

Ramsey-type results for semi-algebraic relations

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Abstract

A k -ary semi-algebraic relation E on \mathbb{R}^d is a subset of \mathbb{R}^{kd} , the set of k -tuples of points in \mathbb{R}^d , which is determined by a finite number of polynomial equations and inequalities in kd real variables. The description complexity of such a relation is at most t if the number of polynomials and their degrees are all bounded by t . A set $A \subset \mathbb{R}^d$ is called homogeneous if all or none of the k -tuples from A satisfy E . A large number of geometric Ramsey-type problems and results can be formulated as questions about finding large homogeneous subsets of sets in \mathbb{R}^d equipped with semi-algebraic relations.

In this paper we study Ramsey numbers for k -ary semi-algebraic relations of bounded complexity and give matching upper and lower bounds, showing that they grow as a tower of height $k - 1$. This improves on a direct application of Ramsey's theorem by one exponential and extends a result of Alon, Pach, Pinchasi, Radoičić, and Sharir, who proved this for $k = 2$. We apply our results to obtain new estimates for some geometric Ramsey-type problems relating to order types and one-sided sets of hyperplanes. We also study the off-diagonal case, achieving some partial results.

1 Introduction

Background and motivation. The term “Ramsey theory” refers to a large body of deep results in mathematics which have a common theme: “Every large system contains a large well organized subsystem.” This is an area in which a great variety of techniques from many branches of mathematics are used and whose results are important not only to graph theory and combinatorics but also to logic, analysis, number theory, computer science, and geometry.

The *Ramsey number* $R(n)$ is the least integer N such that every red-blue coloring of the edges of the complete graph K_N on N vertices contains a monochromatic complete subgraph K_n on n vertices. Ramsey's theorem [25], in its simplest form, states that $R(n)$ exists for every n . Celebrated results of Erdős [13] and Erdős and Szekeres [16] imply that $2^{n/2} < R(n) < 2^{2n}$ for every integer

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$n > 2$. Despite much attention over the last 65 years (see, e.g., [8]), the constant factors in the exponents have not been improved.

Although already for graph Ramsey numbers there are significant gaps between the lower and upper bounds, our knowledge of hypergraph Ramsey numbers is even weaker. The Ramsey number $R_k(n)$ is the minimum N such that every red-blue coloring of all unordered k -tuples of an N -element set contains a monochromatic subset of size n , where a set is called monochromatic if all its k -tuples have the same color. Erdős, Hajnal, and Rado [14, 15] showed that there are positive constants c and c' such that

$$2^{cn^2} < R_3(n) < 2^{2^{c'n}}.$$

They also conjectured that $R_3(n) > 2^{2^{cn}}$ for some constant $c > 0$ and Erdős offered a \$500 reward for a proof. For $k \geq 4$, there is also a difference of one exponential between the known upper and lower bounds for $R_k(n)$, namely,

$$\text{twr}_{k-1}(cn^2) \leq R_k(n) \leq \text{twr}_k(c'n),$$

where the tower function $\text{twr}_k(x)$ is defined by $\text{twr}_1(x) = x$ and $\text{twr}_{i+1} = 2^{\text{twr}_i(x)}$.

The study of $R_3(n)$ is particularly important for our understanding of hypergraph Ramsey numbers. Given any lower bound on $R_k(n)$ for $k \geq 3$, an ingenious construction of Erdős and Hajnal, called the *stepping-up lemma* (see [9, 10, 20]), allows us to give a lower bound on $R_{k+1}(n)$ which is exponentially larger than the one on $R_k(n)$. In the other direction, Erdős and Rado [15] came up with a technique that gives an upper bound on $R_{k+1}(n)$ which is exponential in a power of $R_k(n)$. Therefore, closing the gap between the upper and lower bounds for $R_3(n)$ would also close the gap between the upper and lower bounds for $R_k(n)$ for all k . There is some evidence that the growth rate of $R_3(n)$ is indeed double exponential: Erdős and Hajnal (see [9, 20]) constructed a 4-coloring of the triples of the set $[2^{2^{cn}}]$ which does not contain a monochromatic subset of size n . This result is best possible up to the value of the constant c .

Ramsey numbers were first used by Erdős and Szekeres to give a bound on a beautiful geometric question which asks for the smallest integer $ES(n)$ such that any set of $ES(n)$ points in the plane in general position contains n elements in convex position, that is, n points that form the vertex set of a convex polygon? The following argument due to Tarsi shows that

$$ES(n) \leq R_3(n) \leq 2^{2^{c'n}}.$$

Indeed, let $P = \{p_1, \dots, p_N\}$ be an ordered set of $N = R_3(n)$ points in the plane in general position. Color a triple (p_i, p_j, p_k) red if p_i, p_j, p_k appear in clockwise order along the boundary of their convex hull and color it blue otherwise. By the choice of N , there exists a subset $S \subset P$ of size n that is monochromatic. It is easy to see that this monochromatic subset S must be in convex position. However, since the coloring on the triples of P is defined “algebraically”, one might expect that this bound is not tight. Indeed, a much stronger bound on $ES(n)$ was obtained by Erdős and Szekeres [16]: they showed that $ES(n) \leq 2^{2^n}$.

The main result of the present paper is an exponential improvement on the upper bound for hypergraph Ramsey numbers for colorings defined algebraically (a precise definition is given below). In particular, this shows that the Tarsi argument for estimating $ES(n)$ discussed above naturally results in an exponential bound. We also give a construction which implies an almost matching lower bound. The proofs of these results for k -uniform hypergraphs are based on adaptations of the Erdős-Rado upper bound technique and the Erdős-Hajnal stepping-up method. The key step,

when reducing the problem from $(k+1)$ -uniform hypergraphs to k -uniform ones, is to ensure that the algebraic properties of the underlying relations may be preserved.

Ramsey numbers for semi-algebraic relations. A Boolean function $\Phi(X_1, X_2, \dots, X_t)$ is an arbitrary mapping from variables X_1, \dots, X_t , attaining values “true” or “false”, to $\{0, 1\}$. A set $A \subset \mathbb{R}^d$ is *semi-algebraic* if there are polynomials $f_1, f_2, \dots, f_t \in \mathbb{R}[x_1, \dots, x_d]$ and a Boolean function $\Phi(X_1, X_2, \dots, X_t)$ such that

$$A = \left\{ x \in \mathbb{R}^d : \Phi(f_1(x) \geq 0, f_2(x) \geq 0, \dots, f_t(x) \geq 0) = 1 \right\}.$$

We say that a semi-algebraic set has *description complexity at most t* if $d \leq t$, the number of equations and inequalities is at most t , and each polynomial f_i has degree at most t .

Let $F = \{A_1, \dots, A_N\}$ be an ordered family of semi-algebraic sets in \mathbb{R}^d such that each set has bounded description complexity. We denote $\binom{F}{k}$ to be the set of all *ordered* k -tuples $(A_{i_1}, \dots, A_{i_k})$ such that $i_1 < \dots < i_k$. Now let $E \subset \binom{F}{k}$ be a relation on F . Typical examples of a relation E would be k -tuples having a non-empty intersection, or having a hyperplane transversal, or having a clockwise orientation, etc. (see [2, 4]). Since many of these relations can be described as a Boolean combination of a constant number of variables, we will assume that E is semi-algebraic in the following sense. There exists a constant q depending only on the description complexity so that each set $A_i \in F$ can be represented by a point A_i^* in \mathbb{R}^q . Then the relation E is *semi-algebraic with complexity at most t* if there exists a semi-algebraic set $E^* \subset \mathbb{R}^{qk}$ with description complexity t , where

$$E^* = \left\{ (x_1, \dots, x_k) \in \mathbb{R}^{qk} : \Phi(f_1(x_1, \dots, x_k) \geq 0, f_2(x_1, \dots, x_k) \geq 0, \dots, f_t(x_1, \dots, x_k) \geq 0) = 1 \right\}$$

for some polynomials f_1, \dots, f_t of degree at most t and Boolean function Φ and, for $i_1 < \dots < i_k$, we have

$$(A_{i_1}^*, \dots, A_{i_k}^*) \in E^* \subset \mathbb{R}^{qk} \quad \Leftrightarrow \quad (A_{i_1}, \dots, A_{i_k}) \in E. \quad (1)$$

To simplify the presentation, we will always treat our semi-algebraic sets A_1, \dots, A_N as points p_1, \dots, p_N in a higher-dimensional space. Moreover, we will only consider *ordered* point sets $P = \{p_1, \dots, p_N\}$ in \mathbb{R}^d with a semi-algebraic relation $E \subset \binom{P}{k}$. Note that the same results hold for *symmetric* relations on unordered point sets, as the ordering plays no role in this case.

We say that (P, E) is *homogeneous* if $\binom{P}{k} \subset E$ or $E \cap \binom{P}{k} = \emptyset$. We denote by $\text{hom}(P, E)$ the size of the largest homogeneous subfamily of P . Let $R_k^{d,t}(n)$ be the minimum integer N such that every ordered N -element point set P in \mathbb{R}^d equipped with a k -ary semi-algebraic relation $E \subset \binom{P}{k}$ which has complexity at most t satisfies $\text{hom}(P, E) \geq n$. Our first result shows that $R_k^{d,t}(n)$ may be bounded above by an exponential tower of height $k-1$.

Theorem 1.1. *For $k \geq 2$ and $d, t \geq 1$,*

$$R_k^{d,t}(n) \leq \text{twr}_{k-1}(n^{c_1}),$$

where c_1 is a constant that depends only on d, k , and t .

We note that the $k = 2$ case of this result, which will prove crucial for our induction, is due to Alon, Pach, Pinchasi, Radoičić, and Sharir [2].

Adapting the stepping up approach of Erdős and Hajnal, we may also show that, for every k and every sufficiently large d and t , the function $R_k^{d,t}(n)$ does indeed grow as a $(k - 1)$ -fold exponential tower in n .

Theorem 1.2. *For every $k \geq 2$, there exist $d = d(k)$ and $t = t(k)$ such that*

$$R_k^{d,t}(n) \geq \text{twr}_{k-1}(c_2 n),$$

where c_2 is a positive constant that depends only on k .

Notice that in our proof we find it necessary to make d large in terms of k . This is in some sense necessary since a striking result of Bukh and Matoušek [7] implies that the one-dimensional semi-algebraic Ramsey function $R_k^{1,t}(n)$ is at most double exponential, that is, $R_k^{1,t}(n) \leq 2^{2^{cn}}$, where c depends only on k and t . Nevertheless, for $k = 4$, we will show that there is a one-dimensional construction giving the correct double-exponential lower bound.

Applications. Over the past few decades, Ramsey numbers have been applied extensively to give upper bounds on homogeneity problems arising in geometry. For many of these applications, the relations can be defined algebraically and one can obtain an exponential improvement on the bound by using Theorem 1.1. Here we will present two such applications.

Order types. Consider an ordered set $P = \{p_1, p_2, \dots, p_N\}$ of N points in \mathbb{R}^d in general position, that is, no $d + 1$ members lie on a common hyperplane. For a $(d + 1)$ -tuple $(p_{i_1}, \dots, p_{i_{d+1}})$, where $i_1 < \dots < i_{d+1}$, let $M = M(p_{i_1}, \dots, p_{i_{d+1}})$ be the $(d + 1) \times (d + 1)$ matrix with vectors $(1, p_{i_j})$, i.e., 1 followed by the vector of the d coordinates of p_{i_j} , as the columns for $1 \leq j \leq d + 1$ and let $\det(M)$ denote the determinant of the matrix M . We say that $(p_{i_1}, \dots, p_{i_{d+1}})$ has a positive orientation if $\det(M) > 0$, and we say that $(p_{i_1}, \dots, p_{i_{d+1}})$ has a negative orientation if $\det(M) < 0$. Notice that since P is in general position $\det(M) \neq 0$.

Eliáš and Matoušek [12] defined $OT_d(n)$ to be the smallest integer N such that any set of N points in general position in \mathbb{R}^d contains n members such that every $(d + 1)$ -tuple has the same orientation. It was pointed out in [12] that $OT_1(n) = (n - 1)^2 + 1$, $OT_2(n) = 2^{\Theta(n)}$, and, for $d \geq 3$, the bound $OT_d(n) \leq \text{twr}_{d+1}(cn)$ follows from Ramsey's theorem. They also gave a construction showing that $OT_3(n) \geq 2^{2^{\Omega(n)}}$. In Section 4, we prove the following result which improves the upper bound by one exponential. In particular, for $d = 3$, it shows that the growth rate of $OT_3(n)$ is double exponential in n^c .

Theorem 1.3. *For $d \geq 3$, $OT_d(n) \leq \text{twr}_d(n^c)$, where c depends only on d .*

One-sided hyperplanes. Let $H = \{h_1, \dots, h_N\}$ be an ordered set of N hyperplanes in \mathbb{R}^d in general position, that is, every d members in H intersect at a distinct point. We say that H is *one-sided* if the vertex set of the arrangement of H , that is, the set of intersection points, lies completely on one side of the hyperplane $x_d = 0$.

Let $OSH_d(n)$ denote the smallest integer N such that every set of N hyperplanes in \mathbb{R}^d in general position contains n members that are one-sided. Dujmović and Langerman [11] used the existence of $OSH_d(n)$ to prove a ham-sandwich cut theorem for hyperplanes. Matoušek and Welzl [22] observed that $OSH_2(n) = (n - 1)^2 + 1$ and Eliáš and Matoušek [12] noticed that, for $d \geq 3$, $OSH_d(n) \leq \text{twr}_d(cn)$ follows from Ramsey's theorem, where c depends on d . In Section 4, we prove the following result which again improves the upper bound by one exponential.

Theorem 1.4. For $d \geq 3$, $OSH_d(n) \leq \text{twr}_{d-1}(cn^2 \log n)$, where c depends only on d .

Off-diagonal Ramsey numbers for semi-algebraic relations. The Ramsey number $R_k(s, n)$ is the minimum integer N such that every red-blue coloring of the unordered k -tuples on an N -element set contains a red set of size s or a blue set of size n , where a set is called red (blue) if all k -tuples from this set are red (blue). The off-diagonal Ramsey numbers, i.e., $R_k(s, n)$ with s fixed and n tending to infinity, have been intensively studied. For example, it is known [1, 5, 6, 21] that $R_2(3, n) = \Theta(n^2 / \log n)$ and, for fixed $s > 3$,

$$c_1(\log n)^{1/(s-2)} \left(\frac{n}{\log n} \right)^{(s+1)/2} \leq R_2(s, n) \leq c_2 \frac{n^{s-1}}{\log^{s-2} n}. \quad (2)$$

A classical argument of Erdős and Rado [15] (see also [9] for an improvement) demonstrates that

$$R_k(s, n) \leq 2^{\binom{R_{k-1}(s-1, n-1)}{k-1}} + k - 2. \quad (3)$$

Together with the upper bound in (2) this implies, for fixed s , that

$$R_k(s, n) \leq \text{twr}_{k-1}(c_s n^{2s-2k+2} / \log^{2s-2k} n). \quad (4)$$

The bound in [9] roughly improves the exponent of n from $2s - 2k + 2$ to $s - k + 1$. Note that, for fixed s , we get an exponential improvement on the upper bound for $R_k(s, n)$ that follows from using the trivial bound $R_k(s, n) \leq R_k(n)$.

Off-diagonal Ramsey numbers may also be used to give another simple solution to the problem of estimating $ES(n)$. Let P be a set of $N = R_4(5, n)$ points in the plane in general position. We color the 4-element subsets blue if they are in convex position and color them red otherwise. As noticed by Esther Klein, any five points in general position must contain four points in convex position. Hence, there must be a subset $S \subset P$ of size n such that every 4 points in S is in convex position and therefore S must be in convex position. This shows that $ES(n) \leq R_4(5, n)$. Just as before, one might expect that this double-exponential bound is not tight for such an algebraically defined coloring.

Let $R_k^{d,t}(s, n)$ denote the minimum integer N with the property that for any sequence of N points P in \mathbb{R}^d and any k -ary semi-algebraic relation $E \subset \binom{P}{k}$ of complexity at most t , P has s members such that every k -tuple induced by them is in E or P has n members such that no k -tuple induced by them belongs to E . Clearly, for $s \leq n$, we have

$$R_k^{d,t}(s, n) \leq R_k^{d,t}(n) \leq \text{twr}_{k-1}(n^c), \quad (5)$$

which matches the tower height in (4). However, it seems likely that the following stronger bound holds.

Conjecture 1.5. For fixed $k \geq 3, d, t$, and s , $R_k^{d,t}(s, n) \leq \text{twr}_{k-2}(n^c)$, where $c = c(k, d, t, s)$.

The crucial case is when $k = 3$, since a polynomial bound on $R_3^{d,t}(s, n)$ could be used with the adaptation of the Erdős-Rado upper bound argument discussed in this paper to obtain an exponential improvement over the trivial bound in (5) for all k . In Section 5, we prove a somewhat weaker result, giving a quasi-polynomial bound for point sets in one dimension.

Theorem 1.6. For $s \geq 4$, $R_3^{1,t}(s, n) \leq 2^{\log^c n}$, where c depends only on s and t .

Combining Theorem 1.6 with our adaptation of the Erdős-Rado upper bound argument, we obtain the following result.

Corollary 1.7. For $n \geq 4$, $R_4^{1,t}(s, n) \leq 2^{2^{\log^c n}}$, where c depends only on s and t .

For $k \geq 5$, the result of Bukh and Matoušek [7] mentioned earlier implies that $R_k^{1,t}(s, n) \leq R_k^{1,t}(n) \leq 2^{2^{cn}}$, where $c = c(k, t)$.

For general d , we were only able to establish a good lower bound in the following special case. We say that the pair (P, E) is $K_4^{(3)} \setminus e$ -free if every *four* points induce at most *two* 3-tuples that belong to E .

Theorem 1.8. Let $P = \{p_1, \dots, p_N\}$ be a sequence of N points in \mathbb{R}^d with semi-algebraic relation $E \subset \binom{P}{3}$ such that E has complexity at most t . If (P, E) is $K_4^{(3)} \setminus e$ -free, then there exists a subset $P' \subset P$ such that $\binom{P'}{3} \cap E = \emptyset$ and

$$|P'| \geq 2^{\log^c N},$$

where $c < 1$ depends only on d, t .

Organization. In the next section, we will prove Theorem 1.1, our upper bound on Ramsey numbers for semi-algebraic relations. Then, in Section 3, we will prove the matching lower bound, Theorem 1.2. We discuss the short proofs of our applications, Theorems 1.3 and 1.4, in Section 4 and our results on the off-diagonal case, Theorems 1.6 and 1.8, in Section 5. We conclude with some further remarks.

2 Upper bounds

In this section, we prove Theorem 1.1. First we briefly discuss the classic Erdős-Rado argument obtaining the recursive formula

$$R_k(n) \leq 2^{\binom{R_{k-1}(n-1)}{k-1}} + k - 2.$$

Set $N = 2^{\binom{R_{k-1}(n-1)}{k-1}} + k - 2$ and $M = R_{k-1}(n - 1)$. Given a red-blue coloring χ on the k -tuples from $[N]$, Erdős and Rado greedily construct a sequence of distinct vertices v_1, \dots, v_{M+1} such that, for any given $(k - 1)$ -tuple $1 \leq i_1 < \dots < i_{k-1} \leq M$, all k -tuples $\{v_{i_1}, \dots, v_{i_{k-1}}, v_j\}$ with $j > i_{k-1}$ are of the same color, which we denote by $\chi'(v_{i_1}, \dots, v_{i_{k-1}})$. Since $M = R_{k-1}(n - 1)$, there is a monochromatic set of size $n - 1$ in coloring χ' . Together with the vertex v_{M+1} , these form a monochromatic clique of size n in χ . The greedy construction of the sequence v_1, \dots, v_{M+1} is as follows. First, pick $k - 2$ arbitrary vertices v_1, \dots, v_{k-2} and set $S_{k-2} = S \setminus \{v_1, \dots, v_{k-2}\}$. After having picked $\{v_1, \dots, v_r\}$ we also have a subset S_r such that for any $(k - 1)$ -tuple $v_{i_1}, \dots, v_{i_{k-1}}$ with $1 \leq i_1 < \dots < i_{k-1} \leq r$, all k -tuples $\{v_{i_1}, \dots, v_{i_{k-1}}, w\}$ with $w \in S_r$ are the same color. Let v_{r+1} be an arbitrary vertex in S_r . Let us call two elements $x, y \in S_r \setminus \{v_{r+1}\}$ *equivalent* if for every $(k - 1)$ -tuple $T \subset \{v_1, \dots, v_{r+1}\}$ we have $\chi(T \cup \{x\}) = \chi(T \cup \{y\})$. By the greedy construction, x and y are equivalent if and only if for every $(k - 2)$ -tuple $T \subset \{v_1, \dots, v_r\}$ we have $\chi(T \cup \{v_{r+1}, x\}) = \chi(T \cup \{v_{r+1}, y\})$. Therefore, there are $\binom{r}{k-2}$ possible choices for T and hence

there are at most $2^{\binom{r}{k-2}}$ equivalence classes. We set S_{r+1} to be the largest of those classes. Finally, we set $\chi'(v_{i_1}, \dots, v_{i_{k-2}}, v_{r+1}) = \chi(v_{i_1}, \dots, v_{i_{k-2}}, v_{r+1}, w)$ where $w \in S_{r+1}$. As N is large enough so that S_M is nonempty, we can indeed construct the desired sequence of vertices.

There are two ways we improve the Erdős-Rado approach for semi-algebraic relations. Suppose that the k -tuples which are colored red under χ correspond to a semi-algebraic relation E_1 with bounded description complexity and let E_2 be the relation containing those $(k-1)$ -tuples which are colored red by χ' . The main improvement comes from showing that E_2 will also be semi-algebraic with bounded description complexity. Therefore, we can obtain by induction on k an exponential improvement, starting with the result of Alon et al. [2] as the base case $k = 2$. A further improvement can be made by the observation that $S_r \setminus \{v_{r+1}\}$ does not need to be partitioned into $2^{\binom{r}{k-2}}$ equivalence classes. Instead, we can apply the Milnor-Thom Theorem (stated below) to partition $S_r \setminus \{v_{r+1}\}$ into at most $O(r^{dk})$ equivalence classes with the desired properties, where the implied constant depends on the description complexity of E_1 and the uniformity k .

Let f_1, \dots, f_r be d -variate real polynomials with zero sets Z_1, \dots, Z_r . A vector $\sigma \in \{-1, 0, +1\}^r$ is a *sign pattern* of p_1, \dots, p_r if there exists an $x \in \mathbb{R}^d$ such that the sign of $p_j(x)$ is σ_j for all $j = 1, \dots, r$. The Milnor-Thom theorem (see [4, 23, 24, 27]) bounds the number of cells in the arrangement of the zero sets Z_1, \dots, Z_r and, consequently, the number of possible sign patterns.

Theorem 2.1 (Milnor-Thom). *Let f_1, \dots, f_r be d -variate real polynomials of degree at most D . The number of cells in the arrangement of their zero sets $Z_1, \dots, Z_r \subset \mathbb{R}^d$ and, consequently, the number of sign patterns of f_1, \dots, f_r is at most*

$$\left(\frac{50Dr}{d}\right)^d$$

for $r \geq d \geq 2$.

Theorem 1.1 easily follows from the following recursive formula.

Theorem 2.2. *Set $M = R_{k-1}^{d,t}(n-1)$. Then, for every $k \geq 3$,*

$$R_k^{d,t}(n) \leq 2^{C_1 M \log M},$$

where $C_1 = C_1(k, d, t)$.

Proof. Let $P = \{p_1, \dots, p_N\}$ be a set of $N = 2^{C_1 M \log M}$ points in \mathbb{R}^d with semi-algebraic relation $E_1 \subset \binom{P}{k}$ such that E_1 has complexity at most t and where C_1 is a constant that will be specified later. As mentioned earlier, we can represent E_1 by the set $E_1^* \subset \mathbb{R}^{dk}$ that satisfies (1). Since E_1^* is semi-algebraic with complexity at most t , there exist polynomials $f_1, f_2, \dots, f_t \in \mathbb{R}[x_1, \dots, x_{dk}]$ of degree at most t and a Boolean function Φ such that

$$E_1^* = \left\{x \in \mathbb{R}^{dk} : \Phi(f_1(x) \geq 0, f_2(x) \geq 0, \dots, f_t(x) \geq 0) = 1\right\}.$$

In what follows, we will recursively construct a sequence of points q_1, \dots, q_r from P and a subset $S_r \subset P$, where $r = k-2, k-1, \dots, M+1$, such that the following holds. Every $(k-1)$ -tuple $(q_{i_1}, \dots, q_{i_{k-1}}) \subset \{q_1, \dots, q_{r-1}\}$ with $i_1 < i_2 < \dots < i_{k-1}$ has the property that either $(q_{i_1}, \dots, q_{i_{k-1}}, q) \in E_1^*$ for every point $q \in \{q_j : i_{k-1} < j \leq r\} \cup S_r$ or $(q_{i_1}, \dots, q_{i_{k-1}}, q) \notin E_1^*$ for every point $q \in \{q_j : i_{k-1} < j \leq r\} \cup S_r$. Also,

$$|S_r| \geq \frac{N}{C_2^r r^{dkr}} - r,$$

where $C_2 = C_2(k, d, t)$ is a constant depending only on k, d , and t . Furthermore, for $i < j$, q_i comes before q_j in the original ordering and every point in S_r comes after q_r in the original ordering.

We start by selecting the $k-2$ points $\{q_1, \dots, q_{k-2}\} = \{p_1, \dots, p_{k-2}\}$ from P and setting $S_{k-2} = P \setminus \{p_1, \dots, p_{k-2}\}$. After obtaining $\{q_1, \dots, q_r\}$ and S_r , we define q_{r+1} and S_{r+1} as follows. Let q_{r+1} be the smallest indexed element in S_r and fix a $(k-2)$ -tuple $(q_{i_1}, q_{i_2}, \dots, q_{i_{k-2}}) \subset \{q_1, \dots, q_r\}$. Then, for each j such that $1 \leq j \leq t$, we define d -variate polynomials $h_j \in \mathbb{R}[x_1, \dots, x_d]$ such that, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,

$$h_j(x) = f_j(q_{i_1}, q_{i_2}, \dots, q_{i_{k-2}}, q_{r+1}, x).$$

After doing this for each $(k-2)$ -tuple in $\{q_1, \dots, q_r\}$, we have generated at most $t \binom{r}{k-2}$ zero sets in \mathbb{R}^d . By the Milnor-Thom theorem, the number of cells in the arrangement of these zero sets is at most $C_2 r^{dk}$, where $C_2 = C_2(k, d, t)$. By the pigeonhole principle, there exists a cell $\Delta \subset \mathbb{R}^q$ that contains at least $(|S_r| - 1)/C_2 r^{dk}$ points of S_r . The h_i have the same sign pattern for each point in Δ . In other words, for any fixed $(k-1)$ -tuple $(q_{i_1}, \dots, q_{i_{k-1}}) \subset \{q_1, \dots, q_{r+1}\}$, we have either

$$(q_{i_1}, \dots, q_{i_{k-1}}, p_l) \in E_1^* \quad \forall p_l \in \Delta$$

or

$$(q_{i_1}, \dots, q_{i_{k-1}}, p_l) \notin E_1^* \quad \forall p_l \in \Delta.$$

Let S_{r+1} be the set of points p_l in the cell Δ . Then we have the recursive formula

$$|S_{r+1}| \geq \frac{|S_r| - 1}{C_2 r^{dk}}.$$

Substituting in the lower bound on $|S_r|$, we obtain the desired bound

$$|S_{r+1}| \geq \frac{2^{C_1 M \log M}}{(C_2)^{r+1} (r+1)^{dk(r+1)}} - (r+1).$$

This shows that we can construct the sequence q_1, \dots, q_{r+1} and the set S_{r+1} with the desired properties.

Since $C_1 = C_1(k, d, t)$ is sufficiently large and $M = R_{k-1}^{d,t}(n-1)$, we have

$$|S_M| \geq 1.$$

Hence, $\{q_1, \dots, q_{M+1}\}$ is well defined for $M = R_{k-1}^{d,t}(n-1)$. Set $F = \{q_1, \dots, q_M\}$. We define the semi-algebraic set $E_2^* \subset \mathbb{R}^{q(k-1)}$ by

$$E_2^* = \{x \in \mathbb{R}^{q(k-1)} : \Phi(f_1(x, q_{M+1}) \geq 0, \dots, f_t(x, q_{M+1}) \geq 0) = 1\},$$

and define the relation $E_2 \subset \binom{F}{k-1}$ by

$$E_2 = \left\{ (q_{i_1}, \dots, q_{i_{k-1}}) \in \binom{F}{k-1} : (q_{i_1}, \dots, q_{i_{k-1}}) \in E_2^* \right\}.$$

Therefore, E_2 is a semi-algebraic relation with complexity at most t . By the definition of the function $R_{k-1}^{d,t}(n-1)$, there exist $n-1$ points $\{q_{i_1}, \dots, q_{i_{n-1}}\} \subset F$ such that every $(k-1)$ -tuple belongs to E_2 or no such $(k-1)$ -tuple belongs to E_2 . Recall that by the construction of F , if $(q_{i_1}, \dots, q_{i_{k-1}}) \in$

E_2^* then $(q_{i_1}, \dots, q_{i_{k-1}}, q_j) \in E_1^*$ for every $j > i_{k-1}$. Therefore every k -tuple of $\{q_{i_1}, \dots, q_{i_{n-1}}\} \cup \{q_{M+1}\}$ belongs to E_1 or no such k -tuple belongs to E_1 . Hence, the set $\{q_{i_1}, \dots, q_{i_{n-1}}\} \cup \{q_{M+1}\}$ is homogeneous and this completes the proof. \square

3 Lower bounds

In this section, we will prove Theorem 1.2. For every $n \geq 1$ and $k \geq 3$, we will construct an N -element point set $P_k(N)$ in \mathbb{R}^d with semi-algebraic relation $E_k \subset \binom{P_k(N)}{k}$, where $N = \text{twr}_{k-1}(n)$, $d = 2^{k-3}$, and $E_k^* \subset \mathbb{R}^{dk}$ is a semi-algebraic set with complexity at most $t = t(k)$ such that

$$\text{hom}(P_k(N), E_k) = O(n),$$

where the implied constant depends only on k .

3.1 Base case $k = 3$

Let $P_3(2^n) = \{1, 2, \dots, 2^n\} \subset \mathbb{R}$ with relation $E_3 \subset \binom{P_3(2^n)}{3}$, where

$$E_3^* = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < x_2 < x_3, x_1 + x_3 - 2x_2 \geq 0\}.$$

Clearly, E_3^* has bounded complexity. Now we claim that $\text{hom}(P_3(2^n), E_3) \leq n + 1$. Indeed, let $H = \{p_1, p_2, \dots, p_r\} \subset P_3(2^n)$ be a homogeneous subset such that $p_1 < p_2 < \dots < p_r$.

Case 1. Suppose that $\binom{H}{3} \subset E_3$. Then $p_{i+1} \geq 2p_i - p_1$ for all $1 \leq i \leq r - 1$. Since $p_1 \geq 1$ and $p_2 \geq p_1 + 1$, we have by induction

$$p_i \geq 2^{i-2} + p_1 \geq 2^{i-2} + 1.$$

Hence, if $r = n + 2$, we have

$$2^n + 1 \leq p_r \leq 2^n,$$

which is a contradiction.

Case 2. Suppose that $\binom{H}{3} \cap E_3 = \emptyset$. Then, for $1 \leq i < j < k \leq r$, we have

$$p_i + p_k - 2p_j < 0,$$

which implies

$$(2^n - p_i + 1) + (2^n - p_k + 1) - 2(2^n - p_j + 1) > 0.$$

Set $q_{r-j+1} = 2^n - p_j + 1$ for $1 \leq j \leq r$. Then we have

$$1 \leq q_1 < q_2 < \dots < q_r \leq 2^n,$$

and $q_{i+1} \geq 2q_i - q_1$. By the same argument as above, the set $\{q_1, \dots, q_r\}$ must satisfy $r < n + 2$. This completes the proof.

3.2 The Erdős-Hajnal stepping-up lemma

For $k \geq 3$, we will adapt the Erdős-Hajnal stepping-up lemma to construct the point set $P_k(N)$ and the relation $E_k \subset \binom{P_k(N)}{k}$. Before we describe this procedure, we will briefly sketch the classic Erdős-Hajnal stepping-up lemma (see also [9],[10],[20]).

Let $k \geq 3$ and suppose that $E_1 \subset \binom{[N]}{k}$ is a relation on $[N]$ such that $\text{hom}([N], E_1) < n$. The *stepping-up lemma* uses E_1 to define a relation $E_2 \subset \binom{[2^N]}{k+1}$ with the properties listed below. We refer to E_2 as the step-up relation of E_1 .

For any $a \in [2^N]$, write $a - 1 = \sum_{i=0}^{N-1} a(i)2^i$ with $a(i) \in \{0, 1\}$ for each i . For $a \neq b$, let $\delta(a, b) = i + 1$ where i is the largest value for which $a(i) \neq b(i)$. Notice that

Property A: $\delta(a, b) \neq \delta(b, c)$ for every triple $a < b < c$,

Property B: for $a_1 < \dots < a_n$, $\delta(a_1, a_n) = \max_{1 \leq i \leq n-1} \delta(a_i, a_{i+1})$.

Given any $(k+1)$ -tuple $a_1 < a_2 < \dots < a_{k+1}$ of $[2^N]$, consider the integers $\delta_i = \delta(a_i, a_{i+1})$, $1 \leq i \leq k$. If $\delta_1, \dots, \delta_k$ form a monotone sequence, then let $(a_1, a_2, \dots, a_{k+1}) \in E_2$ if and only if $(\delta_1, \delta_2, \dots, \delta_k) \in E_1$.

Now we have to decide if the $(k+1)$ -tuple $(a_1, \dots, a_{k+1}) \in E_2$ in the case when $\delta_1, \dots, \delta_k$ is not monotone. We say that i is a *local minimum* if $\delta_{i-1} > \delta_i < \delta_{i+1}$, a *local maximum* if $\delta_{i-1} < \delta_i > \delta_{i+1}$, and a *local extremum* if it is either a local minimum or a local maximum. This is well defined by Property A. If δ_2 is a local minimum, then set $(a_1, \dots, a_{k+1}) \in E_2$. If δ_2 is a local maximum, then set $(a_1, \dots, a_{k+1}) \notin E_2$. All remaining edges will not be in E_2 .

Lemma 3.1. *If $\text{hom}([N], E_1) < n$, then the step-up relation E_2 satisfies $\text{hom}([2^N], E_2) < 2n + k - 4$.*

Proof. Suppose for contradiction that the set of vertices $S = \{a_1, \dots, a_{2n+k-4}\}$ is homogeneous with respect to E_2 , where $a_1 < \dots < a_{2n+k-4}$. Without loss of generality, we can assume that $\binom{S}{k+1} \subset E_2$. Set $\delta_i = \delta(a_i, a_{i+1})$.

Case 1. Suppose that there exists a j such that $\delta_j, \delta_{j+1}, \dots, \delta_{j+n-1}$ forms a monotone sequence. First assume that

$$\delta_j > \delta_{j+1} > \dots > \delta_{j+n-1}.$$

Since $\text{hom}([N], E_1) < n$, there exists a subsequence $j \leq i_1 < \dots < i_k \leq j + n - 1$ such that $(\delta_{i_1}, \dots, \delta_{i_k}) \notin E_1$. But then the $(k+1)$ -tuple $(a_{i_1}, \dots, a_{i_k}, a_{i_k+1}) \notin E_2$. Indeed, by Property B,

$$\delta(a_{i_h}, a_{i_h+1}) = \delta(a_{i_h}, a_{i_h+1}) = \delta_{i_h}.$$

Therefore, since $\delta_{i_1}, \dots, \delta_{i_k}$ form a monotone sequence and $(\delta_{i_1}, \dots, \delta_{i_k}) \notin E_1$, we have that the $(k+1)$ -tuple $(a_{i_1}, \dots, a_{i_k}, a_{i_k+1}) \notin E_2$, contradicting that $\binom{S}{k+1} \subset E_2$. A similar argument holds if $\delta_j < \delta_{j+1} < \dots < \delta_{j+n-1}$.

Case 2. Since $\delta_1, \dots, \delta_{2n-2}$ does not have a monotone subsequence of length n , it must have at least two local extrema. Since between any two local minimums there must be a local maximum, this implies that there exists a local maximum among $\delta_1, \dots, \delta_{2n-2}$ and, therefore, a $(k+1)$ -tuple not in E_2 , contradicting our assumption. \square

3.3 Stepping up algebraically

We will now adapt the Erdős-Hajnal stepping-up lemma to our semi-algebraic framework. First we need some definitions. For a point $p \in \mathbb{R}^d$, we let $B(p, \epsilon)$ be the closed ball in \mathbb{R}^d of radius ϵ centered at p . For any two points $p_1, p_2 \in \mathbb{R}^d$, where $p_1 = (a_{1,1}, a_{1,2}, \dots, a_{1,d})$ and $p_2 = (a_{2,1}, a_{2,2}, \dots, a_{2,d})$, we write $p_1 \prec p_2$ if $a_{1,i} < a_{2,i}$ for all $1 \leq i \leq d$. We say that the set of points $\{p_1, \dots, p_N\}$ is *increasing*, if

$$p_1 \prec p_2 \prec \dots \prec p_N.$$

For $r > 0$, we say that $\{p_1, \dots, p_N\}$ is ϵ -*increasing* if, for $1 \leq i \leq N$, $q_i \in B(p_i, \epsilon)$ implies that

$$q_1 \prec q_2 \prec \dots \prec q_N.$$

For any two points $q_1, q_2 \in \mathbb{R}^{2d}$, where

$$q_1 = (x_1, y_1, x_2, y_2, \dots, x_d, y_d) \quad \text{and} \quad q_2 = (x'_1, y'_1, x'_2, y'_2, \dots, x'_d, y'_d),$$

we define the *slope* $\sigma(q_1, q_2)$ of q_1, q_2 to be

$$\sigma(q_1, q_2) = \left(\frac{y'_1 - y_1}{x'_1 - x_1}, \dots, \frac{y'_i - y_i}{x'_i - x_i}, \dots, \frac{y'_d - y_d}{x'_d - x_d} \right) \in \mathbb{R}^d.$$

Thus, the i th coordinate of $\sigma(q_1, q_2)$ is the slope of the line through the points (x_i, y_i) and (x'_i, y'_i) in \mathbb{R}^2 .

Let $P_k(N) = \{p_1, \dots, p_N\}$ be a set of N points in \mathbb{R}^d with relation $E_k \subset \binom{P_k(N)}{k}$ such that E_k^* is a semi-algebraic set in \mathbb{R}^{dk} with complexity at most t . We say that $(P_k(N), E_k)$ is ϵ -*deep* if moving any point in $P_k(N)$ by a distance at most ϵ will not change the relation E_k . More precisely, $(P_k(N), E_k)$ is ϵ -*deep* if, for every $(p_{i_1}, \dots, p_{i_k}) \in E_k$,

$$(q_{i_1}, \dots, q_{i_k}) \in E_k^* \quad \text{for all} \quad q_{i_1} \in B(p_{i_1}, \epsilon), q_{i_2} \in B(p_{i_2}, \epsilon), \dots, q_{i_k} \in B(p_{i_k}, \epsilon)$$

and, for every $(p_{i_1}, \dots, p_{i_k}) \notin E_k$,

$$(q_{i_1}, \dots, q_{i_k}) \notin E_k^* \quad \text{for all} \quad q_{i_1} \in B(p_{i_1}, \epsilon), q_{i_2} \in B(p_{i_2}, \epsilon), \dots, q_{i_k} \in B(p_{i_k}, \epsilon).$$

With these definitions in hand, our algebraic stepping-up lemma is now as follows.

Lemma 3.2 (Stepping up). *For $k \geq 3$ and $\epsilon > 0$, let $P_k(N), E_k$, and E_k^* be as above and such that $P_k(N)$ is ϵ -increasing and $(P_k(N), E_k)$ is ϵ -deep. Then there exists $\epsilon_1 > 0$ such that there is an ϵ_1 -increasing point set $P_{k+1}(2^N)$ of 2^N points in \mathbb{R}^{2d} and a semi-algebraic set E_{k+1}^* in $\mathbb{R}^{2d(k+1)}$ with complexity $t_1 = t_1(t, k)$ for which $E_{k+1} \subset \binom{P_{k+1}(2^N)}{k+1}$ is the step-up relation of E_k and $(P_{k+1}(2^N), E_{k+1})$ is ϵ_1 -deep.*

From Lemmas 3.1 and 3.2, we have the following immediate corollary.

Corollary 3.3. $\text{hom}(P_{k+1}(2^N), E_{k+1}) \leq 2 \cdot \text{hom}(P_k(N), E_k) + k - 4$.

Proof of Lemma 3.2: Construction of $P_{k+1}(2^N)$. Given an ϵ -increasing point set $P_k(N) = \{p_1, \dots, p_N\}$ in \mathbb{R}^d , we will construct an ϵ_1 -increasing set $P_{k+1}(2^N) = \{q_1, \dots, q_{2^N}\}$ of 2^N points in \mathbb{R}^{2d} as follows. For each $p_i \in P_k(N)$, we denote $p_i = (a_{i,1}, \dots, a_{i,d})$. The construction is done by induction on N . For the base case $N = 1$, $p_1 = (a_{1,1}, a_{1,2}, \dots, a_{1,d})$ and $P_{k+1}(2) = \{q_1, q_2\}$, where

$$q_1 = (0, 0, \dots, 0) \quad \text{and} \quad q_2 = (1, a_{1,1}, 1, a_{1,2}, \dots, 1, a_{1,d}).$$

For $N \geq 2$, set $B_1 = B((0, 0, \dots, 0), \epsilon_2) \subset \mathbb{R}^{2d}$, $B_2 = B((1, a_{N,1}, 1, a_{N,2}, \dots, 1, a_{N,d}), \epsilon_2) \subset \mathbb{R}^{2d}$, where ϵ_2 is sufficiently small so that for any two points $q_1^* \in B_1, q_2^* \in B_2$, we have

$$\sigma(q_1^*, q_2^*) \in B(p_N, \epsilon/2).$$

Given the ϵ -increasing point set $P_k(N-1) = \{p_1, \dots, p_{N-1}\} \subset \mathbb{R}^d$, we inductively construct two small dilated copies of $P_{k+1}(2^{N-1})$, $Q_1 = \{q_1, \dots, q_{2^{N-1}}\}$ and $Q_2 = \{q_{2^{N-1}+1}, \dots, q_{2^N}\}$, and translate them so that they lie inside B_1 and B_2 respectively. Hence, the slope of any two points in Q_i is preserved for $i \in \{1, 2\}$. Then $P_{k+1}(2^N) = Q_1 \cup Q_2$. Since $P_k(N-1)$ is ϵ -increasing, for $r \in \{1, 2\}$, we have Q_r is ϵ_1 -increasing for some $\epsilon_1 > 0$. Therefore, $P_{k+1}(2^N)$ is ϵ_1 -increasing.

Now we make the following key observation on the point set $P_{k+1}(2^N)$.

Observation 3.4. *If $q_i \prec q_j$, then the point $\sigma(q_i, q_j) \in B(p_r, \epsilon/2) \subset \mathbb{R}^d$, where $r = \delta(i, j)$ and δ is defined as in Section 3.2.*

Proof. This can be seen by induction on N . For $i < j$, if $q_i \in Q_1, q_j \in Q_2$, then $\sigma(q_i, q_j) \in B(p_N, \epsilon/2)$ and $\delta(i, j) = N$. If $q_i, q_j \in Q_1$ or $q_i, q_j \in Q_2$, then by the induction hypothesis and since the copies are slope preserving, we have

$$\sigma(q_i, q_j) \in B(p_r, \epsilon/2)$$

where $p_r \in \{p_1, \dots, p_{N-1}\}$ is such that $r = \delta(i, j)$. □

Construction of E_{k+1}^* . We define the semi-algebraic set $E_{k+1}^* \subset \mathbb{R}^{2d(k+1)}$ by

$$E_{k+1}^* = \left\{ (x_1, \dots, x_{k+1}) \in \mathbb{R}^{2d(k+1)} : x_i \in \mathbb{R}^{2d}, x_1 \prec x_2 \prec \dots \prec x_{k+1}, C_1 \vee C_2 \vee C_3 \right\},$$

where conditions C_1, C_2, C_3 are defined below.

$$C_1 : [\sigma(x_1, x_2) \succ \sigma(x_2, x_3)] \wedge [\sigma(x_3, x_4) \succ \sigma(x_2, x_3)].$$

$$C_2 : [\sigma(x_1, x_2) \prec \sigma(x_2, x_3) \prec \dots \prec \sigma(x_k, x_{k+1})] \wedge [(\sigma(x_1, x_2), \sigma(x_2, x_3), \dots, \sigma(x_k, x_{k+1})) \in E_k^*].$$

$$C_3 : [\sigma(x_1, x_2) \succ \sigma(x_2, x_3) \succ \dots \succ \sigma(x_k, x_{k+1})] \wedge [(\sigma(x_k, x_{k+1}), \dots, \sigma(x_2, x_3), \sigma(x_1, x_2)) \in E_k^*].$$

Notice that E_{k+1} is the step-up relation of E_k . Indeed, let $(q_{i_1}, \dots, q_{i_{k+1}})$ be a $(k+1)$ -tuple of points in $P_{k+1}(2^N)$ such that $i_1 < \dots < i_{k+1}$. If $\sigma(q_{i_1}, q_{i_2}) \prec \dots \prec \sigma(q_{i_k}, q_{i_{k+1}})$, then

$$(q_{i_1}, \dots, q_{i_{k+1}}) \in E_{k+1}^* \Leftrightarrow (\sigma(q_{i_1}, q_{i_2}), \dots, \sigma(q_{i_k}, q_{i_{k+1}})) \in E_k^*.$$

By Observation 3.4 and since $(P_k(N), E_k)$ is ϵ -deep, this happens if and only if $(p_{r_1}, \dots, p_{r_k}) \in E_k$, where $r_l = \delta(i_l, i_{l+1})$ for $1 \leq l \leq k$. The same is true if $\sigma(q_{i_1}, q_{i_2}) \succ \dots \succ \sigma(q_{i_k}, q_{i_{k+1}})$. If $\sigma(q_{i_1}, q_{i_2}) \succ \sigma(q_{i_2}, q_{i_3}) \prec \sigma(q_{i_3}, q_{i_4})$, then $(q_{i_1}, \dots, q_{i_{k+1}}) \in E_{k+1}^*$ by condition C_1 and we have $\delta(i_1, i_2) > \delta(i_2, i_3) < \delta(i_3, i_4)$ by Observation 3.4. Finally, if $(q_{i_1}, \dots, q_{i_{k+1}})$ does not satisfy C_1, C_2, C_3 , then $(q_{i_1}, \dots, q_{i_{k+1}}) \notin E_{k+1}^*$.

Although each coordinate of σ is a rational function over 4 variables, by clearing denominators in the defining inequalities for E_{k+1}^* , we get that E_{k+1}^* is a semi-algebraic set with description complexity at most c where $c = c(k, t)$.

Observation 3.5. $(P_{k+1}(2^N), E_{k+1})$ is ϵ_1 -deep.

Proof. Suppose $(q_{i_1}, q_{i_2}, \dots, q_{i_{k+1}}) \in E_{k+1}$ with $q_{i_1} \prec \dots \prec q_{i_{k+1}}$ and let $q_i^* \in B(q_i, \epsilon_1)$ for $1 \leq i \leq k+1$. By Observation 3.4, $\sigma(q_{i_j}, q_{i_{j+1}}) \in B(p_l, \epsilon/2)$ for some $p_l \in P_k(N)$. By making ϵ_1 sufficiently small, we have $q_{i_1}^* \prec \dots \prec q_{i_{k+1}}^*$ and

$$\sigma(q_{i_j}^*, q_{i_{j+1}}^*) \in B(p_l, \epsilon).$$

Since $(P_k(N), E_k)$ is ϵ -deep, $(q_{i_1}^*, \dots, q_{i_{k+1}}^*) \in E_{k+1}^*$. By a similar argument, if $(q_{i_1}, \dots, q_{i_{k+1}}) \notin E_{k+1}$, then $(q_{i_1}^*, \dots, q_{i_{k+1}}^*) \notin E_{k+1}^*$. \square

Notice that our construction of $(P_3(2^n), E_3)$ is $(1/10)$ -increasing, $(1/10)$ -deep, and E_3^* has constant description complexity. Applying Lemma 3.2 and Corollary 3.3 inductively on k completes the proof of Theorem 1.2.

3.4 A construction in one dimension

A recent result of Bukh and Matoušek [7] shows that one can not keep stepping up in one dimension to another construction in one dimension. Their result says that there exists a constant $c = c(k, t)$ such that $R_k^{1,t}(n) \leq 2^{2^{cn}}$. They also showed that their result is tight by giving a matching lower bound in the case $k = 5$. Here, we give another matching construction for the case $k = 4$. This is the smallest possible uniformity for such a tight construction as, for $k \leq 3$, Theorem 1.1 gives a single exponential upper bound.

The idea of the construction is to start with the point set $P_3(2^n)$ and relation $E_3 \subset \binom{P_3(2^n)}{3}$ from Subsection 3.1 and then to apply the Erdős-Hajnal stepping-up lemma with a sufficiently large base. After stepping up, we obtain a point set P with 2^{2^n} points in \mathbb{R} with relation E_3 in the “exponent”. Since the only operation in E_3^* is addition, the step-up relation can be defined by multiplication and hence remains semi-algebraic. We now formalize this idea.

The construction: Let $P_3(2^n)$ and E_3^* be as in Section 3. Recall that $(P_3(2^n), E_3)$ is $(1/10)$ -deep. We will step up by considering a point set P on 2^{2^n} points, where for any $p \in P$ we have $p - 1 = \sum_{i=0}^{2^n-1} p(i)b^i$ with $p(i) \in \{0, 1\}$. Here b is a sufficiently large constant to be determined later. For $p > q$, let $\delta(p, q) = \log_b(p - q)$. We will choose b sufficiently large so that for any $p > q$, if i is the largest integer such that $p(i) \neq q(i)$, then

$$i - \frac{1}{10} < \delta(p, q) < i + \frac{1}{10}.$$

Hence, i is the closest integer to δ and this integer satisfies Properties A and B from Section 3.2. Now we define the relation $E \subset \binom{P}{4}$ by the semi-algebraic set

$$E^* = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 < x_2 < x_3 < x_4, C_1 \vee C_2 \vee C_3\},$$

where conditions C_1, C_2, C_3 are defined below.

$$C_1 : [\delta(x_1, x_2) > \delta(x_2, x_3)] \wedge [\delta(x_3, x_4) > \delta(x_2, x_3)].$$

$$C_2 : [\delta(x_1, x_2) < \delta(x_2, x_3) < \delta(x_3, x_4)] \wedge [(\delta(x_1, x_2), \delta(x_2, x_3), \delta(x_3, x_4)) \in E_3^*].$$

$$C_3 : [\delta(x_1, x_2) > \delta(x_2, x_3) > \delta(x_3, x_4)] \wedge [(\delta(x_3, x_4), \delta(x_2, x_3), \delta(x_1, x_2)) \in E_3^*].$$

Notice that these conditions can be rewritten as

$$C_1 : [x_2 - x_1 > x_3 - x_2] \wedge [x_4 - x_3 > x_3 - x_2].$$

$$C_2 : [x_2 - x_1 < x_3 - x_2 < x_4 - x_3] \wedge [(x_2 - x_1)(x_4 - x_3) \geq (x_3 - x_2)^2].$$

$$C_3 : [x_2 - x_1 > x_3 - x_2 > x_4 - x_3] \wedge [(x_2 - x_1)(x_4 - x_3) \geq (x_3 - x_2)^2].$$

Therefore, E^* is semi-algebraic with constant description complexity. Using the fact that (P_3, E_3) is $(1/10)$ -deep, a similar argument to the previous subsection shows that E^* is the step-up relation of E_3^* . Corollary 3.3 then implies that (P, E) does not contain a homogeneous subset of size $2n + 1$.

4 Applications

Let us recall that $OT_d(n)$ is the smallest integer N such that any set of N points in general position in \mathbb{R}^d contains n members such that every $(d + 1)$ -tuple has the same orientation. The proof of Theorem 1.3 giving an upper bound on $OT_d(n)$ follows quickly from Theorem 1.1.

Proof of Theorem 1.3. Let $P = \{p_1, p_2, \dots, p_N\}$ be an ordered point set of N points in \mathbb{R}^d such that P is in general position. Let $E \subset \binom{P}{d+1}$ be a relation on P such that $(p_{i_1}, \dots, p_{i_{d+1}}) \in E$ if $(p_{i_1}, \dots, p_{i_{d+1}})$ has a positive orientation. Then

$$E^* = \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d(d+1)} : x_i \in \mathbb{R}^d, \det(M(x_1, \dots, x_{d+1})) > 0\}.$$

Thus, E^* is a semi-algebraic set in $\mathbb{R}^{d(d+1)}$ with description complexity at most $t = t(d)$. Hence, the statement follows from Theorem 1.1. \square

Recall that $OSH_d(n)$ is the smallest integer N such that every set of N hyperplanes in \mathbb{R}^d in general position contains n members that are one-sided, where a set of hyperplanes H is one-sided if the vertex set of the arrangement of H lies completely on one side of the hyperplane $x_d = 0$. We obtain a stronger bound for Theorem 1.4 by deriving a recursive formula, similar to the one in Theorem 2.2. Since each hyperplane $h_i \in H$ is specified by the linear equation

$$a_{i,1}x_1 + \dots + a_{i,d}x_d = b_i,$$

we can represent $h_i \in H$ by the point $p_i \in \mathbb{R}^{d+1}$ where $p_i = (a_{i,1}, \dots, a_{i,d}, b_i)$ and define a relation $E \subset \binom{P}{d}$. However, for sake of clarity, we will simply define $E \subset \binom{H}{d}$, where $(h_{i_1}, \dots, h_{i_d}) \in E$ if the

point $h_{i_1} \cap \dots \cap h_{i_d}$ lies above the hyperplane $x_d = 0$ (i.e. the d -th coordinate of the intersection point is positive). Clearly, E is a semi-algebraic relation with complexity at most $t = t(d)$.

Since $OSH_2(n) = (n-1)^2 + 1$, Theorem 1.4 follows immediately from the next theorem.

Theorem 4.1. *For $d \geq 3$, let $M = OSH_{d-1}(n-1)$. Then $OSH_d(n) \leq 2^{C_2 M \log M}$, where C_2 depends only on d .*

Proof. Set $N = 2^{C_2 M \log M}$, where C_2 is a sufficiently large constant that depends only on d . Let $H = \{h_1, \dots, h_N\}$ and $E \subset \binom{H}{d}$ be as defined above, and let h_0 be the hyperplane $x_d = 0$. We now follow the proof of Theorem 2.2 to the point where we obtain the sequence $q_1, \dots, q_{M+1} \in H$ such that every $(d-1)$ -tuple $(q_{i_1}, \dots, q_{i_{d-1}}) \subset \{q_1, \dots, q_M\}$ has the property that either $(q_{i_1}, \dots, q_{i_{d-1}}, q_j) \in E$ for all $i_{d-1} < j \leq M+1$ or $(q_{i_1}, \dots, q_{i_{d-1}}, q_j) \notin E$ for all $i_{d-1} < j \leq M+1$.

For each hyperplane $q_i \in \{q_1, \dots, q_M\}$, let $q_i^* = q_i \cap q_{M+1}$ and set $F = \{q_1^*, \dots, q_M^*\} \subset q_{M+1}$. Hence, F is a family of M $(d-2)$ -dimensional hyperplanes in q_{M+1} . Since q_{M+1} is isomorphic to \mathbb{R}^{d-1} and $M = OSH_{d-1}(n-1)$, there exist $n-1$ members $F' = \{q_{i_1}^*, \dots, q_{i_{n-1}}^*\} \subset F$ such that the vertex set of the arrangement of F' lies completely on one side of the $(d-2)$ -dimensional hyperplane $h_0 \cap q_{M+1}$ in q_{M+1} . Let $H' \subset H$ be the hyperplanes corresponding to F' in \mathbb{R}^d . By the construction of $H' \subset \{q_1, \dots, q_M\}$, the vertex set of the arrangement of $H' \cup \{q_{M+1}\}$ lies on one side of the hyperplane h_0 . \square

5 The off-diagonal case

In this section we prove Theorem 1.6, giving an upper bound on $R_3^{1,t}(s, n)$, and Theorem 1.8. We first list several results that we will use.

Lemma 5.1 (Erdős-Szekeres [16]). *Given a sequence of $N = (n-1)^2 + 1$ distinct real numbers p_1, p_2, \dots, p_N , there exists a subsequence $p_{i_1}, p_{i_2}, \dots, p_{i_n}$ of length n such that either $p_{i_1} < p_{i_2} < \dots < p_{i_n}$ or $p_{i_1} > p_{i_2} > \dots > p_{i_n}$.*

The next lemma is a combinatorial reformulation of another classical theorem due to Erdős and Szekeres [16]. A transitive 2-coloring of the triples of $[N]$ is a 2-coloring, say with colors red and blue, such that, for $i_1 < i_2 < i_3 < i_4$, if triples (i_1, i_2, i_3) and (i_2, i_3, i_4) are red (blue), then (i_1, i_2, i_4) and (i_1, i_3, i_4) are also red (blue).

Lemma 5.2 (Fox et al. [18]). *Let $N_3(s, n)$ denote the minimum integer N such that, for every transitive 2-coloring on the triples of $[N]$, there exists a red clique of size s or a blue clique of size n . Then*

$$N_3(s, n) = \binom{s+n-4}{s-2} + 1.$$

The following lemma is the $k = 2$ case of Theorem 1.1, first proved by Alon, Pach, Pinchasi, Radoičić, and Sharir.

Lemma 5.3 (Alon et al. [2]). *Let P be a sequence of N points in \mathbb{R}^d and let $E \subset \binom{P}{2}$ be a semi-algebraic relation on P with description complexity at most t . Then there exists a subset $P' \subset P$ with at least N^α elements such that either every pair of distinct elements of P' belong to E or no such pair belongs to E , where $\alpha > 0$ depends only on t and d .*

The following result, due to Fox, Gromov, Lafforgue, Naor, and Pach, tells us that if many triples of a point set P satisfy a semi-algebraic relation E then there is a large tripartite 3-uniform hypergraph all of whose edges are in E .

Lemma 5.4 (Fox et al. [17]). *Let P be a sequence of N points in \mathbb{R}^d and let $E \subset \binom{P}{3}$ be a semi-algebraic relation on P with description complexity at most t . If $|E| \geq \epsilon \binom{N}{3}$, then there exist disjoint subsets $P_1, P_2, P_3 \subset P$ such that $|P_i| \geq \epsilon^{c_3} N$ and, for all $p_1 \in P_1$, $p_2 \in P_2$, and $p_3 \in P_3$, $(p_1, p_2, p_3) \in E$, where c_3 depends only on t and d .*

The following lemma of Spencer is now an exercise in *The Probabilistic Method* (see [3]).

Lemma 5.5 (Spencer [26]). *Let $H = (V, E)$ be a 3-uniform hypergraph on N vertices. If $E(H) \geq N/3$, then there exists a subset $S \subset V(H)$ such that S is an independent set and*

$$|S| \geq \frac{2N}{3} \left(\frac{N}{3|E(H)|} \right)^{1/2}.$$

The last lemma on our list is an old theorem due to Sturm (see [4]). Let $g(x)$ be a polynomial in x with real coefficients. We say that the sequence of polynomials $g_0(x), g_1(x), \dots, g_t(x), g_{t+1}(x) = 0$ is a *Sturm sequence* for $g(x)$ if

$$g_0(x) = g(x), g_1(x) = g'(x), \text{ and } g_i(x) = -\text{rem}(g_{i-2}, g_{i-1}) \quad \text{for } i \geq 2,$$

where $\text{rem}(g_{i-2}, g_{i-1})$ denotes the remainder of $g_{i-2}(x)/g_{i-1}(x)$. Since the degrees of polynomials g_i are strictly decreasing, there is always an index t such that $g_{t+1}(x) = 0$ and hence $g_{t-1}(x) = q_t(x)g_t(x)$ for some non-zero polynomial $q_t(x)$.

Lemma 5.6 (Sturm). *Let $g(x)$ be a polynomial in x with real coefficients and let $g_0(x), \dots, g_t(x), g_{t+1}(x) = 0$ be the Sturm sequence for $g(x)$. Suppose $g(a), g(b) \neq 0$ for $a < b$. Then the number of distinct real roots of $g(x)$ in the open interval (a, b) is $\sigma(a) - \sigma(b)$, where $\sigma(\xi)$ denotes the number of sign changes (ignoring zeros) in the sequence $g_0(\xi), g_1(\xi), \dots, g_t(\xi)$.*

Let $P = \{p_1, \dots, p_N\}$ be an ordered set of N distinct real numbers. By Lemma 5.1, one can always find a subset $P' \subset P$ of size \sqrt{N} such that the elements of P are either increasing or decreasing. If necessary, by a change of variables we can assume that the elements of P' are increasing. Since this is a negligible loss, we will now only consider increasing point sets.

Let $P = \{p_1, \dots, p_N\}$ be an increasing sequence of N distinct real numbers and let $E \subset \binom{P}{3}$ be a semi-algebraic relation on P such that

$$E^* = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < x_2 < x_3, \Phi(f_1(x_1, x_2, x_3), \dots, f_t(x_1, x_2, x_3)) = 1\},$$

where the f_i are polynomials of degree at most t and Φ is a Boolean function.

The *domain* of P is the open interval (p_1, p_N) . For each pair $p_i, p_j \in P$ with $i < j$, we write $\mathcal{P}(p_i, p_j)$ for the set of non-zero univariate polynomials

$$f_l(x_1, p_i, p_j), f_l(p_i, x_2, p_j), f_l(p_i, p_j, x_3),$$

for $1 \leq l \leq t$. We say that (P, E) has at most r roots within its domain if, for any pair $p_i, p_j \in P$, the univariate polynomials in $\mathcal{P}(p_i, p_j)$ have at most r distinct real roots in total inside the interval (p_1, p_N) . Note that $|\mathcal{P}(p_i, p_j)| \leq 3t$ and $r \leq 3t^2$. We say that (P, E) is $K_s^{(3)}$ -free if every collection of s points in P contains a triple not in E . Theorem 1.6 follows immediately from the observation above and the following theorem.

Theorem 5.7. *Let $P = \{p_1, \dots, p_N\}$ be an increasing sequence of N distinct points in \mathbb{R} and let $E \subset \binom{P}{3}$ be a semi-algebraic relation on P such that E has complexity at most t and (P, E) has at most r roots within its domain. If (P, E) is $K_s^{(3)}$ -free, then there exists a subfamily $P' \subset P$ such that $\binom{P'}{3} \cap E = \emptyset$ and*

$$|P'| \geq e^{\alpha^{\epsilon(r+s)}(\log N)^\epsilon},$$

where $0 < \epsilon, \alpha < 1$ depend only on t .

Proof. The proof is by induction on N , r , and s . The base cases are $s = 3$, $r = 0$, or $N \leq (6t^2)^{2/\alpha}$. When $N \leq (6t^2)^{2/\alpha}$, the statement holds trivially for sufficiently small ϵ . If $s = 3$, then again the statement follows immediately by taking $P' = P$. For $r = 0$, notice that E and $\overline{E} = \binom{P}{3} \setminus E$ are both transitive. Indeed, let $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4} \in P$ be such that $p_{i_1} < \dots < p_{i_4}$ and $(p_{i_1}, p_{i_2}, p_{i_3}), (p_{i_2}, p_{i_3}, p_{i_4}) \in E$. Since the sign pattern of each univariate polynomial in $\mathcal{P}(p_{i_1}, p_{i_2}) \cup \mathcal{P}(p_{i_3}, p_{i_4})$ does not change inside the interval (p_1, p_N) , this implies that $(p_{i_1}, p_{i_2}, p_{i_4}), (p_{i_1}, p_{i_3}, p_{i_4}) \in E$. Likewise, if $(p_{i_1}, p_{i_2}, p_{i_3}), (p_{i_2}, p_{i_3}, p_{i_4}) \notin E$, then we must have $(p_{i_1}, p_{i_2}, p_{i_4}), (p_{i_1}, p_{i_3}, p_{i_4}) \notin E$. Since (P, E) is $K_s^{(3)}$ -free, by Lemma 5.2, there exists a subset $P_0 \subset P$ such that $|P_0| \geq \Omega(N^{1/(s-2)})$ and $\binom{P_0}{3} \cap E = \emptyset$.

Now assume that the statement holds if $r' \leq r$, $s' \leq s$, $N' \leq N$ and not all three inequalities are equalities. Let $f_1, \dots, f_t \in \mathbb{R}[x_1, x_2, x_3]$ be polynomials of degree at most t such that

$$E^* = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < x_2 < x_3, \Phi(f_1(x_1, x_2, x_3), \dots, f_t(x_1, x_2, x_3)) = 1\}.$$

By applying Lemma 5.3 twice, first fixing point p_1 and then fixing point p_N , there exists a subset $P_1 \subset P$ of size N^{α_1} , where $\alpha_1 > 0$ depends only on t , such that, for $j_1 < j_2$,

$$(p_1, p_{j_1}, p_{j_2}) \in E \text{ for all } p_{j_1}, p_{j_2} \in P_1 \setminus \{p_1\} \quad \text{or} \quad (p_1, p_{j_1}, p_{j_2}) \notin E \text{ for all } p_{j_1}, p_{j_2} \in P_1 \setminus \{p_1\},$$

and

$$(p_{j_1}, p_{j_2}, p_N) \in E \text{ for all } p_{j_1}, p_{j_2} \in P_1 \setminus \{p_N\} \quad \text{or} \quad (p_{j_1}, p_{j_2}, p_N) \notin E \text{ for all } p_{j_1}, p_{j_2} \in P_1 \setminus \{p_N\}.$$

We call an ordered triple (p_i, p_j, p_m) *bad* if there exists a polynomial $f \in \mathcal{P}(p_i, p_j)$ such that f has a root at p_m or if there exists a polynomial $f \in \mathcal{P}(p_j, p_m)$ such that f has a root at p_i . Since $\mathcal{P}(p_i, p_j) \cup \mathcal{P}(p_j, p_m)$ gives rise to at most $6t^2$ distinct roots, P_1 has at most $3t^2|P_1|^2$ bad triples. By Lemma 5.5, there exists a subset $P_2 \subset P_1$ such that $|P_2| \geq N^{\alpha_2}$, where $\alpha_2 > 0$ depends only on t , and P_2 has no bad triple.

Let $P_2 = \{q_1, \dots, q_{N^{\alpha_2}}\}$ with $q_1 < \dots < q_{N^{\alpha_2}}$. We now partition $P_2 = Q_1 \cup \dots \cup Q_M$ into $M = \sqrt{N^{\alpha_2}}$ parts, such that

$$Q_i = \left\{ q_j : (i-1)\sqrt{N^{\alpha_2}} + 1 \leq j \leq i\sqrt{N^{\alpha_2}} \right\}.$$

Let I_i be the domain of Q_i . We define the relation $E_i \subset \binom{Q_i}{2}$ on Q_i , where $(q_{j_1}, q_{j_2}) \in E_i$ if the non-zero univariate polynomials in $\mathcal{P}(q_{j_1}, q_{j_2})$ have (in total) strictly less than r roots inside the open interval I_i .

For $l \in \{1, \dots, t\}$, the Euclidean Algorithm implies that the univariate polynomial $f_l(x_1, x_2, x_3)$ in x_1 has a Sturm sequence of length at most t . The same is true for the univariate polynomials $f_l(x_1, x_2, x_3)$ in x_2 and x_3 . Since there are no bad triples in P_2 , we can apply Sturm's Lemma

5.6, which tells us that E_i depends only on the polynomials f_1, \dots, f_t , their Sturm sequences, the endpoints of I_i , and $r \leq 6t^2$. Hence, E_i is semi-algebraic with complexity at most $t' = t'(t)$. By Lemma 5.3, there exists $S_i \subset Q_i$ such that $|S_i| \geq N^\alpha$, where $\alpha > 0$ depends only on t , and either

$$\binom{S_i}{2} \subset E_i \quad \text{or} \quad \binom{S_i}{2} \cap E_i = \emptyset.$$

We may assume that $\alpha < \alpha_2/24$. If $\binom{S_i}{2} \subset E_i$ for some i , then (S_i, E) has at most $r-1$ roots within its domain. By the induction hypothesis, there exists a subset $P_3 \subset S_i$ such that $\binom{P_3}{3} \cap E = \emptyset$ and

$$|P_3| \geq e^{\alpha^\epsilon(r-1+s)(\log N^\alpha)^\epsilon} = e^{\alpha^\epsilon(r+s)(\log N)^\epsilon},$$

and we are done. Therefore, we can assume that $\binom{S_i}{2} \cap E_i = \emptyset$ for all i . Hence, for any $q_{j_1}, q_{j_2} \in S_i$ with $j_1 < j_2$, all r roots (within the interval (p_1, p_N)) of the univariate polynomials in $\mathcal{P}(q_{j_1}, q_{j_2})$ lie inside I_i . Now we make the following observation.

Observation 5.8. *For any two parts S_{i_1} and S_{i_2} , where $i_1 < i_2$, either $(q_{j_1}, q_{j_2}, q_{j_3}) \in E$ for all $q_{j_1}, q_{j_2} \in S_{i_1}$ and $q_{j_3} \in S_{i_2}$ or $(q_{j_1}, q_{j_2}, q_{j_3}) \notin E$ for all $q_{j_1}, q_{j_2} \in S_{i_1}$ and $q_{j_3} \in S_{i_2}$. Likewise, either $(q_{j_1}, q_{j_2}, q_{j_3}) \in E$ for all $q_{j_1} \in S_{i_1}$ and $q_{j_2}, q_{j_3} \in S_{i_2}$ or $(q_{j_1}, q_{j_2}, q_{j_3}) \notin E$ for all $q_{j_1} \in S_{i_1}$ and $q_{j_2}, q_{j_3} \in S_{i_2}$.*

Proof. We first prove the first part of the statement. Since all r roots of the non-zero univariate polynomials in $\mathcal{P}(q_{j_1}, q_{j_2})$ lie inside the interval I_{i_1} and q_{j_3}, p_N lie to the left of I_{i_1} , we have that $f_l(q_{j_1}, q_{j_2}, q_{j_3})$ and $f_l(q_{j_1}, q_{j_2}, p_N)$ have the same sign for all $1 \leq l \leq t$. Therefore

$$(q_{j_1}, q_{j_2}, q_{j_3}) \in E \Leftrightarrow (q_{j_1}, q_{j_2}, p_N) \in E.$$

Since our sets are subsets of P_1 , we have either $(q_{j_1}, q_{j_2}, q_{j_3}) \in E$ for all $q_{j_1}, q_{j_2} \in S_{i_1}$ and $q_{j_3} \in S_{i_2}$ or $(q_{j_1}, q_{j_2}, q_{j_3}) \notin E$ for all $q_{j_1}, q_{j_2} \in S_{i_1}$ and $q_{j_3} \in S_{i_2}$. The second part of the statement follows by the same argument. \square

If there exist two parts S_{i_1}, S_{i_2} with $i_1 < i_2$ such that $(q_{j_1}, q_{j_2}, q_{j_3}) \in E$ for all $q_{j_1}, q_{j_2} \in S_{i_1}$ and $q_{j_3} \in S_{i_2}$, then (S_{i_1}, E) is $K_{s-1}^{(3)}$ -free. By the induction hypothesis, there exists a subset $P_4 \subset S_{i_1}$ such that $\binom{P_4}{3} \cap E = \emptyset$ and

$$|P_4| \geq e^{\alpha^\epsilon(r+s-1)(\log N^\alpha)^\epsilon} = e^{\alpha^\epsilon(r+s)(\log N)^\epsilon}.$$

The same is true if $(q_{j_1}, q_{j_2}, q_{j_3}) \in E$ for all $q_{j_1} \in S_{i_1}$ and $q_{j_2}, q_{j_3} \in S_{i_2}$. Therefore, we can assume that for $i_1 < i_2$ and $j_1 < j_2 < j_3$,

$$(q_{j_1}, q_{j_2}, q_{j_3}) \notin E \quad \text{for all} \quad q_{j_1}, q_{j_2} \in S_{i_1}, \quad q_{j_3} \in S_{i_2} \quad \text{and for all} \quad q_{j_1} \in S_{i_1}, \quad q_{j_2}, q_{j_3} \in S_{i_2}. \quad (6)$$

Set $S = S_1 \cup \dots \cup S_M$ and recall that $M = \sqrt{N^{\alpha_2}}$ and $|S_i| = N^\alpha$. For $i_1 < i_2$, let $q_{j_1} \in S_{i_1}$ and $q_{j_2} \in S_{i_2}$. Then we say that the unordered triple (q_{j_1}, q_{j_2}, S_i) is *homogeneous* if

1. for $i > i_2$, $(q_{j_1}, q_{j_2}, q_i) \in E$ for all $q_i \in S_i$ or $(q_{j_1}, q_{j_2}, q_i) \notin E$ for all $q_i \in S_i$, or
2. for $i_1 < i < i_2$, $(q_{j_1}, q_i, q_{j_2}) \in E$ for all $q_i \in S_i$ or $(q_{j_1}, q_i, q_{j_2}) \notin E$ for all $q_i \in S_i$, or
3. for $i < i_1 < i_2$, $(q_i, q_{j_1}, q_{j_2}) \in E$ for all $q_i \in S_i$ or $(q_i, q_{j_1}, q_{j_2}) \notin E$ for all $q_i \in S_i$.

Since q_{j_1}, q_{j_2} give rise to at most $3t$ polynomials of degree at most t , there are at most $3t^2$ sets S_i such that (q_{j_1}, q_{j_2}, S_i) is not homogeneous. Indeed, the domain of every such S_i must contain a root of one of these polynomials and the total number of roots is at most $3t^2$.

We pick b distinct members of the collection $\{S_1, \dots, S_M\}$ uniformly at random. Let X denote the number of non-homogeneous triples (q_{j_1}, q_{j_2}, S_i) , where S_i is a set from our randomly chosen collection and q_{j_1} and q_{j_2} also lie in distinct sets from this collection. Since $|S_i| = N^\alpha$, then

$$\mathbb{E}[X] \leq \binom{M}{2} (N^\alpha)^2 3t^2 \left(\frac{b}{M}\right)^3 \leq \frac{2t^2 b^3 N^{2\alpha}}{M}.$$

By setting $b = M^{1/9}$ and since $\alpha < \alpha_2/24$, we have

$$\mathbb{E}[X] \leq \frac{2t^2}{M^{1/2}} = \frac{2t^2}{N^{\alpha_2/4}} \leq \frac{2t^2}{N^{6\alpha}}.$$

Since $N > (6t^2)^{2/\alpha}$, we have $\mathbb{E}[X] < 1$. Hence, there exists a subset $T \subset S$ such that $T = T_1 \cup \dots \cup T_b$ where $b = N^{\alpha_2/18}$, $|T_i| = N^\alpha$, and, for any q_{j_1}, q_{j_2} from distinct subsets, (q_{j_1}, q_{j_2}, T_i) is homogeneous. Therefore, we obtain the following.

Observation 5.9. *For parts $T_{i_1}, T_{i_2}, T_{i_3}$, where $i_1 < i_2 < i_3$, either $(q_{j_1}, q_{j_2}, q_{j_3}) \in E$ for all $q_{j_1} \in T_{i_1}, q_{j_2} \in T_{i_2}, q_{j_3} \in T_{i_3}$ or $(q_{j_1}, q_{j_2}, q_{j_3}) \notin E$ for all $q_{j_1} \in T_{i_1}, q_{j_2} \in T_{i_2}, q_{j_3} \in T_{i_3}$.*

Proof. Let $q_{j_1} \in T_{i_1}, q_{j_2} \in T_{i_2}$, and $q_{j_3} \in T_{i_3}$ be such that $(q_{j_1}, q_{j_2}, q_{j_3}) \in E$. It suffices to show that for $a_1 \in T_{i_1}, a_2 \in T_{i_2}$, and $a_3 \in T_{i_3}$ we have $(a_1, a_2, a_3) \in E$. Since $(q_{j_1}, q_{j_2}, T_{i_3})$ is homogeneous, we have $(q_{j_1}, q_{j_2}, a_3) \in E$. Likewise, since (q_{j_1}, T_{i_2}, a_3) is homogeneous, we have $(q_{j_1}, a_2, a_3) \in E$. Finally, since (T_{i_1}, a_2, a_3) is homogeneous, we have $(a_1, a_2, a_3) \in E$. If $(q_{j_1}, q_{j_2}, q_{j_3}) \notin E$, then by the same argument we have $(a_1, a_2, a_3) \notin E$. \square

Let T' be a point set formed by selecting one point from each T_i . Then, by applying the induction hypothesis on (T', E) we can find a set of indices J such that $|J| \geq e^{\alpha^{\epsilon(r+s)}(\log N^{\alpha_2/18})^\epsilon}$ and for every $j_1 < j_2 < j_3$ in J all triples with one vertex in each T_{j_i} do not satisfy E . Applying the induction hypothesis again to every $(T_j, E), j \in J$ we obtain a collection of subsets $U_j \subset T_j, |U_j| \geq e^{\alpha^{\epsilon(r+s)}(\log N^\alpha)^\epsilon}$ such that $\binom{U_j}{3} \cap E = \emptyset$. Let $P' = \cup_{j \in J} U_j$. Then, by (6), this subset satisfies $\binom{P'}{3} \cap E = \emptyset$. Moreover,

$$\begin{aligned} |P'| &\geq e^{\alpha^{\epsilon(r+s)}(\log N^{\alpha_2/18})^\epsilon} e^{\alpha^{\epsilon(r+s)}(\log N^\alpha)^\epsilon} \\ &\geq e^{\alpha^{\epsilon(r+s)}(\log N^\alpha)^\epsilon} e^{\alpha^{\epsilon(r+s)}(\log N^\alpha)^\epsilon} \\ &\geq e^{\alpha^{\epsilon(r+s)}(\log N)^\epsilon}, \end{aligned}$$

for sufficiently small $\epsilon = \epsilon(t)$ such that $2\alpha^\epsilon \geq 1$. \square

Proof of Theorem 1.8: The proof is by induction on N . The base case $N \leq 2^{6c_3}$ is trivial for sufficiently small $c = c(d, t)$, where c_3 is the constant from Lemma 5.4. Furthermore, c will be sufficiently small so that for all $N > 2^{6c_3}$,

$$(\log N - 3c_3 \log^c N)^c \geq \log^c N - 1. \quad (7)$$

Set $\alpha = 2^{-3 \log^c N}$. Now assume that the statement holds for all $N' < N$, where $N > 2^{6c3}$. The proof falls into two cases.

Case 1. Suppose $|E| \leq \alpha \binom{N}{3}$. Then, by Lemma 5.5, there exists $P' \subset P$ such that $P' \cap E = \emptyset$ and

$$|P'| \geq \frac{2}{3}N \left(\frac{N}{3\alpha \binom{N}{3}} \right)^{1/2} \geq \frac{2\sqrt{2}}{3} 2^{(3/2) \log^c N} \geq 2^{\log^c N},$$

so we are done.

Case 2. Suppose $|E| > \alpha \binom{N}{3}$. By Lemma 5.4, there exist disjoint subsets $P_1, P_2, P_3 \subset P$ such that $|P_i| \geq \alpha^{c3}N$ and, for every $p_1 \in P_1$, $p_2 \in P_2$, and $p_3 \in P_3$, we have $(p_1, p_2, p_3) \in E$. Since (P, E) is $K_4^{(3)} \setminus e$ -free, we have $(p_1, p_2, p_3) \notin E$ for every $p_1, p_2 \in P_1$ and $p_3 \in P_2$ and, likewise, for every $p_1 \in P_1$ and $p_2, p_3 \in P_2$. By applying the induction hypothesis on (P_1, E) and (P_2, E) , there exists $P' \subset P_1 \cup P_2$ such that $P' \cap E = \emptyset$ and

$$|P'| \geq 2 \cdot \left(2^{\log^c(\alpha^{c3}N)} \right) = 2^{1+(\log N - 3c3 \log^c N)^c}.$$

By (7), we have $|P'| \geq 2^{\log^c N}$. □

6 Concluding remarks

1. We showed that for every integer $k \geq 2$ there exist $d = d(k)$ and $t = t(k)$ such that $R_k^{d,t}(n) \geq \text{twr}_{k-1}(c_2 n)$, where c_2 depends on k . Our construction gives dimension $d = 2^{k-3}$. Instead of starting from the construction in Subsection 3.1, we could start from the construction in Subsection 3.4 to get dimension $d = 2^{k-4}$ for $k \geq 4$. It would be interesting to see if one could give constructions for all $k \geq 5$ in, say, two dimensions. The result of Bukh and Matoušek mentioned in the introduction, that $R_k^{1,t}(n)$ is at most double exponential for fixed k and t , shows that this cannot be done in one dimension.

2. It would be very interesting if one could improve the bounds in the off-diagonal case. The crucial case is when $k = 3$ and we conjecture that $R_3^{d,t}(s, n) = O(n^c)$, where c depends only on d, t and s . Notice that this would give another proof that $ES(n)$ is at most exponential in a power of n . Indeed, by the argument discussed in the introduction, there exists a t such that $ES(n) \leq R_4^{2,t}(5, n)$. A careful analysis of the proof of Theorem 2.2 gives $R_k^{d,t}(s, n) \leq 2^{C_1 M \log M}$, where $M = R_{k-1}^{d,t}(s-1, n-1)$ and $C_1 = C_1(d, k, t)$. Therefore, $R_3^{d,t}(4, n) = O(n^c)$ implies that

$$ES(n) \leq R_4^{2,t}(5, n) \leq 2^{C_2 n^c \log n},$$

where $C_2 = C_2(t)$.

3. A lower bound for the off-diagonal case may also be achieved by using the semi-algebraic version of the stepping-up lemma. In particular, we may show that for every integer $k \geq 3$ there exist $d = d(k), t = t(k)$, and $s = s(k)$ such that $R_k^{d,t}(s, n) \geq \text{twr}_{k-2}(c'n)$, where c' depends only on k . The original stepping-up lemma gives $s(k) \leq 2^{k-1} - k + 3$ and a recent variant [10] gives the linear bound $s(k) \leq \lceil \frac{5}{2}k \rceil - 3$ for $k \geq 4$. However, we conjecture that the result should already be true with $s = k + 1$.

4. Let us remark that our results and their proofs generalize to multiple relations (colors). More precisely, let $R_{k,q}^{d,t}(n)$ be the least integer N such that any set of N points in \mathbb{R}^d with semi-algebraic

relations $E_1, \dots, E_q \subset \binom{P}{k}$, where each E_i has complexity at most t , contains n members such that every k -tuple belongs to E_i for some fixed i or no such k -tuple belongs to E_i for all i . Then

$$R_{k,q}^{d,t}(n) \leq \text{twr}_{k-1}(n^c),$$

where $c = c(d, k, t, q)$. In particular, for $q \geq 3$, corresponding to $q + 1 \geq 4$ colors, this is a much smaller bound than in the general case, as $R_k(n, n, n, n) = \text{twr}_k(\Theta(n))$, where $R_k(n, n, n, n)$ is the 4-color Ramsey number.

5. One can also prove the following strengthening of Theorem 1.2 for four or more colors. For any $k \geq 2$ and $\alpha > 0$, there exist $d = d(k, \alpha)$, $t = t(k, \alpha)$, and $c' = c'(k, \alpha)$ such that, for $q \geq 3$, $R_{k,q}^{d,t}(n) \geq \text{twr}_{k-1}(c'n^\alpha)$. The key extra tool is a variant of the stepping-up lemma (see [20]) which allows one to step up from $k = 2$ to $k = 3$ at the cost of doubling the number of colors. For our base case, we need to know that for any $\alpha > 0$ there exist d and t such that, for n sufficiently large, $R_2^{d,t}(n) \geq n^\alpha$. This follows from the following famous construction of Frankl and Wilson [19]. Let p be a prime and let $r = p^2 - 1$. The vertices of our graph G are all subsets of size r from $\{1, \dots, m\}$ and two vertices are adjacent if the corresponding sets have intersection of size congruent to $-1 \pmod{p}$. Let $n = \binom{m}{p-1}$. It was shown in [19] that this graph has $\binom{m}{r} = \Omega(n^{p+1})$ vertices and no clique or independent set of size larger than n . This graph can be realized in dimension $d = r$ using a semi-algebraic relation of description complexity $t = r^2$. Every vertex of G will correspond to a vector (x_1, x_2, \dots, x_r) where $x_1 < x_2 < \dots < x_r$ are the elements of the corresponding subset of $\{1, \dots, m\}$ in increasing order. For any two vertices (x_1, \dots, x_r) and (y_1, \dots, y_r) the Boolean function will check whether the number of equalities $x_i = y_j, 1 \leq i, j \leq r$ (i.e., the size of the intersection) is congruent to $-1 \pmod{p}$. The lower bound on $R_{k,q}^{d,t}(n)$ now follows by first stepping-up from $k = 2$ to $k = 3$, thereby doubling the number of colors to 4, and then applying the method of Section 3 for higher uniformities, which does not increase the number of colors. It would be interesting to know whether a similar improved result holds for the 2-color case, $q = 1$.

6. For $k \geq 4$, the upper bound in Lemma 3.1 and Corollary 3.3 can be improved by roughly a factor of 2 using a slight variation of the stepping-up lemma from [10]. In particular, starting with a 4-uniform construction of order N with no homogeneous set of order n , such as the one in Subsection 3.4, we get a construction of order $\text{twr}_{k-3}(N)$ with no homogeneous set of order $n + \lceil 5k/2 \rceil - 10$. This follows from Theorem 3 in [10].

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