# ALMOST OPTIMAL UNRESTRICTED FAST JOHNSON-LINDENSTRAUSS TRANSFORM 

NIR AILON AND EDO LIBERTY


#### Abstract

The problems of random projections and sparse reconstruction have much in common and individually received much attention. Surprisingly, until now they progressed in parallel and remained mostly separate. Here, we employ new tools from probability in Banach spaces that were successfully used in the context of sparse reconstruction to advance on an open problem in random pojection. In particular, we generalize and use an intricate result by Rudelson and Vershynin for sparse reconstruction which uses Dudley's theorem for bounding Gaussian processes. Our main result states that any set of $N=\exp (\tilde{O}(n))$ real vectors in $n$ dimensional space can be linearly mapped to a space of dimension $k=O(\log N$ polylog$(n))$, while (1) preserving the pairwise distances among the vectors to within any constant distortion and (2) being able to apply the transformation in time $O(n \log n)$ on each vector. This improves on the best known $N=\exp \left(\tilde{O}\left(n^{1 / 2}\right)\right)$ achieved by Ailon and Liberty and $N=\exp \left(\tilde{O}\left(n^{1 / 3}\right)\right)$ by Ailon and Chazelle. The dependence in the distortion constant however is believed to be suboptimal and subject to further investigation. For constant distortion, this settles the open question posed by these authors up to a polylog $(n)$ factor while considerably simplifying their constructions.


## 1. Introduction

Designing computationally efficient transformations that reduce dimensionality of data while approximately preserving its metric information lies at the heart of many problems. While in compressed sensing such techniques are sought for sparse data in a real or complex metric space (with respect to some basis), in random projections, following the seminal work of Johnson and Lindenstrauss, one seeks to reduce dimension of any set of finite data. 1 In both applications, random matrices of a suitable size [1] 2] (4) result in optimal construction [5] in the parameters $n$ (the original dimension), $k$ (the target dimension), $N$ (the number of input vectors) and $\delta$ (the distortion). However, these constructions' resulting running time complexity, measured as number of operations needed in order to map a vector, is suboptimal.

A major open question is that of designing such matrix distributions that can be applied efficiently to any vector, with optimal dependence in the parameters $n, k, N$ and $\delta$. Applications for such transformations were found e.g. in designing fast approximation algorithms for solving large scale linear algebraic operations (e.g. 6],

[^0][7) The two lines of work, though sharing much in common, have mostly progressed in parallel. Here we combine recent work on bounds for sparse reconstruction to improve bounds of Ailon and Chazelle [8, 9] and Ailon and Liberty Liberty [10] on fast random projections, also known as Fast Johnson-Lindenstrauss transformations. The new bounds allow obtaining the well known Fast Johnson-Lindenstrauss Transform for finite sets of bounded cardinality $N=\exp (\tilde{O}(n))$ where $n$ is the original dimension. The best known so far was obtained by Ailon and Liberty for sets of size up to $N=\exp \left\{\tilde{O}\left(n^{1 / 2}\right)\right\}$ The latter improved on Ailon and Chazelle's original bound of $N=\exp \left\{O\left(n^{1 / 3}\right)\right\}$, which initiated the construction of Fast Johnson-Lindenstrauss Transforms. We also mention Dasgupta et al.'s work [11] on construction of Johnson-Linenstrauss random matrices which can be more efficiently applied to sparse vectors, with applications in the streaming model, and Ailon et al's work [12] on design of Johnson-Lindenstrauss matrices that run in linear time under certain assumptions on various norms of the input vectors.

The transformation we derive here is a composition of two random matrices: A random sign matrix and a random selection of a suitable number $k$ of rows from a Fourier matrix, where $k=O\left(\delta^{-4}(\log N) \operatorname{poly} \log (n)\right)$, and $\delta$ is the tolerated distortion level. The result, for constant $\delta$, is believed to be suboptimal within the $\operatorname{poly} \log (n)$ factor in the target dimension $k$. The running time of performing the transformation on a vector is dominated by the $O(n \log n)$ of the Fast Fourier Transform, and is believed to be optimal. The possibility of obtaining such a running time for fixed distortion was left as an open problem in Ailon and Chazelle and Ailon and Liberty's work, and here we resolve it up to a factor of $\operatorname{poly} \log (n)$. The dependence on the constant $\delta$ is also believed to be suboptimal, and the "correct" dependence shoould be $\delta^{-2}$. The question of improving this dependence is left as an open problem.

The use of a combination of random sign matrices and various forms of subsampled Fourier matrices was also used in the work of Ailon and Chazelle [8] and later Ailon and Liberty [10], as well as that of Matousek [13]. Here we obtain improved analysis using recent work by Rudelson and Vershynin for sparse reconstruction 14.
1.1. Restricted Isometry. An underlying idea common to both random projections and sparse reconstruction is the preservation of metric information under a dimension reducing transformation. In sparse reconstruction theory, this property is known as restricted isometry [15] [16]. A matrix $\Phi$ is a restricted isometry with sparseness paramater $r$ if for some $\delta>0$,

$$
\begin{equation*}
\forall r \text {-sparse } y \in \mathbb{R}^{n} \quad(1-\delta)\|y\|_{2}^{2} \leq\|\Phi y\|_{2}^{2} \leq(1+\delta)\|y\|_{2}^{2} \tag{1.1}
\end{equation*}
$$

By $r$-sparse $y$ we mean vectors in $\mathbb{R}^{n}$ with all but at most $r$ coordinates zero. It was shown in [15] that the restricted isometry property is sufficient for the purpose of perfect reconstruction of sparse vectors, compressed sensing being one of the prominent applications.

In [17], Rudelson and Vershynin construct a distribution over $k \times n$ matrices $\Phi$ such that, with high probability, $\Phi$ has the restricted isometry property with sparseness parameter $r$ and arbitrarily small $\delta>0$ In their analysis,

[^1]$k=O\left(\delta^{-2} r \log (n) \cdot \log ^{2}(r) \log (r \log n)\right)$ and $\Phi$ can be applied (to a given vector $x)$ in running time $O(n \log n)$. Assuming $r$ polynomial in $n$, this takes the simpler form of $k=O\left(\delta^{-2} r \log ^{4} n\right)$ In fact, $\Phi$ is (up to a constant) nothing other than a random choice of $k$ rows from the (unnormalized) Hadamard matrix, defined as $\Psi_{\omega, t}=(-1)^{\langle\omega, t\rangle}$, where $\langle\cdot, \cdot\rangle$ is the dot product over the binary field, $n$ is assumed to be a power of 2 and $\omega, t$ are thought of as $\log n$ dimensional vectors over the binary field in an obvious way 5 As a corollary of the result, one obtains a universal matrix for reconstructing sparse signals, which can be applied to a vector in time $O(n \log n)$. The conjecture is that the same distribution with $k=O\left(\delta^{-2} r \log n\right)$ should work as well, but this is a major open question beyond the scope of this work. For an excellent survey explaining how restricted isometry can be used for sparse reconstruction, and why designing such matrices with good computational properties is important we refer the readers to [18] and to references therein.

Independently, Ailon and Chazelle [8] and Ailon and Liberty [10] were interested in constructing a distribution of $k \times n$ matrices $\Phi$ such that for any set $Y \subseteq \mathbb{R}^{n}$ of cardinality $N$, one gets

$$
\begin{equation*}
\forall y \in Y(1-\delta)\|y\|_{2}^{2} \leq\|\Phi y\|_{2}^{2} \leq(1+\delta)\|y\|_{2}^{2} \tag{1.2}
\end{equation*}
$$

with constant probability. Additionally, the number of steps required for applying $\Phi$ on any given $x$ is $O(n \log n)$. In their result $k$ was taken as $O\left(\delta^{-2} \log N\right)$, which is also essentially the best possible [5]. Unfortunately, both results break down when $k=\Omega\left(n^{1 / 2}\right) 6^{6}$ Assuming the tolerance parameter $\delta$ fixed, this limitation can be rephrased as follows: The techniques fail when the number of vectors $N$ is in $\exp \left\{\Omega\left(n^{1 / 2}\right)\right\}$.

In both Ailon and Chazelle [8] and Ailon and Liberty's [10] results, as well as in previous work [1] [2] [3] [13] 4] the bounds (1.2) are obtained by proving strong tail bounds on the distribution of the estimator $\|\Phi y\|_{2}$, and then applying a simple union bound on the finite collection $Y$. It is worth a moment's thought to realize that Ailon and Chazelle's result as well as that of Ailon and Liberty can be used for restricted isometry as well. Indeed, a simple epsilon-net argument for the set of $r$-sparse vectors can turn that set into a finite set of $\exp \{O(r \log n)\}$ vectors, on which a union bound can be applied. However, the current limitation of random projections mentioned above will limit $r$ to be in $n^{O(1 / 2-\mu)}$ (for arbitrarily small $\mu$ ). Interestingly, Rudelson and Vershynin's result does not break down for $r$ polynomial in $n$. A careful inspection of their techniques reveals that instead of union bounding on a finite set of strongly concentrated random variables, they use a result due to Dudley to bound extreme values of Gaussian processes. Can this idea be used to improve [8] and [10]? Intuitively there is no reason why a result which is designed for preserving the metric of sparse vectors should help with preserving the metric of any finite set of vectors. It turns out, luckily, that such a reduction can be

[^2]done, though not in an immediate way. A suitable generalization of Rudelson and Vershynin's result (Section 24), combined with Ailon and Chazelle [8] and Ailon and Liberty's [10] method of random sign matrix preconditioning achieves this in Section 3
1.2. Notation. In what follows, we fix $N$ to denote the cardinality of a set $Y$ of vectors in $\mathbb{R}^{n}$, where $n$ is fixed. We also fix a distortion parameter $\delta \in(0,1 / 2$ ], and define $k$ to be an integer in $\Theta\left(\delta^{-4}(\log N)\left(\log ^{4} n\right)\right)$.

Now let $\Phi$ be a random $k \times n$ matrix obtained by picking $k$ random rows from the unnormalized $n \times n$ Hadamard matrix (the Euclidean norm of each column of $\Phi$ is $\sqrt{k})$. Let $\Omega$ denote the probability space for the choice of $\Phi$.

Let $b$ denote a uniformly chosen vector in $\{-1,1\}^{n}$, and let $\Gamma$ denote the probability space on the choice of $b$. For a vector $y \in \mathbb{R}^{n}$, we denote by $D_{y}$ the diagonal $n \times n$ matrix with the coordinates of $y$ on the diagonal. For a real matrix, $\|\cdot\|$ denotes its spectral norm and $(\cdot)^{t}$ its transpose. For a set $T \subseteq\{1, \ldots n\}$, we let $\mathrm{Id}_{T}$ denote the diagonal matrix with $\operatorname{Id}_{T}(i, i)=1$ if $i \in T$, and 0 otherwise. For a vector $y \in \mathbb{R}^{n}$, let $\operatorname{supp}(y)$ denote the support of $y$, namely, its set of nonzero coordinates. For a number $p \geq 1$, let $B_{p} \subseteq \mathbb{R}^{n}$ denote the set of vectors $y \in \mathbb{R}^{n}$ with $\|y\|_{p} \leq 1$ and $\alpha B_{p}$ as the set of vectors $y \in \mathbb{R}^{n}$ for which $\|y\|_{p} \leq \alpha$.

## 2. Restricted isometry result generalization

We follow the main path of Rudelson et al. in 17 to prove a more general formulation of their main theorem which is more suitable for us here.

Theorem 2.1. [Derived from Rudelson and Vershynin[17]] Let $\alpha>0$ be any real number. Define $E_{\alpha}$ as

$$
\begin{equation*}
E_{\alpha}=E_{\Omega}\left[\sup _{y \in B_{2} \cap \alpha B_{\infty}}\left\|D_{y}^{2}-\frac{1}{k} D_{y} \Phi^{t} \Phi D_{y}\right\|\right] \tag{2.1}
\end{equation*}
$$

Then for some global $C_{1}>0$,

$$
\begin{equation*}
E_{\alpha} \leq \frac{C_{1} \log ^{3 / 2}(n) \log ^{1 / 2}(k)}{\sqrt{k}}\left(E_{\alpha}+\alpha^{2}\right)^{1 / 2} \tag{2.2}
\end{equation*}
$$

In particular, if $\frac{\left(\log ^{3 / 2} n\right)\left(\log ^{1 / 2} k\right)}{\sqrt{k}}=O(\alpha)$, then

$$
\begin{equation*}
E_{\alpha}=O\left(\frac{\alpha\left(\log ^{3 / 2} n\right)\left(\log ^{1 / 2} k\right)}{\sqrt{k}}\right) \tag{2.3}
\end{equation*}
$$

The proof we present is an adaptation of the proof of Theorem 3.4 in [17] to a more general setting. In fact, the latter theorem [17] can be obtained as an easy consequence of theorem 2.1) by replacing $\sup _{y \in B_{2} \cap \alpha B_{\infty}}$ in (2.1) by $\sup _{y \in \frac{1}{\sqrt{r}} Y_{r}}$ where $Y_{r} \subseteq \mathbb{R}^{n}$ is defined as the set of vectors with at most $r$ coordinates equalling 1 and the remaining coordinates zero. Indeed, $\frac{1}{\sqrt{r}} Y_{r} \subseteq B_{2} \cap r^{-1 / 2} B_{\infty}$. We can therefore conclude that for $\alpha=\frac{1}{\sqrt{r}}$, by definition,

$$
E_{\Omega}\left[\sup _{y \in \frac{1}{\sqrt{r}} Y_{r}}\left\|D_{y}^{2}-\frac{1}{k} D_{y} \Phi^{t} \Phi D_{y}\right\|\right] \leq E_{\alpha}
$$

If we also assume that $k=\Theta\left(r \log ^{4} n\right)$, then (2.3) will hold, from which we conclude that

$$
\begin{equation*}
E_{\Omega}\left[\sup _{y \in \frac{1}{\sqrt{r}} Y_{r}}\left\|D_{y}^{2}-\frac{1}{k} D_{y} \Phi^{t} \Phi D_{y}\right\|\right] \leq O\left(\frac{\left(\log ^{3 / 2} n\right)\left(\log ^{1 / 2} k\right)}{\sqrt{r k}}\right) \tag{2.4}
\end{equation*}
$$

Now we notice that $D_{y}=\frac{1}{\sqrt{r}} \operatorname{Id}_{\operatorname{supp} y}$, where for a set of indexes $T$ the diagonal matrix $\operatorname{Id}_{T}$ (as defined in [17]) has 1 in diagonal position $i$ if and only if $i \in T$. Using this observation and multiplying (2.4) by $r$ we conclude that

$$
E_{\Omega}\left[\sup _{|T| \leq r}\left\|\operatorname{Id}_{T}-\frac{1}{k} \operatorname{Id}_{T} \Phi^{t} \Phi \mathrm{Id}_{T}\right\|\right] \leq O\left(\frac{\sqrt{r}\left(\log ^{3 / 2} n\right)\left(\log ^{1 / 2} k\right)}{\sqrt{k}}\right)
$$

which is exactly the main result of Rudelson and Vershynin in [17] for restricted isometry.

The proof of Theorem 2.1 below points out the necessary changes to the proof of Theorem 3.4 in [17]. The difference between the theorems is that in our case, the supremum in the definition of $E_{\alpha}$ is taken not only over the set of sparse vectors, but over a richer set. It turns out however that [17] uses sparsity in a very limited way: In fact, the dominating effect of sparsity there is obtained using the fact that the $L_{1}$ norm of a sparse vector is small, compared to its $L_{2}$ norm. These arguments appear at the very end of their proof. For the sake of contributing to the self containment of the paper we walk through the main milestones of the proof of Theorem 3.4 in [17], and point out the changes necessary for our purposes. The reader is nevertheless encouraged to refer to the enlightening exposition in [17] first.

Proof. Clearly $E\left[\frac{1}{k} D_{y} \Phi^{t} \Phi D_{y}\right]=D_{y}^{2}$. We define new independent random i.i.d. variables $\epsilon_{1}, \ldots, \epsilon_{n}$ obtaining each the values $+1,-1$ with equal probability. Let $\Pi$ denote the probability space for $\epsilon_{1}, \ldots, \epsilon_{n}$. It suffices to prove (using a symmetrization argument, see Lemma 6.3 in [19]) that
$E_{\Omega \times \Pi}\left[\sup _{y \in B_{2} \cap \alpha B_{\infty}}\left\|\frac{1}{k} \sum_{i=1}^{k} \epsilon_{i}\left(x_{i} D_{y}\right)^{t}\left(x_{i} D_{y}\right)\right\|\right] \leq \frac{2 C_{1}\left(\log ^{3 / 2} n\right)\left(\log ^{1 / 2} k\right)}{\sqrt{k}}\left(E_{\alpha}+\alpha^{2}\right)^{1 / 2}$,
where $x_{i}$ is the (random) $i$ 'th row of $\Phi_{k}$. To that end, as claimed in 17] (Lemma $3.5)$, if we can show that for any fixed choice of $\Phi$,

$$
\begin{equation*}
E_{\Pi}\left[\sup _{y \in B_{2} \cap \alpha B \infty}\left\|\sum_{i=1}^{k} \epsilon_{i}\left(x_{i} D_{y}\right)^{t}\left(x_{i} D_{y}\right)\right\|\right] \leq k_{1} \sup _{y \in B_{2} \cap \alpha B_{\infty}}\left\|\sum_{i=1}^{k}\left(x_{i} D_{y}\right)^{t}\left(x_{i} D_{y}\right)\right\|^{1 / 2} \tag{2.6}
\end{equation*}
$$

for some number $k_{1}$, then by taking $E_{\Omega}$ on both sides and using Jensen's inequality (to swap $(\cdot)^{1 / 2}$ on the RHS with $E_{\Omega}$ ) and the triangle inequality, the conclusion would be that

$$
\begin{equation*}
E_{\alpha} \leq \frac{2 k_{1}}{\sqrt{k}}\left(E_{\alpha}+\left\|D_{y}^{2}\right\|\right)^{1 / 2} \tag{2.7}
\end{equation*}
$$

Since $\left\|D_{y}^{2}\right\|=\|y\|_{\infty}^{2} \leq \alpha$, we would get the stated result. It thus suffices to prove (2.6) with $k_{1}=O\left(\left(\log ^{3 / 2} n\right)\left(\log ^{1 / 2} k\right)\right)$. To do so, 17] continue by replacing the $k$ binary random variables $\epsilon_{1}, \ldots, \epsilon_{k}$ in (2.6) with $k$ Gaussian random variables
$g_{1}, \ldots, g_{k}$ using a comparison principle (inequality (4.8) in [19]), reducing the problem to that of bounding the expected extreme value of a Gaussian process. Using Dudley's inequality (Theorem 11.17 in [19]), as Rudelson and Vershynin do, one concludes that (2.6) will hold with $k_{1}$ taken as:

$$
\begin{equation*}
\int_{0}^{\infty} \log ^{1 / 2} \mathcal{N}\left(B,\|\cdot\|_{X}, u\right) d u \tag{2.8}
\end{equation*}
$$

where:

- For a norm $\|\cdot\|_{\star}$, a set $S$ and number $u, \mathcal{N}\left(S,\|\cdot\|_{\star}, u\right)$ denotes the minimal number of balls of radius $u$ in norm $\|\cdot\|_{\star}$ centered in points of $S$ needed to cover the set $S$,
- $B$ is defined as $\cup_{y \in B_{2} \cap \alpha B_{\infty}} B_{y}$, where $B_{y}=\left\{D_{y} z: z \in B_{2}\right\}$, and
- $\|x\|_{X}=\max _{i \leq k}\left|\left\langle x_{i}, x\right\rangle\right|$, where we remind the reader that $x_{i}$ is the $i^{\prime}$ th row of $\Phi$.
Rudelson and Vershynin derive bounds on $\mathcal{N}\left(B_{R V},\|\cdot\|_{X}, u\right)$ for small $u$ and for large $u$ separately, where in their case $B_{R V}$ was the set of $r$-sparse vectors of Euclidean norm 1 (denoted by $D_{2}^{r, n}$ in [17]). The sparsity of the vectors in the set $B_{R V}$ is used in both derivations, as follows:
- For large $u$, they use containment argument (11) in 17, asserting that $B_{R V} \subseteq \sqrt{r} B_{1}$. Note that by Cauchy Schwartz and the definition of $B$, $B \subseteq B_{1}$ hence we "gain" a factor of $\sqrt{r}$ when deriving $k_{1}$.
- For small $u$, inequality (13) in [17] asserts that $\mathcal{N}\left(B_{R V},\|\cdot\|_{X}, u\right) \leq d(n, r)(1+$ $2 / u)^{r}$, where $d(n, r)$ is the number of ways to choose $r$ elements from a set of $n$ elements. Since the best sparseness we can assume for vectors in $B$ here is trivially $n$, we replace the expression $d(n, r)$ with $d(n, n)=1$, and $(1+2 / u)^{r}$ with $(1+2 / u)^{n} 7$
Rudelson and Vershynin then derive a bound for $\int_{0}^{\infty} \mathcal{N}^{1 / 2}\left(B_{R V},\|\cdot\|_{X}, u\right) d u$ by balancing the two bounds at $u=1 / \sqrt{r}$. In our case we balance at $u=1 / \sqrt{n}$. The net result will lead to a $k_{1}$ which is as the one in the statement of Lemma 3.5 [17], except that the $\sqrt{r}$ will disappear and $\log r$ will be replaced by $\log n$. The conclusion is that we can take $k_{1}$ to be

$$
k_{1}=O((\log n)(\sqrt{\log n})(\log k))=O\left(\left(\log ^{3 / 2} n\right)(\log k)\right)
$$

as required.

## 3. Random Projections

Our main result claims that the same construction used by Rudelson et al. also gives improved bounds for random projections. In what follows, we fix $r$ to be $\left\lceil\delta^{-2} \log N\right\rceil$ and $\alpha$ to be $1 / \sqrt{r}$. Additionally, we assume that $\Phi$ is such that

$$
\begin{equation*}
E_{\alpha}=O\left(\alpha^{2}\right) \tag{3.1}
\end{equation*}
$$

Indeed, Theorem 2.1 guarantees that this holds with probability at least 0.99 in $\Omega$.

[^3]Theorem 3.1. Let $Y \subseteq B_{2}$ denote a set of cardinality $N$, and let $\Phi$ satisfy (3.1). With probability at least 0.98 (in $\Gamma$ ) we have the following uniform bound for all $y \in Y$ :

$$
1-O(\delta) \leq\left\|\frac{1}{\sqrt{k}} \Phi D_{y} b\right\| \leq 1+O(\delta)
$$

We provide some intuition for the proof. We split our input vectors $Y$ into sums of two vectors, one of which is $r$-sparse and the other with $\ell_{\infty}$ norm bounded by $1 / \sqrt{r}$. We use Rudelson et al.'s original result for the sparse part and our generalization of it (Theorem[2.1), together with Talagrand's measure concentration theorem for the $\ell_{\infty}$-bounded part.

Proof. Let $r$ and $\alpha$ be defined as in Section 2 For each $y \in Y$ we write $y=\hat{y}+\check{y}$, where $\hat{y}$ is the restriction of $y$ to its $r$ largest (in absolute value) coordinates and $\check{y}$ is the restriction to its remaining coordinates. Note that $\|y\|^{2}=\|\hat{y}\|^{2}+\|\check{y}\|^{2}$ and that $\hat{y}$ is $r$-sparse and that $\|\check{y}\|_{\infty} \leq \alpha$.

$$
\left\|\frac{1}{\sqrt{k}} \Phi D_{y} b\right\|^{2}=\left\|\frac{1}{\sqrt{k}} \Phi D_{\hat{y}} b\right\|^{2}+\left\|\frac{1}{\sqrt{k}} \Phi D_{\check{y}} b\right\|^{2}+\frac{2}{k} b^{t} D_{\hat{y}} \Phi^{t} \Phi D_{\check{y}} b .
$$

For the first term we have $\left\|\frac{1}{\sqrt{k}} \Phi D_{\hat{y}} b\right\|^{2}=\|\hat{y}\|^{2}+O(\delta)$ from Theorem 2.1 and the fact that $\hat{y}$ is $r$-sparse.

In what follows we will use the bound on $\|\check{y}\|_{\infty}$ to show that with high probability, for all $y \in Y,\left\|\frac{1}{\sqrt{k}} \Phi D_{\check{y}} b\right\|^{2}=\|\check{y}\|^{2}+O(\delta)$. A similar argument will bound the cross product $\frac{2}{k} b^{t} D_{\hat{y}} \Phi^{t} \Phi D_{\check{y}} b$. Combining the three gives the desired result that $\left\|\frac{1}{\sqrt{k}} \Phi D_{y} b\right\|^{2}=\|y\|^{2}+O(\delta)$.

We start by analyzing the measure concentration properties of $\left\|\frac{1}{\sqrt{k}} \Phi D_{\check{y}} b\right\|^{2}$. Let $X_{\check{y}}$ be the Rademacher random variable defined by

$$
X_{\check{y}}=\left\|\frac{1}{\sqrt{k}} \Phi D_{\check{y}} b\right\| .
$$

Let $\mu_{\check{y}}$ denote a median of $X_{\check{y}}$. By Talagrand [19], we have that for all $t>0$,

$$
\begin{align*}
\operatorname{Pr}\left[X_{\check{y}}>\mu_{\check{y}}+t\right] & \leq \exp \left\{-C_{2} t^{2} / \sigma_{\check{y}}^{2}\right\}  \tag{3.2}\\
\operatorname{Pr}\left[X_{\check{y}}<\mu_{\check{y}}-t\right] & \leq \exp \left\{-C_{2} t^{2} / \sigma_{\check{y}}^{2}\right\} \tag{3.3}
\end{align*}
$$

for some global $C_{2}$, where $\sigma_{\check{y}}=\left\|\frac{1}{\sqrt{k}} \Phi D_{\check{y}}\right\|$. By the triangle inequality and Equation (3.1) we have $\sigma_{\check{y}}^{2}=\left\|\frac{1}{k} D_{\check{y}} \Phi^{t} \Phi D_{\check{y}}-D_{\check{y}}^{2}+D_{\check{y}}^{2}\right\| \leq \alpha^{2}+\left\|D_{\check{y}}^{2}\right\|$. Clearly $\left\|D_{\check{y}}\right\|=$ $\|\check{y}\|_{\infty} \leq \alpha$. Hence, $\sigma_{\check{y}}^{2}=O\left(\alpha^{2}\right)$. From the fact that $E\left[X_{\check{y}}^{2}\right]=\|\check{y}\|^{2}$ and using Appendix A and (3.2)-(3.3) We conclude that $\|\check{y}\|-O\left(\sigma_{\check{y}}\right) \leq \mu_{\check{y}} \leq\|\check{y}\|+O\left(\sigma_{\check{y}}\right)$. Hence, again using (3.2)-(3.3) and union bounding over the $N$ vectors in $Y$, we conclude that with probability 0.99 , uniformly for all $y \in Y$ :

$$
\|\check{y}\|-O(\delta) \leq \frac{1}{\sqrt{k}}\left\|\Phi D_{\check{y}} b\right\| \leq\|\check{y}\|+O(\delta)
$$

We now bound the cross term $Z=\frac{1}{k} b^{t} D_{\hat{y}} \Phi^{t} \Phi D_{\check{y}} b$ ( $y$ is now held fixed). By disjointness of $\operatorname{supp}(\hat{y})$ and $\operatorname{supp}(\hat{y}), E[Z]=0$. Decompose $b$ into $\check{b}+\hat{b}$, where $\operatorname{supp}(\breve{b})=\operatorname{supp}(\check{y})$ and $\operatorname{supp}(\hat{b})=\operatorname{supp}(\hat{y})$. For any fixed $\hat{b}$, the function $Z$ is linear
(and hence convex) in $\check{b}$. Also for all possible values $\hat{b}^{\prime}$ of $\hat{b}, E\left[Z \mid \hat{b}=\hat{b}^{\prime}\right]=0$. Hence, again by Talagrand,

$$
\begin{align*}
\operatorname{Pr}\left[Z>\mu_{\hat{b}^{\prime}}+t\right] & \leq \exp \left\{-C_{2} t^{2} / \sigma_{\hat{b}^{\prime}}^{2}\right\}  \tag{3.4}\\
\operatorname{Pr}\left[Z<\mu_{\hat{b}^{\prime}}-t\right] & \leq \exp \left\{-C_{2} t^{2} / \sigma_{\hat{b}^{\prime}}^{2}\right\} \tag{3.5}
\end{align*}
$$

where $\mu_{\hat{b}}^{\prime}$ is a median of $\left(Z \mid \hat{b}=\hat{b}^{\prime}\right)$, and $\sigma_{\hat{b}^{\prime}}=\left\|\frac{1}{k}\left(\hat{b}^{\prime}\right)^{t} D_{\hat{y}} \Phi^{t} \Phi D_{\tilde{y}}\right\|$. Clearly,

$$
\begin{aligned}
\sigma_{\hat{b}^{\prime}} & \leq\left\|\frac{1}{\sqrt{k}}\left(\hat{b}^{\prime}\right)^{t} D_{\hat{y}} \Phi^{t}\right\| \cdot\left\|\frac{1}{\sqrt{k}} \Phi D_{\check{y}}\right\| \\
& =O\left(\|\hat{y}\| \sigma_{\check{y}}\right)=O\left(\sigma_{\check{y}}\right)=O(\alpha) .
\end{aligned}
$$

Again using Appendix A and $E\left[Z \mid \hat{b}=\hat{b}^{\prime}\right]=0$ gives that $\left|\mu_{\hat{b}}^{\prime}\right|=O(\alpha)$, and again we conclude using a union bound that with probability at least 0.99 , uniformly for all $y \in Y,\left|\frac{1}{k} b^{t} D_{\hat{y}} \Phi^{t} \Phi D_{\check{y}} b\right|=O(\delta)$.

Tying it all together, we conclude that with probability at least 0.98 , uniformly for all $y \in Y$,

$$
\begin{aligned}
\frac{1}{k}\left\|\Phi D_{y} b\right\|^{2} & =\frac{1}{k}\left\|\Phi D_{\check{y}} b\right\|^{2}+\frac{1}{k}\left\|\Phi D_{\hat{y}} b\right\|^{2}+2 b^{t} D_{y^{H}} \Phi^{t} \Phi D_{\check{y}} b \\
& =\|y\|^{2}+O(\delta)
\end{aligned}
$$

as required.

## 4. Conclusions

The obvious problems left open are those of (1) improving the dependence of $k$ in $\delta$ (from $\delta^{-4}$ to $\delta^{-2}$ ) and (2) removing the dependence of $k$ in $\operatorname{polylog}(n)$. Other directions of research include not only reducing the computational efficiency of random dimension reduction, but also the amount of randomness needed for the construction.

## Acknowledgements

We thank Emmanuel Candes for helpful discussions.

## References

[1] W. B. Johnson and J. Lindenstrauss. Extensions of Lipschitz mappings into a Hilbert space. Contemporary Mathematics, 26:189-206, 1984.
[2] P. Frankl and H. Maehara. The Johnson-Lindenstrauss lemma and the sphericity of some graphs. Journal of Combinatorial Theory Series A, 44:355-362, 1987.
[3] S. DasGupta and A. Gupta. An elementary proof of the Johnson-Lindenstrauss lemma. Technical Report, UC Berkeley, 99-006, 1999.
[4] Dimitris Achlioptas. Database-friendly random projections: Johnson-Lindenstrauss with binary coins. J. Comput. Syst. Sci., 66(4):671-687, 2003.
[5] Noga Alon. Problems and results in extremal combinatorics-I. Discrete Mathematics, 273(1-3):31-53, 2003.
[6] Tamás Sarlós. Improved approximation algorithms for large matrices via random projections. In Proceedings of the 47 th Annual IEEE Symposium on Foundations of Computer Science (FOCS), Berkeley, CA, 2006.
[7] Franco Woolfe, Edo Liberty, Vladimir Rokhlin, and Mark Tygert. A fast randomized algorithm for the approximation of matrices. Applied and Computational Harmonic Analysis, $25(3): 335-366,2008$.
[8] Nir Ailon and Bernard Chazelle. Approximate nearest neighbors and the fast JohnsonLindenstrauss transform. In Proceedings of the 38st Annual Symposium on the Theory of Compututing (STOC), pages 557-563, Seattle, WA, 2006.
[9] Nir Ailon and Bernard Chazelle. Faster dimension reduction. Commun. ACM, 53(2):97-104, 2010.
[10] Nir Ailon and Edo Liberty. Fast dimension reduction using rademacher series on dual bch codes. Discrete Comput. Geom., 42(4):615-630, 2009.
[11] Dasgupta A., Kumar R., and Sarlos T. A sparse johnson-lindenstrauss transform. In Proceedings of the $42 n d$ ACM Symposium on Theorey of Computing (STOC), 2010.
[12] Edo Liberty, Nir Ailon, and Amit Singer. Dense fast random projections and lean walsh transforms. In $A P P R O X-R A N D O M$, pages 512-522, 2008.
[13] J. Matousek. On variants of the Johnson-Lindenstrauss lemma. Private communication, 2006.
[14] Mark Rudelson and Roman Veshynin. Sparse reconstruction by convex relaxation: Fourier and gaussian measurmensts.
[15] Emmanuel J. Candès, Justin K. Romberg, and Terence Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. IEEE Transactions on Information Theory, 52(2):489-509, 2006.
[16] David L. Donoho. Compressed sensing. IEEE Transactions on Information Theory, 52(4):1289-1306, 2006.
[17] Mark Rudelson. Sparse reconstruction by convex relaxation: Fourier and gaussian measurements. In CISS 2006 (40th Annual Conference on Information Sciences and Systems, 2006.
[18] Alfred M. Bruckstein, David L. Donoho, and Michael Elad. From sparse solutions of systems of equations to sparse modeling of signals and images. SIAM Rev., 51(1):34-81, 2009.
[19] M. Ledoux and M. Talagrand. Probability in Banach Spaces: Isoperimetry and Processes. Springer-Verlag, 1991.

## Appendix A.

Fact A.1. For any real valued random variable $Z$ such that for all $t>0$

$$
\begin{align*}
\operatorname{Pr}[Z>\mu+t] & \leq \exp \left\{-c t^{2} / \sigma^{2}\right\}  \tag{A.1}\\
\operatorname{Pr}[Z<\mu-t] & \leq \exp \left\{-c t^{2} / \sigma^{2}\right\}
\end{align*}
$$

we have that $\sqrt{E\left(Z^{2}\right)}-O(\sigma) \leq \mu \leq \sqrt{E\left(Z^{2}\right)}+O(\sigma)$.
Proof. Define the variable $Z^{\prime}=(Z-\mu) / \sigma$.

$$
\begin{aligned}
E\left[Z^{\prime}\right] \leq E\left[\left|Z^{\prime}\right|\right] & \leq \sum_{i=1}^{\infty} i \operatorname{Pr}\left(i-1 \leq\left|Z^{\prime}\right| \leq i\right) \\
& \leq \sum_{i=1}^{\infty} i \operatorname{Pr}\left(\left|Z^{\prime}\right| \geq i-1\right) \leq 2 \sum_{i=1}^{\infty} i \exp \left\{-c(i-1)^{2}\right\}=O(1)
\end{aligned}
$$

Clearly, $E\left[Z^{\prime}\right]=O(1)$ gives $E(Z)=\mu+O(\sigma)$. In the same way we get $E\left[Z^{\prime 2}\right]=$ $O(1)$. Thus, $E\left[Z^{2}\right]-2 \mu E[Z]+\mu^{2}=O\left(\sigma^{2}\right)$ and $E\left[Z^{2}\right]=(\mu \pm O(\sigma))^{2}$

Technion, Haifa, Israel
E-mail address: nailon@gmail.com
Yahoo! Research, Haifa, Israel
E-mail address: edo@yahoo-inc.com


[^0]:    Date: May 30, 2010.
    Nir Ailon's affiliation: Technion, Haifa, Israel, nailon@gmail.com.
    Edo Liberty's affiliation: Yahoo! Research, Haifa, Israel edo@yahoo-inc.com.
    ${ }^{1}$ The term "random projections" describes Johnson and Lindenstrauss' s original construction and became synonymous with the process of approximate metric preserving dimension reduction using randomized linear mappings. However, these linear mappings need not be (and indeed are usually not) projections in the linear algebraic sense of the word.

[^1]:    ${ }^{2}$ The notation $\tilde{O}(\cdot)$ suppresses arbitrarily small polynomial coefficients and polylogarithmic factors.
    ${ }^{3}$ Their analysis is done over the complex field, but we restrict the discussion to the reals here.

[^2]:    ${ }^{4}$ In their work, the dependence of $k$ on $\delta$ is not analyzed because $\delta$ is assumed to be fixed (for sparse signal reconstruction purposes, this dependence is not important). It is not hard to derive the quadratic dependence of $k$ in $\delta^{-1}$ from their work.
    ${ }^{5}$ Rudelson and Vershynin use the complex Discrete Fourier Transform matrix, but their analysis does not change when using the Hadamard matrix.
    ${ }^{6}$ Ailon and Chazelle 8 and Ailon and Liberty 10 used $d$ to denote the data dimension, $n$ its cardinality and $\varepsilon$ the sought distortion bound. Here we follow Rudelson and Vershynin's convention using $n$ to denote the dimension and $\delta$ the distortion bound. We now use $N$ to denote the data cardinality.

[^3]:    ${ }^{7}$ To be exact, in [17] they use the expression $(1+2 K / u)^{r}$ and not $(1+2 / u)^{r}$, but the parameter $K$ in their work can be taken as 1 for our purposes.

