-s_0]}, \dots, \mathbf{u}_n^{[s-s_n]}] = (\mathbf{R})$  and  $\mathbf{R}$  is irreducible,  $\mathbf{R}$  divides  $H$ . It follows that  $\deg(\mathbf{R}, \mathbf{u}_i^{[s-s_i]}) \leq \deg(H, \mathbf{u}_i^{[s-s_i]}) \leq \deg(G, \cup_k \mathbf{v}_{ik}) = \sum_{k=0}^{s-s_i} \deg(G, \mathbf{v}_{ik}) = \sum_{k=0}^{s-s_i} \mathcal{M}((\mathcal{Q}_{jl})_{j \neq i, 0 \leq l \leq s-s_j}, \mathcal{Q}_{i0}, \dots, \mathcal{Q}_{i,k-1}, \mathcal{Q}_{i,k+1}, \dots,$

$\mathcal{Q}_{i,s-s_i})$ . □

As a corollary, we give another degree bound for differential resultant by using Bézout bound, which is better than the bound given in Theorem 6.7 in that only the degrees of  $\mathbb{P}_i$  in  $\mathbb{Y}$  are considered in the bound.

**Corollary 6.19** *Let  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ) be generic differential polynomials in  $\mathbb{Y} = \{y_1, \dots, y_n\}$  with order  $s_i$ , degree  $m_i$  and coefficients  $\mathbf{u}_i$  respectively. Let  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  be the differential resultant of  $\mathbb{P}_0, \dots, \mathbb{P}_n$ . Denote  $s = \sum_{i=0}^n s_i$ . Then for each  $i \in \{0, 1, \dots, n\}$ ,  $\deg(\mathbf{R}, \mathbf{u}_i) \leq \frac{s-s_i+1}{m_i} \prod_{j=0}^n m_j^{s-s_j+1}$ .*

*Proof:* Follow the notations in the proof of Theorem 6.18. Since  $\{\mathcal{B}_{ik} : 0 \leq i \leq n; 0 \leq k \leq s-s_i\}$  is an essential set, for any fixed  $k \in \{0, \dots, s-s_i\}$ , the polynomials in  $S_k$  together generate an ideal of dimension zero in  $\mathbb{Y}^{[s]}$ . By lemma 6.17,  $\mathcal{M}((\mathcal{Q}_{jl})_{j \neq i, 0 \leq l \leq s-s_j}, \mathcal{Q}_{i0}, \dots, \mathcal{Q}_{i,k-1}, \mathcal{Q}_{i,k+1}, \dots, \mathcal{Q}_{i,s-s_i}) \leq \frac{1}{m_i} \prod_{j=0}^n m_j^{s-s_j+1}$ . Hence, by Theorem 6.18,

$$\begin{aligned} \deg(\mathbf{R}, \mathbf{u}_i) &\leq \sum_{k=0}^{s-s_i} \mathcal{M}((\mathcal{Q}_{jl})_{j \neq i, 0 \leq l \leq s-s_j}, \mathcal{Q}_{i0}, \dots, \mathcal{Q}_{i,k-1}, \mathcal{Q}_{i,k+1}, \dots, \mathcal{Q}_{i,s-s_i}) \\ &\leq \sum_{k=0}^{s-s_i} \frac{1}{m_i} \prod_{j=0}^n m_j^{s-s_j+1} = \frac{s-s_i+1}{m_i} \prod_{j=0}^n m_j^{s-s_j+1}. \end{aligned}$$

□

**Example 6.20** *Consider two generic differential polynomials of order one and degree two in one indeterminate  $y$ :*

$$\begin{aligned} \mathbb{P}_0 &= u_{00} + u_{01}y + u_{02}y' + u_{03}y^2 + u_{04}yy' + u_{05}(y')^2, \\ \mathbb{P}_1 &= u_{10} + u_{11}y + u_{12}y' + u_{13}y^2 + u_{14}yy' + u_{15}(y')^2. \end{aligned}$$

*Then the degree bound given by Theorem 6.5 is  $\deg(\mathbf{R}) \leq (2+1)^4 = 81$ . The degree bound given by Corollary 6.19 is  $\deg(\mathbf{R}, \mathbf{u}_0) \leq 2^4 = 16$  and hence  $\deg(\mathbf{R}) \leq 32$ . The degree bound  $\deg(\mathbf{R}, \mathbf{u}_0)$  given by Theorem 6.18 is  $\mathcal{M}(\mathcal{Q}_{10}, \mathcal{Q}_{11}, \mathcal{Q}_{00}) + \mathcal{M}(\mathcal{Q}_{10}, \mathcal{Q}_{11}, \mathcal{Q}_{01}) = 4 + 6 = 10$  and consequently  $\deg(\mathbf{R}) \leq 20$ , where  $\mathcal{Q}_{01} = \mathcal{Q}_{10} = \text{conv}\{(0, 0, 0), (2, 0, 0), (0, 2, 0)\}$ ,  $\mathcal{Q}_{01} = \mathcal{Q}_{11} = \text{conv}\{(0, 0, 0), (2, 0, 0), (0, 2, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\}$ , and  $\text{conv}(\cdot)$  means taking the convex hull in  $\mathbb{R}^3$ .*

We will end this section by giving Algorithm **DResultant** to compute differential resultant based on the degree bound given in Theorem 6.18.

**Theorem 6.21** *Let  $\mathbb{P}_0, \dots, \mathbb{P}_n$  be a generic differential polynomial system of the form (30). Denote  $s = \sum_{i=0}^n \text{ord}(\mathbb{P}_i, \mathbb{Y})$  and  $m = \max_{i=0}^n \deg(\mathbb{P}_i, \mathbb{Y})$ . Algorithm **DResultant** computes the differential resultant  $\mathbf{R}$  of  $\mathbb{P}_0, \dots, \mathbb{P}_n$  with the following complexities:*

- 1) *In terms of  $\deg(\mathbf{R})$ , the algorithm needs at most  $O((ns+n)^{2.376} [m \deg(\mathbf{R})]^{O(\ln(s+1))})$   $\mathbb{Q}$ -arithmetic operations where  $l = \max_{i=0}^n \binom{m_i+n(s_i+1)}{n(s_i+1)}$  is the size of system  $\mathbb{P}_i$ .*



---

**Algorithm 3 — DResultant**( $\mathbb{P}_0, \dots, \mathbb{P}_n$ )

---

**Input:** A generic differential polynomial system  $\mathbb{P}_0, \dots, \mathbb{P}_n$ .

**Output:** The differential resultant  $\mathbf{R}(\mathbf{u}_0, \dots, \mathbf{u}_n)$  of  $\mathbb{P}_0, \dots, \mathbb{P}_n$ .

1. For  $i = 0, \dots, n$ , set  $s_i = \text{ord}(\mathbb{P}_i)$ ,  $m_i = \deg(\mathbb{P}_i, \mathbb{Y})$  and  $\mathbf{u}_i = \text{coeff}(\mathbb{P}_i)$ .
2. Set  $\mathbf{R} = 0$ ,  $s = \sum_{i=0}^n s_i$ ,  $m = \max_i \{m_i\}$ ,  $d = n + 1$ ,  $U = \cup_{i=0}^n \mathbf{u}_i^{[s-s_i]}$ .
3. While  $\mathbf{R} = 0$  do
  - 3.1. Set  $\mathbf{R}_0$  to be a homogenous GPol of degree  $d$  in  $U$ .
  - 3.2. Set  $\mathbf{c}_0 = \text{coeff}(\mathbf{R}_0, U)$ .
  - 3.3. Set  $G_{ik}$  ( $i = 0, \dots, n; k = 0, \dots, s - s_i$ ) to be GPols of degree  $(m + 1)d - m_i - 1$  in  $\mathbb{Y}^{[s]}, U$ .
  - 3.4. Set  $\mathbf{c}_{ik} = \text{coeff}(G_{ik}, \mathbb{Y}^{[s]} \cup U)$ .
  - 3.5. Set  $\mathcal{P}$  to be the set of coefficients of  $\mathbf{R}_0(\mathbf{u}_0, \dots, \mathbf{u}_n) - \sum_{i=0}^n \sum_{k=0}^{s-s_i} G_{ik} \mathbb{P}_i^{(k)}$  as a polynomial in  $\mathbb{Y}^{[s]}, U$ .
  - 3.6. Solve the linear equation  $\mathcal{P} = 0$  in variables  $\mathbf{c}_0$  and  $\mathbf{c}_{ik}$ .
  - 3.7. If  $\mathbf{c}_0$  has a nonzero solution, then substitute it into  $\mathbf{R}_0$  to get  $\mathbf{R}$  and go to Step 4, else  $\mathbf{R} = 0$ .
  - 3.8.  $d := d + 1$ .
4. Return  $\mathbf{R}$ .

/\*/ GPol stands for generic algebraic polynomial.

/\*/  $\text{coeff}(P, U)$  returns the set of coefficients of  $P$  as an ordinary polynomial in variables  $U$ .

---

2) The algorithm needs at most  $O((ns + n)^{2.376} [mD]^{O(\ln(s+1))})$   $\mathbb{Q}$ -arithmetic operations, where  $D$  is the degree bound of  $\mathbf{R}$  given by Theorem 6.18.

*Proof:* The algorithm terminates by Theorem 6.18, and returns a differential polynomial  $P$  in  $(\mathbb{P}_0^{[s-s_0]}, \dots, \mathbb{P}_n^{[s-s_n]}) \cap \mathbb{Q}[\mathbf{u}_0^{[s-s_0]}, \dots, \mathbf{u}_n^{[s-s_n]}]$  with the smallest degree, which is exactly the differential resultant.

We will estimate the complexity of the algorithm below. Denote  $l_i = |\mathbf{u}_i| = \binom{m_i + n(s_i + 1)}{n(s_i + 1)}$  ( $i = 0, \dots, n$ ), and  $l = \max_{i=0}^n l_i$ . So  $|U| = \sum_{i=0}^n l_i (s - s_i + 1)$ . In each loop of Step 3, the complexity of the algorithm is clearly dominated by Step 3.5., where we need to solve a system of linear equations  $\mathcal{P} = 0$  over  $\mathbb{Q}$  in  $\mathbf{c}_0$  and  $\mathbf{c}_{ik}$ . It is easy to show that  $|\mathbf{c}_0| = \binom{d + |U| - 1}{|U| - 1}$  and  $|\mathbf{c}_{ik}| = \binom{(m+1)d - m_i - 1 + |U| + n(s+1)}{|U| + n(s+1)}$ . Then  $\mathcal{P} = 0$  is a linear equation system with  $N = |\mathbf{c}_0| + \sum_{i=0}^n \sum_{k=0}^{s-s_i} |\mathbf{c}_{ik}|$  variables and  $M = \binom{(m+1)d + |U| + n(s+1)}{|U| + n(s+1)}$  equations. To solve it, we need at most  $(\max\{M, N\})^\omega$  arithmetic operations over  $\mathbb{Q}$ , where  $\omega$  is the matrix multiplication exponent and the currently best known  $\omega$  is 2.376.

Step 3 may loop from  $d = n + 1$  to  $\deg(\mathbf{R}) \leq D$ , where  $D$  is the degree bound of  $\deg(\mathbf{R})$  given by Theorem 6.18. The whole algorithm needs at most

$$\begin{aligned}
 \sum_{d=n+1}^{\deg(\mathbf{R})} (\max\{M, N\})^{2.376} &\leq O((ns + n)^{2.376} [m \deg(\mathbf{R})]^{O(\ln(s+1))}) \\
 &\leq O((ns + n)^{2.376} [mD]^{O(\ln(s+1))})
 \end{aligned}$$

arithmetic operations over  $\mathbb{Q}$ . In the above inequalities, we assume that  $(m+1)\deg(\mathbf{R}) \geq l(ns+n+1) + n(s+1)$ , which is generally true. Otherwise,  $m\deg(\mathbf{R})$  need to be replaced by  $lns$  to give a single exponential complexity bound. Our complexity also assumes an  $O(1)$ -complexity cost for all field operations over  $\mathbb{Q}$ . Thus, the complexity follows.  $\square$

**Remark 6.22** *One might suggest to use an approach similar to Algorithm **DResultant** to compute the sparse differential resultant and to obtain a complexity bound similar to that given in Theorem 6.21. In this way, a differential polynomial  $P \in \text{sat}(\mathbf{R})$  will be obtained, which cannot be proved to be  $\mathbf{R}$  due to the reason that  $P$  might have a higher order and a lower degree than that of  $\mathbf{R}$ .*

With Theorem 6.15, Theorem 1.3 is proved.

## 7 Conclusion

In this paper, we first introduce the concepts of Laurent differential polynomials and Laurent differentially essential systems, and give a criterion for Laurent differentially essential systems in terms of their supports. Then the sparse differential resultant for Laurent differentially essential system is defined and its basic properties are proved, such as the differential homogeneity, necessary and sufficient conditions for the existence of solutions, differential toric variety, and the Poisson-type product formulas. Furthermore, order and degree bounds for the sparse differential resultant are given. Based on these bounds, an algorithm to compute the sparse differential resultant is proposed, which is single exponential in terms of the order, the number of variables, and the size of the Laurent differentially essential system.

In the rest of this section, we propose several questions for further study.

It is useful to represent the sparse differential resultant as the quotient of two determinants, as done in [11, 14] in the algebraic case. In the differential case, we do not have such formulas, even in the simplest case of the resultant for two generic differential polynomials in one variable. The treatment in [6] is not complete. For instance, let  $f, g$  be two generic differential polynomials in one variable  $y$  with order one and degree two. Then, the differential resultant for  $f, g$  defined in [6] is zero, because all elements in the first column of the matrix  $M(\delta, n, m)$  in [6, p.543] are zero. Although using the idea of Dixon resultants, the algorithm in [54] does not give a matrix representation for the differential resultant.

From (28), a natural idea to find a matrix representation is trying to define the sparse differential resultant as the algebraic sparse resultant of  $\mathbb{P} = \{\mathbb{P}_i^{(k)} (i = 0, \dots, n, k = 0, \dots, h_i)\}$  considered as Laurent polynomials in  $y_l^{(j)}$ , which will lead to a matrix representation for the sparse differential resultant. As far as we know, this is actually very difficult even in the case of the resultant for two generic differential polynomials in one variable.

The degree of the algebraic sparse resultant is equal to the mixed volume of certain polytopes generated by the supports of the polynomials [40] or [19, p.255]. A similar degree bound is given in Theorem 1.3 for the differential resultant. We conjecture that the bound given in Theorem 1.3 is also a degree bound for the sparse differential resultant.

There exist very efficient algorithms to compute algebraic sparse resultants [13, 14, 15], which are based on matrix representations for the resultant. How to apply the principles

behind these algorithms to compute sparse differential resultants is an important problem. A reasonable goal is to find an algorithm whose complexity depends on  $\deg(\mathbf{R})$ , but not on the worst case bound of  $\deg(\mathbf{R})$  which is the case in Algorithm **SDResultant**.

In the algebraic case, it is shown that the sparse polynomials  $\mathbb{P}_i$  ( $i = 0, \dots, n$ ) can be re-parameterized to a new system  $\mathbb{S}_i$  ( $i = 0, \dots, n$ ) with the help of the Newton polygon associated with  $\mathbb{P}_i$  such that vanishing of the sparse resultant gives a sufficient and necessary condition for  $\mathbb{S}_i$  ( $i = 0, \dots, n$ ) to have solutions in  $\mathbb{C}^N$ , where  $\mathbb{C}$  is the field of complex numbers [10, page 312]. It is interesting to extend this result to the differential case. To do that we need a deeper study of differential toric variety introduced in Section 5.3.

As a less important problem, we guess that assuming the first condition in Theorem 5.32, the second condition  $\mathbf{e}_j \in \text{Span}_{\mathbb{Z}}\{\alpha_{ik} - \alpha_{i0} : k = 1, \dots, l_i; i = 0, \dots, n\}$  is also a necessary condition for the system  $\mathbb{P}_i$  to have a unique solution under the condition of  $\mathbf{R} = 0$  in the generic case.

## References

- [1] L. M. Berkovich and V. G. Tsirulik. Differential Resultants and Some of their Applications. *Differentsial'nye Uravneniya*, 22(5), 750-757, 1986.
- [2] D. N. Bernshtein. The Number of Roots of a System of Equations. *Functional Anal. Appl.*, 9(3), 183-185, 1975.
- [3] A. Buium and P. J. Cassidy. Differential Algebraic Geometry and Differential Algebraic Groups. In H. Bass et al eds, *Selected Works of Ellis Kolchin, with Commentary*, 567-636, American Mathematical Society, Providence, RI, 1998.
- [4] W. D. Brownawell. Bounds for the Degrees in the Nullstellensatz. *The Annals of Mathematics*, 126(3), 577-591, 1987.
- [5] J. F. Canny. Generalized Characteristic Polynomials. *Journal of Symbolic Computation*, 9, 241-250, 1990.
- [6] G. Carrà-Ferro. A Resultant Theory for the Systems of Two Ordinary Algebraic Differential Equations. *Applicable Algebra in Engineering, Communication and Computing*, 8, 539-560, 1997.
- [7] M. Chardin. Differential Resultants and Subresultants. *Fundamentals of Computation Theory*, LNCS, Vol. 529, 180-189, Springer, Berlin, 1991.
- [8] S. C. Chou and X. S. Gao. Automated Reasoning in Differential Geometry and Mechanics: I. An Improved Version of Ritt-Wu's Decomposition Algorithm. *Journal of Automated Reasoning*, 10, 161-172, 1993.
- [9] R. M. Cohn. Order and Dimension. *Proc. Amer. Math. Soc.*, 87(1), 1983.
- [10] D. Cox, J. Little, D. O'Shea. *Using Algebraic Geometry*. Springer, 1998.

- [11] C. D’Andrea. Macaulay Style Formulas for Sparse Resultants. *Trans. of Amer. Math. Soc.*, 354(7), 2595-2629, 2002.
- [12] D. Eisenbud, F. O. Schreyer, and J. Weyman. Resultants and Chow Forms via Exterior Syzygies. *Journal of Amer. Math. Soc.*, 16(3), 537-579, 2004.
- [13] I. Z. Emiris. On the Complexity of Sparse Elimination. *J. Complexity*, 12, 134-166, 1996.
- [14] I. Z. Emiris and J. F. Canny. Efficient Incremental Algorithms for the Sparse Resultant and the Mixed Volume. *Journal of Symbolic Computation*, 20(2), 117-149, 1995.
- [15] I. Z. Emiris and V. Y. Pan. Improved Algorithms for Computing Determinants and Resultants. *Journal of Complexity*, 21, 43-71, 2005.
- [16] X. S. Gao, W. Li, C. M. Yuan. Intersection Theory in Differential Algebraic Geometry: Generic Intersections and the Differential Chow Form. Accepted by *Trans. of Amer. Math. Soc.*, 1-58. Also in arXiv:1009.0148v2.
- [17] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. Generalized Euler Integrals and A-hypergeometric Functions. *Advances in Mathematics*, 84, 255-271, 1990.
- [18] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky. Newton Polytopes of the Classical Resultant and Discriminant, *Advances in Mathematics*, 84(2), 237-254, 1990.
- [19] I. M. Gelfand, M. Kapranov, and A. V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants*. Boston, Birkhäuser, 1994.
- [20] O. Golubitsky, M. Kondratieva, A. Ovchinnikov, A. Szanto. A Bound for Orders in Differential Nullstellensatz. *Journal of Algebra*, 322, 3852-3877, 2009.
- [21] J. Heintz. Definability and Fast Quantifier Elimination in Algebraically Closed Fields. *Theoret. Comput. Sci.*, 24, 239-277, 1983.
- [22] W. V. D. Hodge and D. Pedoe. *Methods of Algebraic Geometry, Volume I*. Cambridge Univ. Press, 1968.
- [23] W. V. D. Hodge and D. Pedoe. *Methods of Algebraic Geometry, Volume II*. Cambridge Univ. Press, 1968.
- [24] H. Hong. Ore Subresultant Coefficients in Solutions. *Applicable Algebra in Engineering, Communication and Computing*, 12(5), 421-428, 2001.
- [25] G. Jeronimo and J. Sabia. On the Number of Sets Definable by Polynomials. *Journal of Algebra*, 227, 633-644, 2000.
- [26] J. P. Jouanolou. Le Formalisme du Résultant. *Advances in Mathematics*, 90(2), 117-263, 1991.
- [27] J. P. Jouanolou. Formes D’inertie et Résultant: un Formulaire. *Advances in Mathematics*, 126(2), 119-250, 1997.

- [28] M. Kapranov, B. Sturmfels, and A Zelevinsky. Chow Polytopes and General Resultants. *Duke Math. J.*, 67, 189-218, 1992.
- [29] E. R. Kolchin. *Differential Algebra and Algebraic Groups*. Academic Press, New York and London, 1973.
- [30] E. R. Kolchin. Extensions of Differential Fields, I. *Annals of Mathematics*, 43 , 724-729, 1942.
- [31] E. R. Kolchin. Extensions of Differential Fields, III. *Bull. Amer. Math. Soc.*, 53 , 397-401, 1947.
- [32] E. R. Kolchin. A Problem on Differential Polynomials. *Contemporary Mathematics*, 131, 449-462, 1992.
- [33] E. R. Kolchin. Differential Equations in a Projective Space and Linear Dependence over a Projective Variety. In *Contributions to Analysis: A Colletion of Papers Dedicated to Lipman Bers*, Academic Press, 195-214, 1974.
- [34] B. A. Lando. Jacobi's Bound for the Order of Systems of First Order Differential Equations. *Trans. Amer. Math. Soc.* 152, 119-135, 1970.
- [35] D. Lazard. Grönber Basis. Gaussian Elimination and Resolution of systems of Algebraic Equations. In *Proc. Eurocal 83*, vol. 162 of *Lect. Notes in Comp. Sci*, 146-157, 1983.
- [36] W. Li and X. S. Gao. Differential Chow Form for Projective Differential Variety. In arXiv:1107.3205v1, 2011.
- [37] W. Li, X. S. Gao, C. M. Yuan. Sparse Differential Resultant. In *Proc. ISSAC 2011, San Jose, CA, USA*, 225-232, ACM Press, New York, 2011.
- [38] Z. Li. A Subresultant Theory for Linear Differential, Linear Difference and Ore Polynomials, with Applications. PhD thesis, Johannes Kepler University, 1996.
- [39] O. Ore. Formale Theorie der Linearen Differentialgleichungen. *Journal für die reine und angewandte Mathematik*, 167, 221-234, 1932.
- [40] P. Pedersen and B. Sturmfels. Product Formulas for Resultants and Chow Forms. *Mathematische Zeitschrift*, 214(1), 377-396, 1993.
- [41] P. Philippon. Critères pour L'indpendance Algbrique. *Inst. Hautes Études Sci. Publ. Math.*, 64, 5-52, 1986.
- [42] J. Renegar. On the Computational Complexity and Geometry of the First-order Theory of the Reals, Part I. *Journal of Symbolic Computation*, 13(3), 255-299, 1992.
- [43] J. F. Ritt. Jacobi's Problem on the Order of a System of Differential Equations. *The Annals of Mathematics, Second Series*, 36(2), 303-312, 1935.
- [44] J. F. Ritt. *Differential Equations from the Algebraic Standpoint*. Amer. Math. Soc., New York, 1932.

- [45] J. F. Ritt. *Differential Algebra*. Amer. Math. Soc., New York, 1950.
- [46] S. L. Rueda and J. R. Sendra. Linear Complete Differential Resultants and the Implicitization of Linear DPPEs. *Journal of Symbolic Computation*, 45(3), 324-341, 2010.
- [47] S. L. Rueda. Linear Sparse Differential Resultant Formulas. *arXiv:1112.3921v2*, 2011.
- [48] W. Y. Sit. The Ritt-Kolchin Theory for Differential Polynomials. In *Differential Algebra and Related Topics*, 1-70, World Scientific, 2002.
- [49] B. Sturmfels. Sparse Elimination Theory. In *Computational Algebraic Geometry and Commutative Algebra*, 264-298, Cambridge University Press, 1993.
- [50] B. Sturmfels. On The Newton Polytope of the Resultant. *Journal of Algebraic Combinatorics*, 3, 207-236, 1994.
- [51] W. Vogel. *Lectures on Results on Bezout's Theorem*. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.
- [52] A. Weil. *Foundations of Algebraic Geometry*. Amer. Math. Soc., New York, 1946.
- [53] W. T. Wu. On the Foundation of Algebraic Differential Polynomial Geometry. *Sys. Sci. & Math. Sci.*, 2(4), 289-312, 1989.
- [54] L. Yang, Z. Zeng, W. Zhang. Differential Elimination with Dixon Resultants. Accepted by *Applied Mathematics and Computation*, <http://dx.doi.org/10.1016/j.amc.2012.04.036>, 2012.
- [55] D. Zwillinger. *Handbook of Differential Equations*. Academic Press, San Diego, USA, 1998.