# Sparse Difference Resultant* 

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#### Abstract

In this paper, the concept of sparse difference resultant for a Laurent transformally essential system of difference polynomials is introduced and a simple criterion for the existence of sparse difference resultant is given. The concept of transformally homogenous polynomial is introduced and the sparse difference resultant is shown to be transformally homogenous. It is shown that the vanishing of the sparse difference resultant gives a necessary condition for the corresponding difference polynomial system to have non-zero solutions. The order and degree bounds for sparse difference resultant are given. Based on these bounds, an algorithm to compute the sparse difference resultant is proposed, which is single exponential in terms of the number of variables, the Jacobi number, and the size of the Laurent transformally essential system. Furthermore, the precise order and degree, a determinant representation, and a Poisson-type product formula for the difference resultant are given.


Keywords. Sparse difference resultant, difference resultant, Laurent transformally essential system, Jacobi number, single exponential algorithm.

## 1 Introduction

The resultant, which gives conditions for an over-determined system of polynomial equations to have common solutions, is a basic concept in algebraic geometry and a powerful tool in elimination theory [3, 8, [10, 19, 21, 33]. The concept of sparse resultant originated from the work of Gelfand, Kapranov, and Zelevinsky on generalized hypergeometric functions, where the central concept of $\mathcal{A}$-discriminant is studied [18]. Kapranov, Sturmfels, and Zelevinsky introduced the concept of $\mathcal{A}$-resultant [22]. Sturmfels further introduced the general mixed sparse resultant and gave a single exponential algorithm to compute the sparse resultant [33, 34. Canny and Emiris showed that the sparse resultant is a factor of the determinant of a Macaulay style matrix and gave an efficient algorithm to compute the sparse resultant based on this matrix representation [12]. A determinant representation for the sparse resultant was given by D'Andrea [9. Recently, a rigorous definition for the differential resultant of $n+1$ generic differential polynomials in $n$ variables was presented [16] and also the theory of sparse

[^0]differential resultants for Laurent differentially essential systems was developed [26, 27]. It is meaningful to generalize the theory of sparse resultant to difference polynomial systems.

In this paper, the concept of sparse difference resultant for a Laurent transformally essential system consisting of $n+1$ Laurent difference polynomials in $n$ difference variables is introduced and its basic properties are proved. A criterion is given to check whether a Laurent difference system is essential in terms of their supports, which is conceptually and computationally simpler than the naive approach based on the characteristic set method. The concept of transformally homogeneous is introduced and it is proved that the sparse difference resultant is transformally homogeneous. It is shown that the vanishing of the sparse difference resultant gives a necessary condition for the corresponding difference polynomial system to have nonzero solutions, which is also sufficient in certain sense. It is also shown that the sparse difference resultant is equal to the algebraic sparse resultant of a generic sparse polynomial system, and hence has a determinant representation.

We also give order and degree bounds for the sparse difference resultant. It is shown that the order and effective order of the sparse difference resultant can be bounded by the Jacobi number of the corresponding difference polynomial system and the degree can be bounded by a Bezout type bound. Based on these bounds, an algorithm is given to compute the sparse difference resultant. The complexity of the algorithm in the worst case is single exponential of the form $O\left(m^{O\left(n l J^{2}\right)}(n J)^{O(l J)}\right)$, where $n, m, J$, and $l$ are the number of variables, the degree, the Jacobi number, and the size of the Laurent transformally essential system, respectively.

For the difference resultant, which is non-sparse, more and better properties are proved including its precise order and degree, a determinant representation, and a Poisson-type product formula.

Although most properties for sparse difference resultants and difference resultants are similar to their differential counterparts given in [26, [27, 16], some of them are quite different in terms of descriptions and proofs due to the distinct nature of the differential and difference operators. Firstly, the definition for difference resultant is more subtle than the differential case as illustrated by Problem [3.16 in this paper. Secondly, the criterion for Laurent transformally essential systems given in Section 3.3 is quite different and much simpler than its differential counterpart given in [27]. Also, determinant representations for the sparse difference resultant and the difference resultant are given in Section 5 and Section 7 , but such a representation is still not known for differential resultants [38, 30, 31]. Finally, there does not exist a definition for homogeneous difference polynomials, and the definition we give in this paper is different from its differential counterpart 25 .

The rest of the paper is organized as follows. In Section 2, we prove some preliminary results. In Section 3, we first introduce the concepts of Laurent difference polynomials and Laurent transformally essential systems, and then define the sparse difference resultant for Laurent transformally essential systems. Basic properties of sparse difference resultant are proved in Section 4. In Section 5, the sparse difference resultant is shown to be the algebraic sparse resultant for certain generic polynomial system. In Section 6, we present an algorithm to compute the sparse difference resultant. In Section 7 , we introduce the notion of difference resultant and prove its basic properties. In Section 8, we conclude the paper by proposing several problems for future research. An extended abstract of this paper appeared in the proceedings of ISSAC2013 [28]. Section 4.4 and Section 5 are newly added.

## 2 Preliminaries

In this section, some basic notations and preliminary results in difference algebra will be given. For more details about difference algebra, please refer to [6, 20, 24, 36].

### 2.1 Difference polynomial ring

An ordinary difference field $\mathcal{F}$ is a field with a third unitary operation $\sigma$ satisfying that for any $a, b \in \mathcal{F}, \sigma(a+b)=\sigma(a)+\sigma(b), \sigma(a b)=\sigma(a) \sigma(b)$, and $\sigma(a)=0$ if and only if $a=0$. We call $\sigma$ the transforming operator of $\mathcal{F}$. If $a \in \mathcal{F}, \sigma(a)$ is called the transform of $a$ and is denoted by $a^{(1)}$. And for $n \in \mathbb{Z}^{+}, \sigma^{n}(a)=\sigma^{n-1}(\sigma(a))$ is called the $n$-th transform of $a$ and denoted by $a^{(n)}$, with the usual assumption $a^{(0)}=a$. By $a^{[n]}$ we mean the set $\left\{a, a^{(1)}, \ldots, a^{(n)}\right\}$. If $\sigma^{-1}(a)$ is defined for each $a \in \mathcal{F}$, we say that $\mathcal{F}$ is inversive. All difference fields in this paper are assumed to be inversive. A typical example of difference field is $\mathbb{Q}(x)$ with $\sigma(f(x))=f(x+1)$.

Let $S$ be a subset of a difference field $\mathcal{G}$ which contains $\mathcal{F}$. We will denote respectively by $\mathcal{F}[S], \mathcal{F}(S), \mathcal{F}\{S\}$, and $\mathcal{F}\langle S\rangle$ the smallest subring, the smallest subfield, the smallest difference subring, and the smallest difference subfield of $\mathcal{G}$ containing $\mathcal{F}$ and $S$. If we denote $\Theta(S)=\left\{\sigma^{k} a \mid k \geq 0, a \in S\right\}$, then we have $\mathcal{F}\{S\}=\mathcal{F}[\Theta(S)]$ and $\mathcal{F}\langle S\rangle=\mathcal{F}(\Theta(S))$.

A subset $\mathcal{S}$ of a difference extension field $\mathcal{G}$ of $\mathcal{F}$ is said to be transformally dependent over $\mathcal{F}$ if the set $\left\{\sigma^{k} a \mid a \in \mathcal{S}, k \geq 0\right\}$ is algebraically dependent over $\mathcal{F}$, and is said to be transformally independent over $\mathcal{F}$, or to be a family of difference indeterminates over $\mathcal{F}$ in the contrary case. In the case $\mathcal{S}$ consists of one element $\alpha$, we say that $\alpha$ is transformally algebraic or transformally transcendental over $\mathcal{F}$, respectively. The maximal subset $\Omega$ of $\mathcal{G}$ which are transformally independent over $\mathcal{F}$ is said to be a transformal transcendence basis of $\mathcal{G}$ over $\mathcal{F}$. We use $\sigma$.tr. $\operatorname{deg} \mathcal{G} / \mathcal{F}$ to denote the difference transcendence degree of $\mathcal{G}$ over $\mathcal{F}$, which is the cardinal number of $\Omega$. Considering $\mathcal{F}$ and $\mathcal{G}$ as ordinary algebraic fields, we denote the algebraic transcendence degree of $\mathcal{G}$ over $\mathcal{F}$ by $\operatorname{tr} . \operatorname{deg} \mathcal{G} / \mathcal{F}$.

Now suppose $\mathbb{Y}=\left\{y_{1}, y_{2}, \ldots, y_{n}\right\}$ is a set of difference indeterminates over $\mathcal{F}$. The elements of $\mathcal{F}\{\mathbb{Y}\}=\mathcal{F}\left[y_{j}^{(k)}: j=1, \ldots, n ; k \in \mathbb{N}_{0}\right]$ are called difference polynomials over $\mathcal{F}$ in $\mathbb{Y}$, and $\mathcal{F}\{\mathbb{Y}\}$ itself is called the difference polynomial ring over $\mathcal{F}$ in $\mathbb{Y}$. A difference ideal $\mathcal{I}$ in $\mathcal{F}\{\mathbb{Y}\}$ is an ordinary algebraic ideal which is closed under transforming, i.e. $\sigma(\mathcal{I}) \subset \mathcal{I}$. If $\mathcal{I}$ also has the property that $a^{(1)} \in \mathcal{I}$ implies that $a \in \mathcal{I}$, it is called a reflexive difference ideal. A prime difference ideal is a difference ideal which is prime as an ordinary algebraic polynomial ideal. For convenience, a prime difference ideal is assumed not to be the unit ideal in this paper. If $S$ is a finite set of difference polynomials, we use $(S)$ and $[S]$ to denote the algebraic ideal and the difference ideal in $\mathcal{F}\{\mathbb{Y}\}$ generated by $S$.

An $n$-tuple over $\mathcal{F}$ is an $n$-tuple of the form $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ where the $a_{i}$ are selected from a difference overfield of $\mathcal{F}$. For a difference polynomial $f \in \mathcal{F}\left\{y_{1}, \ldots, y_{n}\right\}$, a is called a difference zero of $f$ if when substituting $y_{i}^{(j)}$ by $a_{i}^{(j)}$ in $f$, the result is 0 . An $n$-tuple $\eta$ is called a generic zero of a difference ideal $\mathcal{I} \subset \mathcal{F}\{\mathbb{Y}\}$ if for any polynomial $P \in \mathcal{F}\{\mathbb{Y}\}$ we have $P(\eta)=0 \Leftrightarrow P \in \mathcal{I}$. It is well known that

Lemma 2.1 [6, p.77] A difference ideal possesses a generic zero if and only if it is a reflexive prime difference ideal other than the unit ideal.

Let $\mathcal{I}$ be a reflexive prime difference ideal and $\eta$ a generic zero of $\mathcal{I}$. The dimension of $\mathcal{I}$ is defined to be $\sigma \cdot \operatorname{tr} . \operatorname{deg} \mathcal{F}\langle\eta\rangle / \mathcal{F}$.

Given two $n$-tuples $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ and $\overline{\mathbf{a}}=\left(\bar{a}_{1}, \ldots, \bar{a}_{n}\right)$ over $\mathcal{F} . \overline{\mathbf{a}}$ is called a specialization of a over $\mathcal{F}$, or a specializes to $\overline{\mathbf{a}}$, if for any difference polynomial $P \in \mathcal{F}\{\mathbb{Y}\}, P(\mathbf{a})=0$ implies that $P(\overline{\mathbf{a}})=0$. The following property about difference specialization will be needed in this paper.

Lemma 2.2 Let $P_{i}(\mathbb{U}, \mathbb{Y}) \in \mathcal{F}\langle\mathbb{Y}\rangle\{\mathbb{U}\}(i=1, \ldots, m)$ where $\mathbb{U}=\left(u_{1}, \ldots, u_{r}\right)$ and $\mathbb{Y}=$ $\left(y_{1}, \ldots, y_{n}\right)$ are sets of difference indeterminates. If $P_{i}(\mathbb{U}, \mathbb{Y})(i=1, \ldots, m)$ are transformally dependent over $\mathcal{F}\langle\mathbb{U}\rangle$, then for any difference specialization $\mathbb{U}$ to $\overline{\mathbb{U}}$ which are elements in $\mathcal{F}$, $P_{i}(\overline{\mathbb{U}}, \mathbb{Y})(i=1, \ldots, m)$ are transformally dependent over $\mathcal{F}$.

Proof: It suffices to show the case $r=1$. Denote $u=u_{1}$. Since $P_{i}(u, \mathbb{Y})(i=1, \ldots, m)$ are transformally dependent over $\mathcal{F}\langle u\rangle$, there exist natural numbers $s$ and $l$ such that $\mathbb{P}_{i}^{(k)}(u, \mathbb{Y})(k \leq s)$ are algebraically dependent over $\mathcal{F}\left(u^{(k)} \mid k \leq s+l\right)$. When $u$ specializes to $\bar{u} \in \mathcal{F}, u^{(k)}(k \geq 0)$ are correspondingly algebraically specialized to $\bar{u}^{(k)} \in \mathcal{F}$. By [37, p.161], $\mathbb{P}_{i}^{(k)}(\bar{u}, \mathbb{Y})(k \leq s)$ are algebraically dependent over $\mathcal{F}$. Thus, $P_{i}(\bar{u}, \mathbb{Y})(i=1, \ldots, m)$ are transformally dependent over $\mathcal{F}$. I

### 2.2 Characteristic set for a difference polynomial system

In this section, we prove several preliminary results about the characteristic set for a difference polynomial system. For details on difference characteristic set method, please refer to [17.

Let $f$ be a difference polynomial in $\mathcal{F}\{\mathbb{Y}\}$. The order of $f$ w.r.t. $y_{i}$ is defined to be the greatest number $k$ such that $y_{i}^{(k)}$ appears effectively in $f$, denoted by ord $\left(f, y_{i}\right)$. And if $y_{i}$ does not appear in $f$, then we set ord $\left(f, y_{i}\right)=-\infty$. The order of $f$ is defined to be $\max _{i} \operatorname{ord}\left(f, y_{i}\right)$, that is, $\operatorname{ord}(f)=\max _{i} \operatorname{ord}\left(f, y_{i}\right)$.

A ranking $\mathscr{R}$ is a total order over $\Theta(\mathbb{Y})=\left\{\sigma^{k} y_{i} \mid 1 \leq i \leq n, k \geq 0\right\}$, which satisfies the following properties:

1) $\sigma(\theta)>\theta$ for all derivatives $\theta \in \Theta(\mathbb{Y})$.
2) $\theta_{1}>\theta_{2} \Longrightarrow \sigma\left(\theta_{1}\right)>\sigma\left(\theta_{2}\right)$ for $\theta_{1}, \theta_{2} \in \Theta(\mathbb{Y})$.

Let $f$ be a difference polynomial in $\mathcal{F}\{\mathbb{Y}\}$ and $\mathscr{R}$ a ranking endowed on it. The greatest $y_{j}^{(k)}$ w.r.t. $\mathscr{R}$ which appears effectively in $f$ is called the leader of $p$, denoted by $\operatorname{ld}(f)$ and correspondingly $y_{j}$ is called the leading variable of $f$, denoted by $\operatorname{lvar}(f)=y_{j}$. The leading coefficient of $f$ as a univariate polynomial in $\operatorname{ld}(f)$ is called the initial of $f$ and is denoted by $\mathrm{I}_{f}$.

Let $p$ and $q$ be two difference polynomials in $\mathcal{F}\{\mathbb{Y}\} . q$ is said to be of higher rank than $p$ if

1) $\operatorname{ld}(q)>\operatorname{ld}(p)$, or
2) $\operatorname{ld}(q)=\operatorname{ld}(p)=y_{j}^{(k)}$ and $\operatorname{deg}\left(q, y_{j}^{(k)}\right)>\operatorname{deg}\left(p, y_{j}^{(k)}\right)$.

Suppose $\operatorname{ld}(p)=y_{j}^{(k)}$. Then $q$ is said to be reduced w.r.t. $p$ if $\operatorname{deg}\left(q, y_{j}^{(k+l)}\right)<\operatorname{deg}\left(p, y_{j}^{(k)}\right)$ for all $l \in \mathbb{N}_{0}$.

A finite chain of nonzero difference polynomials $\mathcal{A}=A_{1}, \ldots, A_{m}$ is said to be an ascending chain if

1) $m=1$ and $A_{1} \neq 0$ or
2) $m>1, A_{j}>A_{i}$ and $A_{j}$ is reduced w.r.t. $A_{i}$ for $1 \leq i<j \leq m$.

Let $\mathcal{A}=A_{1}, A_{2}, \ldots, A_{t}$ be an ascending chain with $\mathrm{I}_{i}$ as the initial of $A_{i}$, and $f$ any difference polynomial. Then there exists an algorithm, which reduces $f$ w.r.t. $\mathcal{A}$ to a polynomial $r$ that is reduced w.r.t. $\mathcal{A}$, satisfying the relation

$$
\prod_{i=1}^{t} \prod_{k=0}^{d_{i}}\left(\sigma^{k} I_{i}\right)^{e_{i k}} \cdot f \equiv r, \bmod [\mathcal{A}]
$$

where the $e_{i k}$ are nonnegative integers. The difference polynomial $r$ is called the difference remainder of $f$ w.r.t. $\mathcal{A}$ [17].

Let $\mathcal{A}$ be an ascending chain. Denote $\mathbb{I}_{\mathcal{A}}$ to be the minimal multiplicative set containing the initials of elements of $\mathcal{A}$ and their transforms. The saturation ideal of $\mathcal{A}$ is defined to be

$$
\operatorname{sat}(\mathcal{A})=[\mathcal{A}]: \mathbb{I}_{\mathcal{A}}=\left\{p: \exists h \in \mathbb{I}_{\mathcal{A}} \text {, s.t. hp } \in[A]\right\} .
$$

And the algebraic saturation ideal of $\mathcal{A}$ is $\operatorname{asat}(\mathcal{A})=(\mathcal{A}): \mathrm{I}_{\mathcal{A}}$, where $\mathrm{I}_{\mathcal{A}}$ is the minimal multiplicative set containing the initials of elements of $\mathcal{A}$.

An ascending chain $\mathcal{C}$ contained in a difference polynomial set $\mathcal{S}$ is said to be a characteristic set of $\mathcal{S}$, if $\mathcal{S}$ does not contain any nonzero element reduced w.r.t. $\mathcal{C}$. A characteristic set $\mathcal{C}$ of a difference ideal $\mathcal{J}$ reduces all elements of $\mathcal{J}$ to zero.

Let $\mathcal{A}$ be a characteristic set of a reflexive prime difference ideal $\mathcal{I}$. We rewrite $\mathcal{A}$ in the following form

$$
\mathcal{A}=\left\{\begin{array}{l}
A_{11}, \ldots, A_{1 k_{1}} \\
\ldots \\
A_{p 1}, \ldots, A_{p k_{p}}
\end{array}\right.
$$

where $\operatorname{lvar}\left(A_{i j}\right)=y_{c_{i}}$ for $j=1, \ldots, k_{i}$ and $\operatorname{ord}\left(A_{i j}, y_{c_{i}}\right)<\operatorname{ord}\left(A_{i l}, y_{c_{i}}\right)$ for $j<l$. In terms of the characteristic set of the above form, $p$ is equal to the codimension of $\mathcal{I}$, that is $n-\operatorname{dim}(\mathcal{I})$. Unlike the differential case, here even though $\mathcal{I}$ is of codimension one, there may be more than one difference polynomials in a characteristic set of $\mathcal{I}$ as shown by the following example.

Example 2.3 Let $A_{11}=\left(y_{1}^{(1)}\right)^{2}+y_{1}^{2}+1, A_{12}=y_{1}^{(2)}-y_{1}$. Then $\mathcal{I}=\left[A_{11}, \mathcal{A}_{12}\right]$ is a reflexive prime difference ideal whose characteristic set is $\mathcal{A}=A_{11}, A_{12}$ and $\mathcal{I}=\operatorname{sat}(\mathcal{A})$ [17]. Note that $\left[A_{11}\right]$ is not a prime difference ideal, because $\sigma\left(A_{11}\right)-A_{11}=\left(y_{1}^{(2)}-y_{1}\right)\left(y_{1}^{(2)}+y_{1}\right) \in\left[A_{11}\right]$ and both $y_{1}^{(2)}-y_{1}$ and $y_{1}^{(2)}+y_{1}$ are not in $\left[A_{11}\right]$.

Now we proceed to show that a property of uniqueness still exists in characteristic sets of a reflexive prime difference ideal in some sense. Firstly, we need several algebraic results.

Let $\mathcal{B}=B_{1}, \ldots, B_{m}$ be an algebraic triangular set in $\mathcal{F}\left[x_{1}, \ldots, x_{n}\right]$ with $\operatorname{lvar}\left(B_{i}\right)=y_{i}$ and $U=\left\{x_{1}, \ldots, x_{n}\right\} \backslash\left\{y_{1}, \ldots, y_{m}\right\}$. We assume $U<y_{1}<y_{2}<\ldots<y_{m}$. A polynomial $f$ is said to be invertible w.r.t. $\mathcal{B}$ if $\left(f, B_{1}, \ldots, B_{s}\right) \cap K[U] \neq\{0\}$ where $\operatorname{lvar}(f)=\operatorname{lvar}\left(B_{s}\right)$. We call $\mathcal{B}$ a regular chain if for each $i>1$, the initial of $B_{i}$ is invertible w.r.t. $B_{1}, \ldots, B_{i-1}$. For a regular chain $\mathcal{B}$, we say that $f$ is invertible w.r.t. asat $(\mathcal{B})$ if $(f$, asat $(\mathcal{B})) \cap \mathcal{F}[U] \neq\{0\}$. The next two lemmas use the notations introduced in this paragraph.

Lemma 2.4 Let $\mathcal{B}$ be a regular chain in $\mathcal{F}\left[x_{1}, \ldots, x_{n}\right]$. If $\sqrt{\operatorname{asat}(\mathcal{B})}=\bigcap_{i=1}^{m} \mathcal{P}_{i}$ is an irredundant prime decomposition of $\sqrt{\operatorname{asat}(\mathcal{B})}$, then a polynomial $f$ is invertible w.r.t. $\operatorname{asat}(\mathcal{B})$ if and only if $f \notin \mathcal{P}_{i}$ for all $i=1, \ldots, m$.

Proof: Since $\sqrt{\operatorname{asat}(\mathcal{B})}=\bigcap_{i=1}^{m} \mathcal{P}_{i}$ is an irredundant prime decomposition of $\sqrt{\operatorname{asat}(\mathcal{B})}, U$ is a parametric set of $\mathcal{P}_{i}$ for each $i$ by paper [15]. And for prime ideals $\mathcal{P}_{i}, f \notin \mathcal{P}_{i}$ if and only if $\left(f, \mathcal{P}_{i}\right) \cap \mathcal{F}[U] \neq\{0\}$. If $f$ is invertible w.r.t. asat $(\mathcal{B}),\{0\} \neq(f$, $\operatorname{asat}(\mathcal{B})) \cap \mathcal{F}[U] \subset$ $\left(f, \mathcal{P}_{i}\right) \cap \mathcal{F}[U]$. Thus, $f \notin \mathcal{P}_{i}$ for each $i$. For the other side, suppose $f \notin \mathcal{P}_{i}$ for all $i$, then there exist nonzero polynomials $h_{i}(U)$ such that $h_{i}(U) \in\left(f, \mathcal{P}_{i}\right)$. Thus, there exists $t \in \mathbb{N}$ such that $\left(\prod_{i=1}^{m} h_{i}(U)\right)^{t} \in(f, \operatorname{asat}(\mathcal{B}))$. So $f$ is invertible w.r.t. asat $(\mathcal{B})$.I

Lemma 2.5 [2] Let $\mathcal{B}$ be a regular chain in $\mathcal{F}[U, Y]$. Let $f$ be a polynomial in $\mathcal{F}[U, Y]$ and $L$ in $\mathcal{F}[U] \backslash\{0\}$ such that $L f \in(\mathcal{B})$. Then $f \in \operatorname{asat}(\mathcal{B})$.

Lemma 2.6 Let $A$ be an irreducible difference polynomial in $\mathcal{F}\{\mathbb{Y}\}$ with $\operatorname{deg}\left(A, y_{i_{0}}\right)>0$ for some $i_{0}$. If $f$ is invertible w.r.t. $A^{[k]}=A, A^{(1)}, \ldots, A^{(k)}$ under some ranking $\mathscr{R}$, then $\sigma(f)$ is invertible w.r.t. $A^{[k+1]}=A, \ldots, A^{(k+1)}$. In particular, $A^{[k]}$ is a regular chain for any $k \geq 0$.

Proof: Since as a difference ascending chain, $A$ is coherent and proper irreducible, by Theorem 4.1 in paper [17], $A$ is difference regular. As a consequence, $A^{[k]}$ is regular for any $k \geq 0$.

The following fact is needed to define sparse difference resultant.
Lemma 2.7 Let $\mathcal{I}$ be a reflexive prime difference ideal of codimension one in $\mathcal{F}\{\mathbb{Y}\}$. The first element in any characteristic set of $\mathcal{I}$ w.r.t. any ranking, when taken irreducible, is unique up to a factor in $\mathcal{F}$.

Proof: Let $\mathcal{A}=A_{1}, \ldots, A_{m}$ be a characteristic set of $\mathcal{I}$ w.r.t. some ranking $\mathscr{R}$ with $A_{1}$ irreducible. Suppose $\operatorname{lvar}(\mathcal{A})=y_{1}$. Given another characteristic set $\mathcal{B}=B_{1}, \ldots, B_{l}$ of $\mathcal{I}$ w.r.t. some other ranking $\mathscr{R}^{\prime}$ ( $B_{1}$ is irreducible), we need to show that there exists $c \in \mathcal{F}$ such that $B_{1}=c \cdot A_{1}$. It suffices to consider the case $\operatorname{lvar}(\mathcal{B}) \neq y_{1}$. Suppose $\operatorname{lvar}\left(B_{1}\right)=y_{2}$. Clearly, $y_{2}$ appears effectively in $A_{1}$ for $\mathcal{B}$ reduces $A_{1}$ to 0 . And since $\mathcal{I}$ is reflexive, there exists some $i_{0}$ such that $\operatorname{deg}\left(A_{1}, y_{i_{0}}\right)>0$.

Suppose $\operatorname{ord}\left(A_{1}, y_{2}\right)=o_{2}$. Take another ranking under which $y_{2}^{\left(o_{2}\right)}$ is the leader of $A_{1}$ and we use $\widetilde{A}_{1}$ to distinguish it from the $A_{1}$ under $\mathscr{R}$. By Lemma [2.6, for each $k, A_{1}^{[k]}$ and $\widetilde{A}_{1}^{[k]}$ are regular chains.

Now we claim that asat $\left(A_{1}^{[k]}\right)=\operatorname{asat}\left(\widetilde{A}_{1}^{[k]}\right)$ for any $k$. On the one hand, for any polynomial $f \in \operatorname{asat}\left(A_{1}^{[k]}\right)$, we have $\left(\prod_{i=0}^{k} \sigma^{i}\left(\mathrm{I}_{A_{1}}\right)\right)^{a} f \in\left(A_{1}^{[k]}\right)$. Since $\mathrm{I}_{A_{1}}$ is invertible w.r.t. $\widetilde{A}_{1}$, by Lemma [2.6] $\left(\prod_{i=0}^{k} \sigma^{i}\left(\mathrm{I}_{A_{1}}\right)\right)^{a}$ is invertible w.r.t. $\widetilde{A}_{1}^{k k]}$. Denote the parameters of $\widetilde{A}_{1}^{[k]}$ by $\widetilde{U}$. So there exists a nonzero polynomial $h(\widetilde{U})$ such that $h(\widetilde{U}) \in\left(\left(\prod_{i=0}^{k} \sigma^{i}\left(\mathrm{I}_{A_{1}}\right)\right)^{a}, \widetilde{A}_{1}^{[k]}\right)$. Thus, $h(\widetilde{U}) f \in\left(\widetilde{A}_{1}^{[k]}\right)$. Since $\widetilde{A}_{1}^{[k]}$ is a regular chain, by Lemma [2.5, $f \in \operatorname{asat}\left(\widetilde{A}_{1}^{[k]}\right)$. So asat $\left(A_{1}^{[k]}\right) \subseteq$ $\operatorname{asat}\left(\widetilde{A}_{1}^{[k]}\right)$. Similarly, we can show that asat $\left(\widetilde{A}_{1}^{[k]}\right) \subseteq \operatorname{asat}\left(A_{1}^{[k]}\right)$. Thus, asat $\left(A_{1}^{[k]}\right)=\operatorname{asat}\left(\widetilde{A}_{1}^{[k]}\right)$.

Suppose ord $\left(B_{1}, y_{2}\right)=o_{2}^{\prime}$. Clearly, $o_{2} \geq o_{2}^{\prime}$. We now proceed to show that it is impossible for $o_{2}>o_{2}^{\prime}$. Suppose the contrary, i.e. $o_{2}>o_{2}^{\prime}$. Then $B_{1}$ is invertible w.r.t. asat $\left(\widetilde{A}_{1}^{[k]}\right)$.

Suppose $\sqrt{\operatorname{asat}\left(\widetilde{A}_{1}^{[k]}\right)}=\bigcap_{i=1}^{t} \mathcal{P}_{i}$ is an irredundant prime decomposition. By Lemma [2.4, $B_{1} \notin \mathcal{P}_{i}$ for each $i$. Since $\operatorname{asat}\left(A_{1}^{[k]}\right)=\operatorname{asat}\left(\widetilde{A}_{1}^{[k]}\right)$, using Lemma [2.4 again, $B_{1}$ is invertible w.r.t. $\operatorname{asat}\left(A_{1}^{[k]}\right)$. Thus, there exists a nonzero difference polynomial $H$ with $\operatorname{ord}\left(H, y_{1}\right)<$ $\operatorname{ord}\left(A_{1}, y_{1}\right)$ such that $H \in\left(B_{1}, \operatorname{asat}\left(A_{1}^{[k]}\right)\right) \subset \mathcal{I}$, which is a contradiction. Thus, $o_{2}=o_{2}^{\prime}$. Since $\mathcal{B}$ reduces $A_{1}$ to zero and $A_{1}$ is irreducible, there exists $c \in \mathcal{F}$ such that $B_{1}=c \cdot A_{1}$. I

## 3 Sparse difference resultant

In this section, the concepts of Laurent difference polynomials and Laurent transformally essential systems are first introduced, and then the sparse difference resultant for Laurent transformally essential systems is defined. A criterion for a Laurent polynomial system to be Laurent transformally essential in terms of the support of the given system is also given.

### 3.1 Laurent difference polynomial

Let $\mathcal{F}$ be an ordinary difference field with a transforming operator $\sigma$ and $\mathcal{F}\{\mathbb{Y}\}$ the ring of difference polynomials in the difference indeterminates $\mathbb{Y}=\left\{y_{1}, \ldots, y_{n}\right\}$. Before defining sparse difference resultant, we first introduce the concept of Laurent difference polynomials.

Definition 3.1 A Laurent difference monomial of order $s$ is a Laurent monomial in variables $\mathbb{Y}^{[s]}=\left(y_{i}^{(k)}\right)_{1 \leq i \leq n ; 0 \leq k \leq s}$. More precisely, it has the form $\prod_{i=1}^{n} \prod_{k=0}^{s}\left(y_{i}^{(k)}\right)^{d_{i k}}$ where $d_{i k}$ are integers which can be negative. A Laurent difference polynomial over $\mathcal{F}$ is a finite linear combination of Laurent difference monomials with coefficients in $\mathcal{F}$.

Clearly, the collections of all Laurent difference polynomials form a commutative difference ring under the obvious sum, product, and the usual transforming operator $\sigma$, where all Laurent difference monomials are invertible. We denote the difference ring of Laurent difference polynomials with coefficients in $\mathcal{F}$ by $\mathcal{F}\left\{y_{1}, y_{1}^{-1}, \ldots, y_{n}, y_{n}^{-1}\right\}$, or simply by $\mathcal{F}\left\{\mathbb{Y}, \mathbb{Y}^{-1}\right\}$.

Definition 3.2 For every Laurent difference polynomial $F \in \mathcal{F}\left\{\mathbb{Y}, \mathbb{Y}^{-1}\right\}$, there exists a unique Laurent difference monomial $M$ such that 1) $M \cdot F \in \mathcal{F}\{\mathbb{Y}\}$ and 2) for any Laurent difference monomial $T$ with $T \cdot F \in \mathcal{F}\{\mathbb{Y}\}, T \cdot F$ is divisible by $M \cdot F$ as polynomials. This $M \cdot F$ is defined to be the norm form of $F$, denoted by $\mathrm{N}(F)$. The order and degree of $\mathrm{N}(F)$ is defined to be the order and degree of $F$, denoted by $\operatorname{ord}(F)$ and $\operatorname{deg}(F)$.

In the following, we consider zeros for Laurent difference polynomials.
Definition 3.3 Let $F$ be a Laurent difference polynomial in $\mathcal{F}\left\{\mathbb{Y}, \mathbb{Y}^{-1}\right\}$. An n-tuple ( $a_{1}$, $\left.\ldots, a_{n}\right)$ over $\mathcal{F}$ with each $a_{i} \neq 0$ is called a nonzero difference solution of $F$ if $F\left(a_{1}, \ldots, a_{n}\right)=$ 0.

For an ideal $\mathcal{I} \in \mathcal{F}\left\{\mathbb{Y}, \mathbb{Y}^{-1}\right\}$, the difference zero set of $\mathcal{I}$ is the set of common nonzero difference zeros of all Laurent difference polynomials in $\mathcal{I}$. We will see later in Example 4.5, how nonzero difference solutions are naturally related with the sparse difference resultant.

### 3.2 Definition of sparse difference resultant

In this section, the definition of the sparse difference resultant will be given. Similar to the study of sparse resultants and sparse differential resultants, we first define sparse difference resultants for Laurent difference polynomials whose coefficients are difference indeterminates. Then the sparse difference resultant for a given Laurent difference polynomial system with concrete coefficients is the value which the resultant in the generic case assumes for the given case.

Suppose $\mathcal{A}_{i}=\left\{M_{i 0}, M_{i 1}, \ldots, M_{i l_{i}}\right\}(i=0,1, \ldots, n)$ are finite sets of Laurent difference monomials in $\mathbb{Y}$. Consider $n+1$ generic Laurent difference polynomials defined over $\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}$ :

$$
\begin{equation*}
\mathbb{P}_{i}=\sum_{k=0}^{l_{i}} u_{i k} M_{i k}(i=0, \ldots, n), \tag{1}
\end{equation*}
$$

where all the $u_{i k}$ are transformally independent over the rational number field $\mathbb{Q}$. Denote

$$
\begin{equation*}
\mathbf{u}_{i}=\left(u_{i 0}, u_{i 1}, \ldots, u_{i l_{i}}\right)(i=0, \ldots, n) \text { and } \mathbf{u}=\bigcup_{i=0}^{n} \mathbf{u}_{i} \backslash\left\{u_{i 0}\right\} \tag{2}
\end{equation*}
$$

The number $l_{i}+1$ is called the size of $\mathbb{P}_{i}$ and $\mathcal{A}_{i}$ is called the support of $\mathbb{P}_{i}$. To avoid the triviality, $l_{i} \geq 1(i=0, \ldots, n)$ are always assumed in this paper.

Definition 3.4 A set of Laurent difference polynomials of the form (1) is called Laurent transformally essential if there exist $k_{i}(i=0, \ldots, n)$ with $1 \leq k_{i} \leq l_{i}$ such that $\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle\frac{M_{0 k_{0}}}{M_{00}}\right.$, $\left.\frac{M_{1 k_{1}}}{M_{10}}, \ldots, \frac{M_{n k_{n}}}{M_{n 0}}\right\rangle / \mathbb{Q}=n$. In this case, we also say that $\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}$ form a Laurent transformally essential system.

Although $M_{i 0}$ are used as denominators to define transformally essential system, the following lemma shows that the definition does not depend on the choices of $M_{i 0}$.

Lemma 3.5 The following two conditions are equivalent.

1. There exist $k_{i}(i=0, \ldots, n)$ with $1 \leq k_{i} \leq l_{i}$ such that $\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle\frac{M_{0 k_{0}}}{M_{00}}, \ldots, \frac{M_{n k_{n}}}{M_{n 0}}\right\rangle / \mathbb{Q}$ $=n$.
2. There exist pairs $\left(k_{i}, j_{i}\right)(i=0, \ldots, n)$ with $k_{i} \neq j_{i} \in\left\{0, \ldots, l_{i}\right\}$ such that $\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle\frac{M_{0 k_{0}}}{M_{0 j_{0}}}, \ldots, \frac{M_{n k_{n}}}{M_{n j_{n}}}\right\rangle / \mathbb{Q}=n$.

Proof: Similar to the proof of [27, Lemma 3.7], it can be easily shown.I
Let $\mathbf{m}$ be the set of all difference monomials in $\mathbb{Y}$ and $\left[\mathrm{N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)\right]$ the difference ideal generated by $\mathrm{N}\left(\mathbb{P}_{i}\right)$ in $\mathbb{Q}\left\{\mathbb{Y}, \mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$. Let

$$
\begin{align*}
\mathcal{I}_{\mathbb{Y}, \mathbf{u}} & =\left(\left[\mathbb{N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)\right]: \mathbf{m}\right),  \tag{3}\\
\mathcal{I}_{\mathbf{u}} & =\mathcal{I}_{\mathbb{Y}, \mathbf{u}} \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\} . \tag{4}
\end{align*}
$$

The following result is a foundation for defining sparse difference resultants.

Theorem 3.6 Let $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$ be the Laurent difference polynomials defined in (1). Then the following assertions hold.

1. $\mathcal{I}_{\mathbb{Y}, \mathbf{u}}$ is a reflexive prime difference ideal in $\mathbb{Q}\left\{\mathbb{Y}, \mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$.
2. $\mathcal{I}_{\mathbf{u}}$ is of codimension one if and only if $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$ form a Laurent transformally essential system.

Proof: Let $\eta=\left(\eta_{1}, \ldots, \eta_{n}\right)$ be a sequence of transformally independent elements over $\mathbb{Q}\langle\mathbf{u}\rangle$, where $\mathbf{u}$ is defined in (2). Let

$$
\begin{equation*}
\zeta_{i}=-\sum_{k=1}^{l_{i}} u_{i k} \frac{M_{i k}(\eta)}{M_{i 0}(\eta)}(i=0,1, \ldots, n) \tag{5}
\end{equation*}
$$

We claim that $\theta=\left(\eta ; \zeta_{0}, u_{01}, \ldots, u_{0 l_{0}} ; \ldots ; \zeta_{n}, u_{n 1}, \ldots, u_{n l_{n}}\right)$ is a generic zero of $\mathcal{I}_{\mathbb{Y}, \mathbf{u}}$, which follows that $\mathcal{I}_{\mathbb{Y}, \mathbf{u}}$ is a reflexive prime difference ideal.

Denote $\mathrm{N}\left(\mathbb{P}_{i}\right)=M_{i} \mathbb{P}_{i}(i=0, \ldots, n)$ where $M_{i}$ are Laurent difference monomials. Clearly, $\mathrm{N}\left(\mathbb{P}_{i}\right)=M_{i} \mathbb{P}_{i}$ vanishes at $\theta(i=0, \ldots, n)$. For any $f \in \mathcal{I}_{\mathbb{Y}, \mathbf{u}}$, there exists an $M \in \mathbf{m}$ such that $M f \in\left[\mathrm{~N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(P_{n}\right)\right]$. It follows that $f(\theta)=0$. Conversely, let $f$ be any difference polynomial in $\mathbb{Q}\left\{\mathbb{Y}, \mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$ satisfying $f(\theta)=0$. Clearly, $\mathrm{N}\left(\mathbb{P}_{0}\right), \mathrm{N}\left(\mathbb{P}_{1}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)$ constitute an ascending chain with $u_{i 0}$ as leaders. Let $f_{1}$ be the difference remainder of $f$ w.r.t. this ascending chain. Then $f_{1}$ is free from $u_{i 0}(i=0, \ldots, n)$ and there exist $a, s \in \mathbb{N}$ such that $\left(\prod_{i=0}^{n} \prod_{l=0}^{s}\left(\sigma^{l}\left(M_{i} M_{i 0}\right)\right)\right)^{a} \cdot f \equiv f_{1}, \bmod \left[\mathrm{~N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(P_{n}\right)\right]$. Clearly, $f_{1}(\theta)=0$. Since $f_{1} \in \mathbb{Q}\{\mathbf{u}, \mathbb{Y}\}, f_{1}=0$. Thus, $f \in \mathcal{I}_{\mathbb{Y}, \mathbf{u}}$. So $\mathcal{I}_{\mathbb{Y}, \mathbf{u}}$ is a reflexive prime difference ideal with a generic zero $\theta$.

Consequently, $\mathcal{I}_{\mathbf{u}}=\mathcal{I}_{\mathbb{Y}, \mathbf{u}} \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$ is a reflexive prime difference ideal with a generic zero $\zeta=\left(\zeta_{0}, u_{01}, \ldots, u_{0 l_{0}} ; \ldots ; \zeta_{n}, u_{n 1}, \ldots, u_{n l_{n}}\right)$. From (5), it is clear that $\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\langle\zeta\rangle / \mathbb{Q} \leq$ $\sum_{i=0}^{n} l_{i}+n$. If there exist pairs $\left(i_{k}, j_{k}\right)(k=1, \ldots, n)$ with $1 \leq j_{k} \leq l_{i_{k}}$ and $i_{k_{1}} \neq i_{k_{2}}\left(k_{1} \neq\right.$ $\left.k_{2}\right)$ such that $\frac{M_{i_{1} j_{1}}}{M_{i_{1} 0}}, \ldots, \frac{M_{i_{n j n}}}{M_{i_{n} 0}}$ are transformally independent over $\mathbb{Q}$, then by Lemma 2.2, $\zeta_{i_{1}}, \ldots, \zeta_{i_{n}}$ are transformally independent over $\mathbb{Q}\langle\mathbf{u}\rangle$. It follows that $\sigma \cdot \operatorname{tr} . \operatorname{deg} \mathbb{Q}\langle\zeta\rangle / \mathbb{Q}=$ $\sum_{i=0}^{n} l_{i}+n$. Thus, $\mathcal{I}_{\mathbf{u}}$ is of codimension 1 .

Conversely, let us assume that $\mathcal{I}_{\mathbf{u}}$ is of codimension 1 . That is, $\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\langle\zeta\rangle / \mathbb{Q}=$ $\sum_{i=0}^{n} l_{i}+n$. We want to show that there exist pairs $\left(i_{k}, j_{k}\right)(k=1, \ldots, n)$ with $1 \leq$ $j_{k} \leq l_{i_{k}}$ and $i_{k_{1}} \neq i_{k_{2}}\left(k_{1} \neq k_{2}\right)$ such that $\frac{M_{i_{1} j_{1}}}{M_{i_{1} 0}}, \ldots, \frac{M_{i_{n} j_{n}}}{M_{i_{n} 0}}$ are transformally independent over $\mathbb{Q}$. Suppose the contrary, i.e., $\frac{M_{i_{1} j_{1}}(\eta)}{M_{i_{1} 0}(\eta)}, \ldots, \frac{M_{i_{n j n}}(\eta)}{M_{i_{n} 0}(\eta)}$ are transformally dependent for any $n$ different $i_{k}$ and $j_{k} \in\left\{1, \ldots, l_{i_{k}}\right\}$. Since each $\zeta_{i_{k}}$ is a linear combination of $\frac{M_{i_{k} j_{k}}(\eta)}{M_{i_{k} 0}(\eta)}$ $\left(j_{k}=1, \ldots, l_{i_{k}}\right)$, it follows that $\zeta_{i_{1}}, \ldots, \zeta_{i_{n}}$ are transformally dependent over $\mathbb{Q}\langle\mathbf{u}\rangle$. Thus, we have $\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\langle\zeta\rangle / \mathbb{Q}<\sum_{i=0}^{n} l_{i}+n$, a contradiction to the hypothesis. $\boldsymbol{I}$

Let $\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right]$ be the difference ideal in $\mathbb{Q}\left\{\mathbb{Y}, \mathbb{Y}^{-1} ; \mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$ generated by $\mathbb{P}_{i}$. Then we have

Corollary 3.7 $\mathcal{I}_{\mathbf{u}}=\left[\mathbb{P}_{0}, \mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$ is a reflexive prime difference ideal of codimension one if and only if $\left\{\mathbb{P}_{i}: i=0, \ldots, n\right\}$ is a Laurent transformally essential system.

Proof: It is easy to show that $\left[\mathbb{P}_{0}, \mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}=\mathcal{I}_{\mathbb{Y}, \mathbf{u}} \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}=\mathcal{I}_{\mathbf{u}}$. And the result is a direct consequence of Theorem 3.6. I

Now suppose $\left\{\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right\}$ is a Laurent transformally essential system. Since $\mathcal{I}_{\mathbf{u}}$ defined in (4) is a reflexive prime difference ideal of codimension one, by Lemma 2.7, there exists a unique irreducible difference polynomial $\mathbf{R}\left(\mathbf{u} ; u_{00}, \ldots, u_{n 0}\right)=\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right) \in \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$ such that $\mathbf{R}$ can serve as the first polynomial in each characteristic set of $\mathcal{I}_{\mathbf{u}}$ w.r.t. any ranking endowed on $\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}$. That is, if $u_{i 0}$ appears in $\mathbf{R}$, then among all the difference polynomials in $\mathcal{I}_{\mathbf{u}}, \mathbf{R}$ is of minimal order in $u_{i 0}$ and of minimal degree with the same order.

Now the definition of sparse difference resultant is given as follows:
Definition 3.8 The above $\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right) \in \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$ is defined to be the sparse difference resultant of the Laurent transformally essential system $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$, denoted by $\operatorname{Res}_{\mathcal{A}_{0}}, \ldots, \mathcal{A}_{n}$ or $\operatorname{Res}_{\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}}$. When all the $\mathcal{A}_{i}$ are equal to the same $\mathcal{A}$, we simply denote it by $\operatorname{Res}_{\mathcal{A}}$.

The following lemma gives another description of sparse difference resultant from the perspective of generic zeros.

Lemma 3.9 Let $\zeta_{i}=-\sum_{k=1}^{l_{i}} u_{i k} \frac{M_{i k}(\eta)}{M_{i 0}(\eta)}(i=0,1, \ldots, n)$ be defined as in equation (5), where $\eta=\left(\eta_{1}, \ldots, \eta_{n}\right)$ is a generic zero of $[0]$ over $\mathbb{Q}\langle\mathbf{u}\rangle$. Then among all the polynomials in $\mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$ vanishing at $\left(\mathbf{u} ; \zeta_{0}, \ldots, \zeta_{n}\right), \mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)=\mathbf{R}\left(\mathbf{u} ; u_{00}, \ldots, u_{n 0}\right)$ is of minimal order and degree in each $u_{i 0}(i=0, \ldots, n)$. Equivalently, among all the polynomials in $\mathcal{I}_{\mathbf{u}}$, $\mathbf{R}$ is of minimal order and degree in each $u_{i 0}(i=0, \ldots, n)$.

Proof: It is a direct consequence of Theorem 3.6 and Definition 3.8]
Remark 3.10 From its definition, the sparse difference resultant can be computed as follows. With the characteristic set method given in [17], we can compute a proper irreducible ascending chain $\mathcal{A}$ which is a characteristic set for the difference polynomial system $\left\{\mathbb{P}_{0}, \mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right\}$ under a ranking such that $u_{i j}<y_{k}$. Then the first difference polynomial in $\mathcal{A}$ is the sparse difference resultant. This algorithm does not have a complexity analysis. In Section 5, we will give a single exponential algorithm to compute the sparse difference resultant.

We give several examples which will be used throughout the paper.
Example 3.11 Let $n=1$ and $\mathbb{P}_{0}=u_{00}+u_{01} y_{1}^{2}, \mathbb{P}_{1}=u_{10} y_{1}^{(1)}+u_{11} y_{1}$. Clearly, $\mathbb{P}_{0}, \mathbb{P}_{1}$ are Laurent transformally essential. The sparse difference resultant of $\mathbb{P}_{0}, \mathbb{P}_{1}$ is

$$
\mathbf{R}=u_{10}^{2} u_{01} u_{00}^{(1)}-u_{11}^{2} u_{00} u_{01}^{(1)} .
$$

Example 3.12 Let $n=2$ and the $\mathbb{P}_{i}$ have the form

$$
\mathbb{P}_{i}=u_{i 0} y_{1}^{(2)}+u_{i 1} y_{1}^{(3)}+u_{i 2} y_{2}^{(3)}(i=0,1,2)
$$

It is easy to show that $y_{1}^{(3)} / y_{1}^{(2)}$ and $y_{2}^{(3)} / y_{1}^{(2)}$ are transformally independent over $\mathbb{Q}$. Thus, $\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}$ form a Laurent transformally essential system. The sparse difference resultant is

$$
\mathbf{R}=\operatorname{Res}_{\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}}=\left|\begin{array}{lll}
u_{00} & u_{01} & u_{02} \\
u_{10} & u_{11} & u_{12} \\
u_{20} & u_{21} & u_{22}
\end{array}\right| .
$$

The following example shows that for a Laurent transformally essential system, its sparse difference resultant may not involve the coefficients of some $\mathbb{P}_{i}$.

Example 3.13 Let $n=2$ and the $\mathbb{P}_{i}$ have the form

$$
\mathbb{P}_{0}=u_{00}+u_{01} y_{1} y_{2}, \mathbb{P}_{1}=u_{10}+u_{11} y_{1}^{(1)} y_{2}^{(1)}, \mathbb{P}_{2}=u_{20}+u_{21} y_{2} .
$$

Clearly, $\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}$ form a Laurent transformally essential system. The sparse difference resultant of $\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}$ is

$$
\mathbf{R}=u_{00}^{(1)} u_{11}-u_{01}^{(1)} u_{10},
$$

which is free from the coefficients of $\mathbb{P}_{2}$.
Example 3.13 can be used to illustrate the difference between the differential and difference cases. If $\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}$ in Example 3.13 are differential polynomials, then the sparse differential resultant is $u_{01}^{2} u_{10} u_{20}^{2} u_{21}^{2}-u_{01} u_{00}^{\prime} u_{11} u_{20} u_{21}^{2} u_{20}^{\prime}+u_{00} u_{01}^{\prime} u_{11} u_{20} u_{21}^{2} u_{20}^{\prime}+u_{01} u_{00} u_{11} u_{20}^{2}$ $\left(u_{21}^{\prime}\right)^{2}+u_{00} u_{01} u_{11} u_{21}^{2}\left(u_{20}^{\prime}\right)^{2}-2 u_{01} u_{00} u_{11} u_{20} u_{21} u_{20}^{\prime} u_{21}^{\prime}+u_{01} u_{00}^{\prime} u_{11} u_{20}^{2} u_{21}^{\prime} u_{21}-u_{00} u_{01}^{\prime} u_{11} u_{21} u_{21}^{\prime} u_{20}^{2}$ which contains all the coefficients of $\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}$.

Remark 3.14 When all the $\mathcal{A}_{i}(i=0, \ldots, n)$ are sets of difference monomials, unless explicitly mentioned, we always consider $\mathbb{P}_{i}$ as Laurent difference polynomials. But when we regard $\mathbb{P}_{i}$ as difference polynomials, $\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}$ is also called the sparse difference resultant of the difference polynomials $\mathbb{P}_{i}$ and we call $\mathbb{P}_{i}$ a transformally essential system. In this paper, sometimes we regard $\mathbb{P}_{i}$ as difference polynomials where we will highlight it.

We now define the sparse difference resultant for any set of specific Laurent difference polynomials over a Laurent transformally essential system. For any finite set $\mathcal{A}=$ $\left\{M_{0}, M_{1}, \ldots, M_{l}\right\}$ of Laurent difference monomials in $\mathbb{Y}$, we use

$$
\begin{equation*}
\mathcal{L}(\mathcal{A})=\left\{\sum_{i=0}^{l} a_{i} M_{i}\right\} \tag{6}
\end{equation*}
$$

to denote the set of all Laurent difference polynomials with support $\mathcal{A}$, where the $a_{i}$ are in some difference extension field of $\mathbb{Q}$.

Definition 3.15 Let $\mathcal{A}_{i}=\left\{M_{i 0}, M_{i 1}, \ldots, M_{i l_{i}}\right\}(i=0,1, \ldots, n)$ be a Laurent transformally essential system. Consider $n+1$ Laurent difference polynomials $\left(F_{0}, F_{1}, \ldots, F_{n}\right) \in$ $\prod_{i=0}^{n} \mathcal{L}\left(\mathcal{A}_{i}\right)$. The sparse difference resultant of $F_{0}, F_{1}, \ldots, F_{n}$, denoted as $\operatorname{Res}_{F_{0}, \ldots, F_{n}}$, is obtained by replacing $\mathbf{u}_{i}$ with the corresponding coefficient vector of $F_{i}$ in $\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}\left(\mathbf{u}_{0}\right.$, $\ldots, \mathbf{u}_{n}$ ).

A major unsolved problem about difference resultant is whether $\mathbf{R}$ defined above contains all the information about the elimination ideal $\mathcal{I}_{\mathbf{u}}$ defined in (44). More precisely, we propose the following problem.

Problem 3.16 As shown by Example 2.3, the characteristic set for a reflexive prime difference ideal of codimension one could contain more than one elements. Let $\mathcal{I}_{\mathbf{u}}$ be the ideal defined in (4). Then $\mathcal{I}_{\mathbf{u}}$ is a reflexive prime difference ideal of codimension one and

$$
\begin{equation*}
\mathcal{I}_{\mathbf{u}}=\mathcal{I}_{\mathbb{Y}, \mathbf{u}} \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}=\operatorname{sat}\left(\mathbf{R}, R_{1}, \ldots, R_{m}\right), \tag{7}
\end{equation*}
$$

where $\mathbf{R}$ is the sparse difference resultant of $\left\{\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right\}$ and $\mathbf{R}, R_{1}, \ldots, R_{m}$ is a characteristic set of $\mathcal{I}_{\mathbf{u}}$. We conjecture that $m=0$, or equivalently $\mathcal{I}_{\mathbf{u}}=\operatorname{sat}(\mathbf{R})$. If this is valid, then better properties can be shown for sparse difference resultant as we will explain later. It is easy to check that for Examples 3.11, 3.12, and 3.13, $\mathcal{I}_{\mathbf{u}}=\operatorname{sat}(\mathbf{R})$.

### 3.3 A criterion for Laurent transformally essential system in terms of supports

Let $\mathcal{A}_{i}(i=0, \ldots, n)$ be finite sets of Laurent difference monomials. According to Definition [3.4, in order to check whether they form a Laurent transformally essential system, we need to check whether there exist $M_{i k_{i}}, M_{i j_{i}} \in \mathcal{A}_{i}(i=0, \ldots, n)$ such that $\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle M_{0 k_{0}} / M_{0 j_{0}}\right.$, $\left.\ldots, M_{n k_{n}} / M_{n j_{n}}\right\rangle / \mathbb{Q}=n$. This can be done with the difference characteristic set method given in paper [17. In this section, a criterion will be given to check whether a Laurent difference system is essential in terms of their supports, which is conceptually and computationally simpler than the naive approach based on the characteristic set method.

Let $B_{i}=\prod_{j=1}^{n} \prod_{k=0}^{s}\left(y_{j}^{(k)}\right)^{d_{i j k}}(i=1, \ldots, m)$ be $m$ Laurent difference monomials. We now introduce a new algebraic indeterminate $x$ and let

$$
d_{i j}=\sum_{k=0}^{s} d_{i j k} x^{k}(i=1, \ldots, m, j=1, \ldots, n)
$$

be univariate polynomials in $\mathbb{Z}[x]$. If $\operatorname{ord}\left(B_{i}, y_{j}\right)=-\infty$, then set $d_{i j}=0$. The vector $\left(d_{i 1}, d_{i 2}\right.$, $\left.\ldots, d_{i n}\right)$ is called the symbolic support vector of $B_{i}$. The matrix $M=\left(d_{i j}\right)_{m \times n}$ is called the symbolic support matrix of $B_{1}, \ldots, B_{m}$.

Note that there is a one-to-one correspondence between Laurent difference monomials and their symbolic support vectors, so we will not distinguish these two concepts in case there is no confusion. The same is true for a set of Laurent difference monomials and its symbolic support matrix.

Definition 3.17 A matrix $M=\left(d_{i j}\right)_{m \times n}$ over $\mathbb{Q}[x]$ is called normal upper-triangular of rank $r$ if for each $i \leq r, d_{i i} \neq 0$ and $d_{i, i-k}=0(1 \leq k \leq i-1)$, and the last $m-r$ rows are zero vectors.

A normal upper-triangular matrix is of the following form:

$$
\left(\begin{array}{cccccc}
a_{11} & * & \cdots & * & \cdots & * \\
0 & a_{22} & \cdots & * & \cdots & * \\
\vdots & \vdots & \ddots & & & \vdots \\
0 & 0 & \cdots & a_{r r} & \cdots & * \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \cdots & \cdots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right)
$$

Definition 3.18 $A$ set of Laurent difference monomials $B_{1}, B_{2}, \ldots, B_{m}$ is said to be in $r$ -upper-triangular form if its symbolic support matrix $M$ is a normal upper triangular matrix of rank $r$.

The following lemma shows that it is easy to compute the difference transcendence degree of a set of Laurent difference monomials in upper-triangular form.

Lemma 3.19 Let $B_{1}, \ldots, B_{m}$ be a set of Laurent difference monomials in $r$-upper-triangular form. Then $\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle B_{1}, \ldots, B_{m}\right\rangle / \mathbb{Q}=r$.

Proof: From the structure of the symbolic support matrix, $B_{i}=\prod_{j=i}^{n} \prod_{k \geq 0}\left(y_{j}^{(k)}\right)^{d_{i j k}}(i=$ $1, \ldots, r)$ with $\operatorname{ord}\left(B_{i}, y_{i}\right) \geq 0$ and $B_{r+1}=\cdots=B_{m}=1$. Let $B_{i}^{\prime}=\prod_{j=i}^{r} \prod_{k \geq 0}\left(y_{j}^{(k)}\right)^{d_{i j k}}$. Then

$$
\begin{aligned}
& \sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}, \ldots, B_{m}\right\rangle / \mathbb{Q} \\
= & \sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}, \ldots, B_{r}\right\rangle / \mathbb{Q} \\
\geq & \sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle y_{r+1}, \ldots, y_{n}\right\rangle\left\langle B_{1}, \ldots, B_{r}\right\rangle / \mathbb{Q}\left\langle y_{r+1}, \ldots, y_{n}\right\rangle \\
= & \sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}^{\prime}, \ldots, B_{r}^{\prime}\right\rangle / \mathbb{Q} .
\end{aligned}
$$

So it suffices to prove $\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}^{\prime}, \ldots, B_{r}^{\prime}\right\rangle / \mathbb{Q}=r$.
If $r=1, B_{1}^{\prime}$ is a nonconstant Laurent difference monomial in $y_{1}$, so $\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle B_{1}^{\prime}\right\rangle / \mathbb{Q}=1$. Suppose we have proved for the case $r-1$. Let $B_{i}^{\prime \prime}=\prod_{j=i}^{r-1} \prod_{k \geq 0}\left(y_{j}^{(k)}\right)^{d_{i j k}}$, then by the hypothesis, $\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}^{\prime \prime}, \ldots, B_{r-1}^{\prime \prime}\right\rangle / \mathbb{Q}=r-1$. Since $B_{r}^{\prime} \in \mathbb{Q}\left\{y_{r}\right\}$, we have

$$
\begin{aligned}
r & \geq \sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}^{\prime}, \ldots, B_{r}^{\prime}\right\rangle / \mathbb{Q} \\
& =\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{r}^{\prime}\right\rangle / \mathbb{Q}+\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}^{\prime}, \ldots, B_{r}^{\prime}\right\rangle / \mathbb{Q}\left\langle B_{r}^{\prime}\right\rangle \\
& \geq 1+\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle y_{r}\right\rangle\left\langle B_{1}^{\prime}, \ldots, B_{r-1}^{\prime}\right\rangle / \mathbb{Q}\left\langle y_{r}\right\rangle \\
& =1+\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}^{\prime \prime}, \ldots, B_{r-1}^{\prime \prime}\right\rangle / \mathbb{Q}=r .
\end{aligned}
$$

So $\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle B_{1}, \ldots, B_{m}\right\rangle / \mathbb{Q}=r$. $\boldsymbol{I}$
In the following, we will show that each set of Laurent difference monomials can be transformed to an upper-triangular set with the same difference transcendence degree. Here we use three types of elementary matrix transformations. For a matrix $M$ over $\mathbb{Q}[x]$,

- Type 1 operations consist of interchanging two rows of $M$, say the $i$-th and $j$-th rows, denoted by $r[i, j]$;
- Type 2 operations consist of adding an $f(x)$-multiple of the $j$-th row to the $i$-th row, where $f(x) \in \mathbb{Q}[x]$, denoted by $[i+j(f(x))]$;
- Type 3 operations consist of interchanging two columns, say the $i$-th and $j$-th columns, denoted by $c[i, j]$.

In this section, by elementary transformations, we mean the above three types of transformations.

Let $B_{1}, \ldots, B_{m}$ be Laurent difference monomials and $M$ their symbolic support matrix. Then the above three types of elementary transformations of $M$ correspond to certain transformations of the difference monomials. Indeed, interchanging the $i$-th and the $j$-th rows of $M$ means interchanging $B_{i}$ and $B_{j}$, and interchanging the $i$-th and the $j$-th columns of $M$ means interchanging $y_{i}$ and $y_{j}$ in $B_{1}, \ldots, B_{m}$ (or in the variable order). Multiplying the $i$-th row of $M$ by a polynomial $f(x)=a_{d} x^{d}+a_{d-1} x^{d-1}+\cdots+a_{0} \in \mathbb{Q}[x]$ and adding the result to the $j$-th row means changing $B_{j}$ to $\prod_{k=0}^{d}\left(\sigma^{k} B_{i}\right)^{a_{k}} B_{j}$.

Lemma 3.20 Let $B_{1}, \ldots, B_{m}$ be Laurent difference monomials and $C_{1}, \ldots, C_{m}$ obtained by successive elementary transformations defined above. Then $\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle B_{1}, \ldots, B_{m}\right\rangle / \mathbb{Q}$ $=\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle C_{1}, \ldots, C_{m}\right\rangle / \mathbb{Q}$.

Proof: It suffices to show that Type 2 operations do not change the difference transcendence degree. That is, for $\sum_{i=0}^{d} a_{i} x^{i} \in \mathbb{Q}[x], \sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}, B_{2}\right\rangle / \mathbb{Q}=\sigma \cdot \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle B_{1}, \prod_{k=0}^{d}\left(\sigma^{k} B_{1}\right)^{a_{k}} B_{2}\right\rangle / \mathbb{Q}$.

Suppose $a_{i}=p_{i} / q$ where $p_{i}, q \in \mathbb{Z}^{*}$. Then, clearly, $\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}\right\rangle / \mathbb{Q}=\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle\prod_{k=0}^{d}\right.$ $\left.\left(\sigma^{k} B_{1}\right)^{p_{k}}\right\rangle / \mathbb{Q}$. Thus, $\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}, \prod_{k=0}^{d}\left(\sigma^{k} B_{1}\right)^{a_{k}} B_{2}\right\rangle / \mathbb{Q}=\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle\prod_{k=0}^{d}\left(\sigma^{k} B_{1}\right)^{p_{k}}, \prod_{k=0}^{d=0}\right.$ $\left.\left(\sigma^{k} B_{1}\right)^{p_{k}} B_{2}^{q}\right\rangle / \mathbb{Q}=\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle\prod_{k=0}^{d}\left(\sigma^{k} B_{1}\right)^{p_{k}}, B_{2}^{q}\right\rangle / \mathbb{Q}=\sigma \cdot \operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left\langle B_{1}, B_{2}\right\rangle / \mathbb{Q} . \boldsymbol{I}$

Theorem 3.21 Let $B_{1}, \ldots, B_{m}$ be a set of Laurent difference monomials with symbolic support matrix $M$. Then $\sigma \cdot \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle B_{1}, \ldots, B_{m}\right\rangle / \mathbb{Q}=\operatorname{rk}(M)$.

Proof: By Lemma 3.19 and Lemma 3.20, it suffices to show that $M$ can be reduced to a normal upper-triangular matrix by performing a series of elementary transformations. This can be done since $\mathbb{Q}[x]$ is an Euclidean domain.

Suppose $M=\left(d_{i j}\right) \neq \mathbf{0}_{m \times n}$ and we denote the new matrix obtained after performing elementary transformations also by $M$. Firstly, perform Type 1 and Type 3 operations when necessary to make $d_{11} \neq 0$ have the minimum degree among all $d_{i j}$. Secondly, try to use $d_{11}(x)$ to reduce other elements in the first column to 0 by performing Type 2 operations. Let $d_{k 1} \neq 0$ and suppose $d_{k 1}(x)=d_{11}(x) q(x)+r(x)$ where $\operatorname{deg}(r(x))<\operatorname{deg}\left(d_{11}(x)\right)$. Performing the transformation $[k+1(-q(x))]$ and then the transformation $r[1, k]$ if $r(x) \neq 0$, we obtain a new matrix in which the degree of $d_{11}$ strictly decreases. Repeat this process when necessary, then after a finite number of steps, we obtain a new matrix $M$ such that $d_{k 1}(x)=0$ for $k>1$. That is,

$$
M=\left(\begin{array}{ll}
d_{11} & * \\
\mathbf{0} & M_{1}
\end{array}\right)
$$

Now we repeat the above process for $M_{1}$ and whenever Type 3 operations are performed for $M_{1}$, we assume the same transformations are performed for the whole matrix $M$. In this way, after a finite number of steps, we obtain a normal upper-triangular matrix $M$. I

Remark 3.22 In the proof of Theorem 3.21, the Euclidean algorithm plays a crucial role. That is why we work with $\mathbb{Q}[x]$, even if the symbolic support matrix of $B_{1}, \ldots, B_{m}$ is a matrix over $\mathbb{Z}[x]$.

Example 3.23 Let $B_{1}=y_{1} y_{2}$ and $B_{2}=y_{1}^{(a)} y_{2}^{(b)}$. Then the symbolic support matrix of $B_{1}$ and $B_{2}$ is $M=\left(\begin{array}{cc}1 & 1 \\ x^{a} & x^{b}\end{array}\right)$. Then $\operatorname{rk}(M)=\left\{\begin{array}{ll}1 & \text { if } a=b \\ 2 & \text { if } a \neq b .\end{array}\right.$ Thus, by Theorem 3.21, if $a \neq b, B_{1}$ and $B_{2}$ are transformally independent over $\mathbb{Q}$. Otherwise, they are transformally dependent over $\mathbb{Q}$.

We now extend Theorem 3.21 to generic difference polynomials in (11). Let $I \subseteq\{0, \ldots, n\}$ and for any $i \in I$, let $\beta_{i k}$ be the symbolic support vector of $M_{i k} / M_{i 0}$. Then the vector

$$
w_{i}=\sum_{k=0}^{l_{i}} u_{i k} \beta_{i k}
$$

is called the symbolic support vector of $\mathbb{P}_{i}$ and the matrix $M_{I}$ whose rows are $w_{i}$ for $i \in I$ is called the symbolic support matrix of $\mathbb{P}_{i}$ for $i \in I$. Similar to Theorem 4.17 in [27], we have

Lemma 3.24 Use the notations introduced above. We have $\sigma . \operatorname{tr} . \operatorname{deg} \mathbb{Q}\left\langle\cup_{i \in I} \mathbf{u}_{i}\right\rangle\left\langle\mathbb{P}_{i} / M_{i 0}\right.$ : $i \in I\rangle / \mathbb{Q}\left\langle\cup_{i \in I} \mathbf{u}_{i}\right\rangle=\operatorname{rk}\left(M_{I}\right)$, where $\mathbf{u}_{i}=\left(u_{i 0}, \ldots, u_{i l_{i}}\right)$.

Now, we have the following criterion for Laurent transformally essential system.
Theorem 3.25 Consider the set of generic Laurent difference polynomials defined in (1). The following three conditions are equivalent.

1. $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$ form a Laurent transformally essential system.
2. There exist $M_{i k_{i}}(i=0, \ldots, n)$ with $1 \leq k_{i} \leq l_{i}$ such that the symbolic support matrix of $M_{0 k_{0}} / M_{00}, \ldots, M_{n k_{n}} / M_{n 0}$ is of rank $n$.
3. The rank of $M_{I}$ is equal to $n$, where $I==\{0,1, \ldots, n\}$.

Proof: The equivalence of 1) and 2) is a direct consequence of Theorem 3.21 and Definition 3.4. The equivalence of 1 ) and 3) follows from Lemma 3.24.

Both Theorem 3.21 and Theorem 3.25 can be used to check whether a system is transformally essential.

Example 3.26 Continue from Example 3.13. Let $B_{0}=M_{01} / M_{00}=y_{1} y_{2}, B_{1}=M_{11} / M_{10}=$ $y_{1}^{(1)} y_{2}^{(1)}$, and $B_{2}=M_{21} / M_{20}=y_{2}$. Then the symbolic support matrix for $\left\{B_{0}, B_{2}\right\}$ is $M=$ $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$. We have $\operatorname{rk}(M)=2$ and by Theorem 3.21, the system $\mathbb{P}=\left\{\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}\right\}$ is transformally essential. Also, the symbolic support matrix for $\mathbb{P}$ is $M_{\mathbb{P}}=\left(\begin{array}{cc}u_{01} & u_{01} \\ u_{11} x & u_{11} x \\ 0 & u_{21}\end{array}\right)$. We have $\operatorname{rk}\left(M_{\mathbb{P}}\right)=2$ and by Theorem 3.25, $\mathbb{P}$ is transformally essential.

We will end this section by introducing a new concept, namely super-essential systems, through which one can identify certain $\mathbb{P}_{i}$ such that their coefficients will not occur in the sparse difference resultant. This will lead to the simplification in the computation of the
resultant. Let $\mathbb{T} \subset\{0,1, \ldots, n\}$. We denote by $\mathbb{P}_{\mathbb{T}}$ the Laurent difference polynomial set consisting of $\mathbb{P}_{i}(i \in \mathbb{T})$, and $M_{\mathbb{P}_{\mathbb{T}}}$ its symbolic support matrix. For a subset $\mathbb{T} \subset\{0,1, \ldots, n\}$, if $\operatorname{card}(\mathbb{T})=\operatorname{rk}\left(\mathrm{M}_{\mathbb{P}_{\mathbb{T}}}\right)$, then $\mathbb{P}_{\mathbb{T}}$, or $\left\{\mathcal{A}_{i}: i \in \mathbb{T}\right\}$, is called a transformally independent set.

Definition 3.27 Let $\mathbb{T} \subset\{0,1, \ldots, n\}$. Then we call $\mathbb{T}$ or $\mathbb{P}_{\mathbb{T}}$ super-essential if the following conditions hold: (1) $\operatorname{card}(\mathbb{T})-r k\left(M_{\mathbb{P}_{\mathbb{T}}}\right)=1$ and (2) $\operatorname{card}(\mathbb{J})=\operatorname{rk}\left(\mathrm{M}_{\mathbb{P}_{\mathbb{J}}}\right)$ for each proper subset $\mathbb{J}$ of $\mathbb{T}$.

Note that super-essential systems are the difference analogue of essential systems introduced in paper 34 and also that of rank essential systems introduced in 27] . Using this definition, we have the following property, which is similar to Corollary 1.1 in [34.

Theorem 3.28 If $\left\{\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right\}$ is a Laurent transformally essential system, then for any $\mathbb{T} \subset\{0,1, \ldots, n\}, \operatorname{card}(\mathbb{T})-\operatorname{rk}\left(\mathrm{M}_{\mathbb{P}_{\mathbb{T}}}\right) \leq 1$ and there exists a unique $\mathbb{T}$ which is super-essential. In this case, the sparse difference resultant of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$ involves only the coefficients of $\mathbb{P}_{i}(i \in \mathbb{T})$.

Proof: Since $n=\operatorname{rk}\left(M_{\mathbb{P}}\right) \leq \operatorname{rk}\left(M_{\mathbb{P}_{\mathbb{T}}}\right)+\operatorname{card}(\mathbb{P})-\operatorname{card}\left(\mathbb{P}_{\mathbb{T}}\right)=\mathrm{n}+1+\operatorname{rk}\left(\mathrm{M}_{\mathbb{P}_{\mathbb{T}}}\right)-\operatorname{card}(\mathbb{T})$, we have $\operatorname{card}(\mathbb{T})-r k\left(M_{\mathbb{P}_{\mathbb{T}}}\right) \leq 1$. Since card $(\mathbb{T})-r k\left(M_{\mathbb{P}_{\mathbb{T}}}\right) \geq 0$, for any $\mathbb{T}$, either $\operatorname{card}(\mathbb{T})-\operatorname{rk}\left(M_{\mathbb{P}_{\mathbb{T}}}\right)=0$ or $\operatorname{card}(\mathbb{T})-\operatorname{rk}\left(\mathrm{M}_{\mathbb{P}_{\mathbb{T}}}\right)=1$. From this fact, it is easy to show the existence of a super-essential set $\mathbb{T}$. For the uniqueness, we assume that there exist two subsets $\mathbb{T}_{1}, \mathbb{T}_{2} \subset\{1, \ldots, m\}$ which are super-essential. Then, we have

$$
\begin{aligned}
\operatorname{rk}\left(M_{\mathbb{T}_{1} \cup \mathbb{T}_{2}}\right) & \leq \operatorname{rk}\left(M_{\mathbb{T}_{1}}\right)+\operatorname{rk}\left(M_{\mathbb{T}_{2}}\right)-\operatorname{rk}\left(M_{\mathbb{T}_{\mathbb{T}_{1} \cap \mathbb{T}_{2}}}\right) \\
& =\operatorname{card}\left(\mathbb{T}_{1}\right)-1+\operatorname{card}\left(\mathbb{T}_{2}\right)-1-\operatorname{card}\left(\mathbb{T}_{1} \cap \mathbb{T}_{2}\right) \\
& =\operatorname{card}\left(\mathbb{T}_{1} \cup \mathbb{T}_{2}\right)-2,
\end{aligned}
$$

which is a contradiction.
Let $\mathbb{T}$ be a super-essential set. Similar to the proof of Theorem 3.6, it is easy to show that $\left[\mathbb{P}_{i}\right]_{i \in \mathbb{T}} \cap \mathbb{Q}\left\{\mathbf{u}_{i}\right\}_{i \in \mathbb{T}}$ is of codimension one, which means that the sparse difference resultant of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$ only involves the coefficients of $\mathbb{P}_{i}(i \in \mathbb{T})$.

Remark 3.29 If $\mathbb{P}_{\mathbb{T}}$ is the super-essential subsystem of a Laurent transformally essential system $\mathbb{P}=\left\{\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right\}$, then clearly $\left[\mathbb{P}_{\mathbb{T}}\right] \cap \mathbb{Q}\left\{\mathbf{u}_{i}: i \in \mathbb{T}\right\}=[\mathbb{P}] \cap \mathbb{Q}\left\{\mathbf{u}_{i}: i \in \mathbb{T}\right\}=\operatorname{sat}(\mathbf{R}, \ldots)$. For convenience, sometimes we will not distinguish $\mathbb{P}$ and $\mathbb{P}_{\mathbb{T}}$ and also call $\mathbf{R}$ the sparse difference resultant of $\mathbb{P}_{\mathbb{T}}$.

Using this property, one can determine which polynomial is needed for computing the sparse difference resultant, which will eventually reduce the computation complexity.

Example 3.30 Continue from Example 3.13. It is easy to show that $\mathbb{P}=\left\{\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}\right\}$ is a Laurent transformally essential system and $\mathbb{P}_{0}, \mathbb{P}_{1}$ constitute a super-essential system. Recall that the sparse difference resultant of $\mathbb{P}$ is free from the coefficients of $\mathbb{P}_{2}$.

## 4 Basic properties of sparse difference resultant

In this section, we will prove some basic properties for the sparse difference resultant.

### 4.1 Sparse difference resultant is transformally homogeneous

We first introduce the concept of transformally homogeneous polynomials.
Definition 4.1 $A$ difference polynomial $f \in \mathcal{F}\left\{y_{0}, \ldots, y_{n}\right\}$ is called transformally homogeneous if for a new difference indeterminate $\lambda$, there exists a difference monomial $M(\lambda)$ in $\lambda$ such that $f\left(\lambda y_{0}, \ldots, \lambda y_{n}\right)=M(\lambda) p\left(y_{0}, \ldots, y_{n}\right)$. If $\operatorname{deg}(M(\lambda))=m$, $f$ is called transformally homogeneous of degree $m$.

The difference analogue of Euler's theorem related to homogeneous polynomials is valid.

Lemma 4.2 $f \in \mathcal{F}\left\{y_{0}, y_{1}, \ldots, y_{n}\right\}$ is transformally homogeneous if and only if for each $r \in \mathbb{N}_{0}$, there exists $m_{r} \in \mathbb{N}_{0}$ such that

$$
\sum_{i=0}^{n} y_{i}^{(r)} \frac{\partial f\left(y_{0}, \ldots, y_{n}\right)}{\partial y_{i}^{(r)}}=m_{r} f
$$

That is, $f$ is transformally homogeneous if and only if $f$ is homogeneous in $\left\{y_{1}^{(r)}, \ldots, y_{n}^{(r)}\right\}$ for any $r \in \mathbb{N}_{0}$.
Proof: " $\Longrightarrow$ " Denote $\mathbb{Y}=\left(y_{0}, \ldots, y_{n}\right)$ temporarily. Suppose $f$ is transformally homogeneous. That is, there exists a difference monomial $M(\lambda)=\prod_{r=0}^{r_{0}}\left(\lambda^{(r)}\right)^{m_{r}}$ such that $f(\lambda \mathbb{Y})=$ $M(\lambda) f(\mathbb{Y})$. Then $\sum_{i=0}^{n} y_{i}^{(r)} \frac{\partial f}{\partial y_{i}^{(r)}}(\lambda \mathbb{Y})=\sum_{i=0}^{n} \frac{\partial f}{\partial y_{i}^{(r)}}(\lambda \mathbb{Y}) \frac{\partial\left(\lambda y_{i}\right)^{(r)}}{\partial \lambda^{(r)}}=\frac{\partial f(\lambda \mathbb{Y})}{\partial \lambda^{(r)}}=\frac{\partial M(\lambda) f(\mathbb{Y})}{\partial \lambda^{(r)}}=$ $\frac{m_{r} M(\lambda)}{\lambda^{(r)}} f(\mathbb{Y})$. Substitute $\lambda=1$ into the above equality, we have $\sum_{i=0}^{n} y_{i}^{(r)} \frac{\partial f}{\partial y_{j}^{(r)}}=m_{r} f$.
" $\Longleftarrow "$ Suppose $\operatorname{ord}(f, \mathbb{Y})=r_{0}$. Then for each $r \leq r_{0}, \lambda^{(r)} \frac{\partial f(\lambda \mathbb{Y})}{\partial \lambda^{(r)}}=\lambda^{(r)} \sum_{i=0}^{n} y_{i}^{(r)} \frac{\partial f}{\partial y_{i}^{(r)}}(\lambda \mathbb{Y})$ $=\sum_{i=0}^{n}\left(\lambda y_{i}\right)^{(r)} \frac{\partial f}{\partial y_{i}^{(r)}}(\lambda \mathbb{Y})=m_{r} f(\lambda \mathbb{Y})$. So $f(\lambda \mathbb{Y})$ is homogeneous of degree $m_{r}$ in $\lambda^{(r)}$. Thus, $f(\lambda \mathbb{Y})=f\left(\lambda y_{0}, \ldots, \lambda y_{n} ; \lambda^{(1)} y_{0}^{(1)}, \ldots, \lambda^{(1)} y_{n}^{(1)} ; \ldots ; \lambda^{\left(r_{0}\right)} y_{0}^{\left(r_{0}\right)}, \ldots, \lambda^{\left(r_{0}\right)} y_{n}^{\left(r_{0}\right)}\right)=$ $\prod_{r=0}^{r_{0}}\left(\lambda^{(r)}\right)^{m_{r}} f(\mathbb{Y})$. Thus, $f$ is transformally homogeneous. I

Sparse difference resultants have the following property.
Theorem 4.3 The sparse difference resultant is transformally homogeneous in each $\mathbf{u}_{i}$ which is the coefficient set of $\mathbb{P}_{i}$.

Proof: Suppose $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=h_{i} \geq 0$. Follow the notations used in Theorem 3.6. By Lemma 3.9, $\mathbf{R}\left(\mathbf{u} ; \zeta_{0}, \ldots, \zeta_{n}\right)=0$. Differentiating this identity w.r.t. $u_{i j}^{(k)}\left(j=1, \ldots, l_{i}\right)$ respectively, due to (5) we have

$$
\begin{equation*}
\overline{\frac{\partial \mathbf{R}}{\partial u_{i j}^{(k)}}}+\overline{\frac{\partial \mathbf{R}}{\partial u_{i 0}^{(k)}}}\left(-\frac{M_{i j}(\eta)}{M_{i 0}(\eta)}\right)^{(k)}=0 \tag{8}
\end{equation*}
$$

In the above equations, $\overline{\frac{\partial \mathbf{R}}{\partial u_{i j}^{(k)}}}\left(k=0, \ldots, h_{i} ; j=0, \ldots, l_{i}\right)$ are obtained by replacing $u_{i 0}$ by $\zeta_{i}(i=0,1, \ldots, n)$ in each $\frac{\partial \mathbf{R}}{\partial u_{i j}^{(k)}}$ respectively.

Multiplying (8) by $u_{i j}^{(k)}$ and for $j$ from 1 to $l_{i}$, adding them together, we get $\zeta_{i}^{(k)} \overline{\frac{\partial \mathbf{R}}{\partial u_{i 0}^{(k)}}}+$ $\sum_{j=1}^{l_{i}} u_{i j}^{(k)} \frac{\overline{\partial \mathbf{R}}}{\partial u_{i j}^{(k)}}=0$. So the difference polynomial $f_{k}=\sum_{j=0}^{l_{i}} u_{i j}^{(k)} \frac{\partial \mathbf{R}}{\partial u_{i j}^{(k)}}$ vanishes at $\left(\zeta_{0}, \ldots, \zeta_{n}\right)$. Since $\operatorname{ord}\left(f_{k}, u_{i 0}\right) \leq \operatorname{ord}\left(\mathbf{R}, u_{i 0}\right)$ and $\operatorname{deg}\left(f_{k}\right)=\operatorname{deg}(\mathbf{R})$, by Lemma 3.9, there exists an $m_{k} \in \mathbb{Z}$ such that $f_{k}=m_{k} \mathbf{R}$. Thus, by Lemma 4.2, $\mathbf{R}$ is transformally homogeneous in $\mathbf{u}_{i}$.

### 4.2 Condition for existence of nonzero solutions

In this section, we will first give a condition for a system of Laurent difference polynomials to have nonzero solutions in terms of sparse difference resultant, and then study the structures of nonzero solutions.

To be more precise, we first introduce some notations. Let $\mathcal{A}=\left\{M_{0}, M_{1}, \ldots, M_{l}\right\}$ be a Laurent monomial set. Then, there is a one to one correspondence between $\mathcal{L}(\mathcal{A})$ defined in (6) and $\mathcal{E}^{l+1}$ where $\mathcal{E}$ is some difference extension field of $\mathbb{Q}$. For $F=\sum_{i=0}^{l} c_{i} M_{i} \in \mathcal{L}(\mathcal{A})$ where $c_{i} \in \mathcal{E}$, denote the coefficient vector of $F$ by $\mathbb{C}(F)=\left(c_{0}, \ldots, c_{l}\right) \in \mathcal{E}^{l+1}$. Conversely, for any $\mathbf{c}=\left(c_{0}, \ldots, c_{l}\right) \in \mathcal{E}^{l+1}$, denote the corresponding Laurent difference polynomial by $\mathbb{L}(\mathbf{c})=\sum_{i=0}^{l} c_{i} M_{i}$.

Let $\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}$ be a Laurent transformally essential system of Laurent monomial sets. By $\left(F_{0}, \ldots, F_{n}\right) \in \mathcal{L}\left(\mathcal{A}_{0}\right) \times \cdots \times \mathcal{L}\left(\mathcal{A}_{n}\right)$, we always mean that there exists a common difference extension field $\mathcal{E}$ such that $\mathbb{C}\left(F_{i}\right) \in \mathcal{E}^{l_{i}+1}(i=0, \ldots, n)$. Clearly, each element $\left(F_{0}, \ldots, F_{n}\right) \in$ $\mathcal{L}\left(\mathcal{A}_{0}\right) \times \cdots \times \mathcal{L}\left(\mathcal{A}_{n}\right)$ can be represented by one and only one element $\left(\mathbb{C}\left(F_{0}\right), \ldots, \mathbb{C}\left(F_{n}\right)\right) \in$ $\hat{\mathcal{E}}=\mathcal{E}^{l_{0}+1} \times \cdots \times \mathcal{E}^{l_{n}+1}$. Let $\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)$ be the set consisting of points $\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \in \hat{\mathcal{E}}$ such that the corresponding $\mathbb{L}\left(\mathbf{v}_{i}\right)=0(i=0, \ldots, n)$ have nonzero solutions. That is,

$$
\begin{align*}
\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)=\bigcup_{\mathcal{E}} & \left\{\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \in \hat{\mathcal{E}}: \mathbb{L}\left(\mathbf{v}_{0}\right)=\cdots=\mathbb{L}\left(\mathbf{v}_{n}\right)=0\right. \\
& \text { have a common nonzero solution }\} . \tag{9}
\end{align*}
$$

Note that the sparse resultant $\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ has $L=\sum_{i=0}^{n}\left(l_{i}+1\right)$ variables. In this section, each element $\mathbf{v} \in \mathcal{E}^{L}$ is naturally treated as an element $\mathbf{v}=\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \in \mathcal{E}^{l_{0}+1} \times \cdots \times \mathcal{E}^{l_{n}+1}$ and $\mathbf{R}(\mathbf{v})=\mathbf{R}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$. In this way, $Z_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)$ and $\mathbb{V}\left(\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}\right)$ are in the same affine space $\mathcal{E}^{L}$ for any $\mathcal{E}$.

The following result shows that the vanishing of sparse difference resultant gives a necessary condition for the existence of nonzero solutions.

Lemma $4.4 Z_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right) \subseteq \mathbb{V}\left(\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}\right)$.
Proof: Let $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$ be a generic Laurent transformally essential system corresponding to $\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}$ with coefficient vectors $\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}$. By Definition 3.8, Res $\mathcal{A}_{0}, \ldots, \mathcal{A}_{n} \in\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right]$ $\cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$. For any point $\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \in Z_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)$, let $\left(\overline{\mathbb{P}}_{0}, \ldots, \overline{\mathbb{P}}_{n}\right) \in \mathcal{L}\left(\mathcal{A}_{0}\right) \times$ $\cdots \times \mathcal{L}\left(\mathcal{A}_{n}\right)$ be the difference polynomial system represented by $\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$. Since $\overline{\mathbb{P}}_{0}, \ldots, \overline{\mathbb{P}}_{n}$ have a nonzero common solution, $\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}$ vanishes at $\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$.

Example 4.5 Continue from Example 3.12. Suppose $\mathcal{F}=\mathbb{Q}(x)$ and $\sigma f(x)=f(x+1)$. In this example, we have $\operatorname{Res}_{\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}} \neq 0$. But $y_{1}=0, y_{2}=0$ constitute a zero solution of $\mathbb{P}_{0}=\mathbb{P}_{1}=\mathbb{P}_{2}=0$. This shows that Lemma 4.4 is not correct if we do not consider nonzero
solutions. This example also shows why we need to consider nonzero difference solutions, or equivalently why we consider Laurent difference polynomials instead of the usual difference polynomials.

The following theorem shows that a particular principal component of the sparse difference resultant gives a sufficient and necessary condition for a Laurent transformally essential system to have nonzero solutions in certain sense.

Theorem 4.6 Let $\mathcal{I}_{\mathbf{u}}=\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}=\operatorname{sat}\left(\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}, R_{1}, \ldots, R_{m}\right)$ as defined in (7). Let $\overline{\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)}$ be the Cohn topological closur』 of $\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)$. Then $\overline{\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)}=\mathbb{V}\left(\operatorname{sat}\left(\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}, R_{1}, \ldots, R_{m}\right)\right)$.
Proof: Similarly to the proof of Lemma4.4, we can show that $\mathcal{I}_{\mathbf{u}}$ vanishes at $\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)$. So $\overline{\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)} \subseteq \mathbb{V}\left(\operatorname{sat}\left(\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}, R_{1}, \ldots, R_{m}\right)\right)$.

For the other direction, follow the notations in the proof of Theorem3.6. By Theorem 3.6, $\left[\mathrm{N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)\right]: \mathbf{m}$ is a reflexive prime difference ideal with a generic point $(\eta, \zeta)$ where $\eta=\left(\eta_{1}, \ldots, \eta_{n}\right)$ is a generic point of $[0]$ over $\mathbb{Q}\left\langle\left(u_{i k}\right)_{i=0, \ldots, n ; k \neq 0}\right\rangle$ and $\zeta=\left(\zeta_{0}, u_{01}, \ldots, u_{0 l_{0}} ; \ldots\right.$; $\left.\zeta_{n}, u_{n 1}, \ldots, u_{n l_{n}}\right)$. Let $\left(F_{0}, \ldots, F_{n}\right) \in \mathcal{L}\left(\mathcal{A}_{0}\right) \times \cdots \times \mathcal{L}\left(\mathcal{A}_{n}\right)$ be a set of Laurent difference polynomials represented by $\zeta$. Clearly, $\eta$ is a nonzero solution of $F_{i}=0$. Thus, $\zeta \in$ $\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right) \subset \overline{\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)}$. Since $\zeta$ is a generic point of $\operatorname{sat}\left(\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}, R_{1}, \ldots, R_{m}\right)$. It follows that $\mathbb{V}\left(\operatorname{sat}\left(\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}, R_{1}, \ldots, R_{m}\right)\right) \subseteq \overline{\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)}$. As a consequence, the theorem is proved.

Remark 4.7 If Problem 3.16 can be solved positively, then the vanishing of $\operatorname{sat}(\mathbf{R})$ also gives a sufficient condition for $\mathbb{P}_{0}=\cdots=\mathbb{P}_{n}=0$ to have a nonzero solution in the sense of Cohn topological closure. That is, $\overline{\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)}=\mathbb{V}(\operatorname{sat}(\mathbf{R}))$.

The following example shows that the vanishing of the sparse difference resultant is not a sufficient condition for the given system to have common nonzero solutions.

Example 4.8 Continue from Example 3.11. Suppose $\overline{\mathbb{P}}_{0}=y_{1}^{2}-4, \overline{\mathbb{P}}_{1}=y_{1}^{(1)}+y_{1}$. Clearly, $\operatorname{Res}\left(\overline{\mathbb{P}}_{0}, \overline{\mathbb{P}}_{1}\right)=0$ but $\overline{\mathbb{P}}_{0}=\overline{\mathbb{P}}_{1}=0$ has no solution. Note that in this example, Problem 3.16 has a positive answer, that is, $\mathcal{I}_{\mathbf{u}}=\operatorname{sat}(\mathbf{R})$. Theorem 4.6 shows that $\mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)$ is dense in $\mathbb{V}(\operatorname{sat}(\mathbf{R}))$. This example shows that for certain $\mathcal{A}_{i}, \mathcal{Z}_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)$ is a proper subset of $\mathbb{V}(\operatorname{sat}(\mathbf{R}))$.

The following lemma reflects the structures of the nonzero solutions.
Lemma 4.9 Use the notations in (1). Let $\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}$ be a Laurent transformally essential system and $\mathbf{R}=\operatorname{Res}_{\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}}$. Then there exists a $\tau$ such that $\operatorname{deg}\left(\mathbf{R}, u_{\tau 0}\right)>0$. Suppose $\overline{\mathbb{P}}_{i}=0$ is a system represented by $\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \in Z_{0}\left(\mathcal{A}_{0}, \ldots, \mathcal{A}_{n}\right)$ and $\frac{\partial \mathbf{R}}{\partial u_{\tau 0}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \neq 0$. If $\xi$ is a common nonzero difference solution of $\overline{\mathbb{P}}_{i}=0(i=0, \ldots, n)$, then for each $j$, we have

$$
\begin{equation*}
\frac{M_{\tau j}(\xi)}{M_{\tau 0}(\xi)}=\frac{\partial \mathbf{R}}{\partial u_{\tau j}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) / \frac{\partial \mathbf{R}}{\partial u_{\tau 0}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \tag{10}
\end{equation*}
$$

[^1]Proof: Since $\mathcal{I}_{\mathbb{Y}, \mathbf{u}}=\left[\mathrm{N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)\right]: \mathbf{m}$ is a reflexive prime difference ideal and $\mathbf{R} \in \mathcal{I}_{\mathbb{Y}, \mathbf{u}}$, there exists some $\tau$ and $j$ such that $\operatorname{deg}\left(\mathbf{R}, u_{\tau j}\right)>0$. By equation (8), $\operatorname{deg}\left(\mathbf{R}, u_{\tau 0}\right)>0$ and for each $j=1, \ldots, l_{0}$, the polynomial $\frac{\partial \mathbf{R}}{\partial u_{\tau 0}} M_{\tau} M_{\tau j}-\frac{\partial \mathbf{R}}{\partial u_{\tau j}} M_{\tau} M_{\tau 0} \in \mathcal{I}_{\mathbb{Y}, \mathbf{u}}$, where $\mathrm{N}\left(\mathbb{P}_{i}\right)=$ $M_{i} \mathbb{P}_{i}(i=0, \ldots, n)$. Thus, if $\xi$ is a common nonzero difference solution of $\overline{\mathbb{P}}_{i}=0$, then $\frac{\partial \mathbf{R}}{\partial u_{\tau 0}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \cdot M_{\tau j}(\xi)-\frac{\partial \mathbf{R}}{\partial u_{\tau j}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) M_{\tau 0}(\xi)=0$. Since $\frac{\partial \mathbf{R}}{\partial u_{\tau 0}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \neq 0$, (10) follows. I

The following result gives a condition for the system to have a unique solution.
Corollary 4.10 Assume that 1) for each $j=1, \ldots, n$, there exists $d_{j i k} \in \mathbb{Z}$ such that $y_{j}=\prod_{i=0}^{n} \prod_{k=0}^{l_{i}}\left(\frac{M_{i k}}{M_{i 0}}\right)^{d_{j i k}}$ and 2) for each $i$ and $k, \operatorname{deg}\left(\mathbf{R}, u_{i k}\right)>0$. Suppose $\overline{\mathbb{P}}_{i}=0$ is a specialized system represented by $\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$ with $\mathbf{R}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)=0$ and $\frac{\partial \mathbf{R}}{\partial u_{i k}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \neq$ $0\left(i=0, \ldots, n ; k=0, \ldots, l_{i}\right)$. Then the system $\overline{\mathbb{P}}_{i}=0(i=0, \ldots, n)$ could have at most one nonzero solution. Furthermore, if Problem 3.16 has a positive answer, that is, $\mathcal{I}_{\mathbf{u}}=\operatorname{sat}(\mathbf{R})$, then the system $\overline{\mathbb{P}}_{i}=0(i=0, \ldots, n)$ has a unique nonzero solution.
Proof: Suppose $\xi$ is a nonzero solution of $\overline{\mathbb{P}}_{i}=0$. By (10), for each $j, \prod_{i=0}^{n} \prod_{k=0}^{l_{i}}\left(\frac{M_{i k}(\xi)}{M_{i 0}(\xi)}\right)^{d_{j i k}}$ $=\prod_{i=0}^{n} \prod_{k=0}^{l_{i}}\left(\overline{\frac{\partial \mathbf{R}}{\partial u_{i k}}} / \overline{\frac{\partial \mathbf{R}}{\partial u_{i 0}}}\right)^{d_{j i k}}=\xi_{j} \neq 0$, where $\frac{\overline{\partial \mathbf{R}}}{\partial u_{i k}}=\frac{\partial \mathbf{R}}{\partial u_{i k}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$. That is, $\xi$ is uniquely determined by $\mathbf{R}$ and $\mathbf{v}_{i}$. Suppose $\mathcal{I}_{\mathbf{u}}=\operatorname{sat}(\mathbf{R})$. Let $\mathcal{I}_{\mathbb{Y}, \mathbf{u}}=\left[N\left(\mathbb{P}_{0}\right), \ldots, N\left(\mathbb{P}_{n}\right)\right]: \mathbf{m}$. Similar to the proof of Lemma 4.9, $T_{j}=\prod_{i=0}^{n} \prod_{k=0}^{l_{i}} M_{i k}^{d_{j i k}} y_{j}-\prod_{i=0}^{n} \prod_{k=0}^{l_{i}} M_{i 0}^{d_{j i k}} \in \mathcal{I}_{\mathbb{Y}, \mathbf{u}}$. Since $\mathcal{I}_{\mathbf{u}}=\operatorname{sat}(\mathbf{R}), \mathcal{A}=\left\{\mathbf{R}, T_{1}, \ldots, T_{n}\right\}$ is a characteristic set for $\mathcal{I}_{\mathbf{u}}$ and $\mathcal{I}_{\mathbf{u}}=\operatorname{sat}(\mathcal{A})$, from which we can deduce that $y_{j}=\prod_{i=0}^{n} \prod_{k=0}^{l_{i}}\left(\frac{\overline{\partial \mathbf{R}}}{\partial u_{i k}} / \frac{\partial \mathbf{R}}{\partial u_{i 0}}\right)^{d_{j i k}}, j=1, \ldots, n$ constitute a nonzero solution of $\overline{\mathbb{P}}_{i}=0$. $\boldsymbol{I}$

Example 4.11 Let $n=2$ and the $\mathbb{P}_{i}$ have the form

$$
\mathbb{P}_{0}=u_{00}+u_{01} y_{1} y_{2}, \mathbb{P}_{1}=u_{10}+u_{11} y_{1} y_{2}^{(1)}, \mathbb{P}_{2}=u_{20}+u_{21} y_{2}
$$

Clearly, $\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}$ form a super-essential system and the sparse difference resultant of $\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}$ is $\mathbf{R}=u_{21} u_{20}^{(1)} u_{11} u_{00}-u_{21}^{(1)} u_{20} u_{01} u_{10}$. Moreover, $\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}$ satisfy the conditions of Corollary 4.10, so given a specialized system $\overline{\mathbb{P}}_{i}$ with $\mathbf{R}\left(\mathbf{v}_{0}, \mathbf{v}_{1}, \mathbf{v}_{2}\right)=0$ and $\frac{\partial \mathbf{R}}{\partial u_{i k}}\left(\mathbf{v}_{0}, \mathbf{v}_{1}, \mathbf{v}_{2}\right) \neq 0(i=$ $0,1,2 ; k=0,1)$, the system $\overline{\mathbb{P}}_{i}=0(i=0, \ldots, n)$ have a unique nonzero solution $y_{2}=-\frac{v_{20}}{v_{21}}$ and $y_{1}=-\frac{v_{00}}{v_{01} y_{2}}=\frac{v_{00} v_{21}}{v_{01} v_{20}}$.

### 4.3 Order bound in terms of Jacobi number

In this section, we will give an order bound for the sparse difference resultant in terms of the Jacobi number of the given system.

Consider a generic Laurent transformally essential system $\left\{\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right\}$ defined in (11) with $\mathbf{u}_{i}=\left(u_{i 0}, u_{i 1}, \ldots, u_{i l_{i}}\right)$ being the coefficient vector of $\mathbb{P}_{i}(i=0, \ldots, n)$. Suppose $\mathbf{R}$ is the sparse difference resultant of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$. Denote ord $\left(\mathbf{R}, \mathbf{u}_{i}\right)$ to be the maximal order of $\mathbf{R}$ in $u_{i k}\left(k=0, \ldots, l_{i}\right)$, that is, $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=\max _{k} \operatorname{ord}\left(\mathbf{R}, u_{i k}\right)$. If $\mathbf{u}_{i}$ does not occur in $\mathbf{R}$, then set $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=-\infty$. Firstly, we have the following result.

Lemma 4.12 For fixed $i$ and $s$, if there exists $k_{0}$ such that $\operatorname{deg}\left(\mathbf{R}, u_{i k_{0}}^{(s)}\right)>0$, then for all $k \in\left\{0,1, \ldots, l_{i}\right\}, \operatorname{deg}\left(\mathbf{R}, u_{i k}^{(s)}\right)>0$. In particular, if $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=h_{i} \geq 0$, then $\operatorname{ord}\left(\mathbf{R}, u_{i k}\right)=$ $h_{i}\left(k=0, \ldots, l_{i}\right)$.
Proof: Firstly, for each $k \in\left\{1, \ldots, l_{i}\right\}$, by differentiating $\mathbf{R}\left(\mathbf{u} ; \zeta_{0}, \ldots, \zeta_{n}\right)=0$ w.r.t. $u_{i k}^{(s)}$, we have $\frac{\partial \mathbf{R}}{\partial u_{i k}^{(s)}}\left(\mathbf{u}, \zeta_{0}, \ldots, \zeta_{n}\right)+\frac{\partial \mathbf{R}}{\partial u_{i 0}^{(s)}}\left(\mathbf{u}, \zeta_{0}, \ldots, \zeta_{n}\right)\left(-\frac{M_{i k}(\eta)}{M_{i 0}(\eta)}\right)^{(s)}=0$. If $k_{0}=0$, then $\frac{\partial \mathbf{R}}{\partial u_{i 0}^{\mathrm{s})}}$ is a nonzero difference polynomial not vanishing at $\left(\mathbf{u}, \zeta_{0}, \ldots, \zeta_{n}\right)$ by lemma 3.9. So $\frac{\partial \mathbf{R}}{\partial u_{i k}^{(s)}} \neq 0$. Thus, $\operatorname{deg}\left(\mathbf{R}, u_{i k}^{(s)}\right)>0$ for each $k$. If $k_{0} \neq 0$, then $\frac{\partial \mathbf{R}}{\partial u_{i k_{0}}^{(s)}}\left(\mathbf{u}, \zeta_{0}, \ldots, \zeta_{n}\right) \neq 0$ and $\frac{\partial \mathbf{R}}{\partial u_{i 0}^{(s)}} \neq 0$ follows. So by the case $k_{0}=0$, for all $k, \operatorname{deg}\left(\mathbf{R}, u_{i k}^{(s)}\right)>0$.

In particular, if $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=h_{i} \geq 0$, then there exists some $k_{0}$ such that $\operatorname{deg}\left(\mathbf{R}, u_{i k_{0}}^{\left(h_{i}\right)}\right)>$ 0 . Thus, for each $k=0, \ldots, l_{i}, \operatorname{deg}\left(\mathbf{R}, u_{i k}^{\left(h_{i}\right)}\right)>0$ and $\operatorname{ord}\left(\mathbf{R}, u_{i k}\right)=h_{i}$ follows. I

Let $A=\left(a_{i j}\right)$ be an $n \times n$ matrix where $a_{i j}$ is an integer or $-\infty$. A diagonal sum of $A$ is any sum $a_{1 \sigma(1)}+a_{2 \sigma(2)}+\cdots+a_{n \sigma(n)}$ with $\sigma$ a permutation of $1, \ldots, n$. If $A$ is an $m \times n$ matrix with $k=\min \{m, n\}$, then a diagonal sum of $A$ is a diagonal sum of any $k \times k$ submatrix of $A$. The Jacobi number of a matrix $A$ is the maximal diagonal sum of $A$, denoted by $\operatorname{Jac}(A)$. Refer to [7, [20] for the concept of Jacobi number and its relation with the order of a difference system.

Let $s_{i j}=\operatorname{ord}\left(\mathrm{N}\left(\mathbb{P}_{i}\right), y_{j}\right)(i=0, \ldots, n ; j=1, \ldots, n)$ and $s_{i}=\operatorname{ord}\left(\mathrm{N}\left(\mathbb{P}_{i}\right)\right)$. We call the $(n+1) \times n$ matrix $A=\left(s_{i j}\right)$ the order matrix of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$. By $A_{\hat{i}}$, we mean the submatrix of $A$ obtained by deleting the $(i+1)$-th row from $A$. We use $\mathbb{P}$ to denote the set $\left\{\mathrm{N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)\right\}$ and by $\mathbb{P}_{\hat{i}}$, we mean the set $\mathbb{P} \backslash\left\{\mathrm{N}\left(\mathbb{P}_{i}\right)\right\}$. We call $J_{i}=\operatorname{Jac}\left(A_{\hat{i}}\right)$ the Jacobi number of the system $\mathbb{P}_{\hat{i}}$, also denoted by $\operatorname{Jac}\left(\mathbb{P}_{\hat{i}}\right)$. Before giving an order bound for sparse difference resultant in terms of the Jacobi numbers, we first list several lemmas.

Given a vector $\vec{K}=\left(k_{0}, k_{1}, \ldots, k_{n}\right) \in \mathbb{N}_{0}^{n+1}$, we can obtain a prolongation of $\mathbb{P}$ :

$$
\begin{equation*}
\mathbb{P}^{[\vec{K}]}=\bigcup_{i=0}^{n} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{\left[k_{i}\right]} \tag{11}
\end{equation*}
$$

Let $t_{j}=\max \left\{s_{0 j}+k_{0}, s_{1 j}+k_{1}, \ldots, s_{n j}+k_{n}\right\}$. Then $\mathbb{P}^{[\vec{K}]}$ is contained in $\mathbb{Q}\left[\mathbf{u}^{[\vec{K}]}, \mathbb{Y}^{[\vec{K}]}\right]$, where $\mathbf{u}^{[\vec{K}]}=\cup_{i=0}^{n} \mathbf{u}_{i}^{\left[k_{i}\right]}$ and $\mathbb{Y}\left[{ }^{[\vec{K}]}=\cup_{j=1}^{n} y_{j}^{\left[t_{j}\right]}\right.$.

Denote $\nu\left(\mathbb{P}^{[\vec{K}]}\right)$ to be the number of $\mathbb{Y}$ and their transforms appearing effectively in $\mathbb{P}^{[\vec{K}]}$. In order to derive a difference relation among $\mathbf{u}_{i}(i=0, \ldots, n)$ from $\mathbb{P}^{[\vec{K}]}$, a sufficient condition is

$$
\begin{equation*}
\left|\mathbb{P}^{[\vec{K}]}\right| \geq \nu\left(\mathbb{P}^{[\vec{K}]}\right)+1 . \tag{12}
\end{equation*}
$$

Note that $\nu\left(\mathbb{P}^{[\vec{K}]}\right) \leq\left|\mathbb{Y}^{[\vec{K}]}\right|=\sum_{j=1}^{n}\left(t_{j}+1\right)$. Thus, if $\left|\mathbb{P}^{[\vec{K}]}\right| \geq \mathbb{Y}^{[\vec{K}]}+1$, or equivalently,

$$
\begin{equation*}
k_{0}+k_{1}+\cdots+k_{n} \geq \sum_{j=1}^{n} \max \left(s_{0 j}+k_{0}, s_{1 j}+k_{1}, \ldots, s_{n j}+k_{n}\right) \tag{13}
\end{equation*}
$$

is satisfied, then so is the inequality (12).

Lemma 4.13 Let $\mathbb{P}$ be a Laurent transformally essential system and $\vec{K}=\left(k_{0}, k_{1}, \ldots, k_{n}\right) \in$ $\mathbb{N}_{0}^{n+1}$ a vector satisfying (13). Then $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right) \leq k_{i}$ for each $i=0, \ldots, n$.

Proof: Denote $\mathbf{m}^{[\vec{K}]}$ to be the set of all monomials in variables $\mathbb{Y}[\vec{K}]$. Let $\mathcal{I}=\left(\mathbb{P}^{[\vec{K}]}\right): \mathbf{m}^{[\vec{K}]}$ be an ideal in the polynomial ring $\mathbb{Q}\left[\mathbb{Y}^{[\vec{K}]}, \mathbf{u}^{[\vec{K}]}\right]$. Denote $U=\mathbf{u}^{[\vec{K}]} \backslash \cup_{i=0}^{n} u_{i 0}^{\left[k_{i}\right]}$. Let $\zeta_{i l}=$ $-\left(\sum_{k=1}^{l_{i}} u_{i k} M_{i k} / M_{i 0}\right)^{(l)}$ for $i=0,1, \ldots, n ; l=0,1, \ldots, k_{i}$. Denote $\zeta=\left(U, \zeta_{0 k_{0}}, \ldots, \zeta_{00}, \ldots\right.$, $\left.\zeta_{n k_{n}}, \ldots, \zeta_{n 0}\right)$. It is easy to show that $\left(\mathbb{Y}^{[\vec{K}]}, \zeta\right)$ is a generic zero of $\mathcal{I}$. Let $\mathcal{I}_{1}=\mathcal{I} \cap \mathbb{Q}\left[\mathbf{u}^{[\mid \vec{K}]}\right]$. Then $\mathcal{I}_{1}$ is a prime ideal with a generic zero $\zeta$. Since $\mathbb{Q}(\zeta) \subset \mathbb{Q}(\mathbb{Y}[\vec{K}], U), \operatorname{Codim}\left(\mathcal{I}_{1}\right)=$ $|U|+\sum_{i=0}^{n}\left(k_{i}+1\right)-\operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}(\zeta) / \mathbb{Q} \geq|U|+\left|\mathbb{P}^{[\vec{K}]}\right|-\operatorname{tr} . \operatorname{deg} \mathbb{Q}\left(\mathbb{Y}^{[\vec{K}]}, U\right) / \mathbb{Q}=\left|\mathbb{P}^{[\vec{K}]}\right|-|\mathbb{Y}[\vec{K}]| \geq 1$. Thus, $\mathcal{I}_{1} \neq\{0\}$. Suppose $f$ is a nonzero polynomial in $\mathcal{I}_{1}$. Clearly, $\operatorname{ord}\left(f, \mathbf{u}_{i}\right) \leq k_{i}$ and $f \in[\mathbb{P}]: \mathbf{m} \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$. By Lemma 3.9 and Lemma 4.12, ord $\left(\mathbf{R}, \mathbf{u}_{i}\right) \leq \operatorname{ord}\left(f, \mathbf{u}_{i}\right) \leq k_{i}$. I

Lemma 4.14 [27, Lemma 5.6] Let $\mathbb{P}$ be a system with $J_{i} \geq 0$ for each $i=0, \ldots, n$. Then $k_{i}=J_{i}(i=0, \ldots, n)$ satisfy (13) in the equality case.

Corollary 4.15 Let $\mathbb{P}$ be a Laurent transformally essential system and $J_{i} \geq 0$ for each $i=0, \ldots, n$. Then $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right) \leq J_{i}(i=0, \ldots, n)$.

Proof: It is a direct consequence of Lemma 4.13 and Lemma 4.14. I
The above corollary shows that when all the Jacobi numbers are not less that 0 , then Jacobi numbers are order bounds for the sparse difference resultant. In the following, we deal with the remaining case when some $J_{i}=-\infty$. To this end, two more lemmas are needed.

Lemma 4.16 [7, 23] Let $A$ be an $m \times n$ matrix whose entries are 0 's and 1 's. Let $\operatorname{Jac}(A)=$ $J<\min \{m, n\}$. Then $A$ contains an $a \times b$ zero sub-matrix with $a+b=m+n-J$.

Lemma 4.17 Let $\mathbb{P}$ be a Laurent transformally essential system with the following $(n+1) \times n$ order matrix

$$
A=\left(\begin{array}{cc}
A_{11} & (-\infty)_{r \times t} \\
A_{21} & A_{22}
\end{array}\right),
$$

where $r+t \geq n+1$. Then $r+t=n+1$ and $\operatorname{Jac}\left(A_{22}\right) \geq 0$. Moreover, when regarded as difference polynomials in $y_{1}, \ldots, y_{r-1},\left\{\mathbb{P}_{0}, \ldots, \mathbb{P}_{r-1}\right\}$ is a Laurent transformally essential system.

Proof: The proof is similar to [27, Lemma 5.9].I
Theorem 4.18 Let $\mathbb{P}$ be a Laurent transformally essential system and $\mathbf{R}$ the sparse difference resultant of $\mathbb{P}$. Then

$$
\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)= \begin{cases}-\infty & \text { if } \quad J_{i}=-\infty \\ h_{i} \leq J_{i} & \text { if } \quad J_{i} \geq 0\end{cases}
$$

Proof: Corollary 4.15 proves the case when $J_{i} \geq 0$ for each $i$. Now suppose there exists at least one $i$ such that $J_{i}=-\infty$. Without loss of generality, we assume $J_{n}=-\infty$ and let
$A_{n}=\left(s_{i j}\right)_{0 \leq i \leq n-1 ; 1 \leq j \leq n}$ be the order matrix of $\mathbb{P}_{\hat{n}}$. By Lemma4.16, we can assume that $A_{n}$ is of the following form

$$
A_{n}=\left(\begin{array}{cc}
A_{11} & (-\infty)_{r \times t} \\
\bar{A}_{21} & \bar{A}_{22}
\end{array}\right),
$$

where $r+t \geq n+1$. Then the order matrix of $\mathbb{P}$ is equal to

$$
A=\left(\begin{array}{cc}
A_{11} & (-\infty)_{r \times t} \\
A_{21} & A_{22}
\end{array}\right)
$$

Since $\mathbb{P}$ is Laurent transformally essential, by Lemma4.17, $r+t=n+1$ and $\operatorname{Jac}\left(A_{22}\right) \geq 0$. Moreover, considered as difference polynomials in $y_{1}, \ldots, y_{r-1}, \widetilde{\mathbb{P}}=\left\{p_{0}, \ldots, p_{r-1}\right\}$ is Laurent transformally essential and $A_{11}$ is its order matrix. Let $\widetilde{J}_{i}=\operatorname{Jac}\left(\left(A_{11}\right)_{\hat{i}}\right)$. By applying the above procedure when necessary, we can suppose that $\widetilde{J}_{i} \geq 0$ for each $i=0, \ldots, r-1$. Since $[\mathbb{P}] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}=[\widetilde{\mathbb{P}}] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{r-1}\right\}, \mathbf{R}$ is also the sparse difference resultant of the system $\widetilde{\mathbb{P}}$ and $\mathbf{u}_{r}, \ldots, \mathbf{u}_{n}$ will not occur in $\mathbf{R}$. By Corollary 4.15, ord $\left(\mathbf{R}, \mathbf{u}_{i}\right) \leq \widetilde{J}_{i}$. Since $J_{i}=\operatorname{Jac}\left(A_{22}\right)+\widetilde{J}_{i} \geq \widetilde{J}_{i}$ for $0 \leq i \leq r-1$, ord $\left(\mathbf{R}, \mathbf{u}_{i}\right) \leq J_{i}$ for $0 \leq i \leq r-1$ and $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=-\infty$ for $i=r, \ldots, n$. $\boldsymbol{I}$

Example 4.19 Let $n=2$ and

$$
\mathbb{P}_{0}=u_{00}+u_{01} y_{1} y_{1}^{(1)}, \mathbb{P}_{1}=u_{10}+u_{11} y_{1}, \mathbb{P}_{2}=u_{10}+u_{11} y_{2}^{(1)}
$$

The sparse resultant is $\mathbf{R}=u_{00} u_{11} u_{11}^{(1)}+u_{01} u_{10} u_{10}^{(1)}$. In this example, the order matrix of $\mathbb{P}$ is $A=\left(\begin{array}{cc}1 & -\infty \\ 0 & -\infty \\ -\infty & 1\end{array}\right)$. Thus $J_{0}=1, J_{1}=2, J_{2}=-\infty . \quad$ And $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{0}\right)=0<$ $J_{0}, \operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{1}\right)=1<J_{1}, \operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{2}\right)=-\infty$.

Corollary 4.20 Let $\mathbb{P}$ be super-essential. Then $J_{i} \geq 0$ for $i=0, \ldots, n$ and $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right) \leq J_{i}$.
Proof: From the proof of Theorem 4.18, if $J_{i}=-\infty$ for some $i$, then $\mathbb{P}$ contains a proper transformally essential subsystem, which contradicts to Theorem 3.28. Therefore, $J_{i} \geq 0$ for $i=0, \ldots, n$. $\boldsymbol{I}$

We conclude this section by giving two improved order bounds based on the Jacobi bound given in Theorem 4.18.

For each $j \in\{1, \ldots, n\}$, let $\underline{o}_{j}=\min \left\{k \in \mathbb{N}_{0} \mid \forall i\right.$ s.t. $\left.\operatorname{deg}\left(\mathbb{N}\left(\mathbb{P}_{i}\right), y_{j}^{(k)}\right)>0\right\}$. In other words, $\underline{o}_{j}$ is the smallest number such that $y_{j}^{\left(\underline{o}_{j}\right)}$ occurs in $\left\{\mathrm{N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)\right\}$. Let $B=\left(s_{i j}-\underline{o}_{j}\right)$ be an $(n+1) \times n$ matrix. We call $\bar{J}_{i}=\operatorname{Jac}\left(B_{\hat{i}}\right)$ the modified Jacobi number of the system $\mathbb{P}_{\hat{i}}$. Denote $\underline{\gamma}=\sum_{j=1}^{n} \underline{o}_{j}$. Clearly, $\bar{J}_{i}=J_{i}-\underline{\gamma}$. Then we have the following result.

Theorem 4.21 Let $\mathbb{P}$ be a Laurent transformally essential system and $\mathbf{R}$ the sparse difference resultant of $\mathbb{P}$. Then

$$
\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=\left\{\begin{array}{lll}
-\infty & \text { if } & J_{i}=-\infty \\
h_{i} \leq J_{i}-\underline{\gamma} & \text { if } \quad J_{i} \geq 0
\end{array}\right.
$$

Proof: The proof is similar to [27, Theorem 5.13]. I
Now, we assume that $\mathbb{P}$ is a Laurent transformally essential system which is not superessential. Let $\mathbf{R}$ be the sparse difference resultant of $\mathbb{P}$. We will give a better order bound for $\mathbf{R}$. By Theorem 3.28, $\mathbb{P}$ contains a unique super-essential sub-system $\mathbb{P}_{\mathbb{T}}$. Without loss of generality, suppose $\mathbb{T}=\{0, \ldots, r\}$ with $r<n$. Let $A_{\mathbb{T}}$ be the order matrix of $\mathbb{P}_{\mathbb{T}}$ and for $i=0, \ldots, r$, let $\left(A_{\mathbb{T}}\right)_{\hat{i}}$ be the matrix obtained from $A_{\mathbb{T}}$ by deleting the $(i+1)$-th row. Note that $\left(A_{\mathbb{T}}\right)_{\hat{i}}$ is an $r \times n$ matrix. Then we have the following result.

Theorem 4.22 With the above notations, we have

$$
\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)= \begin{cases}h_{i} \leq \operatorname{Jac}\left(\left(A_{\mathbb{T}}\right)_{\hat{i}}\right) & i=0, \ldots, r, \\ -\infty & i=r+1, \ldots, n .\end{cases}
$$

Proof: Similarly to the proof of [27, Theorem 5.16], it can be proved. $\boldsymbol{I}$
Example 4.23 Continue from Example 4.19. In this example, $\mathbb{T}=\{0,1\}$. Then $A_{\mathbb{T}}=$ $\binom{1}{0}$. Thus $\operatorname{Jac}\left(\left(A_{\mathbb{T}}\right)_{\hat{0}}\right)=0, \operatorname{Jac}\left(\left(A_{\mathbb{T}}\right)_{\hat{1}}\right)=1$. For this example, the exact bounds are given: $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{0}\right)=0=\operatorname{Jac}\left(\left(A_{\mathbb{T}}\right)_{\hat{0}}\right), \operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{1}\right)=1=\operatorname{Jac}\left(\left(A_{\mathbb{T}}\right)_{\hat{1}}\right), \operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{2}\right)=-\infty$.

### 4.4 Effective order bound in terms of Jacobi number

In this section, we give an improved Jacobi-type bound for the effective order and order of the sparse difference resultant.

For a difference polynomial $f \in \mathcal{F}\left\{y_{1}, \ldots, y_{n}\right\}$ and an arbitrary variable $y_{i}$, the least order of $f$ w.r.t. $y_{i}$ is $\operatorname{Lord}\left(f, y_{i}\right)=\min \left\{k \mid \operatorname{deg}\left(f, y_{i}^{(k)}\right)>0\right\}$ and the effective order of $f$ w.r.t. $y_{i}$ is $\operatorname{Eord}\left(f, y_{i}\right)=\operatorname{ord}\left(f, y_{i}\right)-\operatorname{Lord}\left(f, y_{i}\right)$. And if $y_{i}$ does not appear in $f$, then set $\operatorname{Eord}\left(f, y_{i}\right)=-\infty$. Let $\mathbf{R}$ be the sparse difference resultant of a Laurent transformally essential system $\left\{\mathbb{P}_{0}, \mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right\}$ of the form (11). By Lemma 4.12, $u_{i 0}^{(s)}$ effectively appears in $\mathbf{R}$ if and only if $u_{i k}^{(s)}$ effectively appears in $\mathbf{R}$ for each $k \in\left\{0, \ldots, l_{i}\right\}$. Thus, we can define $\operatorname{Lord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=\operatorname{Lord}\left(\mathbf{R}, u_{i 0}\right)$ and $\operatorname{Eord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)-\operatorname{Lord}\left(\mathbf{R}, \mathbf{u}_{i}\right)$ whenever $\mathbf{u}_{i}$ effectively appears in $\mathbf{R}$.

For further discussion, suppose $\mathbb{P}_{\mathbb{T}}$ is the super-essential subsystem of $\left\{\mathbb{P}_{0}, \mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right\}$. Without loss of generality, assume $\mathbb{T}=\{0,1, \ldots, p\}$. For each $i \in\{0, \ldots, p\}$, let $\underline{s}_{i}=$ $\min _{j=1}^{n}\left\{\operatorname{Lord}\left(\mathbb{P}_{i}, y_{j}\right) \mid \operatorname{Lord}\left(\mathbb{P}_{i}, y_{j}\right) \neq-\infty\right\}$ and $\underline{s}=\sum_{i=0}^{p} \underline{s}_{i}$. Let $\widetilde{J}_{i}=J_{i}-\underline{s}+\underline{s}_{i}$. Then,

Theorem 4.24 The effective order of $\mathbf{R}$ in $\mathbf{u}_{i}$ is bounded by $\widetilde{J}_{i}$ for each $0 \leq i \leq p$.
Proof: Let $m=\max _{i=0}^{p} \underline{\underline{s}}_{i}$. Consider the following difference system

$$
\mathcal{P}_{1}=\left\{\mathbb{P}_{0}^{\left(m-\underline{s}_{0}\right)}, \ldots, \mathbb{P}_{p}^{\left(m-\underline{s}_{p}\right)}\right\}
$$

which is also super-essential. Suppose $\mathbf{R}_{1}$ is the sparse difference resultant of $\mathcal{P}_{1}$. Clearly, $\mathbf{R}_{1} \in \mathcal{I}_{\mathbf{u}}=\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{p}\right] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{p}\right\}$, so ord $\left(\mathbf{R}_{1}, \mathbf{u}_{i}\right) \geq \operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)$ for each $i \in\{0, \ldots, p\}$. Since $y_{j}^{[m-1]}(j=1, \ldots, n)$ do not occur in $\mathcal{P}_{1}$, by replacing $y_{j}^{(t)}(j=1, \ldots, n)$ by $z_{j}^{(t-m)}$ in $\mathcal{P}_{1}$,
we obtain a new system $\mathcal{P}_{2}$. It is clear that $\mathbf{R}_{1}$ is also the sparse difference resultant of $\mathcal{P}_{2}$. By Theorem 4.22, $\operatorname{Eord}\left(\mathbf{R}_{1}, \mathbf{u}_{i}\right) \leq \widetilde{J}_{i}$ and ord $\left(\mathbf{R}_{1}, \mathbf{u}_{i}\right) \leq \widetilde{J}_{i}+m-\underline{s}_{i}$ for each $i \in\{0, \ldots, p\}$.

Let $h_{i}=\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)$ and $o_{i}=\operatorname{Lord}\left(\mathbf{R}, \mathbf{u}_{i}\right)$. We need to show that $h_{i}-o_{i} \leq \widetilde{J}_{i}$ holds for each $i \in\{0, \ldots, p\}$. Suppose the contrary, i.e. there exists some $i_{0} \in\{0, \ldots, p\}$ such that $\operatorname{Eord}\left(\mathbf{R}, \mathbf{u}_{i_{0}}\right)=h_{i_{0}}-o_{i_{0}}>\widetilde{J}_{i_{0}}$.

Suppose $\bar{h}_{i_{0}}=\operatorname{ord}\left(\mathbf{R}_{1}, \mathbf{u}_{i_{0}}\right)$ and $\bar{o}_{i_{0}}=\operatorname{Lord}\left(\mathbf{R}_{1}, \mathbf{u}_{i_{0}}\right)$. Then, $\bar{h}_{i_{0}} \geq h_{i_{0}}$ and $\operatorname{Eord}\left(\mathbf{R}_{1}, \mathbf{u}_{i_{0}}\right)$ $=\bar{h}_{i_{0}}-\bar{o}_{i_{0}} \leq \widetilde{J}_{i_{0}}<h_{i_{0}}-o_{i_{0}}$. Clearly, $\sigma^{\bar{h}_{0}} u_{i 0}$ appears effectively in both $\sigma^{\bar{h}_{i_{0}}-h_{i_{0}}} \mathbf{R}$ and $\mathbf{R}_{1}$. Let $B_{1}$ be the Sylvester resultant of $\sigma^{\bar{h}_{i_{0}}-h_{i}} \mathbf{R}$ and $\mathbf{R}_{1}$ w.r.t. $\sigma^{\bar{h}_{i_{0}}} u_{i 0}$. We claim that $B_{1} \neq 0$. Suppose the contrary, then we have $\sigma^{\bar{h}_{i_{0}}-h_{i}} \mathbf{R} \mid \mathbf{R}_{1}$, for $\mathbf{R}$ is irreducible. This is impossible since $\sigma^{\bar{h}_{i_{0}}-h_{i_{0}}+o_{i}} u_{i 0}$ appears effectively in $\sigma^{\bar{h}_{i_{0}}-h_{i_{0}}} \mathbf{R}$ while not in $\mathbf{R}_{1}$ for $\bar{h}_{i_{0}}-h_{i_{0}}+o_{i_{0}}<\bar{o}_{i_{0}}$.

Let $\widetilde{h}_{i_{0}}=\operatorname{ord}\left(B_{1}, u_{i_{0} 0}\right)$ and $\widetilde{o}_{i_{0}}=\operatorname{Lord}\left(B_{1}, u_{i_{0} 0}\right)$. Since $B_{1}$ is the resultant of $\sigma^{\bar{h}_{i_{0}}-h_{i_{0}}} \mathbf{R}$ and $\mathbf{R}_{1}, \widehat{h}_{i_{0}}<\bar{h}_{i_{0}}$ and $\widetilde{o}_{i_{0}} \geq \bar{h}_{i_{0}}-h_{i_{0}}+o_{i_{0}}$. Then $h_{i_{0}}-\widetilde{o}_{i_{0}}<\bar{h}_{i_{0}}-\left(\bar{h}_{i_{0}}-h_{i_{0}}+o_{i_{0}}\right)=h_{i_{0}}-o_{i_{0}}$. Since $B_{1} \in \mathcal{I}_{\mathbf{u}}$, by Lemma 3.9, $\operatorname{ord}\left(B_{1}, u_{i_{0} 0}\right) \geq \operatorname{ord}\left(\mathbf{R}, u_{i_{0} 0}\right)$. Repeat the above procedure for $B_{1}$ and $\sigma^{\widetilde{h}_{i_{0}}-h_{i 0}} \mathbf{R}$, we obtain a nonzero difference polynomial $B_{2} \in \mathcal{I}_{\mathbf{u}}$ and $\operatorname{ord}\left(B_{2}, u_{i_{0} 0}\right)<$ $\operatorname{ord}\left(B_{1}, u_{i_{0} 0}\right)$. Continue the procedure in this way, one can finally obtain a nonzero $B_{l} \in \mathcal{I}_{\mathbf{u}}$ such that $\operatorname{ord}\left(B_{l}, u_{i_{0} 0}\right)<\operatorname{ord}\left(\mathbf{R}, u_{i_{0} 0}\right)$ which contradicts to Lemma 3.9.

I
By the proof of the above theorem, the order of $\mathbf{R}_{1}$ with respect to $\mathbf{u}_{i}$ is bounded by $\widetilde{J}_{i}+m-\underline{s}_{i}$. Thus, we have the following new order bound for $\mathbf{R}$.

Corollary 4.25 Let $\mathbf{R}$ and $\widetilde{J}_{i}(i=0, \ldots, p)$ be defined as above. Then the order of $\mathbf{R}$ in $\mathbf{u}_{i}$ is bounded by $\underline{J_{i}}=\widetilde{J}_{i}+m-\underline{s}_{i}=J_{i}-\underline{s}+m$ for each $0 \leq i \leq p$ where $m=\max _{i=0}^{p} \underline{s}_{i}$.

Example 4.26 Let $\mathbb{P}_{0}=u_{00}+u_{01} y_{1}+u_{02} y_{2}, \mathbb{P}_{1}=u_{10}+u_{11} y_{1}^{(1)}+u_{12} y_{2}^{(1)}, \mathbb{P}_{2}=u_{20}+u_{21} y_{1}^{(1)}+$ $u_{22} y_{2}^{(1)}$. Then $J_{0}=\bar{J}_{0}=2, J_{1}=\bar{J}_{1}=1, J_{2}=\bar{J}_{2}=1, \widetilde{J}_{0}=\widetilde{J}_{1}=\widetilde{J}_{2}=0$. By corollary 4.255, $\underline{J_{0}}=1, \underline{J_{1}}=0, \underline{J_{2}}=0$. Notice that $\mathbf{R}=\left|\begin{array}{ccc}u_{00}^{(1)} & u_{01}^{(1)} & u_{02}^{(1)} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22}\end{array}\right|$ and $\widetilde{J}_{0}=\widetilde{J}_{1}=\widetilde{J}_{2}=0$, $\underline{J_{0}}=1, \underline{J_{1}}=\underline{J_{2}}=0$ give the exact effective order and order of $\mathbf{R}$ respectively.

## 5 Sparse difference resultant as algebraic sparse resultant

In this section, we will show that the sparse difference resultant is just equal to the algebraic sparse resultant of certain generic sparse polynomial system, which leads to a determinant representation for the sparse difference resultant.

### 5.1 Preliminary on algebraic sparse resultant

We first introduce several basic notions and properties on algebraic sparse resultants which are needed in this paper. For more details about sparse resultant, please refer to [18, 33].

Let $\mathcal{B}_{0}, \ldots, \mathcal{B}_{n}$ be finite subsets of $\mathbb{Z}^{n}$. Assume $\mathbf{0} \in \mathcal{B}_{i}$ and $\left|\mathcal{B}_{i}\right| \geq 2$ for each $i$. For algebraic indeterminates $\mathbb{X}=\left\{x_{1}, \ldots, x_{n}\right\}$ and $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in \mathbb{Z}^{n}$, denote $\mathbb{X}^{\alpha}=\prod_{i=1}^{n} x_{i}^{\alpha_{i}}$.

Let

$$
\begin{equation*}
\mathbb{F}_{i}\left(x_{1}, \ldots, x_{n}\right)=c_{i 0}+\sum_{\alpha \in \mathcal{B}_{i} \backslash\{\mathbf{0}\}} c_{i \alpha} \mathbb{X}^{\alpha}(i=0, \ldots, n) \tag{14}
\end{equation*}
$$

be generic sparse Laurent polynomials, where $c_{i \alpha}$ are algebraic indeterminates. We call $\mathcal{B}_{i}$ the support of $\mathbb{F}_{i}$ and $\omega_{i}=\sum_{\alpha \in \mathcal{B}_{i}} c_{i \alpha} \alpha$ is called the symbolic support vector of $\mathbb{F}_{i}$. The smallest convex subset of $\mathbb{R}^{n}$ containing $\mathcal{B}_{i}$ is called the Newton polytope of $\mathbb{F}_{i}$. For any subset $I \subset\{0, \ldots, n\}$, the matrix $M_{I}$ whose row vectors are $\omega_{i}(i \in I)$ is called the symbolic support matrix of $\left\{\mathbb{F}_{i}: i \in I\right\}$. Denote $\mathbf{c}_{i}=\left(c_{i \alpha}\right)_{\alpha \in \mathcal{B}_{i}}$ and $\mathbf{c}_{I}=\cup_{i \in I} \mathbf{c}_{i}$. Similar to the proof of [27. Theorem 4.17] and use the Jacobi criterion for algebraic independence, it can be easily shown that

Lemma 5.1 For any subset $I \subset\{0, \ldots, n\}$, $\operatorname{tr} \cdot \operatorname{deg} \mathbb{Q}\left(\mathbf{c}_{I}\right)\left(\mathbb{F}_{i}: i \in I\right) / \mathbb{Q}\left(\mathbf{c}_{I}\right)=\operatorname{rk}\left(M_{I}\right)$.
Definition 5.2 Follow the notations introduced above.

- A collection of $\left\{\mathbb{F}_{i}\right\}_{i \in I}$ is said to be weak essential if $\operatorname{rk}\left(M_{I}\right)=|I|-1$.
- A collection of $\left\{\mathbb{F}_{i}\right\}_{i \in I}$ is said to be essential if $\operatorname{rk}\left(M_{I}\right)=|I|-1$ and for each proper subset $J$ of $I, \operatorname{rk}\left(M_{J}\right)=|J|$.

Similar to Theorems 3.25 and 3.28 , we have the following two lemmas.

Lemma $5.3\left\{\mathbb{F}_{i}\right\}_{i \in I}$ is weak essential if and only if $\left(\mathbb{F}_{i}: i \in I\right) \cap \mathbb{Q}\left[\mathbf{c}_{I}\right]$ is of codimension one. In this case, there exists an irreducible polynomial $\mathbf{R} \in \mathbb{Q}\left[\mathbf{c}_{I}\right]$ such that $\left(\mathbb{F}_{i}: i \in I\right) \cap \mathbb{Q}\left[\mathbf{c}_{I}\right]=$ $(\mathbf{R})$ and $\mathbf{R}$ is called the sparse resultant of $\left\{\mathbb{F}_{i}: i \in I\right\}$.

Lemma $5.4\left\{\mathbb{F}_{i}\right\}_{i \in I}$ is essential if and only if $\left(\mathbb{F}_{i}: i \in I\right) \cap \mathbb{Q}\left[\mathbf{c}_{I}\right]=(\mathbf{R})$ and $\mathbf{c}_{i}$ appears effectively in $\mathbf{R}$ for each $i \in I$.

Suppose an arbitrary total ordering of $\left\{\mathbb{F}_{0}, \ldots, \mathbb{F}_{n}\right\}$ is given, say $\mathbb{F}_{0}<\mathbb{F}_{1}<\cdots<\mathbb{F}_{n}$. Now we define a total ordering among subsets of $\left\{\mathbb{F}_{0}, \ldots, \mathbb{F}_{n}\right\}$. For any two subsets $\mathcal{D}=$ $\left\{D_{0}, \ldots, D_{s}\right\}$ and $\mathcal{C}=\left\{C_{0}, \ldots, C_{t}\right\}$ where $D_{0}>\cdots>D_{s}$ and $C_{0}>\cdots>C_{t}, \mathcal{D}$ is said to be of higher ranking than $\mathcal{C}$, denoted by $\mathbb{D} \succ \mathcal{C}$, if 1$)$ there exists an $i \leq \min (s, t)$ such that $D_{0}=C_{0}, \ldots, D_{i-1}=C_{i-1}, D_{i}>C_{i}$ or 2$) s>t$ and $D_{i}=C_{i}(i=0, \ldots, t)$. Note that if $\mathcal{D}$ is a proper subset of $\mathcal{C}$, then $\mathcal{C} \succ \mathcal{D}$.

Lemma 5.5 Let $\mathbb{F}=\left\{\mathbb{F}_{i}: i=0, \ldots, n\right\}$ be the system given in (14). Suppose $\operatorname{rk}\left(M_{\mathbb{F}}\right) \leq n$. Then $\mathbb{F}$ has an essential subset with minimal ranking.

Proof: It suffices to show that $\mathbb{F}$ contains an essential subset, for the existence of an essential subset with minimal ranking can be deduced from the fact that " $\succ$ " is a total ordering.

Let $\mathcal{T}_{i}=\mathbb{F} \backslash\left\{\mathbb{F}_{0}, \ldots, \mathbb{F}_{i-1}\right\}(i=1, \ldots, n)$ and $\mathcal{T}_{0}=\mathbb{F}$. We claim that at least one of $\mathcal{T}_{i}$ is weak essential. If $\operatorname{rk}\left(M_{\mathcal{T}_{0}}\right)=n$, we are done. Otherwise, $\operatorname{rk}\left(M_{\mathcal{T}_{0}}\right)<n$. It is clear that $\operatorname{rk}\left(M_{\mathcal{T}_{i}}\right)=\operatorname{rk}\left(M_{\mathcal{T}_{i-1}}\right)$ or $\operatorname{rk}\left(M_{\mathcal{T}_{i}}\right)=\operatorname{rk}\left(M_{\mathcal{T}_{i-1}}\right)-1$ for $i=1, \ldots, n-1$, so when deleting one row from the matrix, the co-rank, i.e. $\left|\mathcal{T}_{i}\right|-\operatorname{rk}\left(M_{\mathcal{T}_{i}}\right)$, will be unchanged or decreased by 1 . Since $\operatorname{rk}\left(M_{\mathcal{T}_{0}}\right)<n$, the co-rank of $M_{\mathcal{T}_{0}}$ is larger than 1 . Since the co-rank of $M_{\mathcal{T}_{n}}$ is 0 , there exists $k \in\{1, \ldots, n-1\}$ such that the co-rank of $M_{\mathcal{T}_{k}}$ is 1 . Then $M_{\mathcal{T}_{k}}$ is weak essential. Now,
let $\mathbf{R}$ be the sparse resultant of $\mathcal{T}_{k}$ and let $\mathcal{C}$ be the set of $\mathbb{T}_{i} \in \mathcal{T}_{k}$ such that the coefficients of $\mathbb{F}_{i}$ occur in $\mathbf{R}$ effectively. Then, $\mathcal{C}$ is an essential subset of $\mathbb{F}$ by Lemma 5.4. I

Lemma 5.6 Suppose $\mathbb{F}_{I}=\left\{\mathbb{F}_{i}: i \in I\right\}$ is an essential system. Then there exist $n-|I|+1$ of the $x_{i}$ such that by setting these $x_{i}$ to 1 , the specialized system $\widetilde{\mathbb{F}}_{I}=\left\{\widetilde{\mathbb{F}}_{i}: i \in I\right\}$ satisfies
(1) $\widetilde{\mathbb{F}}_{I}$ is still essential.
(2) $\operatorname{rk}\left(M_{\widetilde{\mathbb{F}}_{I}}\right)=|I|-1$ is the number of variables in $\widetilde{\mathbb{F}}_{I}$.
(3) $\left(\mathbb{F}_{I}\right) \cap \mathbb{Q}\left[\mathbf{c}_{I}\right]=\left(\widetilde{\mathbb{F}}_{I}\right) \cap \mathbb{Q}\left[\mathbf{c}_{I}\right]$.

Proof: Let $M_{I}=\left(m_{i j}\right)_{|I| \times n}$ be the symbolic support matrix of $\mathbb{F}_{I}$. Since $\mathbb{F}_{I}$ is essential, $M_{I}$ contains a submatrix of rank $|I|-1$. Without loss of generality, we assume the matrix $M_{0}=\left(m_{i j}\right)_{i=1, \ldots,|I|-1 ; j=1, \ldots,|I|-1}$ is of full rank. Then consider the new system $\widetilde{\mathbb{F}}_{I}$ obtained by setting $x_{i}=1(i=|I|, \ldots, n)$ in $\mathbb{F}_{I}$. Since $M_{0}$ is a submatrix of $M_{\widetilde{\mathbb{F}}_{I}}, \widetilde{\mathbb{F}}_{I}$ is weak essential. By Lemma 5.3 , we have $\left(\mathbb{F}_{I}\right) \cap \mathbb{Q}\left[\mathbf{c}_{I}\right]=(\mathbf{R})$ and $\left(\widetilde{\mathbb{F}}_{I}\right) \cap \mathbb{Q}\left[\mathbf{c}_{I}\right]=(\widetilde{\mathbf{R}})$ where $\mathbf{R}, \widetilde{\mathbf{R}}$ are irreducible polynomials in $\mathbb{Q}\left[\mathbf{c}_{I}\right]$. Hence, there exists a monomial $\mathfrak{m} \in \mathbb{Q}\left[x_{1}, \ldots, x_{n}\right]$ such that $\mathfrak{m} \mathbf{R}=\sum Q_{i} \mathbb{F}_{i}$. Set $x_{i}=1(i=|I|, \ldots, n)$, then we have $\widetilde{\mathfrak{m}} \mathbf{R}=\sum \widetilde{Q}_{i} \widetilde{\mathbb{F}}_{i}$. Hence $\mathbf{R} \in(\widetilde{\mathbf{R}})$. Since both $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ are irreducible, $(\widetilde{\mathbf{R}})=(\mathbf{R})$ and (2) follows. Thus, $\mathbf{c}_{i}(i \in I)$ appears effectively in $\widetilde{\mathbf{R}}$, for $\mathbb{F}_{I}$ is essential. By Lemma 5.4, $\widetilde{\mathbb{F}}_{I}$ is essential and (1) is proved. (2) is obvious and the lemma is proved.I

An essential system $\left\{\mathbb{F}_{i}\right\}_{i \in I}$ is said to be variable-essential if there are only $|I|-1$ variables appearing effectively in $\mathbb{F}_{i}$. Clearly, if $\left\{\mathbb{F}_{i}: i=0, \ldots, n\right\}$ is essential, then it is variableessential.

Lemma 5.7 Let $\mathbb{F}=\left\{\mathbb{F}_{i}: i=0, \ldots, n\right\}$ be an essential system of the form (14). Then we can find an invertible variable transformation $x_{1}=\prod_{j=1}^{n} z_{j}^{m_{1 j}}, \ldots, x_{n}=\prod_{j=1}^{n} z_{j}^{m_{n j}}$ for $m_{i j} \in \mathbb{Q}$, such that the image $\mathbb{G}$ of $\mathbb{F}$ under the above transformation is a generic sparse Laurent polynomial system satisfying
(1) $\mathbb{G}$ is essential.
(2) $\operatorname{Span}_{\mathbb{Z}}(\mathcal{B})=\mathbb{Z}^{\mathrm{n}}$, where $\mathcal{B}$ is the set of the supports of all monomials in $\mathbb{G}$.
$(\mathbb{F}) \cap \mathbb{Q}[\mathbf{c}]=(\mathbb{G}) \cap \mathbb{Q}[\mathbf{c}]$.

Proof: This is a direct consequence of the Smith normal form method [4, p. 67]. Also see paper [32] for an alternative proof.I

We call a variable-essential system $\mathbb{F}=\left\{\mathbb{F}_{i}: i=0, \ldots, n\right\}$ strong essential if $\mathbb{F}$ also satisfies condition (2) in Lemma 5.7. Recall that condition (2) is a basic requirement for studying sparse resultant in historic literatures and a strong essential system here is just an essential system as defined in papers [34, 9]. If $\mathbb{F}$ is strong essential, a matrix representation for $\mathbf{R}$ can be derived, that is, $\mathbf{R}$ can be represented as the quotient of the determinants of two matrices as shown in paper [9. Moreover, the exact degree of the sparse resultant $\mathbf{R}$ can be given in terms of mixed volumes [34, famous as the BKK-type degree bound. That is,

Theorem $5.8([34])$ Suppose $\mathbb{F}=\left\{\mathbb{F}_{i}: i=0, \ldots, n\right\}$ is a strong essential system of the form (14). Then, for each $i \in\{0,1, \ldots, n\}$, the degree of the sparse resultant in $\mathbf{u}_{i}$ is a
positive integer, equal to the mixed volume

$$
\mathcal{M}\left(\mathcal{Q}_{0}, \ldots, \mathcal{Q}_{i-1}, \mathcal{Q}_{i+1}, \ldots, \mathcal{Q}_{n}\right)=\sum_{J \subset\{0, \ldots, i-1, i+1, \ldots, n\}}(-1)^{n-|J|} \operatorname{vol}\left(\sum_{j \in J} \mathcal{Q}_{j}\right)
$$

where $\mathcal{Q}_{i}$ is the Newton polytope of $\mathbb{F}_{i}, \operatorname{vol}(\mathcal{Q})$ means the $n$-dimensional volume of $\mathcal{Q} \subset \mathbb{R}^{n}$ and $\mathcal{Q}_{1}+\mathcal{Q}_{2}$ means the Minkowski sum of $\mathcal{Q}_{1}$ and $\mathcal{Q}_{2}$.

### 5.2 Sparse difference resultant as algebraic sparse resultant

With the above preparation, we now give the main theorem of this section.
Theorem 5.9 Let $\mathbf{R}$ be the sparse difference resultant of a Laurent transformally essential system (1). Then we can derive a strong essential generic algebraic sparse polynomial system $\mathcal{S}$ from (1), such that the sparse resultant of $\mathcal{S}$ is equal to $\mathbf{R}$.

Proof: By Theorem 3.28, the system (1) has a unique super-essential subsystem $\mathbb{P}_{\mathbb{T}}$. Without loss of generality, assume $\mathbb{T}=\{0,1, \ldots, p\}$. For each $i \in\{0, \ldots, p\}$, let $k_{i}=\operatorname{Jac}\left(\left(A_{\mathbb{T}}\right)_{\hat{i}}\right)$ as defined in Theorem 4.22 and let $\vec{K}=\left(k_{0}, k_{1}, \ldots, k_{p}\right) \in \mathbb{N}_{0}^{p+1}$. Similar to (11), let

$$
\begin{equation*}
\mathcal{P}=\mathbb{P}^{[\vec{K}]}=\bigcup_{i=0}^{p} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{\left[k_{i}\right]} \tag{15}
\end{equation*}
$$

be the prolongation of $\mathbb{P}_{\mathbb{T}}$ with respect to $\vec{K}$. Note that $|\mathcal{P}|=\sum_{i=0}^{p} k_{i}+p+1$. Regarding $\mathcal{P}$ as a set of algebraic polynomials in $y_{i}^{(j)}$ with coefficients $U=\cup_{i=0}^{n} \mathbf{u}_{i}^{\left[k_{i}\right]}$, then $\mathcal{P}$ is a generic sparse polynomial system.

A total ordering for polynomials in $\mathcal{P}$ is assigned as follows: $\sigma^{k} \mathbb{P}_{i}<\sigma^{l} \mathbb{P}_{j}$ if and only if $i<j$ or $i=j$ and $k<l$. A total ordering $\succ$ among subsets of $\mathcal{P}$ is the same as the one given in Section 5.1. By Theorem 4.22, $\operatorname{rk}\left(M_{\mathcal{P}}\right) \leq \sum_{i=0}^{p} k_{i}+p=|\mathcal{P}|-1$. By Lemma 5.5, we can construct an essential subsystem $\mathcal{P}_{1}$ of $\mathcal{P}$ with minimal ranking. Let $\mathbf{R}_{1}$ be the sparse resultant of $\mathcal{P}_{1}$, that is, $\left(\mathcal{P}_{1}\right) \cap \mathbb{Q}[U]=\left(\mathbf{R}_{1}\right)$.

We claim that $\mathbf{R}_{1}=c \mathbf{R}$ for some $c \in \mathbb{Q}$. Since $\mathbb{P}_{\mathbb{T}}$ is super essential, for each $i \in \mathbb{T}$, $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right) \geq 0$. By Theorem 4.22, $\mathbf{R} \in(\mathcal{P})$. Let $\mathcal{P}_{2}$ be the elements of $\mathcal{P}$ whose coefficients appear effectively in $\mathbf{R}$. By Lemma 5.4, $\mathcal{P}_{2}$ is essential and $\left(\mathcal{P}_{2}\right) \cap \mathbb{Q}[U]=(\mathbf{R})$. Let $k_{1}$ and $k_{2}$ be the largest integers such that $\sigma^{k_{1}} \mathbb{P}_{p} \in \mathcal{P}_{1}$ and $\sigma^{k_{2}} \mathbb{P}_{p} \in \mathcal{P}_{2}$. Since $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ are essential, ord $\left(\mathbf{R}_{1}, \mathbf{u}_{p}\right)=k_{1}$ and $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{p}\right)=k_{2}$. Since $\mathcal{P}_{2} \succ \mathcal{P}_{1}, k_{1} \leq k_{2}$. Since $\mathbf{R}_{1} \in\left(\mathcal{P}_{1}\right) \cap \mathbb{Q}[U] \subset\left[\mathbb{P}_{\mathbb{T}}\right] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{p}\right\}$, by Lemma 3.9, $k_{1} \geq k_{2}$. Hence, $k_{1}=k_{2}$. Since $\mathbf{R}_{1} \in\left[\mathbb{P}_{\mathbb{T}}\right] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{p}\right\}=\operatorname{sat}(\mathbf{R}, \ldots)$ and $\operatorname{ord}\left(\mathbf{R}_{1}, \mathbf{u}_{p}\right)=\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{p}\right), \mathbf{R}_{1}$ is algebraically reduced to zero by $\mathbf{R}$. Since both $\mathbf{R}$ and $\mathbf{R}_{1}$ are irreducible, $\mathbf{R}=c \mathbf{R}_{1}$ where $c \in \mathbb{Q}$.

Apply Lemma[5.6] to $\mathcal{P}_{1}$, we obtain a variable-essential system $\mathcal{P}_{2}$ satisfying $\left(\mathcal{P}_{2}\right) \cap \mathbb{Q}[U]=$ (R). Then apply Lemma 5.7 to $\mathcal{P}_{2}$, we obtain a strong essential generic system $\mathcal{S}$ satisfying $(\mathcal{S}) \cap \mathbb{Q}[U]=(\mathbf{R})$ and the existence of $\mathcal{S}$ is proved.

We will show that $\mathcal{S}$ can be given algorithmically. Through the above procedures, only Lemma 5.5 is not constructive. Since $\mathcal{P}$ contains an essential subsystem, we can simply check each subsystems $\mathcal{S}$ of $\mathcal{P}$ to see whether $\mathcal{S}$ is essential and find the one with minimal ranking. Note that $\mathcal{S}$ is essential if and only if $\operatorname{rk}\left(M_{\mathcal{S}}\right)=|\mathcal{S}|-1$ and any proper subset $\mathcal{C}$ of $\mathcal{S}$ satisfies $\operatorname{rk}\left(M_{\mathcal{C}}\right)=|\mathcal{C}|$.

Example 5.10 Let $n=3$. Denote $y_{i j}=y_{i}^{(j)}$ and let $\mathbb{P}=\left\{\mathbb{P}_{0}, \mathbb{P}_{1}, \mathbb{P}_{2}, \mathbb{P}_{3}\right\}$ where

$$
\begin{aligned}
& \mathbb{P}_{0}=u_{00}+u_{01} y_{11}^{2} y_{21}^{2} y_{3}+u_{02} y_{1}^{2} y_{2} y_{3}, \\
& \mathbb{P}_{1}=u_{10}+u_{11} y_{12}^{4} y_{22}^{4} y_{31}^{2}+u_{12} y_{11}^{2} y_{21} y_{31}, \\
& \mathbb{P}_{2}=u_{20}+u_{21} y_{11}^{2} y_{21}^{2} y_{3}+u_{22} y_{1}^{2} y_{2} y_{3}, \\
& \mathbb{P}_{3}=u_{30}+u_{31} y_{11} y_{3} .
\end{aligned}
$$

It is easy to show that $\mathbb{P}$ is a Laurent transformally essential system and $\mathbb{T}=\{0,1,2\}$. Clearly, $\operatorname{Jac}\left(\left(A_{\mathbb{T}}\right)_{\hat{0}}\right)=3, \operatorname{Jac}\left(\left(A_{\mathbb{T}}\right)_{\hat{1}}\right)=2$ and $\operatorname{Jac}\left(\left(A_{\mathbb{T}}\right)_{\hat{2}}\right)=3$. Using the notations in Theorem 5.9, we have $\mathcal{P}=\left\{\mathbb{P}_{0}^{[3]}, \mathbb{P}_{1}^{[2]}, \mathbb{P}_{2}^{[3]}\right\}$, and we can compute an essential subset $\mathcal{P}_{1}$ with minimal ranking. Here, we have $\mathcal{P}_{1}=\left\{\sigma \mathbb{P}_{0}, \mathbb{P}_{1}, \sigma \mathbb{P}_{2}\right\}$. Using the variable order $y_{11}<y_{12}<y_{21}<y_{22}<y_{31}$ to obtain the symbolic support matrix of $\mathcal{P}_{1}$, the first $2 \times 2$ sub-matrix of $M_{\mathcal{P}_{1}}$ is of rank 2. By the proof of Lemma 5.6. we set $y_{21}, y_{22}, y_{31}$ to 1 to obtain a variable essential system $\mathcal{P}_{2}=\left\{\widetilde{\sigma \mathbb{P}_{0}}, \widetilde{\mathbb{P}_{1}}, \widetilde{\sigma \mathbb{P}_{2}}\right\}$ where

$$
\begin{aligned}
\widetilde{\sigma \mathbb{P}_{0}} & =u_{00}^{(1)}+u_{01}^{(1)} y_{12}^{2}+u_{02}^{(1)} y_{11}^{2}, \\
\widetilde{\mathbb{P}_{1}} & =u_{10}+u_{11} y_{12}^{4}+u_{12} y_{11}^{2}, \\
\widetilde{\sigma \mathbb{P}_{2}} & =u_{20}^{(1)}+u_{21}^{(1)} y_{12}^{2}+u_{22}^{(1)} y_{11}^{2} .
\end{aligned}
$$

Apply Lemma 5.7 to $\mathcal{P}_{2}$, set $z_{1}=y_{11}^{2}, z_{2}=y_{12}^{2}$, we obtain a strong essential generic system $\mathcal{P}_{3}=\left\{Q_{0}, Q_{1}, Q_{2}\right\}$ where

$$
\begin{aligned}
Q_{0} & =u_{00}^{(1)}+u_{01}^{(1)} z_{2}+u_{02}^{(1)} z_{1}, \\
Q_{1} & =u_{10}+u_{11} z_{2}^{2}+u_{12} z_{1}, \\
Q_{2} & =u_{20}^{(1)}+u_{21}^{(1)} z_{2}+u_{22}^{(1)} z_{1} .
\end{aligned}
$$

The sparse resultant of system $\mathcal{P}_{3}$ is $R=u_{10}\left(u_{02}^{(1)} u_{21}^{(1)}-u_{01}^{(1)} u_{22}^{(1)}\right)^{2}+u_{11}\left(u_{00}^{(1)} u_{22}^{(1)}-u_{02}^{(1)} u_{20}^{(1)}\right)^{2}+$ $u_{12}\left(u_{00}^{(1)} u_{21}^{(1)}-u_{01}^{(1)} u_{20}^{(1)}\right)\left(u_{02}^{(1)} u_{21}^{(1)}-u_{01}^{(1)} u_{22}^{(1)}\right)$, which is the sparse difference resultant of $\mathbb{P}$.

The following corollary is a direct consequence of the proof of Theorem 5.9 and paper 99 .

Corollary 5.11 The sparse difference resultant $\mathbf{R}$ of a Laurent transformally essential system (11) can be represented as the quotient of two determinants whose elements are $u_{i j}^{(k)}$ or their sums for certain $i \in\{0, \ldots, n\}, j \in\left\{0, \ldots, l_{i}\right\}$ and $k \in\left\{0, \ldots, J_{i}\right\}$, where $J_{i}$ is the Jacobi number of the system (1) as defined in Section 4.3.

Remark 5.12 It is desirable to derive a degree bound for $\mathbf{R}$ from Theorem 5.9. Let $\mathcal{S}$ be the strong essential set mentioned in the theorem. Then, the degree of $\mathbf{R}$ is equal to the mixed volume of $\mathcal{S}$ by Theorem 5.8. The problem is how to express the mixed volume of $\mathcal{S}$ in terms of certain quantities of $\mathbb{P}_{\mathbb{T}}$ without computing $\mathcal{S}$.

## 6 A single exponential algorithm to compute the sparse difference resultant

In this section, we give an algorithm to compute the sparse difference resultant for a Laurent transformally essential system with single exponential complexity. The idea is to estimate the degree bounds for the resultant and then to use linear algebra to find the coefficients of the resultant.

### 6.1 Degree bound for sparse difference resultant

In this section, we give an upper bound for the degree of the sparse difference resultant, which will be crucial to our algorithm to compute the sparse resultant. Before proposing the main theorem, we first give some algebraic results which will be needed in the proof.

Lemma 6.1 [27, Theorem 6.2] Let $\mathcal{I}$ be a prime ideal in $K\left[x_{1}, \ldots, x_{n}\right]$ and $\mathcal{I}_{k}=\mathcal{I} \cap$ $K\left[x_{1}, \ldots, x_{k}\right]$ for any $1 \leq k \leq n$. Then $\operatorname{deg}\left(\mathcal{I}_{k}\right) \leq \operatorname{deg}(\mathcal{I})$.

Lemma 6.2 [35, Corollary 2.28] Let $V_{1}, \ldots, V_{r} \subset \boldsymbol{P}^{n}(r \geq 2)$ be pure dimensional projective varieties in $\boldsymbol{P}^{n}$. Then

$$
\prod_{i=1}^{r} \operatorname{deg}\left(V_{i}\right) \geq \sum_{C} \operatorname{deg}(C)
$$

where $C$ runs through all irreducible components of $V_{1} \cap \cdots \cap V_{r}$.
Now we are ready to give the main theorem of this section.
Theorem 6.3 Let $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$ be a Laurent transformally essential system of form (11) with $\operatorname{ord}\left(\mathrm{N}\left(\mathbb{P}_{i}\right)\right)=s_{i}$ and $\operatorname{deg}\left(\mathrm{N}\left(\mathbb{P}_{i}\right), \mathbb{Y}\right)=m_{i}$. Suppose $\mathrm{N}\left(\mathbb{P}_{i}\right)=\sum_{k=0}^{t_{i}} u_{i k} N_{i k}$ and $J_{i}$ is the Jacobi number of $\left\{\mathrm{N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)\right\} \backslash\left\{\mathrm{N}\left(\mathbb{P}_{i}\right)\right\}$. Denote $m=\max _{i}\left\{m_{i}\right\}$. Let $\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ be the sparse difference resultant of $\mathbb{P}_{i}(i=0, \ldots, n)$. Suppose $\operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=h_{i}$ for each $i$. Then the following assertions hold:

1) $\operatorname{deg}(\mathbf{R}) \leq \prod_{i=0}^{n}\left(m_{i}+1\right)^{h_{i}+1} \leq(m+1)^{\sum_{i=0}^{n}\left(J_{i}+1\right)}$, where $m=\max _{i}\left\{m_{i}\right\}$.
2) $\mathbf{R}$ has a representation

$$
\begin{equation*}
\prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(N_{i 0}^{(k)}\right)^{\operatorname{deg}(\mathbf{R})} \cdot \mathbf{R}=\sum_{i=0}^{n} \sum_{k=0}^{h_{i}} G_{i k} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{(k)} \tag{16}
\end{equation*}
$$

where $G_{i k} \in \mathbb{Q}\left[\mathbf{u}_{0}^{\left[h_{0}\right]}, \ldots, \mathbf{u}_{n}^{\left[h_{n}\right]}, \mathbb{Y}^{[h]}\right]$ and $h=\max \left\{h_{i}+e_{i}\right\}$ such that $\operatorname{deg}\left(G_{i k} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{(k)}\right) \leq$ $\left[m+1+\sum_{i=0}^{n}\left(h_{i}+1\right) \operatorname{deg}\left(N_{i 0}\right)\right] \operatorname{deg}(\mathbf{R})$.

Proof: In R, let $u_{i 0}$ be replaced by $\left(\mathrm{N}\left(\mathbb{P}_{i}\right)-\sum_{k=1}^{t_{i}} u_{i k} N_{i k}\right) / N_{i 0}$ for each $i=0, \ldots, n$ and let $\mathbf{R}$ be expanded as a difference polynomial in $\mathrm{N}\left(\mathbb{P}_{i}\right)$ and their transforms. Then there exist $a_{i k} \in$ $\mathbb{N}$ and polynomials $G_{i k}$ such that $\prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(N_{i 0}^{(k)}\right)^{a_{i k}} \mathbf{R}=\sum_{i=0}^{n} \sum_{k=0}^{h_{i}} G_{i k} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{(k)}+T$ with
$T \in \mathbb{Q}\{\mathbf{u}, \mathbb{Y}\}$ free from $u_{i 0}$. Since $T \in \mathcal{I}=\left[\mathrm{N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)\right]: \mathbf{m}, T$ vanishes identically, for $\mathcal{I} \cap \mathbb{Q}\{\mathbf{u}, \mathbb{Y}\}=\{0\}$ by Theorem 3.6. Thus,

$$
\prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(N_{i 0}^{(k)}\right)^{a_{i k}} \mathbf{R}=\sum_{i=0}^{n} \sum_{k=0}^{h_{i}} G_{i k} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{(k)}
$$

1) Let $\mathcal{J}=\left(\mathrm{N}\left(\mathbb{P}_{0}\right)^{\left[h_{0}\right]}, \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)^{\left[h_{n}\right]}\right): \mathbf{m}^{[h]}$ be an algebraic ideal in $\mathcal{R}=\mathbb{Q}\left[\mathbb{Y}^{[h]}, \mathbf{u}_{0}^{\left[h_{0}\right]}\right.$, $\left.\ldots, \mathbf{u}_{n}^{\left[h_{n}\right]}\right]$ where $h=\max _{i}\left\{h_{i}+s_{i}\right\}$ and $\mathbf{m}^{[h]}$ is the set of all monomials in $\mathbb{Y}^{[h]}$. Then $\mathbf{R} \in \mathcal{J}$ by the above equality. Let $\eta=\left(\eta_{1}, \ldots, \eta_{n}\right)$ be a generic zero of $[0]$ over $\mathbb{Q}\langle\mathbf{u}\rangle$ and denote $\zeta_{i}=-\sum_{k=1}^{t_{i}} u_{i k} \frac{N_{i k}(\eta)}{N_{i 0}(\eta)}(i=0, \ldots, n)$. It is easy to show that $\mathcal{J}$ is a prime ideal in $\mathcal{R}$ with a generic zero $\left(\eta^{[h]} ; \widetilde{\mathbf{u}}, \zeta_{0}^{\left[h_{0}\right]}, \ldots, \zeta_{n}^{\left[h_{n}\right]}\right)$ and $\mathcal{J} \cap \mathbb{Q}\left[\mathbf{u}_{0}^{\left[h_{0}\right]}, \ldots, \mathbf{u}_{n}^{\left[h_{n}\right]}\right]=(\mathbf{R})$, where $\widetilde{\mathbf{u}}=\cup_{i} \mathbf{u}_{i}^{\left[h_{i}\right]} \backslash\left\{u_{i 0}^{\left[h_{i}\right]}\right\}$. Let $H_{i k}$ be the homogeneous polynomial corresponding to $\mathrm{N}\left(\mathbb{P}_{i}\right)^{(k)}$ with $x_{0}$ the variable of homogeneity. Then $\mathcal{J}^{0}=\left(\left(H_{i k}\right)_{1 \leq i \leq n ; 0 \leq k \leq h_{i}}\right): \widetilde{\mathbf{m}}$ is a prime ideal in $\mathbb{Q}\left[x_{0}, \mathbb{Y}^{[h]}, \mathbf{u}_{0}^{\left[h_{0}\right]}, \ldots, \mathbf{u}_{n}^{\left[h_{n}\right]}\right]$ where $\widetilde{\mathbf{m}}$ is the whole set of monomials in $\mathbb{Y}^{[h]}$ and $x_{0}$. And $\operatorname{deg}\left(\mathcal{J}^{0}\right)=\operatorname{deg}(\mathcal{J})$.

Since $\mathbb{V}\left(\left(H_{i k}\right)_{1 \leq i \leq n ; 0 \leq k \leq h_{i}}\right)=\mathbb{V}\left(\mathcal{J}^{0}\right) \cup \mathbb{V}\left(H_{i k}, x_{0}\right) \bigcup \cup_{j, l} \mathbb{V}\left(H_{i k}, y_{j}^{(l)}\right), \mathbb{V}\left(\mathcal{J}^{0}\right)$ is an irreducible component of $\mathbb{V}\left(\left(H_{i k}\right)_{1 \leq i \leq n ; 0 \leq k \leq h_{i}}\right)$. By Lemma6.2, $\operatorname{deg}\left(\mathcal{J}^{0}\right) \leq \prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(m_{i}+1\right)=$ $\prod_{i=0}^{n}\left(m_{i}+1\right)^{h_{i}+1}$. Thus, $\operatorname{deg}(\mathcal{J}) \leq \prod_{i=0}^{n}\left(m_{i}+1\right)^{h_{i}+1}$. Since $\mathcal{J} \cap \mathbb{Q}\left[\mathbf{u}_{0}^{\left[h_{0}\right]}, \ldots, \mathbf{u}_{n}^{\left[h_{n}\right]}\right]=(\mathbf{R})$, by Lemma 6.1, $\operatorname{deg}(\mathbf{R}) \leq \operatorname{deg}(\mathcal{J}) \leq \prod_{i=0}^{n}\left(m_{i}+1\right)^{h_{i}+1} \leq(m+1)^{\sum_{i=0}^{n}\left(J_{i}+1\right)}$ follows. The last inequality holds because $h_{i} \leq J_{i}$ by Theorem 4.21,
2) To obtain the degree bounds for the above representation of $\mathbf{R}$, that is, to estimate $\operatorname{deg}\left(G_{i k} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{(k)}\right)$ and $a_{i k}$, we take each monomial $M$ in $\mathbf{R}$ and substitute $u_{i 0}$ by $\left(\mathrm{N}\left(\mathbb{P}_{i}\right)-\right.$ $\left.\sum_{k=1}^{l_{i}} u_{i k} N_{i k}\right) / N_{i 0}$ into $M$ and then expand it. To be more precise, we take one monomial $M\left(\mathbf{u} ; u_{00}, \ldots, u_{n 0}\right)=\mathbf{u}^{\gamma} \prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(u_{i 0}^{(k)}\right)^{d_{i k}}$ with $|\gamma|+\sum_{i=0}^{n} \sum_{k=0}^{h_{i}} d_{i k}=\operatorname{deg}(\mathbf{R})$ for an example, where $\mathbf{u}^{\gamma}$ represents a difference monomial in $\mathbf{u}$ and their transforms with exponent vector $\gamma$. Then

$$
M\left(\mathbf{u} ; u_{00}, \ldots, u_{n 0}\right)=\mathbf{u}^{\gamma} \prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(\left(\mathrm{~N}\left(\mathbb{P}_{i}\right)-\sum_{k=1}^{l_{i}} u_{i k} N_{i k}\right)^{(k)}\right)^{d_{i k}} / \prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(N_{i 0}^{(k)}\right)^{d_{i k}} .
$$

When expanded, every term of $\prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(N_{i 0}^{(k)}\right)^{d_{i k}} M$ is of degree bounded by $|\gamma|+$ $\sum_{i=0}^{n} \sum_{k=0}^{h_{i}}\left(m_{i}+1\right) d_{i k} \leq(m+1) \operatorname{deg}(\mathbf{R})$ in $\mathbf{u}_{0}^{\left[h_{0}\right]}, \ldots, \mathbf{u}_{n}^{\left[h_{n}\right]}$ and $\mathbb{Y}^{[h]}$. Suppose $\mathbf{R}=\sum_{M} a_{M} M$ and $a_{i k} \geq \max _{M}\left\{d_{i k}\right\}$. Then

$$
\prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(N_{i 0}^{(k)}\right)^{a_{i k}} \mathbf{R}=\sum_{i=0}^{n} \sum_{k=0}^{h_{i}} G_{i k} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{(k)}
$$

with $\operatorname{deg}\left(G_{i k} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{(k)}\right) \leq(m+1) \operatorname{deg}(\mathbf{R})+\sum_{i=0}^{n} \sum_{k=0}^{h_{i}} \operatorname{deg}\left(N_{i 0}\right) a_{i k}$. Clearly, we can take $a_{i k}=\operatorname{deg}(\mathbf{R})$ and then $\operatorname{deg}\left(G_{i k} \mathrm{~N}\left(\mathbb{P}_{i}\right)^{(k)}\right) \leq\left(m+1+\sum_{i=0}^{n}\left(h_{i}+1\right) \operatorname{deg}\left(N_{i 0}\right)\right) \operatorname{deg}(\mathbf{R})$. Thus, (16) follows. I

For a transformally essential difference polynomial system with degree 0 terms, the second part of Theorem 6.3 can be improved as follows.

Corollary 6.4 Let $\mathbb{P}_{i}=u_{i 0}+\sum_{k=1}^{l_{i}} u_{i k} N_{i k}(i=0, \ldots, n)$ be a transformally essential difference polynomial system with $m=\max _{i}\left\{\operatorname{deg}\left(\mathbb{P}_{i}, \mathbb{Y}\right)\right\}$ and $J_{i}$ the Jacobi number of $\left\{\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right\} \backslash\left\{\mathbb{P}_{i}\right\}$. Let $\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ be the sparse difference resultant of $\mathbb{P}_{i}(i=0, \ldots, n)$. Suppose ord $\left(\mathbf{R}, \mathbf{u}_{i}\right)=$ $h_{i}$ for each $i$ and $h=\max \left\{h_{i}+s_{i}\right\}$. Then $\mathbf{R}$ has a representation

$$
\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)=\sum_{i=0}^{n} \sum_{j=0}^{h_{i}} G_{i j} \mathbb{P}_{i}^{(j)}
$$

where $G_{i j} \in \mathbb{Q}\left[\mathbf{u}_{0}^{\left[h_{0}\right]}, \ldots, \mathbf{u}_{n}^{\left[h_{n}\right]}, \mathbb{Y}^{[h]}\right]$ such that $\operatorname{deg}\left(G_{i j} \mathbb{P}_{i}^{(j)}\right) \leq(m+1) \operatorname{deg}(\mathbf{R}) \leq(m+1)^{\sum_{i=0}^{n}\left(J_{i}+1\right)+1}$.
Proof: It is direct consequence of Theorem 6.3 by setting $N_{i 0}=1$. I
The following result gives an effective difference Nullstellensatz under certain conditions.
Corollary 6.5 Let $f_{0}, \ldots, f_{n} \in \mathcal{F}\left\{y_{1}, \ldots, y_{n}\right\}$ have no common solutions with $\operatorname{deg}\left(f_{i}\right) \leq m$. Let $\operatorname{Jac}\left(\left\{f_{0}, \ldots, f_{n}\right\} \backslash\left\{f_{i}\right\}\right)=J_{i}$. If the sparse difference resultant of $f_{0}, \ldots, f_{n}$ is nonzero, then there exist $H_{i j} \in \mathcal{F}\left\{y_{1}, \ldots, y_{n}\right\}$ s.t. $\sum_{i=0}^{n} \sum_{j=0}^{J_{i}} H_{i j} f_{i}^{(j)}=1$ and $\operatorname{deg}\left(H_{i j} f_{i}^{(j)}\right) \leq(m+$ 1) $\sum_{i=0}^{n}\left(J_{i}+1\right)+1$.

Proof: The hypothesis implies that $\mathbb{P}\left(f_{i}\right)$ form a transformally essential system. Clearly, $\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ has the property stated in Corollary 6.4, where $\mathbf{u}_{i}$ are coefficients of $\mathbb{P}\left(f_{i}\right)$. The result follows directly from Corollary 6.4 by specializing $\mathbf{u}_{i}$ to the coefficients of $f_{i}$. I

### 6.2 A single exponential algorithm to compute sparse difference resultant

If a polynomial $R$ is the linear combination of some known polynomials $F_{i}(i=1, \ldots, s)$, that is $R=\sum_{i=1}^{s} H_{i} F_{i}$, and we know the upper bounds of the degrees of $R$ and $H_{i} F_{i}$, then a general idea to estimate the computational complexity of $R$ is to use linear algebra to find the coefficients of $R$.

For sparse difference resultant, we already have given its degree bound and the degrees of the expressions in the linear combination in Theorem 6.3.

Now, we give the algorithm SDResultant to compute sparse difference resultants based on the linear algebra techniques. The algorithm works adaptively by searching for $\mathbf{R}$ with an order vector $\left(h_{0}, \ldots, h_{n}\right) \in \mathbb{N}_{0}^{n+1}$ with $h_{i} \leq J_{i}$ by Theorem 6.3, Denote $o=\sum_{i=0}^{n} h_{i}$. We start with $o=0$. And for this $o$, choose one vector $\left(h_{0}, \ldots, h_{n}\right)$ at a time. For this $\left(h_{0}, \ldots, h_{n}\right)$, we search for $\mathbf{R}$ from degree $d=1$. If we cannot find an $\mathbf{R}$ with such a degree, then we repeat the procedure with degree $d+1$ until $d>\prod_{i=0}^{n}\left(m_{i}+1\right)^{h_{i}+1}$. In that case, we choose another $\left(h_{0}, \ldots, h_{n}\right)$ with $\sum_{i=0}^{n} h_{i}=o$. But if for all $\left(h_{0}, \ldots, h_{n}\right)$ with $h_{i} \leq J_{i}$ and $\sum_{i=0}^{n} h_{i}=o, \mathbf{R}$ cannot be found, then we repeat the procedure with $o+1$. In this way, we will find an $\mathbf{R}$ with the smallest order satisfying equation (16), which is the sparse resultant.

Theorem 6.6 Let $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$ be a Laurent transformally essential system of form (1). Denote $\mathbb{P}=\left\{\mathrm{N}\left(\mathbb{P}_{0}\right), \ldots, \mathrm{N}\left(\mathbb{P}_{n}\right)\right\}, J_{i}=\operatorname{Jac}\left(\mathbb{P}_{\hat{i}}\right), J=\max _{i} J_{i}$ and $m=\max _{i=0}^{n} \operatorname{deg}\left(\mathbb{P}_{i}, \mathbb{Y}\right)$. Algorithm SDResultant computes the sparse difference resultant $\mathbf{R}$ of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$ with the following complexities:

1) In terms of a degree bound $D$ of $\mathbf{R}$, the algorithm needs at most $O\left(D^{O(l J)}(n J)^{O(l J)}\right)$ $\mathbb{Q}$-arithmetic operations, where $l=\sum_{i=0}^{n}\left(l_{i}+1\right)$ is the size of all $\mathbb{P}_{i}$.

Algorithm 1 - SDResultant $\left(\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right)$
Input: A generic Laurent transformally essential system $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$.
Output: The sparse difference resultant $\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$.

1. For $i=0, \ldots, n$, set $\mathrm{N}\left(\mathbb{P}_{i}\right)=\sum_{k=0}^{l_{i}} u_{i k} N_{i k}$ with $\operatorname{deg}\left(N_{i 0}\right) \leq \operatorname{deg}\left(N_{i k}\right)$.

Set $m_{i}=\operatorname{deg}\left(N\left(\mathbb{P}_{i}\right)\right), m_{i 0}=\operatorname{deg}\left(N_{i 0}\right), \mathbf{u}_{i}=\operatorname{coeff}\left(\mathbb{P}_{i}\right)$ and $\left|\mathbf{u}_{i}\right|=l_{i}+1$.
Set $s_{i j}=\operatorname{ord}\left(\mathbb{N}\left(\mathbb{P}_{i}\right), y_{j}\right), A=\left(s_{i j}\right)$ and compute $J_{i}=\operatorname{Jac}\left(A_{\hat{i}}\right)$.
2. Set $\mathbf{R}=0$, $o=0, m=\max _{i}\left\{m_{i}\right\}$.
3. While $\mathbf{R}=0$ do
3.1. For each $\left(h_{0}, \ldots, h_{n}\right) \in \mathbb{N}_{0}^{n+1}$ with $\sum_{i=0}^{n} h_{i}=o$ and $h_{i} \leq J_{i}$ do
3.1.1. $U=\cup_{i=0}^{n} \mathbf{u}_{i}^{\left[h_{i}\right]}, h=\max _{i}\left\{h_{i}+e_{i}\right\}, d=1$.
3.1.2. While $\mathbf{R}=0$ and $d \leq \prod_{i=0}^{n}\left(m_{i}+1\right)^{h_{i}+1}$ do
3.1.2.1. Set $\mathbf{R}_{0}$ to be a homogeneous GPol of degree $d$ in $U$.
3.1.2.2. Set $\mathbf{c}_{0}=\operatorname{coeff}\left(\mathbf{R}_{0}, U\right)$.
3.1.2.3. Set $H_{i j}\left(i=0, \ldots, n ; j=0, \ldots, h_{i}\right)$ to be GPols of degree $\left[m+1+\sum_{i=0}^{n}\left(h_{i}+1\right) m_{i 0}\right] d-m_{i}-1$ in $\mathbb{Y}^{[h]}, U$.
3.1.2.4. Set $\mathbf{c}_{i j}=\operatorname{coeff}\left(H_{i j}, \mathbb{Y}^{[h]} \cup U\right)$.
3.1.2.5. Set $\mathcal{P}$ to be the set of coefficients of $\prod_{i=0}^{n} \prod_{k=0}^{h_{i}}\left(N_{i 0}^{(k)}\right)^{d} \mathbf{R}_{0}-$ $\sum_{i=0}^{n} \sum_{j=0}^{h_{i}} H_{i j}\left(\mathbb{N}\left(\mathbb{P}_{i}\right)\right)^{(j)}$ as a polynomial in $\mathbb{Y}^{[h]}, U$.
3.1.2.6. Solve the linear equation $\mathcal{P}=0$ in variables $\mathbf{c}_{0}$ and $\mathbf{c}_{i j}$.
3.1.2.7. If $\mathbf{c}_{0}$ has a nonzero solution, then substitute it into $\mathbf{R}_{0}$ to get $\mathbf{R}$ and go to Step 4, else $\mathbf{R}=0$.
3.1.2.8. $\mathrm{d}:=\mathrm{d}+1$.
3.2. $\mathrm{o}:=\mathrm{o}+1$.

## 4. Return R.

/*/ GPol stands for generic algebraic polynomial.
$/^{*} / \operatorname{coeff}(P, V)$ returns the set of coefficients of $P$ as an ordinary polynomial in variables $V$.
2) The algorithm needs at most $O\left(m^{O\left(n l J^{2}\right)}(n J)^{O(l J)}\right) \mathbb{Q}$-arithmetic operations.

Proof: The algorithm finds a difference polynomial $P$ in $\mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$ satisfying equation (16), which has the smallest order and the smallest degree in those with the same order. Existence for such a difference polynomial is guaranteed by Theorem 6.3. By the definition of sparse difference resultant, $P$ must be $\mathbf{R}$.

We will estimate the complexity of the algorithm below. Denote $D$ to be the degree bound of $\mathbf{R}$. By Theorem [6.3, $D \leq(m+1)^{\sum_{i=0}^{n}\left(J_{i}+1\right)}$. In each loop of Step 3, the complexity of the algorithm is clearly dominated by Step 3.1.2, where we need to solve a system of linear equations $\mathcal{P}=0$ over $\mathbb{Q}$ in $\mathbf{c}_{0}$ and $\mathbf{c}_{i j}$. It is easy to show that $\left|\mathbf{c}_{0}\right|=\binom{d+L-1}{L-1}$ and $\left|\mathbf{c}_{i j}\right|=$ $\binom{d_{1}-m_{i}-1+L+n(h+1)}{L+n(h+1)}$, where $L=\sum_{i=0}^{n}\left(h_{i}+1\right)\left(l_{i}+1\right)$ and $d_{1}=\left[m+1+\sum_{i=0}^{n}\left(h_{i}+1\right) m_{i 0}\right] d$. Then $\mathcal{P}=0$ is a linear equation system with $N=\binom{d+L-1}{L-1}+\sum_{i=0}^{n}\left(h_{i}+1\right)\binom{d_{1}-m_{i}-1+L+n(h+1)}{L+n(h+1)}$ variables and $M=\binom{d_{1}+L+n(h+1)}{L+n(h+1)}$ equations. To solve it, we need at most $(\max \{M, N\})^{\omega}$
arithmetic operations over $\mathbb{Q}$, where $\omega$ is the matrix multiplication exponent and the currently best known $\omega$ is 2.376 .

The iteration in Step 3.1.2 may go through 1 to $\prod_{i=0}^{n}\left(m_{i}+1\right)^{h_{i}+1} \leq(m+1)^{\sum_{i=0}^{n}\left(J_{i}+1\right)}$, and the iteration in Step 3.1 at most will repeat $\prod_{i=0}^{n}\left(J_{i}+1\right) \leq(n+1)(J+1)$ times, where $J=\max _{i} J_{i}$. And by Theorem 6.3. Step 3 may loop from $o=0$ to $\sum_{i=0}^{n}\left(J_{i}+1\right)$. The whole algorithm needs at most

$$
\begin{aligned}
& \sum_{o=0}^{\sum_{i=0}^{n}\left(J_{i}+1\right)} \sum_{\substack{h_{i} \leq J_{i} \\
\sum_{i} h_{i}=0}} \sum_{d=1}^{\prod_{i=0}^{n}\left(m_{i}+1\right)^{n_{i}+1}}(\max \{M, N\})^{2.376} \\
& \leq O\left(D^{O(l J)}(n J)^{O(l J)}\right) \leq O\left(m^{O\left(n l J^{2}\right)}(n J)^{O(l J)}\right)
\end{aligned}
$$

arithmetic operations over $\mathbb{Q}$. In the above inequalities, we assume that $(m+1)^{\sum_{i=0}^{n}\left(J_{i}+1\right)+1} \geq$ $l(n+1) J$ and $l \geq(n+1)^{2}$, where $l=\sum_{i=0}^{n}\left(l_{i}+1\right)$. Our complexity assumes an $O(1)$ complexity cost for all field operations over $\mathbb{Q}$. Thus, the complexity follows.।

Remark 6.7 As we indicated at the end of Section 3.3, if we first compute the superessential set $\mathbb{T}$, then the algorithm can be improved by only considering the Laurent difference polynomials $\mathbb{P}_{i}(i \in \mathbb{T})$ in the linear combination of the sparse resultant.

Remark 6.8 Algorithm SDResultant can be improved by using a better search strategy. If $d$ is not big enough, instead of checking $d+1$, we can check $2 d$. Repeating this procedure, we may find a $k$ such that $2^{k} \leq \operatorname{deg}(\mathbf{R}) \leq 2^{k+1}$. We then bisecting the interval $\left[2^{k}, 2^{k+1}\right]$ again to find the proper degree for $\mathbf{R}$. This may lead to a better complexity, which is still single exponential.

For difference polynomials with non-vanishing degree terms, a better degree bound is given in Corollary 6.4. Based on this bound, we can simplify the Algorithm SDResultant to compute the sparse difference resultant by removing the computation for $\mathrm{N}\left(P_{i}\right)$ and $N_{i 0}$ in the first step where $N_{i 0}$ is exactly equal to 1 .

Theorem 6.9 Algorithm SDResultant computes the sparse difference resultant for a transformally essential system $\left\{\mathbb{P}_{i}=u_{i 0}+\sum_{k=1}^{l_{i}} u_{i k} N_{i k}\right\}$ with at most $O\left(n^{3.376} J^{O(n)} m^{O\left(n l J^{2}\right)}\right)$ $\mathbb{Q}$-arithmetic operations.

Proof: Follow the proof process of Theorem 6.6, it can be shown that the complexity is $O\left(n^{3.376} J^{O(n)} m^{O\left(n l J^{2}\right)}\right)$.

## 7 Difference resultant

In this section, we introduce the notion of difference resultant and prove its basic properties.
Definition 7.1 Let $\mathbf{m}_{s, r}$ be the set of all difference monomials in $\mathbb{Y}$ of order $\leq s$ and degree $\leq r$. Let $\mathbf{u}=\left\{u_{M}\right\}_{M \in \mathbf{m}_{s, r}}$ be a set of difference indeterminates over $\mathbb{Q}$. Then, $\mathbb{P}=\sum_{M \in \mathbf{m}_{s, r}} u_{M} M$ is called a generic difference polynomial of order $s$ and degree $r$.

Throughout this section, a generic difference polynomial is assumed to be of degree greater than zero. For any vector $\alpha=\left(a_{1}, \ldots, a_{m}\right) \in \mathbb{Z}^{m}$ and $\mathbb{X}=\left(x_{1}, \ldots, x_{m}\right)$, denote $x_{1}^{a_{1}} x_{2}^{a_{2}} \cdots x_{m}^{a_{m}}$ by $\mathbb{X}^{\alpha}$. Let

$$
\begin{equation*}
\mathbb{P}_{i}=u_{i 0}+\sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{n\left(s_{i}+1\right)} \\ 1<|\alpha|<m_{i}}} u_{i \alpha}\left(\mathbb{Y}^{\left[s_{i}\right]}\right)^{\alpha}(i=0,1, \ldots, n) \tag{17}
\end{equation*}
$$

be $n+1$ generic difference polynomials in $\mathbb{Y}$ of order $s_{i}$, degree $m_{i}$ and coefficients $\mathbf{u}_{i}$. Since $\left\{1, y_{1}, \ldots, y_{n}\right\}$ is contained in the support of each $\mathbb{P}_{i},\left\{\mathbb{P}_{0}, \mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right\}$ is a superessential system and the sparse difference resultant $\operatorname{Res}_{\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ exists. We define $\operatorname{Res}_{\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ to be the difference resultant of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$.

Because each generic difference polynomial $\mathbb{P}_{i}$ contains all difference monomials of order bounded by $s_{i}$ and total degree at most $m_{i}$, the difference resultant is sometimes called the dense difference resultant, in contrary to the sparse difference resultant.

The difference resultant satisfies all the properties we have proved for sparse difference resultants in previous sections. Apart from these, the difference resultant possess other better properties to be given in the rest of this section.

### 7.1 Exact Order and Degree

In this section, we will give the precise order and degree for the difference resultant, which is of BKK-style [1, 8].

Theorem 7.2 Let $\mathbb{P}_{i}(i=0, \ldots, n)$ be generic difference polynomials of form (17) with order $s_{i}$, degree $m_{i}$, and coefficients $\mathbf{u}_{i}$, respectively. Let $\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ be the difference resultant of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$. Denote $s=\sum_{i=0}^{n} s_{i}$. Then $\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ is also the algebraic sparse resultant of $\mathbb{P}_{0}^{\left[s-s_{0}\right]}, \ldots, \mathbb{P}_{n}^{\left[s-s_{n}\right]}$ treated as polynomials in $\mathbb{Y}^{[s]}$, and for each $i \in\{0,1, \ldots, n\}$ and $k=0, \ldots, s-s_{i}$,

$$
\begin{align*}
& \operatorname{ord}\left(\mathbf{R}, \mathbf{u}_{i}\right)=s-s_{i}  \tag{18}\\
& \operatorname{deg}\left(\mathbf{R}, \mathbf{u}_{i}^{(k)}\right)=\mathcal{M}\left(\left(\mathcal{Q}_{j l}\right)_{j \neq i, 0 \leq l \leq s-s_{j}}, \mathcal{Q}_{i 0}, \ldots, \mathcal{Q}_{i, k-1}, \mathcal{Q}_{i, k+1}, \ldots, \mathcal{Q}_{i, s-s_{i}}\right) \tag{19}
\end{align*}
$$

where $\mathcal{Q}_{j l}$ is the Newton polytope of $\mathbb{P}_{j}^{(l)}$ as a polynomial in $\mathbb{Y}^{[s]}$ and $\mathbf{u}_{i}^{(k)}=\left\{u_{i \alpha}^{(k)}, u_{i \alpha} \in \mathbf{u}_{i}\right\}$.
Proof: Regard $\mathbb{P}_{i}^{(k)}\left(i=0, \ldots, n ; k=0, \ldots, s-s_{i}\right)$ as polynomials in the $n(s+1)$ variables $\mathbb{Y}^{[s]}=\left\{y_{1}, \ldots, y_{n}, y_{1}^{(1)}, \ldots, y_{n}^{(1)}, \ldots, y_{1}^{(s)}, \ldots, y_{n}^{(s)}\right\}$, and we denote its support by $\mathcal{B}_{i k}$. Since the coefficients $\mathbf{u}_{i}^{(k)}$ of $\mathbb{P}_{i}^{(k)}$ can be treated as algebraic indeterminates, $\mathbb{P}_{i}^{(k)}$ are generic sparse polynomials with supports $\mathcal{B}_{i k}$, respectively. Now we claim that $\overline{\mathcal{B}}$ is strong essential, that is

C1) $\overline{\mathcal{B}}=\left\{\mathcal{B}_{i k}: 0 \leq i \leq n ; 0 \leq k \leq s-s_{i}\right\}$ is an essential set.
C2) $\overline{\mathcal{B}}=\left\{\mathcal{B}_{i k}: 0 \leq i \leq n ; 0 \leq k \leq s-s_{i}\right\}$ jointly spans the affine lattice $\mathbb{Z}^{n(s+1)}$.
Note that $|\overline{\mathcal{B}}|=n(s+1)+1$. To prove C1), it suffices to show that any $n(s+1)$ distinct $\mathbb{P}_{i}^{(k)}$ are algebraically independent. Without loss of generality, we prove that for a fixed
$l \in\left\{0, \ldots, s-s_{0}\right\}$,

$$
S_{l}=\left\{\left(\mathbb{P}_{i}^{(k)}\right)_{1 \leq i \leq n ; 0 \leq k \leq s-s_{i}}, \mathbb{P}_{0}, \ldots, \mathbb{P}_{0}^{(l-1)}, \mathbb{P}_{0}^{(l+1)}, \ldots, \mathbb{P}_{0}^{\left(s-s_{0}\right)}\right\}
$$

is an algebraically independent set. Clearly, $\left\{y_{j}^{(k)}, \ldots, y_{j}^{\left(s_{i}+k\right)} \mid j=1, \ldots, n\right\}$ is a subset of the support of $\mathbb{P}_{i}^{(k)}$. Now we choose a monomial from each $\mathbb{P}_{i}^{(k)}$ and denote it by $m\left(\mathbb{P}_{i}^{(k)}\right)$. Let
$m\left(\mathbb{P}_{0}^{(k)}\right)=\left\{\begin{array}{ll}y_{1}^{(k)} & 0 \leq k \leq l-1 \\ y_{1}^{\left(s_{0}+k\right)} & l+1 \leq k \leq s-s_{0}\end{array}\right.$ and $m\left(\mathbb{P}_{1}^{(k)}\right)=\left\{\begin{array}{ll}y_{1}^{(l+k)} & 0 \leq k \leq s_{0} \\ y_{2}^{\left(s_{1}+k\right)} & s_{0}+1 \leq k \leq s-s_{1}\end{array}\right.$.
For each $i \in\{2, \ldots, n\}$, let

$$
m\left(\mathbb{P}_{i}^{(k)}\right)=\left\{\begin{array}{ll}
y_{i}^{(k)} & 0 \leq k \leq \sum_{j=0}^{i-1} s_{j} \\
y_{i+1}^{\left(s_{i}+k\right)} & \sum_{j=0}^{i-1} s_{j}+1 \leq k \leq s-s_{i}
\end{array} .\right.
$$

So $m\left(S_{l}\right)$ is equal to $\left\{y_{j}^{[s]}: 1 \leq j \leq n\right\}$, which are algebraically independent over $\mathbb{Q}$. Thus, the $n(s+1)$ members of $S_{l}$ are algebraically independent over $\mathbb{Q}$. For if not, all the $\mathbb{P}_{i}^{(k)}-u_{i 0}^{(k)}\left(\mathbb{P}_{i}^{(k)} \in S_{l}\right)$ are algebraically dependent over $\mathbb{Q}(\mathbf{v})$ where $\mathbf{v}=\cup_{i=0}^{n} \mathbf{u}_{i}^{\left[s-s_{i}\right]} \backslash\left\{u_{i 0}^{\left[s-s_{i}\right]}\right\}$. Now specialize the coefficient of $m\left(\mathbb{P}_{i}^{(k)}\right)$ in $\mathbb{P}_{i}^{(k)}$ to 1 , and all the other coefficients of $\mathbb{P}_{i}^{(k)}-u_{i 0}^{(k)}$ to 0 , by the algebraic version of Lemma [2.2, $\left\{m\left(\mathbb{P}_{i}^{(k)}\right): \mathbb{P}_{i}^{(k)} \in S_{l}\right\}$ are algebraically dependent over $\mathbb{Q}$, which is a contradiction. Thus, claim C1) is proved. Claim C2) follows from the fact that 1 and $\mathbb{Y}^{[s]}$ are contained in the support of $\mathbb{P}_{0}^{\left[s-s_{0}\right]}$.

By C1) and C2), the sparse resultant of $\left(\mathbb{P}_{i}^{(k)}\right)_{0 \leq i \leq n ; 0 \leq k \leq s-s_{i}}$ exists and we denote it by $G$. Then $(G)=\left(\left(\mathbb{P}_{i}^{(k)}\right)_{0 \leq i \leq n ; 0 \leq k \leq s-s_{i}}\right) \cap \mathbb{Q}\left[\mathbf{u}_{0}^{\left[s-s_{0}\right]}, \cdots, \mathbf{u}_{n}^{\left[s-s_{n}\right]}\right]$, and by Theorem [5.8, $\operatorname{deg}\left(G, \mathbf{u}_{i}^{(k)}\right)=\mathcal{M}\left(\left(\mathcal{Q}_{j l}\right)_{j \neq i, 0 \leq l \leq s-s_{j}}, \mathcal{Q}_{i 0}, \ldots, \mathcal{Q}_{i, k-1}, \mathcal{Q}_{i, k+1}, \ldots, \mathcal{Q}_{i, s-s_{i}}\right)$, where $\mathbf{u}_{i}^{(k)}=\left(u_{i 0}^{(k)}\right.$, $\left.\ldots, u_{i \alpha}^{(k)}, \ldots\right)$. The theorem will be proved if we can show that $G=c \cdot \mathbf{R}$ for some $c \in \mathbb{Q}$.

Since $G \in\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right]$ and $\operatorname{ord}\left(G, \mathbf{u}_{i}\right)=s-s_{i}$, by Lemma [3.9, ord $\left(\mathbf{R}, \mathbf{u}_{i}\right) \leq s-s_{i}$ for each $i=0, \ldots, n$. If for some $i$, ord $\left(\mathbf{R}, \mathbf{u}_{i}\right)=h_{i}<s-s_{i}$, then $\mathbf{R} \in\left(\left(\mathbb{P}_{j}^{(k)}\right)_{j \neq i ; 0 \leq k \leq s-s_{j}}, \mathbb{P}_{i}\right.$, $\left.\ldots, \mathbb{P}_{i}^{\left(h_{i}\right)}\right)$, a contradiction to $\left.\mathbf{C} 1\right)$. Thus, ord $\left(\mathbf{R}, \mathbf{u}_{i}\right)=s-s_{i}$ and $\mathbf{R} \in(G)$. Since $\mathbf{R}$ is irreducible, there exists some $c \in \mathbb{Q}$ such that $G=c \cdot \mathbf{R}$. So $\mathbf{R}$ is equal to the algebraic sparse resultant of $\mathbb{P}_{0}^{\left[s-s_{0}\right]}, \ldots, \mathbb{P}_{n}^{\left[s-s_{n}\right]} . \boldsymbol{I}$

As a direct consequence of the above theorem and the determinant representation for algebraic sparse resultant given by D'Andrea 9, we have the following result.

Corollary 7.3 The difference resultant for generic difference polynomials $\mathbb{P}_{i}, i=0, \ldots, n$ can be written as the form $\operatorname{det}\left(M_{1}\right) / \operatorname{det}\left(M_{0}\right)$ where $M_{1}$ and $M_{0}$ are matrixes whose elements are coefficients of $\mathbb{P}_{i}$ and their transforms up to the order $s-s_{i}$ and $M_{0}$ is a minor of $M_{1}$.

Based on the matrix representation given in the above corollary, the efficient algorithms given by Canny, Emiris, and Pan [12, 14] can be used to compute the difference resultant.

Corollary 7.4 The degree of $\mathbf{R}$ in each coefficient set $\mathbf{u}_{i}$ is

$$
\operatorname{deg}\left(\mathbf{R}, \mathbf{u}_{i}\right)=\sum_{k=0}^{s-s_{i}} \mathcal{M}\left(\left(\mathcal{Q}_{j l}\right)_{j \neq i, 0 \leq l \leq s-s_{j}}, \mathcal{Q}_{i 0}, \ldots, \mathcal{Q}_{i, k-1}, \mathcal{Q}_{i, k+1}, \ldots, \mathcal{Q}_{i, s-s_{i}}\right)
$$

and the total degree of $\mathbf{R}$ is

$$
\operatorname{deg}(\mathbf{R})=\sum_{i=0}^{n} \sum_{k=0}^{s-s_{i}} \mathcal{M}\left(\left(\mathcal{Q}_{j l}\right)_{j \neq i, 0 \leq l \leq s-s_{j}}, \mathcal{Q}_{i 0}, \ldots, \mathcal{Q}_{i, k-1}, \mathcal{Q}_{i, k+1}, \ldots, \mathcal{Q}_{i, s-s_{i}}\right)
$$

Remark 7.5 From the proof of Theorem [7.2, we can see that for each $i$ and $0 \leq k \leq s-s_{i}$, $\operatorname{deg}\left(\mathbf{R}, \mathbf{u}_{i}^{(k)}\right)>0$. Furthermore, by Lemma 4.12, $\operatorname{deg}\left(\mathbf{R}, u_{i 0}^{(k)}\right)>0$ and $\operatorname{deg}\left(\mathbf{R}, u_{i \alpha}^{(k)}\right)>0$ for each $\alpha$. In particular, $\operatorname{deg}\left(\mathbf{R}, u_{i 0}\right)>0$ and $\operatorname{deg}\left(\mathbf{R}, u_{i \alpha}\right)>0$.

Example 7.6 Consider two generic difference polynomials of order one and degree two in one indeterminate $y$ :

$$
\begin{aligned}
& \mathbb{P}_{0}=u_{00}+u_{01} y+u_{02} y^{(1)}+u_{03} y^{2}+u_{04} y y^{(1)}+u_{05}\left(y^{(1)}\right)^{2}, \\
& \mathbb{P}_{1}=u_{10}+u_{11} y+u_{12} y^{(1)}+u_{13} y^{2}+u_{14} y y^{(1)}+u_{15}\left(y^{(1)}\right)^{2} .
\end{aligned}
$$

Then the degree bound given by Theorem 6.3 is $\operatorname{deg}(\mathbf{R}) \leq(2+1)^{4}=81$. By Theo$\operatorname{rem}$ [7.2, $\operatorname{deg}\left(\mathbf{R}, \mathbf{u}_{0}\right)=\mathcal{M}\left(\mathcal{Q}_{10}, \mathcal{Q}_{11}, \mathcal{Q}_{00}\right)+\mathcal{M}\left(\mathcal{Q}_{10}, \mathcal{Q}_{11}, \mathcal{Q}_{01}\right)=8+8=16$ and consequently $\operatorname{deg}(\mathbf{R})=32$, where $\mathcal{Q}_{00}=\mathcal{Q}_{10}=\operatorname{conv}\{(0,0,0),(2,0,0),(0,2,0)\}, \mathcal{Q}_{01}=\mathcal{Q}_{11}=$ conv $\{(0,0,0),(0,2,0),(0,0,2)\}$, and $\operatorname{conv}(\cdot)$ means taking the convex hull in $\mathbb{R}^{3}$. By the proof of Theorem [7.2, $\mathbf{R}$ is the sparse resultant of $\mathbb{P}_{0}, \sigma\left(\mathbb{P}_{0}\right), \mathbb{P}_{1}, \sigma\left(\mathbb{P}_{1}\right)$.

### 7.2 Poisson-type product formula

In this section, we will give a Poisson-type product formula for difference resultant.
Let $\widetilde{\mathbf{u}}=\cup_{i=0}^{n} \mathbf{u}_{i} \backslash\left\{u_{00}\right\}$ and $\mathbb{Q}\langle\widetilde{\mathbf{u}}\rangle$ be the transformally transcendental extension of $\mathbb{Q}$ in the usual sense. Let $\mathbb{Q}_{0}=\mathbb{Q}\langle\widetilde{\mathbf{u}}\rangle\left(u_{00}, \ldots, u_{00}^{\left(s-s_{0}-1\right)}\right)$. Here, $\mathbb{Q}_{0}$ is not necessarily a difference overfield of $\mathbb{Q}$, for the transforms of $u_{00}$ are not defined. In the following, we will follow Cohn [5] to obtain algebraic extensions $\mathcal{G}_{i}$ of $\mathbb{Q}_{0}$ and define transforming operators to make $\mathcal{G}_{i}$ difference fields. Consider $\mathbf{R}$ as an irreducible algebraic polynomial $r\left(u_{00}^{\left(s-s_{0}\right)}\right)$ in $\mathbb{Q}_{0}\left[u_{00}^{\left(s-s_{0}\right)}\right]$. In a suitable algebraic extension field of $\mathbb{Q}_{0}, r\left(u_{00}^{\left(s-s_{0}\right)}\right)=0$ has $t_{0}=\operatorname{deg}\left(r, u_{00}^{\left(s-s_{0}\right)}\right)=$ $\operatorname{deg}\left(\mathbf{R}, u_{00}^{\left(s-s_{0}\right)}\right)$ roots $\gamma_{1}, \ldots, \gamma_{t_{0}}$. Thus

$$
\begin{equation*}
\mathbf{R}\left(\mathbf{u}_{0}, \mathbf{u}_{1}, \ldots, \mathbf{u}_{n}\right)=A \prod_{\tau=1}^{t_{0}}\left(u_{00}^{\left(s-s_{0}\right)}-\gamma_{\tau}\right) \tag{20}
\end{equation*}
$$

where $A \in \mathbb{Q}_{0}$. Let $\mathcal{I}_{\mathbf{u}}=\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$. By the definition of the difference resultant, $\mathcal{I}_{\mathbf{u}}$ is an essential reflexive prime difference ideal in the decomposition of $\{\mathbf{R}\}$ which is not held by any difference polynomial of order less than $s-s_{0}$ in $u_{00}$. Suppose $\mathbf{R}, \mathbf{R}_{1}, \mathbf{R}_{2}, \ldots$ is a basic sequenc $\varepsilon^{2}$ of $\mathbf{R}$ corresponding to $\mathcal{I}_{\mathbf{u}}$. That is, $\mathcal{I}_{\mathbf{u}}=\bigcup_{k \geq 0} \operatorname{asat}\left(\mathbf{R}, \mathbf{R}_{1}, \ldots, \mathbf{R}_{k}\right)$. Regard all the $\mathbf{R}_{i}$ as algebraic polynomials over the coefficient field $\mathbb{Q}\langle\widetilde{\mathbf{u}}\rangle$. Denote $\gamma_{\tau 0}=\gamma_{\tau}$. Clearly, $u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau 0}$ is a generic zero of asat $(\mathbf{R})$. Suppose $\gamma_{\tau i}(i \leq k)$ are found in some

[^2]algebraic extension field of $\mathbb{Q}_{0}$ such that $u_{00}^{\left(s-s_{0}+i\right)}=\gamma_{\tau i}(0 \leq i \leq k)$ is a generic zero of $\operatorname{asat}\left(\mathbf{R}, \mathbf{R}_{1}, \ldots, \mathbf{R}_{k}\right)$. Then let $\gamma_{\tau, k+1}$ be an element such that $u_{00}^{\left(s-s_{0}+i\right)}=\gamma_{\tau i}(0 \leq i \leq k+1)$ is a generic zero of $\operatorname{asat}\left(\mathbf{R}, \mathbf{R}_{1}, \ldots, \mathbf{R}_{k}, \mathbf{R}_{k+1}\right)$. Clearly, $\gamma_{\tau, k+1}$ is also algebraic over $\mathbb{Q}_{0}$. Let $\mathcal{G}_{\tau}=\mathbb{Q}\langle\widetilde{\mathbf{u}}\rangle\left(u_{00}, \ldots, u_{00}^{\left(s-s_{0}-1\right)}, \gamma_{\tau}, \gamma_{\tau 1}, \ldots\right)$. Clearly, $\mathcal{G}_{\tau}$ is an algebraic extension of $\mathbb{Q}_{0}$ and $\mathcal{G}_{\tau}$ is algebraically isomorphic to the quotient field of $\mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\} / \mathcal{I}_{\mathbf{u}}$. Since the quotient field of $\mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\} / \mathcal{I}_{\mathbf{u}}$ is also a difference field, we can introduce a transforming operator $\sigma_{\tau}$ into $\mathcal{G}_{\tau}$ to make it a difference field such that the above isomorphism becomes a difference one. That is, $\left.\sigma_{\tau}\right|_{\mathbb{Q}_{0}}=\left.\sigma\right|_{\mathbb{Q}_{0}}$ and
\[

\sigma_{\tau}^{k}\left(u_{00}\right)= $$
\begin{cases}u_{00}^{(k)} & 0 \leq k \leq s-s_{0}-1 \\ \gamma_{\tau, k-s-s_{0}} & k \geq s-s_{0}\end{cases}
$$
\]

In this way, $\left(\mathcal{G}_{\tau}, \sigma_{\tau}\right)$ is a difference field.
Let $F$ be a difference polynomial in $\mathbb{Q}\left\{\mathbf{u}_{0}, \mathbf{u}_{1}, \ldots, \mathbf{u}_{n}\right\}=\mathbb{Q}\left\{\widetilde{\mathbf{u}}, u_{00}\right\}$. For convenience, by the symbol $\left.F\right|_{u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}}$, we mean substituting $u_{00}^{\left(s-s_{0}+k\right)}$ by $\sigma_{\tau}^{k} \gamma_{\tau}=\gamma_{\tau k}(k \geq 0)$ into $F$. Similarly, by saying $F$ vanishes at $u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}$, we mean $\left.F\right|_{u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}}=0$. The following lemma is a direct consequence of the above discussion.

Lemma 7.7 $F \in \mathcal{I}_{\mathbf{u}}$ if and only if $F$ vanishes at $u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}$.
Proof: Since $\mathcal{I}_{\mathbf{u}}=\bigcup_{k \geq 0}$ asat $\left(\mathbf{R}, \mathbf{R}_{1}, \ldots, \mathbf{R}_{k}\right)$ and $u_{00}^{\left(s-s_{0}+i\right)}=\gamma_{\tau i}(0 \leq i \leq k)$ is a generic zero of $\operatorname{asat}\left(\mathbf{R}, \mathbf{R}_{1}, \ldots, \mathbf{R}_{k}\right)$, the lemma follows. $\boldsymbol{I}$

Remark 7.8 In order to make $\mathcal{G}_{\tau}$ a difference field, we need to introduce a transforming operator $\sigma_{\tau}$ which is closely related to $\gamma_{\tau}$. Since even for a fixed $\tau$, generic zeros of $\operatorname{asat}\left(\mathbf{R}, \mathbf{R}_{1}, \ldots, \mathbf{R}_{k}\right)$ beginning from $u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}$ may not be unique, the definition of $\sigma_{\tau}$ also may not be unique, which is different from the differential case. In fact, it is a common phenomena in difference algebra. Here, we just choose one, for they do not influence the following discussions.

Now we give the following Poisson type formula for the difference resultant.
Theorem 7.9 Let $\mathbf{R}\left(\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right)$ be the difference resultant of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n} . \operatorname{Let} \operatorname{deg}\left(\mathbf{R}, u_{00}^{\left(s-s_{0}\right)}\right)$ $=t_{0}$. Then there exist $\xi_{\tau k}\left(\tau=1, \ldots, t_{0} ; k=1, \ldots, n\right)$ in overfields $\left(\mathcal{G}_{\tau}, \sigma_{\tau}\right)$ of $(\mathbb{Q}\langle\widetilde{\mathbf{u}}\rangle, \sigma)$ such that

$$
\begin{equation*}
\mathbf{R}=A \prod_{\tau=1}^{t_{0}} \mathbb{P}_{0}\left(\xi_{\tau 1}, \ldots, \xi_{\tau n}\right)^{\left(s-s_{0}\right)} \tag{21}
\end{equation*}
$$

where $A \in \mathbb{Q}\left\langle\mathbf{u}_{1}, \ldots, \mathbf{u}_{n}\right\rangle\left[\mathbf{u}_{0}^{\left[s-s_{0}\right]} \backslash u_{00}^{\left(s-s_{0}\right)}\right]$. Note that (21) is formal and should be understood in the following precise meaning: $\mathbb{P}_{0}\left(\xi_{\tau}\right)^{\left(s-s_{0}\right)} \triangleq \sigma^{s-s_{0}} u_{00}+\sigma_{\tau}^{s-s_{0}}\left(\sum_{\alpha \in \mathcal{B}_{0} \backslash\{0\}} u_{0 \alpha}\left(\xi_{\tau}^{\left[s-s_{0}\right]}\right)^{\alpha}\right)$, where $\xi_{\tau}=\left(\xi_{\tau 1}, \ldots, \xi_{\tau n}\right)$.
Proof: By Theorem 4.3, there exists $m \in \mathbb{N}$ such that

$$
u_{00} \frac{\partial \mathbf{R}}{\partial u_{00}}+\sum_{\alpha} u_{0 \alpha} \frac{\partial \mathbf{R}}{\partial u_{0 \alpha}}=m \mathbf{R}
$$

Setting $u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}$ in both sides of the above equation, we have

$$
\left.u_{00} \frac{\partial \mathbf{R}}{\partial u_{00}}\right|_{u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}}+\left.\sum_{\alpha} u_{0 \alpha} \frac{\partial \mathbf{R}}{\partial u_{0 \alpha}}\right|_{u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}}=0
$$

Let $\xi_{\tau \alpha}=\left.\left(\frac{\partial \mathbf{R}}{\partial u_{0 \alpha}} / \frac{\partial \mathbf{R}}{\partial u_{00}}\right)\right|_{u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}}$. Then $u_{00}=-\sum_{\alpha} u_{0 \alpha} \xi_{\tau \alpha}$ with $u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}$. That is, $\gamma_{\tau}=-\sigma_{\tau}^{s-s_{0}}\left(\sum_{\alpha} u_{0 \alpha} \xi_{\tau \alpha}\right)=-\left(\sum_{\alpha} u_{0 \alpha} \xi_{\tau \alpha}\right)^{\left(s-s_{0}\right)}$. Thus,

$$
\mathbf{R}=A \prod_{\tau=1}^{t_{0}}\left(u_{00}+\sum_{\alpha} u_{0 \alpha} \xi_{\tau \alpha}\right)^{\left(s-s_{0}\right)}
$$

Suppose $\mathbb{P}_{0}=u_{00}+\sum_{j=1}^{n} u_{0 j 0} y_{j}+T_{0}$. Let $\xi_{\tau j}=\left.\left(\frac{\partial \mathbf{R}}{\partial u_{0 j 0}} / \frac{\partial \mathbf{R}}{\partial u_{00}}\right)\right|_{u_{00}^{\left(s-s_{0}\right)}=\gamma_{\tau}}(j=1, \ldots, n)$ and $\xi_{\tau}=\left(\xi_{\tau 1}, \ldots, \xi_{\tau n}\right)$. It remains to show that $\xi_{\tau \alpha}=\left(\xi_{\tau}^{[s 0]}\right)^{\alpha}$.

Let $\zeta_{i}=-\sum_{\alpha} u_{i \alpha}\left(\mathbb{Y}^{\left[s_{i}\right]}\right)^{\alpha}(i=0, \ldots, n)$. Clearly, $\zeta=\left(\mathbf{u}, \zeta_{0}, \ldots, \zeta_{n}\right)$ is a generic zero of $\mathcal{I}_{\mathbf{u}}=\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right] \cap \mathbb{Q}\left\{\mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$, where $\mathbf{u}=\cup_{i=1}^{n} \mathbf{u}_{i} \backslash\left\{u_{i 0}\right\}$. For each $\left(\mathbb{Y}^{\left[s_{0}\right]}\right)^{\alpha}=\prod_{j=1}^{n}\left(y_{j}^{(k)}\right)^{d_{j k}}$, by equation (8) , $\left(\mathbb{Y}\left[s^{[s]}\right)^{\alpha}=\overline{\frac{\partial \mathbf{R}}{\partial u_{0 \alpha}}} / \overline{\frac{\partial \mathbf{R}}{\partial u_{00}}}=\prod_{j=1}^{n} \prod_{k=0}^{s_{0}}\left(\left(\overline{\frac{\partial \mathbf{R}}{\partial u_{0 j}}} / \overline{\frac{\partial \mathbf{R}}{\partial u_{00}}}\right)^{(k)}\right)^{d_{j k}}\right.$, where $\overline{\frac{\partial \mathbf{R}}{\partial u_{0} \alpha}}=\left.\frac{\partial \mathbf{R}}{\partial u_{0 \alpha}}\right|_{u_{i 0}=\zeta i}$. So $\frac{\partial \mathbf{R}}{\partial u_{0 \alpha}} \prod_{j=1}^{n} \prod_{k=0}^{s_{0}}\left(\left(\frac{\partial \mathbf{R}}{\partial u_{00}}\right)^{(k)}\right)^{d_{j k}}-\frac{\partial \mathbf{R}}{\partial u_{00}} \prod_{j=1}^{n} \prod_{k=0}^{s_{0}}\left(\left(\frac{\partial \mathbf{R}}{\left.\partial u_{00}\right)^{(k)}}\right)^{d_{j k}} \in \mathcal{I}_{\mathbf{u}}\right.$. By Lemma [7.7, $\xi_{\tau \alpha}=$ $\prod_{j=1}^{n} \prod_{k=0}^{s_{0}}\left(\xi_{\tau j}^{(k)}\right)^{d_{j k}}=\left(\xi_{\tau}^{\left[s_{0}\right]}\right)^{\alpha}$. Thus, (21) follows. I

Theorem 7.10 The points $\xi_{\tau}=\left(\xi_{\tau 1}, \ldots, \xi_{\tau n}\right)\left(\tau=1, \ldots, t_{0}\right)$ in (21) are generic zeros of the difference ideal $\left[\mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right] \subset \mathbb{Q}\left\langle\mathbf{u}_{1}, \ldots, \mathbf{u}_{n}\right\rangle\{\mathbb{Y}\}$.
Proof: Clearly, $\xi_{\tau}$ are $n$-tuples over $\mathbb{Q}\left\langle\mathbf{u}_{1}, \ldots, \mathbf{u}_{n}\right\rangle$. For each $i=1, \ldots, n$, rewrite $\mathbb{P}_{i}=$ $u_{i 0}+\sum_{\alpha} u_{i \alpha} \prod_{j=1}^{n} \prod_{k=1}^{s_{i}}\left(y_{j}^{(k)}\right)^{\alpha_{j k}}$. Since $\zeta_{i}=-\sum_{\alpha} u_{i \alpha} \prod_{j=1}^{n} \prod_{k=1}^{s_{i}}\left(y_{j}^{(k)}\right)^{\alpha_{j k}}$ and $y_{j}=\overline{\frac{\partial \mathbf{R}}{\partial u_{0 j 0}}} / \overline{\frac{\partial \mathbf{R}}{\partial u_{00}}}, \zeta_{i}+$ $\sum_{\alpha} u_{i \alpha} \prod_{j=1}^{n} \prod_{k=1}^{s_{i}}\left(\left(\frac{\overline{\partial \mathbf{R}}}{\partial u_{0 j 0}} / \overline{\partial \mathbf{R}}\right)^{(k)}\right)^{\alpha_{j k}}=0$. Let $a_{j k}=\max _{\alpha} \alpha_{j k}$. Then $u_{i 0} \prod_{j=1}^{n} \prod_{k=1}^{s_{i}}\left(\left(\frac{\partial \mathbf{R}}{\partial u_{00}}\right)^{(k)}\right)^{a_{j k}}+$ $\sum_{\alpha} u_{i \alpha} \prod_{j=1}^{n} \prod_{k=1}^{s_{i}}\left(\left(\frac{\partial \mathbf{R}}{\partial u_{0 j 0}}\right)^{(k)}\right)^{\alpha_{j k}}\left(\left(\frac{\partial \mathbf{R}}{\partial u_{00}}\right)^{(k)}\right)^{a_{j k}-\alpha_{j k}} \in \mathcal{I}_{\mathbf{u}}$. Thus, by Lemma 7.7, $\mathbb{P}_{i}\left(\xi_{\tau}\right)=u_{i 0}+$ $\sum_{\alpha} u_{i \alpha} \prod_{j=1}^{n} \prod_{k=1}^{s_{i}}\left(\xi_{\tau j}^{(k)}\right)^{\alpha_{j k}}=0(i=1, \ldots, n)$.

On the other hand, suppose $F \in \mathbb{Q}\left\langle\mathbf{u}_{1}, \ldots, \mathbf{u}_{n}\right\rangle\{\mathbb{Y}\}$ vanishes at $\xi_{\tau}$. Without loss of generality, suppose $F \in \mathbb{Q}\left\{\mathbf{u}_{1}, \ldots, \mathbf{u}_{n}, \mathbb{Y}\right\}$. Clearly, $\mathbb{P}_{1}, \ldots, \mathbb{P}_{n}$ constitute an ascending chain in $\mathbb{Q}\left\{\mathbf{u}_{1}, \ldots, \mathbf{u}_{n}, \mathbb{Y}\right\}$ with $u_{i 0}$ as leaders. Let $G$ be the difference remainder of $F$ with respect to this ascending chain. Then $G$ is free from $u_{i 0}$ and $F \equiv G \bmod \left[\mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right]$. Then $G\left(\xi_{\tau}\right)=$ $G\left(\widetilde{\mathbf{u}} ; \xi_{\tau 1}, \ldots, \xi_{\tau n}\right)=0$, where $\widetilde{\mathbf{u}}=\cup_{i=1}^{n} \mathbf{u}_{i} \backslash\left\{u_{i 0}\right\}$. So there exist $a_{k} \in \mathbb{N}$ such that $G_{1}=$ $\prod_{k}\left(\left(\frac{\partial \mathbf{R}}{\partial u_{00}}\right)^{(k)}\right)^{a_{k}} G(\widetilde{\mathbf{u}} ; \mathbb{Y}) \in \mathcal{I}_{\mathbf{u}}$. Thus, $G_{1}$ vanishes at $u_{i 0}=\zeta_{i}(i=1, \ldots, n)$ while $\frac{\partial \mathbf{R}}{\partial u_{0} 0}$ does not. It follows that $G(\widetilde{\mathbf{u}} ; \mathbb{Y}) \equiv 0$ and $F \in\left[\mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right]$. So $\xi_{\tau}$ are generic zeros of $\left[\mathbb{P}_{1}, \ldots, \mathbb{P}_{n}\right] \subset$ $\mathbb{Q}\left\langle\mathbf{u}_{1}, \ldots, \mathbf{u}_{n}\right\rangle\{\mathbb{Y}\}$.

By Theorems 7.9] and 7.10, we can see that difference resultants have Poisson-type product formula, which is similar to their algebraic and differential analogues.

We conclude this section by proving the following theorem, which explores the relationship between the difference resultant and the solvability of the given systems.

Theorem 7.11 Let $\mathbf{R}$ be the difference resultant of $\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}$. Suppose when each $\mathbf{u}_{i}$ is specialized to $\mathbf{v}_{i}, \mathbb{P}_{i}$ is specialized to $\overline{\mathbb{P}}_{i}$. If $\overline{\mathbb{P}}_{0}=\cdots=\overline{\mathbb{P}}_{n}=0$ has a common difference solution, then $\mathbf{R}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)=0$. Moreover, if $\mathbf{R}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)=0$ and $\frac{\partial \mathbf{R}}{\partial u_{00}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \neq 0$, then $\overline{\mathbb{P}}_{0}=$ $\cdots=\overline{\mathbb{P}}_{n}=0$ has at most one solution $\left(\bar{y}_{1}, \ldots, \bar{y}_{n}\right)$ with each $\bar{y}_{k}=\left(\frac{\partial \mathbf{R}}{\partial u_{0 k}} / \frac{\partial \mathbf{R}}{\partial u_{00}}\right)\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$, where $u_{0 k}$ is the coefficient of $y_{k}$ in $\mathbb{P}_{0}$.

Proof: Suppose $\mathbb{P}_{i}=u_{i 0}+T_{i}(i=1, \ldots, n)$ and $\mathbf{u}=\cup_{i=0}^{n} \mathbf{u}_{i} \backslash\left\{u_{i 0}\right\}$. Clearly, $\left(\mathbb{Y} ; \mathbf{u},-T_{0}(\mathbb{Y})\right.$, $\ldots,-T_{n}(\mathbb{Y})$ ) is a generic zero of $\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right] \subset \mathbb{Q}\left\{\mathbb{Y} ; \mathbf{u}_{0}, \ldots, \mathbf{u}_{n}\right\}$. Taking the partial derivative of $\mathbf{R}\left(\mathbf{u} ;-T_{0}(\mathbb{Y}), \ldots,-T_{n}(\mathbb{Y})\right)=0$ w.r.t. $\quad u_{0 k}$, we can show that $\frac{\partial \mathbf{R}}{\partial u_{0}} y_{k}-\frac{\partial \mathbf{R}}{\partial u_{0 k}} \in$ $\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right](k=1, \ldots, n)$. If $\overline{\mathbb{P}}_{0}=\cdots=\overline{\mathbb{P}}_{n}=0$ has a common solution $\xi$, then $\left(\xi ; \mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$ is a common solution of $\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right]$. Since $\mathbf{R} \in\left[\mathbb{P}_{0}, \ldots, \mathbb{P}_{n}\right]$, $\mathbf{R}$ must vanish at $\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$. Now suppose $\mathbf{R}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)=0$ and $\frac{\partial \mathbf{R}}{\partial u_{00}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \neq 0$. If $\left(\bar{y}_{1}, \ldots, \bar{y}_{n}\right)$ is a common solution of $\overline{\mathbb{P}}_{i}=0$, then each $\frac{\partial \mathbf{R}}{\partial u_{00}} y_{k}-\frac{\partial \mathbf{R}}{\partial u_{0 k}}$ vanishes at $\left(\bar{y}_{1}, \ldots, \bar{y}_{n} ; \mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$. Thus, $\bar{y}_{k}=\left(\frac{\partial \mathbf{R}}{\partial u_{0 k}} / \frac{\partial \mathbf{R}}{\partial u_{00}}\right)\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$, since $\frac{\partial \mathbf{R}}{\partial u_{00}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \neq 0$. Hence, the second assertion holds. I

Remark 7.12 If Problem 3.16 can be solved positively, then Theorem 7.11 can be strengthened as follows: If $\mathbf{R}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)=0$ and $\frac{\partial \mathbf{R}}{\partial u_{00}}\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right) \neq 0$, then $\overline{\mathbb{P}}_{0}=\cdots=\overline{\mathbb{P}}_{n}=0$ has a unique solution $\left(\bar{y}_{1}, \ldots, \bar{y}_{n}\right)$ with each $\bar{y}_{k}=\left(\frac{\partial \mathbf{R}}{\partial u_{0 k}} / \frac{\partial \mathbf{R}}{\partial u_{00}}\right)\left(\mathbf{v}_{0}, \ldots, \mathbf{v}_{n}\right)$.

## 8 Conclusion and problem

In this paper, we first introduce the concepts of Laurent difference polynomials and Laurent transformally essential systems and give a criterion for a difference polynomial system to be Laurent transformally essential in terms of its supports. Then the sparse difference resultant for a Laurent transformally essential system is defined and its basic properties are proved. Furthermore, order and degree bounds for the sparse difference resultant are given. Based on these bounds, an algorithm to compute the sparse difference resultant is proposed, which is single exponential in terms of the order, the number of variables, and the size of the Laurent transformally essential system. Besides these, the difference resultant is introduced and its basic properties are given, such as its precise order and BKK style degree, determinant representation, and a Poisson-type product formula.

We now propose several questions for further study apart from Problem 3.16.
The degree of the algebraic sparse resultant is equal to the mixed volume of certain polytopes generated by the supports of the polynomials as shown in [29] or [18, p.255]. And Theorem 7.2 shows that the degree of difference resultants is exactly of such BKK-style. It is desirable to obtain such a bound for sparse difference resultant. For more details, see Remark 5.12.

There exist very efficient algorithms to compute algebraic sparse resultants [11, 12, 14, 9, which are based on matrix representations for the resultant. How to apply the principles behind these algorithms to compute sparse difference resultants is an important problem.

Algebraic resultant and sparse resultant have many interesting applications [3, 8, 13, 18]. It is desirable to develop the corresponding theory for difference polynomial systems based difference resultant.

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[^1]:    ${ }^{1}$ For definition, see 36 .

[^2]:    ${ }^{2}$ For the rigorous definition of basic sequence, please refer to [5]. Here, we list its basic properties: i) For each $k \geq 0, \operatorname{ord}\left(\mathbf{R}_{k}, u_{00}\right)=s-s_{0}+k$ and $\mathbf{R}, \mathbf{R}_{1}, \ldots, \mathbf{R}_{k}$ is an irreducible algebraic ascending chain, and ii) $\bigcup_{k \geq 0} \operatorname{asat}\left(\mathbf{R}, \mathbf{R}_{1}, \ldots, \mathbf{R}_{k}\right)$ is a reflexive prime difference ideal.

