A Tight Bound for Black and White Pebbles on the Pyramid

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Abstract. Lengauer and Tarjan proved that the number of black and white pebbles needed to pebble the root of a tree is at least half the number of black pebbles needed to pebble the root. This result is extended to a larger class of acyclic directed graphs including pyramid graphs.

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General Terms: Theory

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1. *Introduction*

Pebbling acyclic directed graphs is a technique that can be used to model space and time requirements of straight-line programs. It has many applications in computer science, including comparisons of programming languages, code generation and optimization for compilers, time–space trade-offs for a wide variety of problems, and as a simulation tool in relating complexity classes. A detailed survey of pebbling results is contained in Pippenger [6]. The concern of this paper is to compare the complexity of two variations of the pebble game, black&white versus black, which correspond, respectively, to the space requirements of nondeterministic and deterministic evaluations of straight-line programs.

Let G be an acyclic directed graph. We call a vertex of G an *input vertex* or simply an *input* if it has no incoming edges. All other vertices are referred to as *noninput vertices.* A vertex x is said to be an *immediate predecessor* of a vertex y if there is an edge from x to y. Furthermore, x is a *predecessor* of y if there is a path from x to y. Finally, a *proper predecessor* of y is any predecessor of y other than y itself. The black&white pebble game is played on an acyclic directed graph according to the following set of rules:

(1) A black pebble may be removed at any time.
(2) A black pebble may be placed on a vertex if all of its immediate predecessors have pebbles.
(3) If all the immediate predecessors of a vertex have pebbles, a black pebble may be slid from one of the immediate predecessors onto the vertex.

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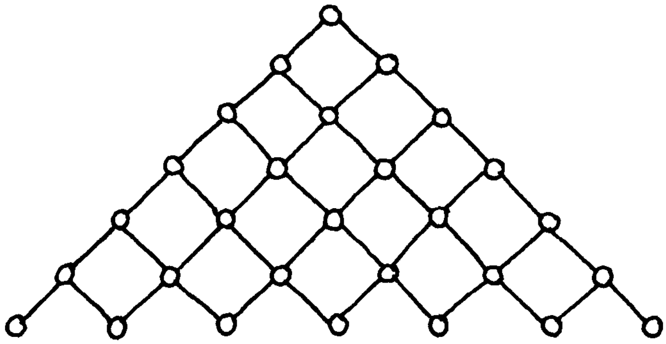


FIG. 1. The pyramid of height 7.

(4) A white pebble may be placed on any vertex at any time.
(5) A white pebble may be removed from a vertex if all of its immediate predecessors have pebbles.
(6) A white pebble may be slid from a vertex onto one of its immediate predecessors if all its other immediate predecessors have pebbles.

A *configuration* (B, W) on a graph G is a pair of disjoint (possibly empty) subsets of the vertices of G, where B and W represent the vertices containing black and white pebbles, respectively. A *black&white strategy* is a finite sequence of configurations, $(B_0, W_0), \ldots, (B_n, W_n)$, such that $(B_0, W_0) = (\varnothing, \varnothing)$, and (B_{i+1}, W_{i+1}) is the result of applying one of the rules (1)–(6) to (B_i, W_i) for each $i < n$. Moreover we say that a strategy *achieves* (B, W) if (B, W) is the last configuration in the sequence. The *space requirement* of a strategy is defined to be the maximum number of pebbles in any configuration in its sequence. The *black&white pebble number* of a vertex is the minimum of the space requirements of strategies that end with the configuration having a single black pebble on that vertex and no other pebbles on the graph. We denote the black&white pebble number of a vertex x by $b\&w(x)$. (This should really be $b\&w(G, x)$ where G is the acyclic directed graph, but we suppress the G for simplicity of notation.) Similarly, the black&white pebble number of a configuration is the minimum of the space requirements of black&white strategies that achieve that configuration.

Analogous definitions are made for the black pebble game, which is played according to rules (1), (2), and (3) above. In particular, we denote the black pebble number of a vertex x by $b(x)$. We always have $b\&w(x) \le b(x)$. Moreover, it is easy to find examples where equality holds (e.g., if G is a directed path). Thus we are interested in proving lower bounds on $b\&w(x)$ in terms of $b(x)$.

A pyramid graph of height m has m levels containing $m, m-1, \ldots, 2, 1$ vertices, respectively, arranged so that each noninput vertex has incoming edges from the two vertices immediately on its left and right in the level below. A pyramid of height 7 is shown in Figure 1. (In this paper our figures always follow the convention that edges are directed upward.) Another way of visualizing pyramid graphs is as triangular fragments of directed two-dimensional rectilinear lattices.

In [1] Cook proved that the black pebble number of the apex of a pyramid of height m is m. Roughly speaking, the purpose of this result was to provide "evidence" that there is a problem that can be solved in polynomial time, but not polylog space. To strengthen this "evidence," Cook and Sethi [2] introduced the black&white pebble game and proved that the black&white pebble number of the apex of a pyramid of height m is at least $\sqrt{m/2} - 1$; thus for x a vertex of a pyramid graph, $b\&w(x) \geq \sqrt{b(x)/2} - 1$. They also conjectured that for pyramid graphs, in fact $b\&w(x) = \Omega(b(x))$.

The first (and only) result bounding $b\&w(x)$ in terms of $b(x)$ for vertices of arbitrary graphs was proved by Meyer auf der Heide [5], who showed that $b\&w(x) \geq \frac{1}{2} + \sqrt{2b(x) - 7/4}$. This can be viewed as a pebbling analog of Savitch's theorem that NSPACE(s) $\subset$ DSPACE(s^2). In the same paper Meyer auf der Heide also proved that for vertices of pyramids, $b\&w(x) \leq \lfloor b(x)/2 \rfloor + 2$. For vertices of trees it can be proved that $b\&w(x) \geq \lfloor b(x)/2 \rfloor + 1$ (this was proved for complete trees by Loui [4] and Meyer auf der Heide [5] independently, and for arbitrary trees by Lengauer and Tarjan [3]). In their paper Lengauer and Tarjan mention pyramid graphs as the natural class of graphs to which their result might be extended.

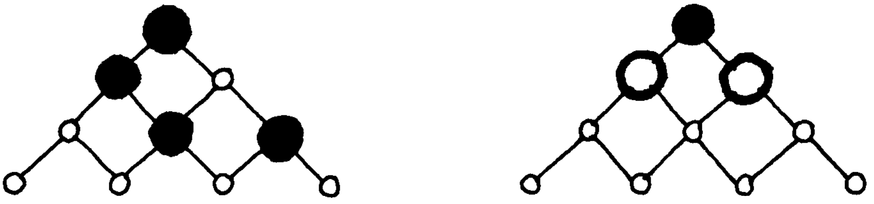
In this paper we extend this lower bound, $b\&w(x) \geq \lfloor b(x)/2 \rfloor + 1$, to spreading graphs, a fairly broad class of graphs that includes pyramid graphs and several natural generalizations. The basic idea is to define a cost function for black and white pebble configurations and then to prove that for spreading graphs this cost function is a lower bound for the black&white pebble number of the configuration. The lower bound on $b\&w(x)$ for vertices of spreading graphs follows almost immediately from this result. Since the definition of spreading graph is somewhat complicated, we leave this till Section 3. Instead, in Section 2, we define the cost function, state a property of spreading graphs, use this property to prove that the cost function is indeed a lower bound and finally obtain the lower bound on the black&white pebble number of vertices of spreading graphs. In Section 3 we prove that spreading graphs possess the property used in Section 2. The final task is to show that pyramids and their generalizations are spreading graphs. Since the definition of spreading graph makes it generally difficult to determine whether a graph is a spreading graph, we introduce nice graphs in Section 4 and prove that they are a subclass of spreading graphs. Since it is very easy to verify that pyramids and their generalizations are nice, this provides a unified method of verifying that a variety of graphs are spreading graphs.

2. *The Lower Bound*

This section highlights the basic structure of the proof, leaving most of the technical details for later. We now introduce terminology and notation that we use to define a cost function for pebble configurations.

We say that a set of vertices Y *blocks* a vertex x if, for every path L from an input to x, we have $Y \cap L \neq \emptyset$. Moreover we say that Y blocks a set of vertices X if Y blocks every vertex in X. It is easily verified that blocking is a transitive property; that is, if Z blocks Y and Y blocks X, then Z blocks X. Thus, for example, if Y blocks every immediate predecessor of a noninput vertex x, then Y blocks x itself. For each vertex set Y we define its measure $m(Y)$ as $m(Y) = \max\{j + 2 \mid Y[j] \mid : Y[j] \neq \emptyset\}$, where for each positive integer j we define $Y[j] = \{x \in Y : b(x) \geq j\}$. Now for each configuration (B, W) we define its cost $c(B, W)$ as $c(B, W) = \min\{\lfloor m(Y)/2 \rfloor : Y \cup W \text{ blocks } B\}$.

We now attempt to provide some intuition as to why $c(B, W)$ is a reasonable candidate for a lower bound on the black&white pebble number of (B, W). Consider the following pebbling strategy for (B, W). First use an optimal black&white pebble strategy to place a black pebble on the vertex in B with the highest black&white pebble number. Leaving that pebble on its vertex, now use an optimal black&white pebble strategy to black pebble the vertex in B with the second highest black&white



(a) (b)

FIG. 2. Example in which the independent pebble strategy is not optimal.

pebble number. Continue to black pebble the vertices in B in decreasing order of black&white pebble number until all have black pebbles, and then place white pebbles on the vertices in W. Ignoring the last part of the strategy in which the vertices in W are pebbled, the number of pebbles needed by this strategy is $\max\{k + |\{x \in B: b\&w(x) \geq k\}| - 1 : \{x \in B: b\&w(x) \geq k\} \neq \varnothing\}$. If $b\&w(x) \geq \lfloor b(x)/2 \rfloor + 1$ for each x in B, then $\lfloor m(B)/2 \rfloor$ is a lower bound for this number. This strategy is unlikely to be optimal for two reasons. First of all, it pebbles each of the black pebbles independently, ignoring the fact that it may be easier to pebble them jointly than separately. Second, it ignores the fact that the presence of the white pebbles on W might make the placing of the black pebbles easier. Examples of these two cases are shown in Figure 2. The definition of $c(B, W)$ avoids these problems by using the blocking concept to simulate the "joint" pebbling of vertices and by allowing the white pebbles to help in the blocking of B.

We now state the property of spreading graphs, which is needed to prove our main result.

Property 2.1 (*proved in Theorem* 3.7). If (B, W) is a configuration on a spreading graph, then there is a vertex set Y such that $Y \cup W$ blocks B, $c(B, W) = \lfloor m(Y)/2 \rfloor$, and either $Y = B$ or $|Y| < |B| + |W|$.

Using Property 2.1 we now prove our main theorem, which, combined with Property 2.1, yields the desired lower bound on $b\&w(x)$ as a corollary.

THEOREM 2.2. *If $(B_0, W_0), (B_1, W_1), \ldots, (B_n, W_n)$ is a black&white pebble strategy on a spreading graph, then for each i we have $c(B_i, W_i) \leq \max_{0 \leq j \leq i}(|B_j| + |W_j|)$.*

PROOF. The proof is by induction on i. It clearly holds for $i = 0$, so assume that $i \geq 1$ and that $c(B_{i-1}, W_{i-1}) \leq \max_{0 \leq j \leq i-1}(|B_j| + |W_j|)$. The remainder of the proof depends on what move led from (B_{i-1}, W_{i-1}) to (B_i, W_i). Let us assume that the move did not consist of placing a black pebble on an input, or of removing a white pebble from an input. In this case we shall see that $c(B_i, W_i) \leq c(B_{i-1}, W_{i-1})$, which clearly completes the proof. To see this, let Y be any set such that $Y \cup W_{i-1}$ blocks B_{i-1}. Obviously it suffices to show that $Y \cup W_i$ blocks B_i. This is immediate from the definitions and the transitivity of blocking, except perhaps for the case that the move consists of the removal or sliding of a white pebble. Let w be the vertex from which the white pebble was removed or slid, and let L be any path from an input to a vertex in B_i (note that $B_i = B_{i-1}$). We show that $L \cap (Y \cup W_i) \neq \varnothing$. Since $L \cap (Y \cup W_{i-1}) \neq \varnothing$ and $W_{i-1} \subset W_i \cup \{w\}$, it suffices to consider the case that $L \cap (Y \cup W_{i-1}) = \{w\}$. Let L' be the subpath of L from the input to w. Now since w is blocked by $B_i \cup W_i$, we have $L' \cap (B_i \cup W_i) \neq \varnothing$. Suppose there is some vertex x in $L' \cap B_i$. Then since x is blocked by $Y \cup$

$W_{i-1}\backslash\{w\}$, we have $L' \cap (Y \cup W_{i-1}\backslash\{w\}) \neq \emptyset$, which contradicts $L \cap (Y \cup W_{i-1}) = \{w\}$. Thus $L' \cap W_i \neq \emptyset$ and hence $L \cap W_i \neq \emptyset$, which completes the proof that $Y \cup W_i$ blocks B_i.

Now suppose that (B_i, W_i) is obtained from (B_{i-1}, W_{i-1}) by placing a black pebble on an input vertex y. By Property 2.1 there exists a set Y such that $c(B_{i-1}, W_{i-1}) = \lfloor m(Y)/2 \rfloor$, $Y \cup W_{i-1}$ blocks B_{i-1}, and $|Y| \leq |B_{i-1}| + |W_{i-1}|$. Let $Y' = Y \cup \{y\}$. Clearly $Y' \cup W_i$ blocks B_i, so by the induction hypothesis our proof will be complete if we can show that $\lfloor m(Y')/2 \rfloor \leq \max\{\lfloor m(Y)/2 \rfloor, |B_i| + |W_i|\}$. We have $Y'[j] = Y[j]$ for each $j > 1$ and $Y'[1] = Y \cup \{y\}$, so $m(Y') = \max\{m(Y), 1 + 2|Y'[1]|\} = \max\{m(Y), 3 + 2|Y|\} \leq \max\{m(Y), 3 + 2(|B_{i-1}| + |W_{i-1}|)\}$. Thus $\lfloor m(Y')/2 \rfloor \leq \max\{\lfloor m(Y)/2 \rfloor, 1 + |B_{i-1}| + |W_{i-1}|\} = \max\{\lfloor m(Y)/2 \rfloor, |B_i| + |W_i|\}$ as desired.

Finally suppose that (B_i, W_i) is obtained from (B_{i-1}, W_{i-1}) by removing a white pebble from an input vertex y. By Property 2.1 there exists a set Y such that $c(B_{i-1}, W_{i-1}) = \lfloor m(Y)/2 \rfloor$, $Y \cup W_{i-1}$ blocks B_{i-1}, and either $Y = B_{i-1}$ or $|Y| < |B_{i-1}| + |W_{i-1}|$. Since $B_i = B_{i-1}$, if $Y = B_{i-1}$ we obviously have $c(B_i, W_i) \leq \lfloor m(Y)/2 \rfloor = c(B_{i-1}, W_{i-1})$ and we are done. On the other hand if $|Y| < |B_{i-1}| + |W_{i-1}|$, let $Y' = Y \cup \{y\}$. Clearly $Y' \cup W_i = Y \cup W_{i-1}$, which blocks B_i. Since $Y'[1] = Y[1] \cup \{y\}$ and $Y'[j] = Y[j]$ for $j > 1$, $\lfloor m(Y')/2 \rfloor \leq \max\{\lfloor m(Y)/2 \rfloor, 1 + |Y|\} \leq \max\{c(B_{i-1}, W_{i-1}), |B_{i-1}| + |W_{i-1}|)$. Thus $c(B_i, W_i) \leq \max\{c(B_{i-1}, W_{i-1}), |B_{i-1}| + |W_{i-1}|)$, completing the proof. □

COROLLARY 2.3. *If x is a vertex of a spreading graph then $b\&w(x) \geq \lfloor b(x)/2 \rfloor + 1$.*

PROOF. By Theorem 2.2 we have $b\&w(x) \geq c(\{x\}, \emptyset)$ so it suffices to show that $c(\{x\}, \emptyset) \geq \lfloor b(x)/2 \rfloor + 1$. By Property 2.1 we can find a set Y such that Y blocks x, $c(\{x\}, \emptyset) = \lfloor m(Y)/2 \rfloor$ and either $Y = \{x\}$ or $|Y| < 1$. Obviously, $|Y| < 1$ is impossible as no vertex can be blocked by the empty set, so we have $c(\{x\}, \emptyset) = \lfloor m(\{x\})/2 \rfloor$. Finally, it is easy to see that for any vertex we have $m(\{x\}) = b(x) + 2$, which completes the proof. □

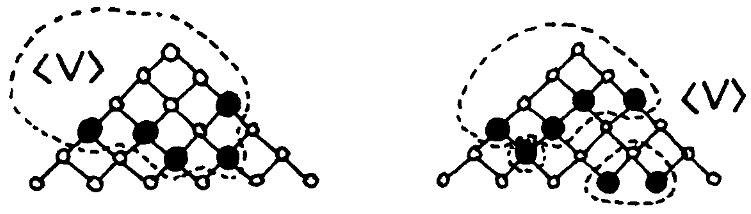
3. *Spreading Graphs*

We start by establishing some facts about minimal blocking sets that lead to a concept of connectedness for minimal blocking sets. Spreading graphs are then defined as graphs whose connected sets have a certain property. To prove property 2.1, we must show that, if Y has minimal cardinality such that $Y \cup W$ blocks B and $c(B, W) = \lfloor m(Y)/2 \rfloor$, then either $Y = B$ or $|Y| < |B| + |W|$. We first prove a somewhat stronger result for the case that $Y \cup W$ is connected, and then obtain the general result by applying the stronger version to the connected components of $Y \cup W$.

We call a set V of vertices *tight* if, for every vertex v in V, the set $V\backslash\{v\}$ does not block v. We use $\langle V \rangle$ to denote the set of vertices that are blocked by V, and, if V is tight, for each x in $\langle V \rangle$ we define $V_x = \{v \in V$: there is a path L from an input to x such that $L \cap V = \{v\}\}$. Note that since V is tight, for each $v \in V$ we have $V_v = \{v\}$.

LEMMA 3.1. *If V is tight and $x \in \langle V \rangle$, then V_x blocks x and V_x is contained in every subset of V that blocks x.*

PROOF. It is obvious that V_x must be contained in any subset of V that blocks x since, for each v in V_x, there is a path L from an input to x such that v is the



(a) (b)

FIG. 3. Tight sets on the pyramid. (a) Connected. (b) Disconnected.

only element of V on L. Now suppose that V_x does not block x, and let L be a path from an input to x such that $L \cap V_x = \varnothing$. Let v be the vertex in $L \cap V$ that is closest to x on L. Since $V \backslash \{v\}$ does not block v, there is a path M from an input to v such that $V \cap M = \{v\}$. Now let L' be the path from an input to x, which begins with M and then continues along L from v to x. Now $L' \cap V = \{v\}$ so $v \in V_x$, a contradiction. □

LEMMA 3.2. *For any set U there is a tight subset V of U such that $\langle V \rangle = \langle U \rangle$.*

PROOF. Let V be a minimal subset of U such that $\langle V \rangle = \langle U \rangle$. It is easy to see that V is tight, since if any vertex v of V is blocked by $V \backslash \{v\}$, then $\langle V \backslash \{v\} \rangle = \langle V \rangle = \langle U \rangle$. □

For each vertex x we use $P(x)$ to denote its set of immediate predecessors and $P^*(x)$ to denote its set of predecessors. A set V is said to be *connected* if it is tight, and if the graph with vertices $\langle V \rangle$ and edges $\{(x, y): x, y \in \langle V \rangle$ and $P^*(x) \cap \langle V_x \rangle \cap P^*(y) \cap \langle V_y \rangle \neq \varnothing\}$ is a connected graph. Figure 3 shows examples of connected and disconnected tight sets on the pyramid.

LEMMA 3.3. *Let V be a tight set and let U be a connected component of the graph on $\langle V \rangle$ defined previously. Then $U = \langle V \cap U \rangle$.*

PROOF. We first show that $U \subset \langle V \cap U \rangle$. For each x in $\langle V \rangle$ and y in V_x, the edge (x, y) is in the graph on $\langle V \rangle$ since $y \in P^*(x) \cap V_x \cap P^*(y) \cap V_y$. Thus, if $x \in U$, then $V_x \subset U$, and hence by Lemma 3.1, $V \cap U$ blocks x, so $x \in \langle V \cap U \rangle$. We now show that $\langle V \cap U \rangle \subset U$. If $x \in \langle V \cap U \rangle$, then by Lemma 3.1 we have $V_x \subset V \cap U$. Now since x and V_x must be in the same component of $\langle V \rangle$ by the previous argument, x is in U also. □

For any set X of vertices we define $b_j(X) = \min\{|Y| : Y$ blocks $X[j]$ and $Y = Y[j]\}$. In other words, $b_j(X)$ is the size of any smallest set Y with the following two properties:

(1) Every vertex in Y has black pebble number at least j.
(2) Y blocks every vertex in X that has black pebble number at least j.

A graph G is called a *spreading graph* if, for every connected set V of G and $j \geq 0$ such that $\langle V \rangle [j] \neq \varnothing$, we have $|V| \geq b_j(\langle V \rangle) + j - \min\{b(v): v \in V\}$. The idea behind this definition is to ensure that if a connected set containing a vertex with a small black pebble number blocks a vertex with a high black pebble number, then the connected set must have a lot of vertices.

The general idea behind the proof of Property 2.1 is as follows. We can use the definition of spreading graph to prove that, if $Y \cup W$ blocks B and $Y \cup W$ is connected, then there is some set Y' such that $Y' \cup W$ blocks B, $m(Y') \leq m(Y)$, and either $Y' = B$ or $|Y'| < |B| + |W|$. Now, if X is a set such that $X \cup W$ blocks B and $c(B, W) = \lfloor m(X)/2 \rfloor$, we may assume by Lemma 3.2 that $X \cup W$ is tight. By applying the statement above to each connected component of $\langle X \cup W \rangle$, we would expect to obtain a set X' with $m(X') \leq m(X)$ and the other desired properties. Note that, since $m(X) = c(B, W)$, this would imply $m(X') = c(B, W)$ also, and we would be done. Unfortunately, this does not work because the function $m(X)$ does not behave nicely with respect to disjoint unions. In fact, it is easy to find disjoint sets of vertices X_1, X_2, and X_3 such that $m(X_1) < m(X_2)$ but $m(X_1 \cup X_3) > m(X_2 \cup X_3)$. For example, let w, x, y, z be distinct vertices with $b(w) = 4$ and $b(x) = b(y) = b(z) = 1$. Then, taking $X_1 = \{x, y\}$, $X_2 = \{w\}$, and $X_3 = \{z\}$, we have $m(X_1) = 5$, $m(X_2) = 6$, $m(X_1 \cup X_3) = 7$, and $m(X_2 \cup X_3) = 6$. This difficulty leads us to the $\ll$ relationship between sets, which characterizes one situation in which $m(X)$ behaves nicely with respect to disjoint union.

We say that $X \ll Y$ if for each j there is some $i \leq j$ such that $j + 2|X[j]| \leq i + 2|Y[i]|$.

LEMMA 3.4. *If $X \cap Z = Y \cap Z = \varnothing$ and $X \ll Y$ then $m(X \cup Z) \leq m(Y \cup Z)$.*

PROOF. It suffices to show that for each j such that $(X \cup Z)[j] \neq \varnothing$, there is some i with $(Y \cup Z)[i] \neq \varnothing$ such that $j + 2|(X \cup Z)[j]| \leq i + 2|(Y \cup Z)[i]|$. Since $X \ll Y$, there exists $i \leq j$ such that $j + 2|X[j]| \leq i + 2|Y[i]|$, and, since $i \leq j$, $|Z[j]| \leq |Z[i]|$ also. Now $j + 2|(X \cup Z)[j]| = j + 2|X[j]| + 2|Z[j]| \leq i + 2|Y[i]| + 2|Z[i]| = i + 2|(Y \cup Z)[i]|$. □

Before proving the stronger version of Property 2.1 for connected blocking sets, we first prove a technical lemma.

LEMMA 3.5. *For every set B we can find a set B' that blocks B such that $|B'[j]| \leq b_j(B)$ for each j, and either $B' = B$ or $|B'| < |B|$.*

PROOF. If $|B[j]| \leq b_j(B)$ for each j with $B[j] \neq \varnothing$, we may take $B' = B$, so let i be minimal such that $|B[i]| > b_i(B)$. Choose a set Y such that Y blocks $B[i]$, $Y = Y[i]$, and $|Y| = b_i(B)$. We now define $B' = Y \cup (B \setminus B[i])$. For $j < i$, we have $|B'[j]| < |B[j]| \leq b_j(B)$ and also $|B'[i]| = |Y| = b_i(B)$. For $j > i$ we have $B'[j] = Y[j]$, so it suffices to prove that, if $j > i$, then $|Y[j]| \leq b_j(B)$. Suppose not and let Y' be a set that blocks $B[j]$ with $Y'[j] = Y'$ and $|Y'| = b_j(B) < |Y[j]|$. We first note that $Y' \cup (Y \setminus Y[j])$ blocks $B[i]$, since, if $x \in B[j]$, then x is blocked by Y'. On the other hand, if $x \in B[i] \setminus B[j]$, then x is blocked by Y, but as $b(x) < j$, we see that x must be blocked by $Y \setminus Y[j]$. Next we observe that $(Y' \cup (Y \setminus Y[j]))[i] = Y' \cup (Y \setminus Y[j])$ since the black pebble number of every vertex in Y' is at least j and $j > i$. However, these observations contradict $|Y| = b_i(B)$, since $|Y'[j]| < |Y[j]|$ implies $|Y' \cup (Y \setminus Y[j])| < |Y|$. □

LEMMA 3.6. *Suppose B, W, and Y are vertex subsets of a spreading graph such that $Y \cup W$ blocks B and $Y \cup W$ is connected. Then there is a set Y' such that $Y' \cup W$ blocks B with $Y' \ll Y$ and either $Y' = B$ or $|Y'| < |B| + |W|$.*

PROOF. We may assume that $|Y| \geq |B| + |W|$, since otherwise we may take Y to be Y'. Let Y' be the B' of Lemma 3.5, that is, Y' blocks B, $|Y'[j]| \leq b_j(B)$ for each j, and either $Y' = B$ or $|Y'| < |B| + |W|$. We must show that $Y' \ll Y$. Let $h = \min\{b(x) : x \in Y \cup W\}$. First suppose that $j \leq h$. Then $j + 2|Y'[j]| \leq j + 2|Y'| \leq j + 2|B| \leq j + 2|Y| = j + 2|Y[j]|$. Now suppose that $j > h$. Note that, since $B \subset \langle Y \cup W \rangle$, we have $b_j(B) \leq b_j(\langle Y \cup W \rangle)$ for each j, and thus, since $Y \cup W$ is a connected subset of the vertices of a spreading graph, we have $j +$

$b_j(B) \leq h + |Y \cup W|$. Thus $j + 2|Y'[j]| \leq j + 2b_j(B) \leq j + b_j(B) + |B| \leq h + |Y \cup W| + |B| \leq h + |Y| + |W| + |B| \leq h + 2|Y| = h + 2|Y[h]|$. □

We are finally ready to complete the proof of Property 2.1.

THEOREM 3.7. *If (B, W) is a configuration on a spreading graph, then there is a vertex set Y such that $Y \cup W$ blocks B, $c(B, W) = \lfloor m(Y)/2 \rfloor$, and either $Y = B$ or $|Y| < |B| + |W|$.*

PROOF. Let Y be a vertex subset such that $Y \cup W$ blocks B, $c(B, W) = \lfloor m(Y)/2 \rfloor$, $|Y|$ is minimal, and, moreover, such that among such subsets $|Y \cap B|$ is maximal. By Lemma 3.2 we may assume without loss of generality that $Y \cup W$ is tight. Let $U_1, \ldots, U_k$ be the connected components of $Y \cup W$, and for each $i = 1, \ldots, k$ let $B_i = B \cap U_i$, $W_i = W \cap U_i$ and $Y_i = Y \cap U_i$. For each i, Lemma 3.3 shows that $Y_i \cup W_i$ is a connected set that blocks B_i. Moreover, since the U_i are disjoint, we have $|B| = \sum |B_i|$, $|W| = \sum |W_i|$, and $|Y| = \sum |Y_i|$. Thus it suffices to show that for each i we either have $Y_i = B_i$ or $|Y_i| < |B_i| + |W_i|$. Suppose $|Y_i| \geq |B_i| + |W_i|$ and $Y_i \neq B_i$. By Lemma 3.6 there exists $Y_i' \ll Y_i$ such that $Y_i' \cup W_i$ blocks B_i and either $Y_i' = B_i$ or $|Y_i'| < |B_i| + |W_i| \leq |Y_i|$. If we define $Y' = Y_i' \cup (Y \backslash Y_i)$, then $m(Y') \leq m(Y)$ by Lemma 3.4, $Y' \cup W$ blocks B, and either $|Y'| < |Y|$ or $|Y' \cap B| > |Y \cap B|$ contradicting the choice of Y. □

4. *Nice Graphs*

To complete the proof of the lower bound on the black&white pebble number of the pyramid graph, we must show that the pyramid graph is a spreading graph. One of the deficiencies of the spreading graph definition is that it is not generally easy to prove that a given graph is a spreading graph. To make our task easier, we introduce the class of nice graphs. The advantage of nice graphs is that, though it will still require some effort to prove that every nice graph "spreads," it is almost trivial to verify that pyramids and their generalizations are nice.

We say that a graph is *nice* if it has the following properties:

Property 4.1. If y and z are immediate predecessors of a vertex x, then $b(y) = b(z)$.

Property 4.2. If y and z are distinct immediate predecessors of a vertex x, then y is not a predecessor of z.

Property 4.3. If $x_1, \ldots, x_k$ are vertices such that x_i is not a predecessor of x_j whenever $i \neq j$, then there exist vertex disjoint paths $L_2, \ldots, L_k$ containing no predecessors of x_1 such that L_i is a path from an input to x_i for each $i = 2, \ldots, k$.

Let us begin by verifying that the pyramid is a nice graph. It obviously satisfies 4.1 and 4.2, so suppose $x_1, \ldots, x_k$ are vertices such that x_i is not a predecessor of x_j whenever $i \neq j$. Let M be the vertical line passing through x_1. If x_i lies on the left (right) of M, we take L_i to be the straight path from an input to x_i running parallel to the left (right) edge of the pyramid. No x_i can actually lie on M since then x_1 would be a predecessor of x_i, or vice versa. It is easy to see that the paths $L_2, \ldots, L_k$ have the desired properties and hence that the pyramid is nice. We show an example of 4.3 on the pyramid in Figure 4.

Let us now consider some generalizations of the pyramid. If we think of the pyramid as a fragment of a two-dimensional rectilinear lattice, a natural choice for a k-dimensional pyramid is the corresponding fragment of a k-dimensional recti-

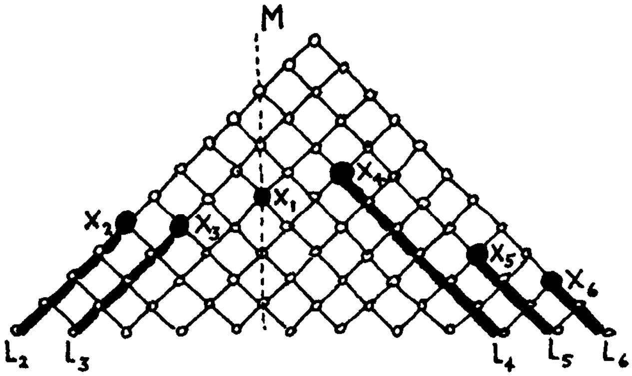


FIG. 4. An example of Property 4.3 on the pyramid.

linear lattice. Figure 5 shows a three-dimensional pyramid. Again it is easy to see that 4.1 and 4.2 are satisfied. To see that 4.3 is also satisfied, one chooses the paths L_i to be parallel to the faces of the k-dimensional subpyramid of predecessors of x_1, and, if this is done with a small amount of care, it is easy to verify that they will be disjoint. There are several other possible k-dimensional generalizations of the pyramid and in all cases they can easily be shown to be nice graphs by similar arguments.

As a final generalization we consider the t-ary pyramid in which each noninput vertex has t incoming edges. Figure 6 shows an example of a 3-ary pyramid. For any t, the same proof as used for the usual (i.e., binary) pyramid shows that t-ary pyramids are nice.

Before abandoning examples of nice graphs, we turn to trees for a moment. It is clear that every tree satisfies 4.2 and 4.3 but most trees do not satisfy 4.1. It is, however, well known (and easy to prove) that, if x is a vertex of a tree, then $b(x) = \max\{j - 1 + |P(x)[j]| : P(x)[j] \neq \emptyset\}$. Suppose $b(x) = j - 1 + k$ where $P(x)[j] = \{x_1, \ldots, x_k\}$. It is easy to see that for each $i = 1, \ldots, k$, by "pruning" the subtree under x_i we can reduce $b(x_i)$ to j. Now, by deleting the subtrees under the sons of x in $P(x) \setminus P(x)[j]$, we have pruned the subtree under x without changing $b(x)$ so that 4.1 is satisfied for this particular vertex x. By applying this procedure repeatedly, it is easy to see that every tree has a nice subtree whose root has the same black pebble number as the root of the original tree, and hence the lower bound for the black&white pebble number of vertices of nice graphs implies the same lower bound for vertices of trees. Although this result for trees was already known [3], it is nice to be able to place this result within the same framework as pyramid graphs.

We now attack the problem of showing that nice graphs are spreading graphs. We begin with an easy lemma. We say that V *minimally blocks* a vertex x if V blocks x but no proper subset of V blocks x.

LEMMA 4.4. *If V minimally blocks a vertex x, then $|V| \geq b(x) + 1 - min\{b(v): v \in V\}$.*

PROOF. Choose z in V such that $b(z) = \min\{b(v): v \in V\}$. The proof is by induction on the length of the longest path from z to x, which we denote by $d(z, x)$. If $z = x$, the lemma is trivially true since $|V| \geq 1$, so assume $z \neq x$. Since V minimally blocks x, there is some path L from an input to x such that $V \cap L = \{z\}$. Let $x_1, \ldots, x_k$ be the immediate predecessors of x ordered so that x_1 is the immediate predecessor of x on L, and let V' be a subset of V that minimally blocks

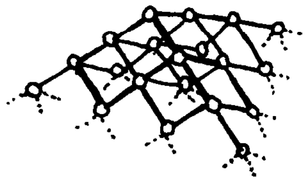


FIG. 5. A three-dimensional pyramid.

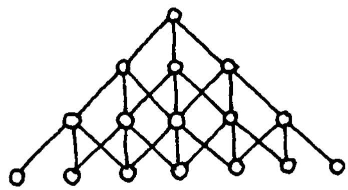


FIG. 6. A 3-ary pyramid.

x_1. Note that $z \in V'$ since there is a path L' (the subpath of L) from an input to x_1 with $L' \cap V = \{z\}$. Now since $d(z, x_1) < d(z, x)$, by the inductive hypothesis we have $|V'| \geq b(x_1) + 1 - b(z)$. Since G is nice, by properties 4.2 and 4.3 it is easy to see that $|V \backslash V'| \geq k - 1$ since as V' minimally blocks x_1, every vertex in V' must be a predecessor of x_1, and V must have at least one vertex on each path L_i for $i = 2, \ldots, k$. Finally since x can be black pebbled by black pebbling each of its immediate predecessors independently and then sliding a pebble onto x, and since by property 4.1 we have $b(x_i) = b(x_1)$ for each i, clearly $b(x) \leq b(x_1) + k - 1$. Combining all this yields $|V| = |V'| + |V \backslash V'| \geq b(x_1) + 1 - b(z) + k - 1 \geq b(x) + 1 - b(z)$. □

If V is tight, we define $V{*}j = \bigcup \{V_x : x \in \langle V \rangle[j]\}$. Notice that $V{*}j[j] = V[j]$, since clearly $V[j] \subset \langle V \rangle[j]$ and for each v in $V[j]$ we have $V_v = \{v\}$.

LEMMA 4.5. *If V is tight and $\langle V \rangle[j] \neq \varnothing$ then $|V{*}j| \geq b_j(\langle V \rangle) + j - \min\{b(v) : v \in V{*}j\}$.*

PROOF. Since $V{*}j$ obviously blocks $\langle V \rangle[j]$, if $\min\{b(v) : v \in V{*}j\} \geq j$, then $|V{*}j| \geq b_j(\langle V \rangle)$, so the result clearly holds. Thus we assume that $\min\{b(v) : v \in V{*}j\} < j$. Let $A = \{x \in \langle V \rangle[j] \backslash V : x$ has an immediate predecessor y with $b(y) < j\}$. We first note that $A \cup V{*}j[j]$ blocks $\langle V \rangle[j]$, since if L is a path from an input to a vertex in $\langle V \rangle[j]$, the first vertex of L that is in $\langle V \rangle$ and has black pebble number at least j must be in $A \cup V[j] = A \cup V{*}j[j]$. Since every vertex in $A \cup V{*}j[j]$ clearly has black pebble number at least j, this shows that $|A \cup V{*}j[j]| \geq b_j(\langle V \rangle)$. It is also easy to see that $V{*}j \backslash V{*}j[j]$ blocks A since for each x in A, $V_x \neq \{x\}$ since $x \notin V$, and by 4.1 every immediate predecessor (and hence every proper predecessor) of x has black pebble number less than j so $V_x \subset V{*}j \backslash V{*}j[j]$. Let $z \in V{*}j$ such that $b(z) = \min\{b(v) : v \in V{*}j\}$, let $x \in \langle V \rangle[j]$ such that $z \in V_x$, let L be a path from an input to x such that $L \cap V = \{z\}$, and let y be the first vertex on L that is in $\langle V \rangle$ and has black pebble number at least j. Then as observed before y is in $A \cup V{*}j[j]$, but, since y is on L and y is not z, y cannot be in V and hence must be in A. Obviously $z \in V_y$, so by Lemma 4.4 we have $|V_y| \geq b(y) + 1 - b(z) \geq j + 1 - b(z)$. Next we observe that, if v and w are distinct vertices of A, then w is not a predecessor of v, since, as we noted before, 4.1 implies that every proper predecessor of v has black pebble number less than j. By 4.3 this implies that $|(V{*}j \backslash V{*}j[j]) \backslash V_y| \geq |A| - 1$. Combining all this together we have $|V{*}j| = |V{*}j[j]| + |V_y| + |(V{*}j \backslash V{*}j[j]) \backslash V_y| \geq (b_j(\langle V \rangle) - |A|) + (j + 1 - b(z)) + (|A| - 1) = b_j(\langle V \rangle) + j - b(z)$. □

LEMMA 4.6. *Suppose V is a tight subset of a nice graph, and $j = j_0 \geq j_1 \geq \cdots \geq j_k$ where $\langle V\rangle[j] \neq \emptyset$ and for $i = 1, \ldots, k$ we have $j_i = min\{b(v) : v \in V{*}j_{i-1}\}$. Then $|V[j_k]| \geq b_j(\langle V\rangle) + j - j_k$.*

PROOF. The proof is by induction on k. For $k = 1$, this follows immediately from Lemma 4.5, so assume $k \geq 2$ and that $|V[j_{k-1}]| \geq b_j(\langle V\rangle) + j - j_{k-1}$. Obviously it suffices to show that $|V[j_k]\backslash V[j_{k-1}]| \geq j_{k-1} - j_k$. We assume that $j_k < j_{k-1}$, since otherwise this trivially holds. Choose $x \in \langle V\rangle[j_{k-1}]$, $z \in V_x$ such that $b(z) = j_k$, and let L be a path from an input to x such that $L \cap V = \{z\}$. If we let y be the first vertex on L that is in $\langle V\rangle$ and has black pebble number at least j_{k-1}, then, as in the preceding proof, it is easy to see that $V_y \subset V[j_k]\backslash V[j_{k-1}]$ and that $z \in V_y$. Thus $|V[j_k]\backslash V[j_{k-1}]| \geq |V_y| \geq j_{k-1} + 1 - j_k$ by Lemma 4.4. □

We are finally ready to prove that nice graphs are spreading.

THEOREM 4.7. *If G is nice, then G is a spreading graph.*

PROOF. Let V be a connected subset of G. We must prove that, for each j such that $\langle V\rangle[j] \neq \emptyset$, we have $|V| \geq b_j(\langle V\rangle) + j - \min\{b(v) : v \in V\}$. By the preceding lemma, since $|V| \geq |V[m]|$ for all m, it suffices to find a sequence $j = j_0 \geq \cdots \geq j_k = \min\{b(v) : v \in V\}$ where for $i = 1, \ldots, k$ we have $j_i = \min\{b(v) : v \in V{*}j_{i-1}\}$. In fact we show that for any $j > \min\{b(v) : v \in V\}$ such that $\langle V\rangle[j] \neq \emptyset$, we have $j > \min\{b(v) : v \in V{*}j\}$, which clearly implies the existence of the desired sequence. Since V is connected, we can find vertices $x(1), \ldots, x(n)$ with $b(x(1)) = \max\{b(v) : v \in \langle V\rangle\}$, $b(x(n)) = \min\{b(v) : v \in V\}$ and $P^*(x(i)) \cap \langle V_{x(i)}\rangle \cap P^*(x(i+1)) \cap \langle V_{x(i+1)}\rangle \neq \emptyset$ for $i = 1, \ldots, n-1$. Let i be maximal such that $b(x(i)) \geq j$. Since $b(x(1)) \geq j > b(x(n))$, this is well defined and $i < n$. Now we have $j > b(x(i+1)) = \max\{b(v) : v \in P^*(x(i+1)) \cap \langle V_{x(i+1)}\rangle\}$. Also, since $P^*(x(i)) \cap \langle V_{x(i)}\rangle \cap P^*(x(i+1)) \cap \langle V_{x(i+1)}\rangle \neq \emptyset$, $\max\{b(v) : v \in P^*(x(i+1)) \cap \langle V_{x(i+1)}\rangle\} \geq \min\{b(v) : v \in P^*(x(i)) \cap \langle V_{x(i)}\rangle\}$. Since $\langle V_{x(i)}\rangle \subset V{*}b(x(i))$, clearly $\min\{b(v) : v \in P^*(x(i)) \cap \langle V_{x(i)}\rangle\} \geq \min\{b(v) : v \in V{*}b(x(i))\}$, and since $b(x(i)) \geq j$, we have $\min\{b(v) : v \in V{*}b(x(i))\} \geq \min\{b(v) : v \in V{*}j\}$. Combining all this yields $j > \min\{b(v) : v \in V{*}j\}$, as desired. □

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