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(Citation)
ACM Communications in Computer Algebra,46(3-4):116-117

(Issue Date)
2012-09

(Resource Type)
conference paper

(Version)
Accepted Manuscript

Backward error analysis of approximate Gröbner basis

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1 Introduction

There are many algorithms for computing approximate Gröbner basis and they can be thought as just one of symbolic-numeric computations for polynomials (see [1] for some note from the methological point of view). However, there is a big difference between approximate Gröbner basis and others that the backward error analyses are naturally given or not (i.e. easy or not). For example, we consider an approximate factorization of the following irreducible polynomial.

$$\tilde{f}(x,y,z) = 81x^4 + 72x^2y^2 + \frac{3}{1292}x^2z^2 - 648x^2 + 16y^4 + \frac{1}{969}y^2z^2 - 288y^2 - \frac{837227}{1292}z^4 - \frac{3}{323}z^2 + 1296.$$

We can have the following factorization $\tilde{f}_1(x,y,z)\tilde{f}_2(x,y,z)$ with tolerance $\varepsilon = 4.54478 \times 10^{-6}$.

$$\tilde{f}_1(x,y,z) = 9.000x^2 + 4.000y^2 - 25.46z^2 - 36.00,\ \tilde{f}_2(x,y,z) = 9.000x^2 + 4.000y^2 + 25.46z^2 - 36.00.$$

The resulting approximate factorization $\tilde{f}_1(x,y,z)\tilde{f}_2(x,y,z)$ can be characterized as the factorization of the following polynomial in the exact sense, by rationalizing the coefficients and multiplying them.

$$f(x,y,z) = 81x^4 + 72x^2y^2 - 648x^2 + 16y^4 - 288y^2 - \frac{78400}{121}z^4 + 1296.$$

In contrast, approximate Gröbner basis does not have this behavior in general. For example, the following $\tilde{G}_{app}$ is the approximate Gröbner basis of the input $\tilde{F}_{app}$ w.r.t. the graded lexicographic order ($x \succ y$), computed by Mathematica (we note that other algorithms also have similar behaviors).

$$\begin{aligned}\tilde{F}_{app} &= \{\tilde{f}_1(\vec{x}) = 0.01084x^3y + 0.891x^3,\ \tilde{f}_2(\vec{x}) = 0.503xy^3 + 0.1129x + 0.02201\},\\ \tilde{G}_{app} &= \{1.0xy^3 + 0.224453x + 0.0437575,\ 1.0x^3 - 7.87965 \times 10^{-8}x^2\}.\end{aligned}$$

However, we have the following result if we compute a Gröbner basis of $\tilde{G}_{app}$ with the rationalized coefficients in the exact sense (we show it in floating-point numbers due to the narrow paper width).

$$G_{app} \approx \{1.23120 \times 10^{30}y^3 + 6.83710 \times 10^{35},\ 3.12500 \times 10^{21}x - 2.46239 \times 10^{14}\}.$$

Moreover, the following G_{ex} is a Gröbner basis of the ideal generated by $\tilde{F}_{app}$ with the rationalized coefficients, which is different from G_{app} and $\tilde{G}_{app}$. We note that the resulting G_{ex} may not the basis we want since the input system may have a priori errors on their coefficients and supports.

$$G_{ex} \approx \{5.55931 \times 10^{17}x - 4.38054 \times 10^{10},\ 271.000y + 22275.0\}.$$

Therefore, the resulting approximate Gröbner basis is not a Gröbner basis and does not generate the given ideal in the exact sense. This behavior is common for algorithms computing approximate Gröbner basis hence we have a very natural question: "What is that we computed?" In this poster, we introduce a proof of concept method to find an exact result from those approximate Gröbner bases over the set of floating-point numbers $\mathbb{F}$, so we can have a backward error analysis. We note that this poster does not introduce any newly defined approximate Gröbner basis. Our aim is just only finding an answer for the above question across several definitions and methods from the exact point of view.

2 Nearby Gröbner basis and system

Problem 1 *For the given $\tilde{F}_{app}, \tilde{G}_{app} \subset \mathbb{F}[\vec{x}]$, compute $F_{cl}, G_{cl} \subset \mathbb{R}[\vec{x}]$ such that G_{cl} is a Gröbner basis of* ideal(F_{cl}) *in the exact sense and F_{cl} and G_{cl} are close to $\tilde{F}_{app}$ and $\tilde{G}_{app}$, respectively.* $\triangleleft$

For this problem we propose the following method: 1) find constraints w.r.t. a close enough exact Gröbner basis G_{cl} of itself to the given approximate Gröbner basis $\tilde{G}_{app}$, 2) solve least squares w.r.t. a close enough system F_{cl} that is a subset of the ideal generated by the resulting exact Gröbner basis G_{cl}, and 3) solve the minimization problem w.r.t. a close enough system F_{cl}.

We show an example of this method for the given $\tilde{F}_{app}$ and $\tilde{G}_{app}$ in the previous section. At first, we construct a set of parametric polynomials:

$$G_{par} = \{g_1(\vec{x}) = xy^3 + p_{12}x + p_{13},\ g_2(\vec{x}) = x^3 + p_{22}x^2\} \subset \mathbb{R}[\vec{p}][\vec{x}].$$

The corresponding constraints ($p_{13} = 0,\ p_{13}p_{22} = 0$) are computed by the monomial reduction:

$$\mathrm{Spoly}(g_1, g_2) = -p_{22}x^2y^3 + p_{12}x^3 + p_{13}x^2 \Longrightarrow \overline{\mathrm{Spoly}(g_1, g_2)}^{G_{par}} = p_{13}x^2 + p_{13}p_{22}x = 0.$$

If we assume that we have $\mathrm{supp}(f_1(\vec{x})) = \{x^3y,\ x^3,\ x^2y,\ x^2\}$ and $\mathrm{supp}(f_2(\vec{x})) = \{xy^3,\ x,\ 1\}$ for $F_{cl} = \{f_1(\vec{x}) = \sum_i h_{1i}(\vec{x})g_i(\vec{x}),\ f_2(\vec{x}) = \sum_i h_{2i}(\vec{x})g_i(\vec{x})\}$. The problem becomes the least squares:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ p_{22} & 0 \\ 0 & p_{22} \end{pmatrix} \overrightarrow{h_1} = \begin{pmatrix} \frac{271}{25000} \\ \frac{891}{1000} \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 \\ p_{12} \\ p_{13} \end{pmatrix} \overrightarrow{h_2} = \begin{pmatrix} \frac{503}{1000} \\ \frac{1129}{10000} \\ \frac{2201}{100000} \end{pmatrix},\ \overrightarrow{h_1} \in \mathbb{R}(\vec{p})^2,\ \overrightarrow{h_2} \in \mathbb{R}(\vec{p})^1.$$

This can be solved by the exact arithmetic (e.g. LUP, LSP or LQUP matrix decompositions) and we have the following F_{cl}.

$$F_{cl} = \{\frac{271x^3y + 22275x^3 + 271p_{22}x^2y + 22275p_{22}x^2}{25000(p_{22}^2+1)},$$
$$\frac{(11290p_{12}+2201p_{13}+50300)xy^3 + (11290p_{12}^2+2201p_{12}p_{13}+50300p_{12})x + (11290p_{12}p_{13}+2201p_{13}^2+50300p_{13})}{100000(p_{12}^2+p_{13}^2+1)}\} \subset \mathbb{R}(\vec{p})[\vec{x}].$$

To get the values of parameters (p_{12}, p_{13}, p_{22}), we solve the following minimization problem.

$$\underset{\vec{p}}{\text{minimize}} \sum_{i=1}^{m} \|f_i(\vec{x}) - \tilde{f}_i(\vec{x})\| \quad \text{subject to } p_{13} = 0 \text{ and } p_{13}p_{22} = 0.$$

Finally, we get the following result which makes us to be able to evaluate the backward error.

$$G_{cl} = \{xy^3 + \frac{1129}{5030}x,\ x^3\},\ F_{cl} = \{\frac{271}{25000}x^3y + \frac{891}{1000}x^3,\ \frac{503}{1000}xy^3 + \frac{1129}{10000}x\}.$$

We note that the author wishes to thank Prof. Kaltofen and Lichtblau for their comments at Hybrid2011 which led me to the problem and this work was supported in part by Japanese MEXT KAKENHI (22700011). The analyses for other algorithms will be given in the poster presentation at ISSAC 2012, computed by the following our preliminary implementation on Mathematica.
(URL: http://wwwmain.h.kobe-u.ac.jp/˜nagasaka/research/snap/issac12.nb)

References

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