

A Game-Theoretic Model Motivated by the DARPA Network Challenge*

Rajesh Chitnis[†] MohammadTaghi Hajiaghayi[†] Jonathan Katz Koyel Mukherjee

January 29, 2013

Abstract

In this paper we propose a game-theoretic model to analyze events similar to the 2009 *DARPA Network Challenge*, which was organized by the Defense Advanced Research Projects Agency (DARPA) for exploring the roles that the Internet and social networks play in incentivizing wide-area collaborations. The challenge was to form a group that would be the first to find the locations of ten moored weather balloons across the United States. We consider a model in which N people (who can form groups) are located in some topology with a fixed coverage volume around each person's geographical location. We consider various topologies where the players can be located such as the Euclidean d -dimension space and the vertices of a graph. A balloon is placed in the space and a group wins if it is the first one to report the location of the balloon. A larger team has a higher probability of finding the balloon, but we assume that the prize money is divided equally among the team members. Hence there is a competing tension to keep teams as small as possible.

Risk aversion is the reluctance of a person to accept a bargain with an uncertain payoff rather than another bargain with a more certain, but possibly lower, expected payoff. In our model we consider the *isoelastic* utility function derived from the Arrow-Pratt measure of relative risk aversion. The main aim is to analyze the structures of the groups in Nash equilibria for our model. For the d -dimensional Euclidean space ($d \geq 1$) and the class of bounded degree regular graphs we show that in any Nash Equilibrium the *richest* group (having maximum expected utility per person) covers a constant fraction of the total volume. The objective of events like the DARPA Network Challenge is to mobilize a large number of people quickly so that they can cover a big fraction of the total area. Our results suggest that this objective can be met under certain conditions.

1 Introduction

With the advent of communication technologies, and the Web in particular, we can now harness the collective abilities of large groups of people to accomplish tasks with unprecedented speed, accuracy, and scale. In the popular culture and the business literature, this process has come to be known as crowdsourcing [11]. Crowdsourcing has been used in various tasks such as labeling of images [23], predicting protein structures [6], and posting and solving Human Intelligence Tasks in Amazon's *Mechanical Turk* [19]. An important class of crowdsourcing problems demand a large recruitment along with an extremely fast execution. Examples of such *time-critical social mobilization* tasks include search-and-rescue operations in the times of disasters, evacuation in the event of terrorist attacks, and distribution of medicines during epidemics. For example, in the aftermath of Hurricane Katrina, amateur radio volunteers played an important

*Dept. of Computer Science, University of Maryland, USA. Email: {rchitnis, hajiagha, jkatz, koyelm}@cs.umd.edu.

[†]Supported in part by NSF CAREER award 1053605, NSF grant CCF-1161626, ONR YIP award N000141110662, DARPA/AFOSR grant FA9550-12-1-0423, a University of Maryland Research and Scholarship Award (RSA).

role by coordinating dispatch of emergency services to isolated areas [13]. “Collaboratition” [15] is a newly coined term to describe a type of crowdsourcing used for those problems which require a **collaborative** effort to be successful, but use **competition** as a motivator for the participation or the performance.

The DARPA Network Challenge: A good example of collaboratition is The 2009 *DARPA Network Challenge* [1], an event organized by the Defense Advanced Research Projects Agency (DARPA) for exploring the roles that the Internet and social networks play in incentivizing wide-area collaborations. Collaboration of efforts was required to complete the challenge quickly and in addition to the competitive motivation of the contest as a whole, the winning team from MIT established what they called a “collaborapetitive” environment to generate participations in their team and found all the ten balloons in less than seven hours. Their strategy in Pickard et al. [18]. Their main focus is on the mechanics of the group formation process in the DARPA Network Challenge, whereas in this paper we try to analyze the structures of the groups which form in Nash Equilibria.

Related Work: Douceur and Moscibroda [8] addressed a problem close to the spirit of the DARPA Network Challenge. They address the problem of motivating people to install and run a distributed service, like peer-to-peer systems, in which the decision and the effort to install a service falls to the individuals rather than to a central planner. Their paper appeared in 2007; two years before the DARPA Network Challenge took place. Their focus is on incentivizing the growth of a single group whereas in this paper we take a bird’s-eye view and try to analyze the structures of the groups in Nash equilibria.

In this paper we focus on analyzing the structures of the groups in Nash equilibria for our model. Some recent results analyze the structures of Nash Equilibria. Some upper and lower bounds are given on the diameter of the Equilibrium graphs in *Basic Network Creation Games* [2]. It was also shown that the equilibrium graphs have polylogarithmic diameter in *Cooperative Network Creation Games* [7]. A well-studied parameter related to Nash equilibria is the *price of anarchy* [12, 17, 21], which is the worst possible ratio of the total cost found by independent selfish behavior and the optimal total cost possible by a centralized, social welfare maximizing solution. However as observed in [2, 7], bounds on the structures of Nash equilibria lead to approximate bounds on the price of anarchy as well but not necessarily the other way around. Therefore trying to analyze the structures of the groups in Nash equilibria is more general than trying to bound the price of anarchy.

Myerson [16] used graph-theoretic ideas to model and analyze games with *partial cooperation structures*. The DARPA Network Challenge is also similar as any groups can possibly form but the geographical locations of the people causes certain group structures to become infeasible. There is an entire body of literature in Economics which is closely related to the model we consider in this paper. There have also been studies on how the rules of coalition formation affect the stability of environmental agreements between countries [9]. Their rules for leaving or entering the coalitions are very similar to the ones we consider in this paper for formation or splitting of the groups. Risk aversion is a natural assumption to make while modeling the behavior of humans. There is a recent paper [4] which gives efficient algorithms for computing truthful mechanisms for risk-averse sellers. Another paper [3] considers scenarios in which the goal is to ensure that information propagates through a large network of nodes. They assume a model where all nodes have the required information to compete which removes the incentive to propagate information. In this paper, we consider the natural assumption of risk aversion which gives a concave utility function (derived from the Arrow-Pratt measure of relative risk aversion). This motivates formation of groups which is consistent with what was observed in the DARPA Challenge. Another paper [5] considers the problem of acquiring information in a strategic networked environment. They show that in the DARPA Network Challenge, the idea to offer split contracts instead of fixed-payment contracts is robust against the selfishness displayed by the participating agents.

Organization of the paper: In Section 2 we describe our model in detail. Then we consider various topologies where our model can be implemented: the one-dimensional (line) space (in Section 4), and more generally the d -dimensional Euclidean space (in Section 5). For both these topologies, in any Nash Equilibrium we show that there always exists a group covering a constant fraction of the total volume. In Section 6 we consider the discrete version of our model, where the players form the vertices of an undirected graph. For the class of bounded-degree regular graphs, we prove that in any Nash Equilibrium there always exists a group covering a constant fraction of the total number of vertices. In contrast, under an assumption that defecting to an empty group is prohibited, we show for every constant $0 < c < 1$ there exist graphs which have a Nash Equilibrium where all groups occupy strictly less than a c -fraction of the total number of vertices.

2 Our Model

We assume there is a set of N players, each covering a region of space within the total volume A . In particular, in the Euclidean space, we assume each player covers a ball of radius one centered at his location; in the discrete case we view the players as occupying the vertices of a graph and assume each player covers himself and his neighbors.

Players are allowed to organize themselves into a collection of disjoint *groups* partitioning the set of players. In this work, we do not consider the precise dynamics of group formation, but instead we focus on analyzing the structures of the groups in Nash equilibria. Once the groups are formed, we envision the balloon being placed in the space. We say the balloon *falls within* a group S if the location of the balloon is in the coverage of S ; a group S *wins* if it is the first one to report the location of the balloon. To model this we assume the probability that the balloon *falls within* a group S is A_S/A , where A_S is the total volume covered by the players in S and A is the total volume. The prize money M is given to the group that wins, and the money received by a group is split equally among all members of that group. We note the balloon can be placed anywhere in the space, and we do not know where it will be placed. Hence the probability of any of the groups (which might form) finding the balloon first is the same and we do not consider this common factor hereafter.

Risk aversion [10, 14, 22, 20] is a concept in psychology, economics, and finance, based on the behavior of humans (especially consumers and investors) whilst exposed to uncertainty. Risk aversion is the reluctance of a person to accept a bargain with an uncertain payoff rather than another bargain with a more certain, but possibly lower, expected payoff [20]. Risk aversion is a natural assumption when we consider money and people: most of us would accept a guaranteed payment of say X dollars than a 50% chance of receiving $2X$ and a 50% chance of getting nothing, especially if X is large (the DARPA Challenge had a prize money of \$40,000). *Constant relative risk aversion* means that the ratio of the increase in the utility to the increase in the risk taken is constant. Assuming that the Arrow-Pratt measure of relative risk aversion is constant, the *isoelastic* utility function for money x is given by $u(x) = \frac{x^{1-r}}{1-r}$ where $0 \leq r < 1$ is the risk aversion factor [10]. For $r = 1$ we take the utility to be the natural logarithm. Here $r = 0$ means there is no risk aversion. For simplicity we scale up everything by a factor of $1 - r$ to get a concave utility function given by $u(x) = x^{1-r}$ where $0 < r < 1$. The *expected* utility for a player who is a member of a group S is given by $p(S) \cdot u(\frac{M}{|S|})$, where $p(S) = \frac{A_S}{A}$ is the probability that the balloon fall within S . Consider two players who have disjoint area of coverage. If they are on their own, then their expected utility is $u = (\frac{|M|}{1})^{1-r} \cdot \frac{a}{A}$ where a is the area they can cover. If they join together to form a group then their expected utility is $u' = (\frac{|M|}{2})^{1-r} \cdot \frac{2a}{A} = 2^r \cdot u > u$ since $1 > r > 0$. Therefore two people whose coverage areas are disjoint will always join together, not matter what the risk aversion factor r is. The intuition is that the value of r affects how much overlapping coverage areas is allowed for it to be beneficial for people to join together. The smaller the value of r the lesser the

overlap must be between the coverage areas of the players for it to make sense for them to merge.

We assume that the balloon is placed in a location covered by at least one player. Given a partition S_1, \dots, S_ℓ of all the players into groups, we now ask whether it forms an equilibrium. More formally, we allow two types of actions:

1. Two groups S_i and S_j can decide to *merge*. We say this operation is *incentivized* only if each player in S_i and S_j would increase their expected utility by merging.
2. A member x of group S_i may *defect* to join a different group S_j . We say this operation is *incentivized* only if both x 's expected utility and the expected utility of each player in S_j increase after the defect.

We say a given partition is a *Nash Equilibrium* if no merge or defect operation is incentivized, i.e., no player can do better by unilaterally changing his group. We could consider a generalized *defect* operation where a subset of a group S_i may leave to join a group S_j . However for the sake of clarity (while still capturing the essence of the model), we consider the *defect* operation where at a given time only a single person can leave his current group to join a new group.

3 Lower Bounds on the Total Prize Money

In this paper we consider the *social welfare* from the viewpoint of the agency which hosts the event described by our model. We now show that the hosting agency needs to offer prize money proportional to the desired size of a largest group or to the desired fraction of the total volume covered if each person must receive a minimum threshold expected utility.

Theorem 3.1. $[\star]$ *If there exists a Nash Equilibrium in which at least one group S covers a λ -fraction of the total volume, and each player in S covers volume V and has an expected utility of at least c , then $M \geq \lambda A c^{\frac{1}{1-r}}$.*

Theorem 3.2. $[\star]$ *If there exists a Nash Equilibrium in which there is at least one group S of size k , and each player in S covers volume V and has an expected utility of at least c , then $M \geq k \left(\frac{cA}{N} \right)^{\frac{1}{1-r}}$.*

In the rest of the paper the total prize money M is not important as we just compare the expected utilities of various group structures to see which ones form Nash equilibria, and hence M cancels out. However the above two theorems imply the hosting agency must spend money M which depends on c and therefore M cannot be arbitrarily small.

4 The One-Dimensional (Line) Case

In this section the players are located along a line. We show for any Nash Equilibrium there is at least one group covering a constant fraction (depending on the risk aversion factor r) of the total length. We assume each person has a coverage length of one on both sides. Recall for each person x in a group S the expected utility is $E[u(x)] = \left(\frac{M}{|S|} \right)^{1-r} \cdot \frac{A_S}{A}$ where M is the total money, A_S is the length covered by group S and A is the total length. We contract the points not covered by any player. Therefore every point in the total length has at least one person whose coverage length contains it.

Lemma 4.1. *For the line case, let S be a richest group in a Nash Equilibrium. Then there is no player $i \notin S$ who can add a length of at least $2(1-r)$ to the length A_S currently covered by S .*

The proofs of the results labeled with \star have been deferred to the Appendix

Proof. Suppose there is a player $i \notin S$, who can add a length of at least $2(1-r)$ to the length covered by S . However, since it is a Nash Equilibrium, either the new expected utility of S on adding this player is less than or equal to the current expected utility of S (hence S would have no incentive in adding the player i) or the player i would not have any incentive to move to S , as the projected new expected utility of i is less than or equal to his current expected utility. Since S is a richest group, both these conditions combine to give:

$$\left(\frac{M}{|S|+1}\right)^{1-r} \cdot \frac{A_S + 2(1-r)}{A} \leq \left(\frac{M}{|S|}\right)^{1-r} \cdot \frac{A_S}{A} \quad (1)$$

As each player has a coverage length of two we have $2|S| \geq A_S$ (equality holds only if the coverage lengths of the members of S are pairwise disjoint). The function $f(x) = \frac{x}{x+1}$ is increasing on $(0, \infty)$ and hence $\frac{|S|}{|S|+1} \geq \frac{\beta}{\beta+1}$ where $\beta = \frac{A_S}{2}$. Combining with Equation 1 gives

$$\left(\frac{\beta}{\beta+1}\right)^{1-r} \leq \left(\frac{|S|}{|S|+1}\right)^{1-r} \leq \frac{A_S}{A_S + 2(1-r)}$$

Rearranging and setting $1-r = \frac{1}{t}$ implies

$$\frac{A_S + 2(1-r)}{A_S} \leq \left(\frac{\beta+1}{\beta}\right)^{\frac{1}{t}} \implies \left(1 + \frac{2(1-r)}{A_S}\right)^t \leq 1 + \frac{1}{\beta}$$

Bernoulli's inequality states if $x, q \in \mathbb{R}$ and $x > -1, q > 1$ then $(1+x)^q > 1+qx$. Applying the inequality for $x = \frac{2(1-r)}{A_S}$ and $q = t = \frac{1}{1-r} > 1$ gives

$$1 + \frac{1}{\beta} = 1 + \frac{2t(1-r)}{A_S} < \left(1 + \frac{2(1-r)}{A_S}\right)^t \leq 1 + \frac{1}{\beta}$$

which is a contradiction. □

Definition 4.2. Given a partition of the people into groups, we say a group G is a **richest** group if its expected utility per person value is at least that of any other group in the partition.

We note that given any partition of people into groups, there always exists at least one richest group. We show the following interesting and unexpected phenomenon: in every Nash Equilibrium any richest group covers a constant fraction of the total volume.

Theorem 4.3. For the line case, in any Nash Equilibrium there is always a group which covers a constant fraction of the total length where the constant is $\frac{1}{1+2(1-r)}$.

Proof. We refer to Figure 1 for the notation used in this proof. Given a Nash Equilibrium, consider any richest group S . Since we contracted the points not covered by anybody, a player $x \notin S$ has coverage length starting from the leftmost point of E_L or E_L might be empty. If E_L is not empty then $|E_L| \leq 2(1-r)$ else x contradicts Lemma 4.1. In either case $|E_L| \leq 2(1-r)$. Similarly there is a player $y \notin S$ whose coverage length ends at the rightmost point of E_R or E_R is empty. In either case $|E_R| \leq 2(1-r)$. Claim is $|I_j| \leq 4(1-r)$ for $1 \leq j \leq m-1$. Suppose $\exists j$ such that $|I_j| > 4(1-r)$. The midpoint of I_j must be covered by a player $z \notin S$ and so this z covers at least half of I_j , i.e., z can offer greater than $2(1-r)$ length to S which contradicts Lemma 4.1. Also $|S_j| \geq 2$ for all $1 \leq j \leq m$ as each S_j is a concatenation of the coverage lengths of one or

more members of S . Therefore $|A_S| = \sum_{j=1}^m |S_j| \geq 2m$ and

$$\begin{aligned}
|A| &= \sum_{j=1}^m |S_j| + \sum_{j=1}^{m-1} |I_j| + |E_L| + |E_R| \\
&\leq \sum_{j=1}^m |S_j| + \left(4(m-1)(1-r)\right) + 2(1-r) + 2(1-r) \\
&\leq \sum_{j=1}^m |S_j| + 4(1-r)m \\
&\leq \sum_{j=1}^m |S_j| + 2(1-r) \sum_{j=1}^m |S_j| \\
&= \left(1 + 2(1-r)\right) \left(\sum_{j=1}^m |S_j|\right)
\end{aligned}$$

Thus $\frac{|A_S|}{|A|} = \frac{\sum_{j=1}^m |S_j|}{|A|} \geq \frac{1}{1 + 2(1-r)}$, i.e., the group S covers a constant fraction of the total length. \square

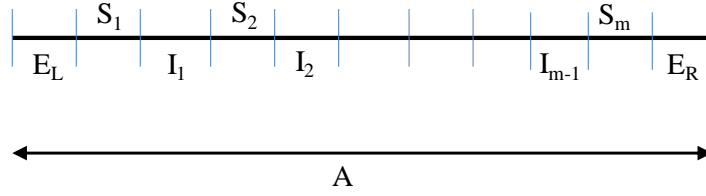


Figure 1: The segments covered by S are S_1, S_2, \dots, S_m . The internal gaps are I_1, I_2, \dots, I_{m-1} . The left and right external gaps are E_L and E_R respectively. The total length A stretches from the left endpoint of E_L to the right endpoint of E_R

5 The Euclidean d -dimensional Case

In this section we consider the case in which the players are located in a Euclidean d -dimensional space and each person covers a unit ball around himself. The next lemma bounds the ratio of volumes of the union of the two families of balls with the same set of centers but different radii.

Lemma 5.1. *Let A be a finite family of balls of radius one in a Euclidean d -dimensional space. Let B be a family of balls with the same set of the centers but radius $t \geq 1$. Let A_U, B_U denote the union of balls in A and B respectively. Then $\text{Vol}(B_U) \leq t^d \cdot \text{Vol}(A_U)$ where $\text{Vol}(A_U), \text{Vol}(B_U)$ denotes the volume of A_U and B_U respectively.*

Proof. Let $C = \{c_1, c_2, \dots, c_n\}$ be the centers of the balls in A . For $x \in B_U$ define $c(x) = \min\{d(c_j, x) \mid 1 \leq j \leq n\}$. Consider the partition P_1, P_2, \dots, P_n of B_U into n parts: $x \in B_U$ is placed in P_i if and only if $i = \min\{j \mid d(x, c_j) = c(x)\}$.

We claim $y \in P_j$ implies $[c_j, y] \in P_j$. Suppose there is a point $z \in [c_j, y]$ such that $z \in P_k$ for $k \neq j$. By the triangle inequality $d(c_k, y) \leq d(c_k, z) + d(z, y) \leq d(c_j, z) + d(z, y) = d(c_j, y)$ where we used $z \in P_k$ implies $d(c_k, z) \leq d(c_j, z)$. So $d(c_k, y) \leq d(c_j, y)$. But $y \in P_j$ implies $d(c_k, y) = d(c_j, y)$, $d(c_k, z) = d(c_j, z)$ and $j < k$. This contradicts the membership of z in P_k . So we can apply homothety: for every $1 \leq i \leq n$ we contract each P_i w.r.t point c_i by a factor of $\frac{1}{t}$ to get a region say P'_i . We note $P'_i \subseteq A_U$ as $x \in P_i$ implies $d(x, c_i) \leq t$ and if we denote by x' the point to which x is mapped under the contraction, then $d(x', c_i) = \frac{1}{t} \cdot d(x, c_i) \leq \frac{1}{t} \cdot t = 1$.

The next claim is $P'_i \cap P'_j = \emptyset$ for any $i \neq j$. Suppose not and say $y \in P'_i \cap P'_j$. Let y_i, y_j be the points in P_i and P_j respectively which get mapped to y under the contraction. Let $d(c_j, y) = \alpha$ and $d(c_i, y) = \beta$. By the triangle inequality we have $d(c_i, y_j) \leq d(c_i, y) + d(y, y_j) = \beta + (t-1)\alpha$. Also $y_j \in P_j$ implies $d(c_i, y_j) \geq d(c_j, y_j) = t\alpha$. So $\beta + (t-1)\alpha \geq d(c_i, y_j) \geq d(c_j, y_j) \geq t\alpha$, i.e., $\beta \geq \alpha$. Similarly we have $\alpha \geq \beta$ which implies $\alpha = \beta$. Therefore $t\alpha = d(c_j, y_j) \leq d(c_i, y_j) \leq d(c_i, y) + d(y, y_j) = \beta + (t-1)\alpha = t\alpha$. Equality in the triangle inequality gives c_i, y_j, y_i are on the same line and $d(c_i, y_j) = d(c_i, y_i)$ which implies $y_i = y_j$ which is a contradiction. So we have the following two conditions :

1. $P'_i \subseteq A_U$ for every $1 \leq i \leq n$.
2. $P'_i \cap P'_j = \emptyset$ for any $i \neq j$.

Therefore, $\text{Vol}(A_U) \geq \sum_{i=1}^n \text{Vol}(P'_i) = \frac{1}{t^d} \cdot \sum_{i=1}^n \text{Vol}(P_i) = \frac{1}{t^d} \cdot \text{Vol}(B_U)$. We note the bound is tight when all the balls in B are disjoint. \square

We now prove a generalization of Lemma 4.1 for a Euclidean d -dimensional space. Let V_d denote the volume of a unit ball in the Euclidean d -dimensional space.

Lemma 5.2. *Let S be a richest group in a Nash Equilibrium. There is no player $i \notin S$ who can add a volume of at least $(1-r)V_d$ to the volume A_S currently covered by S .*

Proof. Suppose there is a player $i \notin S$, who can add a volume of at least $(1-r)V_d$ to the volume covered by S . However, since it is a Nash Equilibrium, either the new expected utility of S on adding this player is less than or equal to the current expected utility of S , hence S would have no incentive in adding the player i . Or else the player i would not have any incentive to move to S , as the projected new expected utility of i is less than or equal to his current expected utility. Since S is a richest group, both these conditions combine to give:

$$\left(\frac{M}{|S|+1} \right)^{1-r} \cdot \frac{A_S + (1-r)V_d}{A} \leq \left(\frac{M}{|S|} \right)^{1-r} \cdot \frac{A_S}{A} \quad (2)$$

As each player has a coverage volume of V_d we have $|S|V_d \geq A_S$ (with equality only if the coverage volumes of the members of S are pairwise disjoint). The function $f(x) = \frac{x}{x+1}$ is increasing on $(0, \infty)$ and hence $\frac{|S|}{|S|+1} \geq \frac{\beta}{\beta+1}$ where $\beta = \frac{A_S}{V_d}$. Combining with Equation 2 gives

$$\left(\frac{\beta}{\beta+1} \right)^{1-r} \leq \left(\frac{|S|}{|S|+1} \right)^{1-r} \leq \frac{A_S}{A_S + (1-r)V_d}$$

Rearranging and setting $1-r = \frac{1}{t}$ implies

$$\frac{A_S + (1-r)V_d}{A_S} \leq \left(\frac{\beta+1}{\beta} \right)^{\frac{1}{t}} \implies \left(1 + \frac{(1-r)V_d}{A_S} \right)^t \leq 1 + \frac{1}{\beta}$$

Bernoulli's inequality states if $x, q \in \mathbb{R}$ and $x > -1, q > 1$ then $(1+x)^q > 1+qx$. Applying the inequality for $x = \frac{(1-r)V_d}{A_S}$ and $q = t = \frac{1}{1-r} > 1$ gives

$$1 + \frac{1}{\beta} = 1 + \frac{t(1-r)V_d}{A_S} < \left(1 + \frac{(1-r)V_d}{A_S} \right)^t \leq 1 + \frac{1}{\beta}$$

which is a contradiction. \square

We first give a simple proof that if S is any richest group in a Nash Equilibrium then $\frac{A_S}{A} \geq \frac{1}{3^d}$ where A_S is the volume covered by the group S and A is the total volume.

Theorem 5.3. *If the players are located in a d -dimensional Euclidean space then in any Nash Equilibrium every richest group covers at least a $\frac{1}{3^d}$ -fraction of the total volume.*

Proof. Let S be a richest group in a Nash Equilibrium and consider a player $x \notin S$. There must exist a member of $s \in S$ such that the distance between x and s is at most two. Otherwise the coverage ball of x is disjoint from the coverage ball of S and thus can contribute a volume of $V_d > (1-r)V_d$ which contradicts Lemma 5.2. So the total volume A is covered by the volume A'_S of the union of the family of balls of radius three centered at the members of S . By Lemma 5.1 we have $\frac{A_S}{A} \geq \frac{A_S}{A'_S} \geq \frac{1}{3^d}$. \square

The above bound of $\frac{1}{3^d}$ is independent of the risk aversion factor r . We now give a better bound which depends on r . To this end, in the following lemma, we show how to bound the volume of intersection of two unit balls in a Euclidean d -dimensional space in terms of the distance between their centers.

Lemma 5.4. *Let B_1 and B_2 be two unit balls in a Euclidean d -dimensional space. For $a \leq 1$, if the distance between the centers of B_1 and B_2 is $2a$, then the volume of intersection of B_1 and B_2 is at most $2(1-a^2)^{\frac{d-1}{2}} \cdot V_{d-1}$ where V_{d-1} is the volume of a unit ball in a Euclidean $(d-1)$ -dimensional space.*

Proof. We assume $a \leq 1$ otherwise the area of intersection is clearly zero. Let c_1, c_2 be the centers of B_1 and B_2 respectively. Then $B_1 \cap B_2$ is contained in the cylinder (say B) of radius $\sqrt{1-a^2}$ and height $2a$ centered at the midpoint of the segment joining c_1 and c_2 . So $\text{Vol}(B_1 \cap B_2) \leq \text{Vol}(B) = 2a(1-a^2)^{\frac{d-1}{2}} V_{d-1} \leq 2(1-a^2)^{\frac{d-1}{2}} \cdot V_{d-1}$ as $a \leq 1$. \square

The next lemma gives a lower bound on the ratio of the volumes of unit balls in Euclidean spaces of consecutive dimensions.

Lemma 5.5. *For $d \geq 2$ let V_{d-1}, V_d be the volumes of unit balls in the $d-1$ and d -dimensional Euclidean spaces respectively. Then $\frac{V_d}{V_{d-1}} \geq \frac{1}{d}$.*

Proof. We prove by induction on d . Base case is $d=2$ and $\frac{V_2}{V_1} = \frac{\pi}{2} \geq \frac{1}{2}$. We use the well-known recurrence relation for V_n : $V_n = \frac{2\pi}{n} \cdot V_{n-2}$. Suppose the hypothesis is true for all $k \leq n-1$. Then we have $\frac{V_n}{V_{n-1}} = \frac{n-1}{n} \cdot \frac{V_{n-2}}{V_{n-3}} \geq \frac{n-1}{n} \cdot \frac{1}{n-2} > \frac{1}{n}$ and so the hypothesis holds true for all $d \geq 2$. \square

We are now ready to give a better bound than $\frac{1}{3^d}$ on the fraction of the total volume covered by any richest group in a Nash Equilibrium.

Theorem 5.6. *If the players are located in a d -dimensional Euclidean space, then in any Nash Equilibrium there always is a group which covers at least a $\frac{1}{(2\delta+1)^d}$ -fraction of the total volume where $\delta = \sqrt{1 - (\frac{r}{2d})^{\frac{2}{d-1}}}$. We note $\frac{1}{(2\delta+1)^d} > \frac{1}{3^d}$ as $\delta < 1$ and therefore this improves on Theorem 5.3.*

Proof. Consider a richest group S in a Nash Equilibrium. By Lemma 5.2, no player outside of S can get his coverage ball to contribute at least $(1-r)V_d$ volume to S , i.e., for every $x \notin S$ there is a player $s \in S$ such that volume of intersection of balls B_x, B_s of x and s respectively is at least rV_d . Let the distance between centers of B_x and B_s be $2a$. Lemma 5.4 gives $2(1-a^2)^{\frac{d-1}{2}} V_{d-1} \geq \text{Vol}(B_s \cap B_x) \geq r \cdot V_d$ which implies $2(1-a^2)^{\frac{d-1}{2}} \geq r \frac{V_d}{V_{d-1}} \geq \frac{r}{d}$ by Lemma 5.5. Rearranging we get $a \leq \sqrt{1 - (\frac{r}{2d})^{\frac{2}{d-1}}} = \text{say } \delta$. So each player not in S is at a distance of at most 2δ from some player of S . Therefore the total volume A is covered by the volume A'_S of the union of the family of balls of radius $2\delta+1$ centered at members of S . By Lemma 5.1 we have $\frac{A_S}{A} \geq \frac{A_S}{A'_S} \geq \frac{1}{(2\delta+1)^d}$. \square

6 The Graph Case

In this section we consider the discrete version of the problem where players form the vertex set of an undirected graph. The coverage of a vertex is its closed neighborhood, i.e., a vertex covers itself and all its neighbors. We assume the same utility function as before: Each member x belonging to a group S has expected utility given by $E[u(x)] = \left(\frac{M}{|S|}\right)^{1-r} \cdot \frac{|A_S|}{|A|}$ where M is the total money, A_S is the union of the closed neighborhoods of the vertices in S and A is the vertex set of the graph. We first show a preliminary lemma which bounds the contribution to a richest group in a Nash Equilibrium by any vertex which is not in the richest group. This lemma can be viewed as a discrete version of Lemma 5.2.

Lemma 6.1. *Let $G = (V, E)$ be an undirected graph with maximum degree Δ . Let S be a richest group in a Nash Equilibrium. Then there is no player $i \notin A_S$ who can add at least $(1-r)(\Delta+1)$ vertices to the set A_S currently covered by S .*

Proof. Suppose a player $i \notin A_S$ can add at least $(1-r)(\Delta+1)$ vertices to the set A_S currently covered by S . However, since it is a Nash Equilibrium, either the new expected utility of S on adding this player is less than or equal to the current expected utility of S , hence S would have no incentive in adding the player i . Or else the player i would not have any incentive to move to S , as the projected new expected utility of i is less than or equal to its current expected utility. Since S is a richest group, both these conditions combine to give:

$$\left(\frac{M}{|S|+1}\right)^{1-r} \cdot \frac{A_S + (1-r)(\Delta+1)}{A} \leq \left(\frac{M}{|S|}\right)^{1-r} \cdot \frac{A_S}{A} \quad (3)$$

As each player has degree at most Δ we have $|S|(\Delta+1) \geq A_S$ (with equality only if the closed neighborhoods of the vertices of S are pairwise disjoint). Since $f(x) = \frac{x}{x+1}$ is an increasing function we have $\frac{|S|}{|S|+1} \geq \frac{\beta}{\beta+1}$ where $\beta = \frac{A_S}{\Delta+1}$. Combining with Equation 3 gives

$$\left(\frac{\beta}{\beta+1}\right)^{1-r} \leq \left(\frac{|S|}{|S|+1}\right)^{1-r} \leq \frac{A_S}{A_S + (1-r)(\Delta+1)}$$

Rearranging and setting $1-r = \frac{1}{t}$ implies

$$\frac{A_S + (1-r)(\Delta+1)}{A_S} \leq \left(\frac{\beta+1}{\beta}\right)^{\frac{1}{t}} \implies \left(1 + \frac{(1-r)(\Delta+1)}{A_S}\right)^t \leq 1 + \frac{1}{\beta}$$

Bernoulli's inequality states if $x, q \in \mathbb{R}$ and $x > -1, q > 1$ then $(1+x)^q > 1+qx$. Applying the inequality for $x = \frac{(1-r)(\Delta+1)}{A_S}$ and $q = t = \frac{1}{1-r} > 1$ gives

$$1 + \frac{1}{\beta} = 1 + \frac{t(1-r)(\Delta+1)}{A_S} < \left(1 + \frac{(1-r)(\Delta+1)}{A_S}\right)^t \leq 1 + \frac{1}{\beta}$$

which is a contradiction. \square

In the next theorem we show if the topology is the class of bounded-degree regular graphs, then in any Nash Equilibrium there always exists a group which covers a constant fraction of the total number of vertices.

Theorem 6.2. *Let $G = (A, E)$ be a f -regular graph. In any Nash Equilibrium there always exists a group covering a constant fraction of the total number of vertices where the constant is $\frac{1}{\frac{f-1}{r(f+1)} + 1}$.*

Proof. Consider a richest group S in a Nash Equilibrium. Let A_S be the vertices covered by S , i.e., A_S is the union of the closed neighborhoods of the vertices of S . Denote by $\overline{A_S}$ the set $A \setminus A_S$. Since the graph is f -regular the size of the closed neighborhood of every vertex is $f+1$. By Lemma 6.1, every vertex $x \notin A_S$ must add less than $(1-r)(f+1)$ vertices to the set A_S . So each vertex in $\overline{A_S}$ has at least $(f+1) - (1-r)(f+1) = r(f+1)$ neighbors in A_S . Let β be the number of edges with one endpoint in A_S and one endpoint in $\overline{A_S}$. Thus $\beta \geq r|\overline{A_S}|(f+1)$. By the definition of A_S as the union of the closed neighborhoods of vertices of S , only the vertices from $A_S \setminus S$ can have edges to $\overline{A_S}$. Each vertex of $A_S \setminus S$ has at least one neighbor in S and so $\beta \leq |A_S|(f-1)$. Combining the two bounds we have $|A_S|(f-1) \geq \beta \geq r|\overline{A_S}|(f+1) = r(|A| - |A_S|)(f+1)$. Letting $\mu = \frac{f-1}{r(f+1)}$ we have $\mu|A_S| \geq |A| - |A_S|$, i.e., $\frac{|A_S|}{|A|} \geq \frac{1}{\mu+1} = \frac{1}{\frac{f-1}{r(f+1)} + 1}$ \square

The general graph case does not seem to be hopeful. Recall in all the three topologies (the one-dimensional (line) space, the d -dimensional Euclidean space and the bounded-degree regular graphs) considered so far, we were able to show the surprising phenomenon that any *richest* group in a Nash Equilibrium covers a constant fraction of the total volume/vertices. We show this approach fails for general graphs, i.e., there exist graphs having a Nash Equilibrium in which no richest group covers a constant fraction of the total number of vertices.

Theorem 6.3. *There exist graphs which have a Nash Equilibrium in which no richest group covers a constant fraction of the total number of vertices.*

Proof. Consider the family of graphs G_z where z is a parameter satisfying the equation

$$\left(\frac{M}{1}\right)^{1-r} \cdot \frac{12z^r}{|G_z|} > \left(\frac{M}{2}\right)^{1-r} \cdot \frac{12z^r + 4}{|G_z|} \quad , i.e., \quad z^r > \frac{1}{3(2^{1-r} - 1)} \quad (4)$$

We now describe the graph G_z : it contains a clique K of size $12z^r$. We say these vertices are of Type I. A path P of length $3z$ is attached to a vertex (say v) of Type I. We call the vertices of the path P (excluding v) as the vertices of Type II.

First we show no two vertices x, y of Type I merge. If $v \notin \{x, y\}$ then x and y will not merge as they both have the same coverage. So without loss of generality let $x = v$. Then the initial expected utility of y is $\left(\frac{M}{1}\right)^{1-r} \cdot \frac{12z^r}{|G_z|}$ and the expected utility of the group $\{x, y\}$ is $\left(\frac{M}{2}\right)^{1-r} \cdot \frac{12z^r + 1}{|G_z|}$. Equation 4 implies $\left(\frac{M}{1}\right)^{1-r} \cdot \frac{12z^r}{|G_z|} > \left(\frac{M}{2}\right)^{1-r} \cdot \frac{12z^r + 4}{|G_z|} > \left(\frac{M}{2}\right)^{1-r} \cdot \frac{12z^r + 1}{|G_z|}$. Hence no two vertices of Type I merge.

We next show no vertex p of Type I merges with a vertex q of Type II. The initial expected utility of p is at least $\left(\frac{M}{1}\right)^{1-r} \cdot \frac{12z^r}{|G_z|}$ with equality if $p \neq v$. The expected utility of the group $\{p, q\}$ will be at most $\left(\frac{M}{2}\right)^{1-r} \cdot \frac{12z^r + 4}{|G_z|}$ as any vertex of Type II can add at most 3 vertices to coverage of a vertex of Type I. Equation 4 implies no vertex of Type I will merge with a vertex of Type II.

So a group in any Nash Equilibrium has to either be a single vertex of Type I or a set of Type II vertices. In the first case the maximum expected utility will be for the group formed by v alone and is given by

$$\left(\frac{M}{1}\right)^{1-r} \cdot \frac{12z^r + 1}{|G_z|} = \text{say } U_1 \quad (5)$$

For the second case if the group consist of b vertices of Type II then the expected utility of this group is $\left(\frac{M}{b}\right)^{1-r} \cdot \frac{C}{|G_z|} = \text{say } U_2$ where C is the coverage of the group. The coverage of any vertex of Type II is at most three implies $C \leq 3b$ with equality only if the coverages of the members of the group are pairwise

disjoint. So $U_2 = \left(\frac{M}{b}\right)^{1-r} \cdot \frac{C}{|G_z|} \leq \left(\frac{M}{b}\right)^{1-r} \cdot \frac{3b}{|G_z|} = (M)^{1-r} \cdot \frac{3b^r}{|G_z|} \leq (M)^{1-r} \cdot \frac{3(3z)^r}{|G_z|} < \left(\frac{M}{1}\right)^{1-r} \cdot \frac{12z^r+1}{|G_z|} = U_1$ as $b \leq 3z$ and $r \in (0, 1)$. Therefore the only richest group in any Nash Equilibrium in G_z is the group $\{v\}$. The fraction of vertices covered by v is $\frac{12z^r+1}{12z^r+3z} < \frac{12z^r+3z^r(2^{1-r}-1)}{12z^r+3z} = \frac{4+(2^{1-r}-1)}{4+z^{1-r}} = \frac{3+2^{1-r}}{3+z^{1-r}}$ which tends to 0 as we increase z since $r \in (0, 1)$ (Equation 4 only imposed a lower bound on z and hence there is no issue with increasing z arbitrarily). So there exist graphs which have a Nash Equilibrium in which no richest group covers a constant fraction of the total number of vertices. \square

Theorem 6.3 implies we need different techniques than the ones used above to resolve the general graph case. However, under the assumption that *defection to an empty group* is not allowed, we can show given any constant $c < 1$ there exists a graph G_c and a Nash Equilibrium in G_c such that each group in the Nash Equilibrium covers strictly less than a c -fraction of the total number of vertices. We now explicitly construct such graphs. Consider the family of graphs $G_{k,\ell}$: it has a clique of size k formed by the vertices $\{v_1, v_2, \dots, v_k\}$. We call these vertices as the **primary** vertices. Each primary vertex v_i has ℓ leaves attached to it. We denote these **secondary** vertices attached to the primary vertex v_i by $L(v_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,\ell}\}$. We note $|G_{k,\ell}| = k + k\ell$.

Lemma 6.4. *If k and ℓ satisfy*

$$k > \ell \left(\frac{2-f(\ell)}{f(\ell)-1} \right) \quad \text{where} \quad f(\ell) = \left(1 + \frac{1}{\ell+1} \right)^{1-r} \quad (6)$$

then the groups S_1, S_2, \dots, S_k defined by $S_i = L(v_i) \cup \{v_i\}$ form a Nash Equilibrium for the graph $G_{k,\ell}$.

Proof. The function $g(x) = \frac{2-x}{x-1}$ is decreasing in the interval $(1, 2^{1-r})$. Also $f(\ell) < 2^{1-r}$ implies

$$g(2^{1-r}) < g(f(\ell)) \quad , i.e., \quad g(2^{1-r}) = \frac{2-2^{1-r}}{2^{1-r}-1} < \frac{2-f(\ell)}{f(\ell)-1} < \frac{k}{\ell} \quad (7)$$

where the last step follows from Equation 6. The quantity $g(2^{1-r})$ is a constant and we denote it by say γ .

We show no two of the groups S_1, S_2, \dots, S_k will merge. The current expected utility of any S_i is $\left(\frac{M}{\ell+1}\right)^{1-r} \cdot \frac{k+\ell}{k+k\ell} = \text{say } u_1$ and if any S_i and S_j merge then the expected utility of the new merged group will be $\left(\frac{M}{2(\ell+1)}\right)^{1-r} \cdot \frac{k+2\ell}{k+k\ell} = \text{say } u_2$. Now

$$\begin{aligned} u_1 > u_2 &\Leftrightarrow \left(\frac{M}{\ell+1}\right)^{1-r} \cdot \frac{k+\ell}{k+k\ell} > \left(\frac{M}{2(\ell+1)}\right)^{1-r} \cdot \frac{k+2\ell}{k+k\ell} \\ &\Leftrightarrow 2^{1-r}(k+\ell) > (k+2\ell) \\ &\Leftrightarrow k > \ell \left(\frac{2-2^{1-r}}{2^{1-r}-1} \right) = \ell \cdot g(2^{1-r}) \end{aligned}$$

which follows from Equation 7. So no two of the groups S_1, S_2, \dots, S_k will merge.

We next show no player will defect from one group to another. The number of players in the new group will be $\ell+2$ and it will have more coverage if a primary vertex defects rather than a secondary vertex. So it is enough to prove the defection of a primary vertex to another group is not possible. Suppose the primary vertex v_i defects to join the group S_j . The initial expected utility of v_i is $\left(\frac{M}{\ell+1}\right)^{1-r} \cdot \frac{k+\ell}{k+k\ell} = \text{say } u_3$ and the

expected utility of the group $\{v_i\} \cup S_j$ is $\left(\frac{M}{\ell+2}\right)^{1-r} \cdot \frac{k+2\ell}{k+k\ell} = \text{say } u_4$. Now

$$\begin{aligned} u_3 > u_4 &\Leftrightarrow \left(\frac{M}{\ell+1}\right)^{1-r} \cdot \frac{k+\ell}{k+k\ell} > \left(\frac{M}{\ell+2}\right)^{1-r} \cdot \frac{k+2\ell}{k+k\ell} \\ &\Leftrightarrow \left(\frac{\ell+2}{\ell+1}\right)^{1-r} \cdot (k+\ell) > (k+2\ell) \\ &\Leftrightarrow f(\ell) \cdot (k+\ell) > (k+2\ell) \\ &\Leftrightarrow k > \ell \left(\frac{2-f(\ell)}{f(\ell)-1} \right) \end{aligned}$$

which holds by Equation 6. Therefore no defection will take place. Since no merging or defection can occur, the groups S_1, S_2, \dots, S_k form a Nash Equilibrium in the graph $G_{k,\ell}$. \square

Equations 6 and 7 (given in proof of Lemma 6.4) do not impose any absolute upper bounds on k or ℓ . We use this fact in the following theorem to show for every positive constant $c < 1$ there exists a graph G_c and a Nash Equilibrium in G_c such that no group in the Nash Equilibrium covers at least a c -fraction of the total number of vertices.

Theorem 6.5. *Given any positive constant $c < 1$, under the assumption that defecting to an empty group is not allowed, there exists a graph G_c and a Nash Equilibrium in G_c such that each group in the Nash Equilibrium covers strictly less than a c -fraction of the total number of vertices.*

Proof. We set G_c to be the graph $G_{k,\ell}$ where k, ℓ will be determined later. Consider the Nash Equilibrium in G_c given by the groups S_1, S_2, \dots, S_k in Lemma 6.4. The fraction of the total number of vertices covered by any group S_i is given by $\frac{k+\ell}{k+k\ell} = \frac{1+\frac{\ell}{k}}{1+\ell} < \frac{1+\frac{1}{\gamma}}{1+\ell}$ where the last step follows from Equation 7. Recalling $\gamma = g(2^{1-r}) = \frac{2-2^{1-r}}{2^{1-r}-1}$ is a constant, we choose ℓ large enough so $c > \frac{1+\frac{1}{\gamma}}{1+\ell}$ which proves our theorem. We need to choose k large enough to satisfy Equation 6 but this is not an issue as we do not have any absolute upper bound constraints on either k or ℓ . \square

7 Conclusions and Open Problems

In this paper we have suggested a game-theoretic model motivated by the DARPA Network Challenge. We analyze the structures of the groups in Nash equilibria. We show for various topologies: a one-dimensional space (line), a d -dimensional Euclidean space, and bounded-degree regular graphs; in any Nash Equilibrium there always exists a group which covers a constant fraction of the total volume. The objective of events like the DARPA Network Challenge is to mobilize a large number of people quickly so that they can cover a big fraction of the total area. Our results suggest that this objective can be met under certain conditions.

However our ideas however do not generalize to all the graphs and we provide explicit examples of graphs for which our techniques fail. Under an additional assumption that defecting to an empty group is not allowed, we show given any constant $c < 1$ there exists a graph G_c and a Nash Equilibrium in G_c where each group in the Nash Equilibrium covers strictly less than a c -fraction of the total number of vertices. It would be interesting to prove Theorem 6.5 without the assumption that defecting to an empty group is prohibited.

References

- [1] The 2009 DARPA Network Challenge (<http://archive.darpa.mil/networkchallenge/>).
URL <http://archive.darpa.mil/networkchallenge/>
- [2] N. Alon, E. D. Demaine, M. Hajiaghayi, T. Leighton, Basic Network Creation Games, in: SPAA, 2010.
- [3] M. Babaioff, S. Dobzinski, S. Oren, A. Zohar, On bitcoin and red balloons, in: ACM Conference on Electronic Commerce, 2012.
- [4] A. Bhargat, T. Chakraborty, S. Khanna, Mechanism design for a risk averse seller, in: WINE, 2012.
- [5] M. Cebrián, L. Coviello, A. Vattani, P. Voulgaris, Finding red balloons with split contracts: robustness to individuals' selfishness, in: STOC, 2012.
- [6] S. Cooper, F. Khatib, A. Treuille, J. Barbero, J. Lee, M. Beenen, A. Leaver-Fay, D. Baker, Z. Popovic, Predicting protein structures with a multiplayer online game, *Nature* 466 (7307) (2010) 756–760.
- [7] E. D. Demaine, M. Hajiaghayi, H. Mahini, M. Zadimoghaddam, The Price of Anarchy in Cooperative Network Creation Games, in: STACS, 2009.
- [8] J. Douceur, T. Moscibroda, Lottery Trees: Motivational Deployment of Networked Systems, *ACM SIGCOMM Computer Communication Review* 37 (4) (2007) 121–132.
- [9] M. Finus, B. Rundshagen, How the rules of coalition formation affect stability of international environmental agreements, *Working Papers* 2003.62, Fondazione Eni Enrico Mattei (Jul. 2003).
URL <http://ideas.repec.org/p/fem/femwpa/2003.62.html>
- [10] C. A. Holt, S. K. Laury, Risk Aversion and Incentive Effects, *The American Economic Review* 92 (5) (2002) 1644–1655.
- [11] J. Howe, *Crowdsourcing: Why the Power of the Crowd Is Driving the Future of Business*, Crown Business, 2008.
- [12] E. Koutsoupias, C. H. Papadimitriou, Worst-case Equilibria, in: STACS, 1999.
- [13] G. Krakow, Ham radio operators to the rescue after Katrina.
URL <http://www.msnbc.msn.com/id/9228945/>
- [14] D. K. Lambert, B. A. McCarl, Risk Modeling Using Direct Solution of Nonlinear Approximations of the Utility Function, *American Journal of Agricultural Economics* 67 (4) (1985) pp. 846–852.
- [15] T. Mohin, .
URL <http://www.forbes.com/sites/forbesleadershipforum/2012/01/18/the-top-10-trends-in-csr-for-2012/>
- [16] R. Myerson, Graphs and Cooperation in Games, *Mathematics of Operations Research* 2 (3) (1977) 225–229.
- [17] C. H. Papadimitriou, Algorithms, games, and the internet, in: STOC, 2001.
- [18] G. Pickard, I. Rahwan, W. Pan, M. Cebrián, R. Crane, A. Madan, A. Pentland, Time Critical Social Mobilization: The DARPA Network Challenge Winning Strategy, *CoRR* abs/1008.3172.

- [19] J. Pontin, Artificial Intelligence, With Help From the Humans , The New York Times.
URL <http://www.nytimes.com/2007/03/25/business/yourmoney/25Stream.html>
- [20] J. Pratt, Risk aversion in the small and in the large, *Econometrica: Journal of the Econometric Society* (1964) 122–136.
- [21] T. Roughgarden, *Selfish Routing and the Price of Anarchy*, The MIT Press, 2005.
- [22] J. Tobin, Liquidity Preference as Behavior Towards Risk, *The Review of Economic Studies* 25 (2) (1958) pp. 65–86.
- [23] L. von Ahn, Games with a Purpose, *IEEE Computer* 39 (6) (2006) 92–94.

A Omitted Proofs from Section 3

Proof of Theorem 3.1

Proof. Let S be a group which covers volume $A_S = \lambda A$, where A is the total volume. Since each member of S contributes volume 1 we have $|S| \geq A_S \geq \lambda A \geq \lambda^{\frac{1}{1-r}+1} A$ (because $\lambda \in (0, 1)$). If each player in S has expected utility at least c then $c \leq E[u(x)] \leq \lambda \cdot \left(\frac{M}{|S|}\right)^{1-r}$, which implies $\left(\frac{M}{|S|}\right)^{1-r} \geq \frac{c}{\lambda}$. So $M \geq |S| \left(\frac{c}{\lambda}\right)^{\frac{1}{1-r}} \geq \lambda^{\frac{1}{1-r}+1} A \left(\frac{c}{\lambda}\right)^{\frac{1}{1-r}} = \lambda A c^{\frac{1}{1-r}}$, proving the lemma. \square

Proof of Theorem 3.2

Proof. Let S be a group of size k . Then we have $N \geq |S| = k$ where N is total number of players. Since each player can contribute at most 1 exclusive volume we have $A_S \leq |S| \leq N$ where A_S is volume covered by S . If $x \in S$ then we have $c \leq E[u(x)] = \left(\frac{M}{|S|}\right)^{1-r} \cdot \frac{A_S}{A} \leq \left(\frac{M}{|S|}\right)^{1-r} \cdot \frac{N}{A}$. Therefore we have $\left(\frac{M}{|G|}\right)^{1-r} \geq \frac{cA}{N}$ which implies $M \geq |G| \left(\frac{cA}{N}\right)^{\frac{1}{1-r}} = k \left(\frac{cA}{N}\right)^{\frac{1}{1-r}}$. \square