Pricing Public Goods for Private Sale

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Abstract

We consider the pricing problem faced by a seller who assigns a price to a good that confers its benefits not only to its buyers, but also to other individuals around them. For example, a snow-blower is potentially useful not only to the household that buys it, but also to others on the same street. Given that the seller is constrained to selling such a (locally) public good via individual private sales, how should he set his prices given the distribution of values held by the agents?

We study this problem as a two-stage game. In the first stage, the seller chooses and announces a price for the product. In the second stage, the agents (each having a private value for the good) decide simultaneously whether or not they will buy the product. In the resulting game, which can exhibit a multiplicity of equilibria, agents must strategize about whether they will themselves purchase the good to receive its benefits.

In the case of a fully public good (where all agents benefit whenever any agent purchases), we describe a pricing mechanism that is approximately revenue-optimal (up to a constant factor) when values are drawn from a regular distribution. We then study settings in which the good is only "locally" public: agents are arranged in a network and share benefits only with their neighbors. We describe a pricing method that approximately maximizes revenue, in the worst case over equilibria of agent behavior, for any d-regular network. Finally, we show that approximately optimal prices can be found for general networks in the special case that private values are drawn from a uniform distribution. We also discuss some barriers to extending these results to general networks and regular distributions.

1 Introduction

Pricing products for sale is an important strategic decision for firms. Based on the demand at different prices, an optimal price should maximize the number of items sold, times the revenue per sold item. A long history of work in economics, and more recently in computer science, studies the problem of finding an optimal price (or, more generally, selling mechanism), given a demand curve or estimate thereof [30, 29, 26, 18].

This view ignores the fact that products frequently exhibit externalities: if a consumer j purchases the product, it may affect the utility of consumer i. These externalities naturally differ in two dimensions: (1) whether they are positive or negative, and (2) whether they affect other consumers when they purchase the product, or when they do not purchase it.

Some of the classical literature in economics [21, 22, 23, 24, 10] focuses on negative externalities experienced by consumer i as a result of j's purchase, regardless of whether i himself purchases. Motivating examples are weapons or powerful competitive technologies. If a competitor j has access to these technologies, it poses an often significant threat to i, and hence, i would be willing to pay significant amounts of money to prevent j from acquiring the product. There has been a recent focus on positive externalities between pairs i, j when both purchase [19, 5, 1, 4, 16, 7]. This type of scenario arises, for instance, for implicit creation of technology standards, where the use of a particular technology (such as an operating system or cell phone plan) becomes more advantageous as others use the same technology. In this context, the focus is often on finding the right "seeds" to create enough implicit peer influence effects; de facto, some users are offered much lower prices to serve as seeds.

In the present paper, we investigate important domains of externalities, and the impact they have on pricing decisions. Our main focus is on *positive externalities from purchasers on non-purchasers*. In other words, when one customer purchases an item, others will derive utility from it, even if they themselves do not purchase it. This is the case commonly known as *public goods* in economics [32, 6, 27]. Public goods arise in many real-world scenarios:

1. If one researcher acquires a useful piece of infrastructure (such as a poster printer), other research groups in the same department profit as well.

2. If one family purchases a useful and expensive gardening tool, its neighbors can borrow the tool and use it as well.

3. If a company finances useful infrastructure in a region, it also makes the region more attractive for other companies. One concrete example is the Wi-Fi networks that Google recently built in Chelsea and in Kansas City [28], which are expected to attract more talent to those areas.

Since the goods described above benefit an entire group of agents, one way of purchasing them would be to gather as a group, purchase a single copy, and split its cost among the group members. This is, however, not always possible due to various reasons: in case (1), regulations might allow a researcher to pay for a printer from his grant budget, but not to pay for it partially; in case (2), the family might consider it impolite to ask each potential borrower of the gardening tool to contribute to it; and in case (3), the companies that will benefit from the infrastructure being in place might be competitors and therefore might be unwilling to cooperate. More fundamentally, it has been long known that rational agents in these types of settings have incentives to misrepresent their true utilities (see, e.g., [32]), a phenomenon colloquially known as free-riding. Hence, it is very common that, despite the public-good nature of these goods, purchases are made privately; that is, one agent purchases the good, incurring the entire cost alone, while benefiting the group as a whole. It is crucial for a seller who is offering the product for sale to take these externality considerations into account.[1] Overall, we would expect the demand for such items to be reduced given that the buyers, taken as a whole, will demand fewer copies.

We model the locally public nature of the good as follows. We consider a graph G that captures the interactions between the buyers. Each buyer has a non-negative valuation drawn independently from a distribution F common to all buyers. If the buyer or one of his neighbors purchases the good, he obtains his valuation as utility; if he was the one purchasing it, then the good's price (set by the seller) is subtracted from his utility. We study the Nash Equilibria of the game described above and the problem faced by the seller of setting a price (based on the graph and the valuation distribution) in order to optimize the revenue at equilibrium.

In investigating this question, we are interested in understanding the influence of the different parameters on the optimal pricing choice. For example, how does the optimal price depend on the topology of the network G? Since it is usually hard for the seller to learn the buyers' social network, is it possible to find a price that will generate approximately optimal revenue for any network; or, if not, a price that depends only on simple statistics about the network, such as its average degree? We are also interested in investigating the power of discriminatory vs. non-discriminatory pricing. Can the seller benefit from setting a different price for each agent? Is there a non-discriminatory price that gives a good approximation with respect to every discriminatory pricing policy?

Negative externalities and the Hipster Game Our framework can also be used to study other types of externalities. Consider a product that serves the role of a fashion statement or status symbol. In that case, it may be essential to the purchaser to be the *only* one with a copy. His utility is, therefore, his valuation if he has the product and no other agent in his neighborhood has it. Otherwise, his utility is zero. We call this the *Hipster Game*. We show that the pricing problem in the Hipster Game is analogous to the problem for public goods, and the same algorithmic and analysis techniques yield essentially identical results.

1.1 Our Contribution

Globally public goods We begin our study by focusing on the complete graph, i.e., the case of *globally public goods*. We are interested in prices which will yield high revenue at equilibrium. One immediate obstacle in this context is that the (Bayesian) purchasing game played by the buyers may have (infinitely) many equilibria. We show that nonetheless, there is a single price p which can be computed explicitly from the agents' value distribution, and which is approximately optimal in the following very strong sense: The revenue under the worst-case equilibrium at price p is within a constant factor of the revenue of the *best* equilibrium for the best general (not necessarily uniform) price vector. In other words, price discrimination can improve revenue by at most a constant factor, even if one is optimistic about the equilibrium that will be reached.

Our analysis draws a relation between our problem and the optimal (Myerson) revenue of a single-item auction among n bidders. The main insight driving our result is that, at equilibrium, the agents aim to make purchasing decisions so that only one agent will buy the product, in expectation. This connection allows us to leverage the rich literature on single-item auctions for our analysis; it also explicates the connection to the Hipster Game, where positive utility can only be derived when exactly one agent obtains the good.

[1] The examples listed above can be considered nearly *pure* public goods, in that the benefits from being the purchaser and being a "neighbor" are very similar. A much larger number of products — such as most entertainment technology — has a significant public component, but also a significant private component. We discuss this interesting extension as a direction for future work in Section 5.

Locally public goods With a solid understanding of globally public goods in place, we next turn our attention to *locally public goods*, which are modeled by arbitrary networks G. At this point, we cannot answer the question of finding optimal prices for arbitrary G. However, we make significant progress on the question, as follows.

First, we consider the case of d-regular graphs G. Here, the results on globally public goods carry over in spirit. However, technically, the assertion is weaker: we show how to explicitly compute a uniform price p which, when offered to all the agents, is guaranteed to obtain a constant fraction of the *worst-case revenue* for any fixed price p'. Remarkably, this price depends only on the degree d and the distribution F, and is independent of the actual graph structure. Notice that the guarantee is weaker than the one for globally public goods in two respects: (1) it only provides guarantees compared to one fixed price, not a price vector with discrimination, and (2) it compares only to the worst-case revenue for these other prices p' (instead of the best-case one). This weaker assertion is inevitable: we show that there exist d-regular graphs in which the gap between the best worst-case revenue and the best revenue in equilibrium is $\Theta(d)$, and similarly, the gap between the worst-case revenue of the best uniform price vector and the best discriminating prices is $\Theta(d)$ as well.

We next consider the case of general graphs. We present evidence that our previous approaches will face inherent difficulties in handling general graphs. In particular, we give an instance of a network such that, for every price, the gap between the best-case and worst-case revenue is $\Theta(n)$. Therefore, approximate optimality of worst-case equilibria cannot be established by bounding best-case revenues. At a minimum, this raises an equilibrium selection problem: which is the right revenue to optimize, and to compare against?

For d-regular graphs, our solution concept is to bound worst-case revenue for the price against the worst-case revenue at other prices p'. We show that for general graphs, this approach faces a fundamental obstacle: approximating worst-case revenue to within a factor $n^{1-\epsilon}$ for a given price is NP-hard, even if F is the uniform distribution. Thus, we do not expect a concise or useful characterization of the approximate worst-case revenue.

Surprisingly, for the specific case of the uniform distribution F, one does not need to be able to compute the objective function in order to *optimize* it: for the uniform distribution F, simply offering a price of $\frac{1}{2}$ guarantees worst-case revenue within a factor at most $4/e$ of optimal. Unfortunately, the analysis techniques for this case rely very specifically on the uniform distribution of valuations; it is an interesting open question whether they can be extended beyond the uniform distribution.

Related Work

Externalities in general, and public goods in particular, have a rich and long history of study in economics. The tension arising from private provisioning of public goods has been realized since the early studies of public goods: Samuelson [32] already noticed that private provisioning will not necessarily achieve a social optimum. (See also the discussion in Chapter 11 of [27].) Implicit in the study of markets for public goods in this literature is the goal of setting the right price, taking into account production costs and utility curves. Our model differs from the classic models in that purchase decisions are binary, whereas traditional models allow agents to choose a continuous level x_i at which to purchase the public good. Each agent's utility in the fully public setting is a function of $\sum_i x_i$, whereas interpreting the x_i as probabilities, the utilities in our setting are of the form $1 - \prod_i (1 - x_i)$. Thus, the analysis techniques commonly used in the literature on public goods do not apply directly in our setting.

The study of private sales of public goods is also present in the classic paper of Bergstrom, Blume and Varian [6] and in work by Allouch [3]. The authors consider a model in which agents need to split an initial endowment of public and private goods. The focus of those papers is to prove existence and uniqueness of equilibria in such games.

In our work, we assume that the good to be allocated is fully public. There is a large body of literature studying the effects of *congestion*, where a good's value to an individual decreases as others use it. Several works study allocation mechanisms to price such congestion effects, going back to the original work of Pigou [31]. For overviews of pricing of congestion in public and club goods, see [14, 20]. In the present work, the good does not become congested; instead, a graph structure specifies which individuals derive utility from the purchases made by others.

A study of locally public goods in the graph-theoretic sense considered here[2] has only been begun much more recently, as part of the recent trend toward studying classic games in a networked setting. (See Galleotti [15] for a general overview.) Specifically, locally public goods have been studied by Bramoullé and Kranton [8] and Bramoullé, Kranton and D'Amours [9]. Bramoullé and Kranton [8, 9] study a setting in which agents decide on a level of effort; an agent's utility grows as a function of the cumulative efforts of himself and all his neighbors in the network. In this sense, the model generalizes the classical public goods model to networks; as we discussed above, in contrast, our model focuses on probabilistic decisions to purchase or not to purchase. One main difference between [8] and our work is that, instead of merely taking the games as given, we seek to engineer the network game by setting parameters (in our case: prices) that will lead to more desirable equilibria (equilibria of higher revenue).

Also closely related to our model is the work of Candogan, Bimpikis and Ozdaglar [11]. This work considers a monopolist who sets prices for agents that are embedded in a network and exhibit positive externalities. Their model differs from ours in three main respects. First, as with the work of Bramoullé et al., the level of consumption in their model is continuous rather than binary. Second, their externality model is different in that an agent's utility is *additive* over the purchases made by his neighbors, whereas in our case, purchases of neighbors are substitutes. Third, they adopt a full-information model, in which the auctioneer knows the demands of the agents, whereas in our model, the agents' values are drawn from a known distribution.

We focus our attention on mechanisms that allocate a (globally or locally) public good by way of *posted prices*. Posted price mechanisms have received significant recent attention in the context of auctions with multiple objects for sale [12, 13] where it has been shown that, in various settings, approximately optimal revenue can be extracted by offering a vector of take-it-or-leave-it prices to each buyer in sequence. Our analysis shares similarities with this line of work: like [12], we relate our pricing problem to a corresponding single-item auction problem. However, unlike [12], setting a posted price in an auction for a *public* good can lead to multiple equilibria of buyer behavior, with different equilibria generating substantially different revenues.

2 Models and Preliminaries

We write $[n] = \{1, 2, \dots, n\}$. Throughout, vectors are denoted by bold face. The buyers form the vertices $V = [n]$ of an undirected graph $G = ([n], E)$. The neighbors of a node $i \in V$ are denoted by $N(i) = \{j \mid (i, j) \in E\}$, with the convention that $i \notin N(i)$. For an event $\mathcal{E}$, we write $\mathbb{1}\{\mathcal{E}\}$ for the function whose value is 1 when $\mathcal{E}$ happens and 0 otherwise.

We are interested in *locally public goods*: goods that let a player derive utility either from being allocated the good, or from having a neighbor who is allocated the good. More formally, we define

[2] Past work on "local public goods" used the term to describe public goods for a community such as a small town. As such, the term corresponds to a fully public (though possibly congestible) good, when the set of individuals under consideration is restricted.

utilities as follows: Each agent i has a private valuation v_i for the good, drawn independently from a common and commonly known atomless distribution F.[3] Since we assume that F is atomless, for every $q \in [0,1]$, there is at least one value of p for which $F(p) = q$. We write $F^{-1}(q) = \min\{p | F(p) = q\}$.

If S is the set of agents allocated the good, and π_i the payment of agent i, then agent i's utility is

$$u_i(S, \pi_i) = \begin{cases} v_i - \pi_i & \text{if } i \in S \text{ or } S \cap N(i) \neq \emptyset \\ -\pi_i & \text{otherwise.} \end{cases}$$

A natural question arises regarding whether agents $i \notin S$ should have non-zero payments, given that they may profit from the allocation to their neighbors. In the present work, we focus on the *private sale* of the good via *posted prices*, i.e., the seller determines the price of the good, and an agent is only charged when purchasing the good. This is the most widely used mechanism for selling goods, public or private.

We remark that since our setting is a single-parameter setting, Myerson's theory of optimal auctions [30] would yield a revenue-optimal mechanism. However, the mechanism does not correspond to private sales since it charges not only the buyers, but also their neighbors who derive benefit from the item.

2.1 Equilibria in the posted-prices game

The pricing decisions can be modeled as a two-stage game. In the first stage, the seller sets a price vector $\mathbf{p}$ to offer the buyers. For most of the paper, and unless specified otherwise, all agents will be offered the same price p. Subsequently, the buyers play a simultaneous Bayesian game. The seller's goal is to choose $\mathbf{p}$ so as to maximize revenue.

We assume that the agents maximize their expected utility. Given a price p_i, a player i will buy if his utility from buying, $v_i - p_i$, exceeds the expected utility from not buying, $v_i \cdot \left(1 - \prod_{j \in N(i)} \mathbb{P}[j \text{ does not buy}]\right)$. At equality, i could randomize between the two strategies, but since we assumed the distribution F to be atomless, equality is an event of probability 0. Thus, each agent will employ a threshold strategy: buy if and only if

$$v_i \geq \frac{p_i}{\prod_{j \in N(i)} \mathbb{P}[j \text{ does not buy}]} =: T_i.$$

Because all other players j also employ threshold strategies, we can write $\mathbb{P}[j \text{ does not buy}] = \mathbb{P}[v_j \leq T_j] = F(T_j)$. Thus, the Nash Equilibria are exactly the threshold vectors $\mathbf{T} = (T_1, \ldots, T_n)$ satisfying the following condition:

$$T_i \cdot \prod_{j \in N(i)} F(T_j) = p_i, \quad \text{for all } i \in V. \tag{1}$$

Given a price vector $\mathbf{p}$, we use $\mathcal{N}_{\mathbf{p}}$ to denote the set of Nash Equilibria $\mathbf{T} = (T_1, \ldots, T_n)$ of the posted prices game with prices $\mathbf{p}$. We prove below that $\mathcal{N}_{\mathbf{p}} \neq \emptyset$. Given a Nash Equilibrium $\mathbf{T} \in \mathcal{N}_{\mathbf{p}}$, the corresponding expected revenue is $\mathcal{R}(\mathbf{p}, \mathbf{T}) = \sum_i p_i \cdot (1 - F(T_i))$.

[3] Some of our preliminary results carry over to the case when buyers have different distributions F_i.

2.1.1 Existence of (possibly multiple) Equilibria

To prove the existence of at least one equilibrium, define

$$B = \left[p_1, \frac{p_1}{\prod_{j\in N(1)} F(p_j)}\right] \times \left[p_2, \frac{p_2}{\prod_{j\in N(2)} F(p_j)}\right] \times \cdots \times \left[p_n, \frac{p_n}{\prod_{j\in N(n)} F(p_j)}\right],$$

and consider the best-response function $\Psi : B \to \mathbb{R}^n_+$, defined as $\Psi_i(\mathbf{T}) = p_i / \prod_{j\in N(i)} F(T_j)$. We claim that $\Psi(\mathbf{T}) \in B$ for all $\mathbf{T} \in B$.

First, notice that for any $\mathbf{T}$, we have $\Psi_i(\mathbf{T}) \geq p_i$. Intuitively, this captures that, regardless of the other players' strategies, no player will ever buy the good for more than his value. On the other hand, because $T_j \geq p_j$ for $\mathbf{T} \in B$, we also get that

$$\Psi_i(\mathbf{T}) = \frac{p_i}{\prod_{j\in N(i)} F(T_j)} \leq \frac{p_i}{\prod_{j\in N(i)} F(p_j)}.$$

Thus, $\Psi : B \to B$ is a continuous function from B to B. So long as the prices are such that $F(p_i) > 0$ for all i, B is compact, and the existence of a fixed point (and thus an equilibrium of the game) follows from Brouwer's Fixed Point Theorem.

If there is one or more agent i with $F(p_i) = 0$, then the following construction proves the existence of an equilibrium. For each agent i with $F(p_i) = 0$, set $T_i = p_i$, and for all neighbors of i, set $T_i = \infty$. In other words, i deterministically buys the good, and i's neighbors never buy the good. It follows directly from the definition of Ψ that the best response for all these agents will be $\Psi_i(\mathbf{T}) = T_i$. Since the agents with $T_i = \infty$ contribute a term $F(\infty) = 1$ to their neighbors' product, the remaining problem remains unchanged if we remove all these agents completely, and focus on the restriction of Ψ to the remaining agents. For those, the previous compactness argument applies.

Remark 1 In general, there could be many equilibria of the game. Even for the special case of $G = K_n$ and $F(x) = x$ for $x \in [0, 1]$, any threshold vector $\mathbf{T}$ with $\prod_i T_i = p$ is a Nash Equilibrium of the posted-prices game with uniform prices p. Thus, there is in general a continuum of equilibria.

2.1.2 Symmetric Equilibria for d-Regular Graphs

When the graph G is d-regular, and the prices offered to the buyers are the same, i.e., $p_i = p$ for all i, then we can show that Ψ also has a symmetric equilibrium. Notice that if $\mathbf{T} = T \cdot \mathbf{1}$, then $\Psi(\mathbf{T}) = p/F(T)^d \cdot \mathbf{1}$, so the best responses will be symmetric. It therefore suffices to study the function $\psi(T) = p/F(T)^d$, and show that it has a fixed point. To see this, observe that the condition for the existence of a symmetric equilibrium is the existence of a threshold T such that $T \cdot F(T)^d = p$. Because $\psi(p) = p/F(p)^d$ and $\psi(p/F(p)^d) \leq p/F(p)^d$, the existence of a fixed point in the interval $[p, p/F(p)^d]$ follows by the intermediate value theorem.

2.2 Hipster Game

In this section, we consider the following variation of the game. In the *Hipster Game*, each agent strives to be unique among his friends, so upon acquiring a good, he only derives value from it if he is the only person in his social network who has this good. More precisely, if S is the set of allocated agents, and π is the vector of payments, then:

$$u_i(S, \pi_i) = \begin{cases} v_i - \pi_i & \text{if } i \in S \text{ and } S \cap N(i) = \emptyset \\ -\pi_i & \text{otherwise.} \end{cases}$$

Notice that this definition of utilities seems to give us a game which is the complete opposite of the Public Goods Game. While the Public Goods Game was an example of positive externalities, the Hipster Game is an example of negative externalities. In fact, this game can be described as a congestion game: the graph nodes are congestable resources, and the resources requested by a player are exactly all nodes in his neighborhood. While the Hipster Game is characterized by negative externalities, it exhibits a very similar equilibrium structure to the Public Goods Game. Player i decides to purchase the good for price p_i if

$$v_i \cdot \mathbb{P}[\text{no agent in } N(i) \text{ buys}] - p_i \geq 0.$$

Therefore, the set of equilibria for this game is composed of threshold strategies for all agents such that the thesholds satisfy $p_i = T_i \cdot \prod_{j \in N(i)} F(T_j)$ for all i.

Thus, the Public Goods Game and the Hipster Game have the same set of equilibria and also the same revenue. (However, they are *not isomorphic*, since the payoff structure is not the same.) We can use this observation to get a crude upper bound on the expected revenue of the Public Goods Game for arbitrary graphs. We note that it is equal to the expected revenue of the Hipster Game, which in turn is at most the expected welfare of the Hipster Game, as each agent must derive non-negative expected utility at equilibrium. The expected welfare of the Hipster Game is at most the expected weight of the maximum weighted independent set with weights v_i drawn i.i.d. from F. Thus, we conclude:

Lemma 2 *The expected revenue from the Public Goods Game is at most the expected weight of the maximum weighted independent set of G with node weights v_i drawn i.i.d. from F.*

2.3 Regularity, Myerson's Lemma and the Prophet Inequality

Much of our analysis will be based on Myerson's Lemma about the optimal selling mechanism, combined with the prophet inequality. We briefly review these concepts here. A more comprehensive exposition can be found in Hartline's lecture notes [18].

Definition 3 (Virtual values and regularity) *Let F be the cumulative distribution function of an atomless distribution on an interval $[a, b]$, and let f be its corresponding density function. The* virtual value function *associated with distribution F is defined as $\phi(x) = x - \frac{1-F(x)}{f(x)}$. The distribution F is* regular *if $\phi(x)$ is non-decreasing.*

Consider a single-agent scenario in which an agent with value v, drawn from F, is made a take-it-or-leave-it offer at price p. The agent will accept the offer iff his value exceeds p, which happens with probability $1 - F(p) =: q$. Therefore, the revenue obtained by posting a price p is $p \cdot (1 - F(p))$, which can be also written in terms of the *quantile space* as $q \cdot F^{-1}(1 - q)$. This motivates the following definition:

Definition 4 (Revenue curve) *The* revenue curve *corresponding to the cumulative distribution function F is a function $R : [0, 1] \to \mathbb{R}_+$, defined by $R(q) = q \cdot F^{-1}(1 - q)$. It specifies the revenue as a function of the* ex ante *probability of sale.*

The derivative of the revenue curve with respect to q is $\frac{dR}{dq}(q) = F^{-1}(1 - q) - \frac{q}{f(F^{-1}(1-q))} = \phi(F^{-1}(1-q))$. Since F^{-1} is monotone non-decreasing, a distribution is regular iff its corresponding revenue curve is concave.

2.3.1 Single-item auctions and Myerson's Lemma

We draw repeatedly on the scenario in which a single item is sold to n agents with valuations drawn i.i.d. from a regular distribution F. A mechanism receives a vector of bids $\mathbf{b} = (b_1, \ldots, b_n)$ and returns an allocation vector $\mathbf{x}(\mathbf{b}) = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ such that $\sum_i x_i \leq 1$, and a payment vector $\pi(\mathbf{b}) \in \mathbb{R}_+^n$. The mechanism is *incentive compatible* if no bidder can benefit from reporting a value other than his true value, i.e., if bidding $b_i = v_i$ is a weakly dominant strategy for each agent i. Myerson [30] established the following lemma, which relates the payments of an incentive compatible mechanism to the expected virtual values:

Lemma 5 (Myerson [30]) *For any incentive compatible mechanism, and any bidder i, $\mathbb{E}_{\mathbf{v}}[\pi_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}}[x_i(\mathbf{v}) \cdot \phi(v_i)]$.*

In particular, it follows from Lemma 5 that the revenue-maximizing incentive compatible mechanism allocates the item entirely to an agent with highest non-negative virtual value. The *n-agent Myerson Revenue* is the optimal revenue that can be obtained in a setting with a single item and n agents, and is given by $\mathcal{R}_n^M = \mathbb{E}[\max_i \phi(v_i)^+]$, where $z^+ = \max(0, z)$. When clear from the context, we drop the subscript n.

Lemma 5 also implies that the optimal mechanism for selling an item to a single agent is a posted price mechanism with price $r = \phi^{-1}(0)$, known as the *Myerson Reserve Price*. It follows that the n-agent Myerson Revenue can be bounded as follows:

$$\mathcal{R}_n^M = \mathbb{E}[\max_i \phi(v_i)^+] \; \leq \; \mathbb{E}[\sum_i \phi(v_i)^+] \; = \; n \cdot r \cdot (1 - F(r)).$$

2.3.2 Posted-Price Mechanisms and the Prophet Inequality

A natural mechanism for selling a good is the *sequential posted prices* mechanism. In round i, if the good has not been sold previously, the mechanism offers the good to agent i at a price of p_i. The revenue obtained by this mechanism is $\sum_{i=1}^n p_i \cdot \mathbb{P}[v_i \geq p_i \text{ and } v_j < p_j \text{ for all } j < i] = \sum_{i=1}^n p_i \cdot (1 - F(p_i)) \cdot \prod_{j<i} F(p_j)$. Because the sequential posted-price mechanism is incentive compatible, and $\mathcal{R}_n^M$ is defined as the optimum expected revenue for any incentive compatible mechanism, we obtain that for any price vector $(p_1, \ldots, p_n)$:

$$\sum_{i=1}^n p_i \cdot (1 - F(p_i)) \cdot \prod_{j<i} F(p_j) \leq \mathcal{R}_n^M.$$

The result known as the *Prophet Inequality* guarantees the existence of a price p^* (called the *prophet price*) such that a sequential posted-price mechanism with uniform price p^* (i.e., where $p_i = p^*$ for all i) generates at least half of the optimal revenue. In other words:

$$\sum_{i=1}^n p^* \cdot (1 - F(p^*)) \cdot F(p^*)^{i-1} \geq \frac{1}{2}\mathcal{R}_n^M.$$

The Prophet Inequality (and its variants) is a powerful tool in optimal stopping theory; it was introduced and applied in algorithmic mechanism design by Hajiaghayi, Kleinberg and Sandholm [17]. See [2] and [25] for recent developments of the topic.

3 Pricing globally public goods

In this section, we focus on the case of a *globally* public good. That is, the underlying network is a clique, $G = K_n$. We assume that the common value distribution F of the agents is atomless and regular. Our main result is the following:

Theorem 6 *In the globally public good setting, let $p = F^{-1}(1 - 1/n) \cdot (1 - 1/n)^{n-1}$. Then, if the price p is offered to all agents, the worst-case revenue among the equilibria $\mathbf{T} \in \mathcal{N}_p$ is at least a constant fraction of the revenue of the best equilibrium for the best (possibly non-uniform) price vector to offer the agents.*

The main insight driving Theorem 6 is that, at equilibrium, the agents aim to make purchasing decisions in such a way that only one agent will buy the product, in expectation. With this in mind, we draw a relationship between the public good pricing problem and a single item auction that attempts to sell a single item to n bidders with value distributions F. We relate the revenue at different price vectors and equilibria in the public good mechanism to the optimal (Myerson) revenue in the single item auction. We can then apply the theory of optimal auctions to guide our choice of pricing in the public good mechanism. We note that similar techniques have been applied in the context of sequential posted pricing for multi-item auctions [12]. However, a novel difficulty that we must overcome is the existence of multiple equilibria of bidder behavior for any given price; we must therefore find a price for which *all* equilibria generate a good approximation to the optimal revenue.

First, in Proposition 7, we show that the revenue of any equilibrium of any mechanism is upper-bounded by $\mathcal{R}_n^M$. Next, Lemma 8 shows that for the price vector $\mathbf{p} = p \cdot \mathbf{1}$, where p is the price specified in the assertion of Theorem 6, the *symmetric* equilibrium is guaranteed to achieve at least a constant fraction of the Myerson Revenue. Finally, in Lemma 9, we show that in *every* equilibrium for this price vector $\mathbf{p}$, the revenue is at least a constant fraction of that of the symmetric equilibrium for this price vector.

A corollary of this analysis is that the ability to price-discriminate does not substantially influence revenue: a uniform price vector can extract a constant fraction of the optimal revenue attainable by any mechanism, and hence any (non-uniform) vector of prices.

We note that while our analysis makes use of a connection to the Myerson optimal auction, offering the Myerson Reserve Price does not necessarily extract a constant fraction of the optimal revenue, even when F is regular. In the appendix A, we provide an example illustrating this revenue gap at the Myerson Reserve Price.

Proposition 7 *Let $\mathbf{p} = (p_1, \ldots, p_n)$ be any price vector, and $\mathbf{T} = (T_1, \ldots, T_n) \in \mathcal{N}_{\mathbf{p}}$ be an arbitrary equilibrium of the public goods selling game with prices $\mathbf{p}$. Then, $\mathcal{R}(\mathbf{p}, \mathbf{T}) \leq \mathcal{R}_n^M$.*

Proof. Using that $p_i = T_i \cdot \prod_{j \neq i} F(T_j)$ by Equation (1), we can bound the revenue as

$$\mathcal{R}(\mathbf{p}, \mathbf{T}) = \sum_i T_i \cdot (1 - F(T_i)) \cdot \prod_{j \neq i} F(T_j) \;\leq\; \sum_i T_i \cdot (1 - F(T_i)) \cdot \prod_{j < i} F(T_j) \;\leq\; \mathcal{R}_n^M.$$

In the last inequality, we used that the sum expresses the expected revenue of the sequential posted price mechanism in which the i^{th} player is offered a price of T_i; therefore, the sum is upper-bounded by the expected revenue of the optimal mechanism for selling a single item. ■

For the remainder of this section, we fix T such that $F(T) = 1 - \frac{1}{n}$, and $p = T \cdot F(T)^{n-1} = T \cdot (1 - 1/n)^{n-1}$. Let $\mathbf{p} = p \cdot \mathbf{1}$ be the vector in which all agents are offered p.

Lemma 8 *Let* $\mathbf{T} = T \cdot \mathbf{1}$ *be the symmetric equilibrium corresponding to* $\mathbf{p}$*. Then,*

$$\mathcal{R}(\mathbf{p}, \mathbf{T}) = n \cdot T \cdot (1 - F(T)) \cdot F(T)^{n-1} \;\geq\; \frac{1}{4} \cdot \mathcal{R}_n^M.$$

Proof. We use a variant of an argument by Chawla, Hartline and Kleinberg [12]. We distinguish between two cases, based on the relation of the Myerson Reserve Price $r = \phi^{-1}(0)$ with T.

1. If $T > r$, we let $\nu = \phi(T) > 0$. We can bound the Myerson Revenue as follows:

$$\begin{aligned}\mathcal{R}_n^M &= \mathbb{E}[\max_i \phi(v_i) \cdot \mathbb{1}\{\max \phi(v_i) \geq 0\}] \\ &\qquad \leq\; \nu \cdot \mathbb{P}[0 \leq \max_i \phi(v_i) \leq \nu] + \mathbb{E}[\max \phi(v_i) \cdot \mathbb{1}\{\max \phi(v_i) \geq \nu\}].\end{aligned}$$

   We bound each term separately. For the first term, we have that $\nu = \phi(T) \leq T$, and

$$\mathbb{P}[0 \leq \max_i \phi(v_i) \leq \nu] \leq \mathbb{P}[\max_i v_i \leq T] \;=\; F(T)^n \;=\; (1 - 1/n)^n \;\leq\; 1/e.$$

   For the second term, we have that

$$\mathbb{E}[\max \phi(v_i) \cdot \mathbb{1}\{\max \phi(v_i) \geq \nu\}] \leq \mathbb{E}[\sum_i \phi(v_i) \mathbb{1}\{\phi(v_i) \geq \nu\}].$$

   By Lemma 5, $\mathbb{E}[\phi(v_i)\mathbb{1}\{\phi(v_i) \geq \nu\}]$ is the revenue of the single-agent mechanism that makes agent i a take-it-or-leave-it offer at price T; therefore,

$$\mathbb{E}[\sum_i \phi(v_i)\mathbb{1}\{\phi(v_i) \geq \nu\}] = \sum_i T \cdot \mathbb{P}[v_i \geq T] \;=\; T \cdot \sum_i (1 - F(T)) \;=\; T,$$

   by definition of T. Combining the bound on the two terms, we get that $\mathcal{R}_n^M \leq T \cdot (1 + 1/e)$. On the other hand, for the symmetric prices and symmetric equilibrium, we have that $\mathcal{R}(\mathbf{p}, \mathbf{T}) = n \cdot T \cdot (1 - F(T)) \cdot F(T)^{n-1} = T \cdot (1 - 1/n)^{n-1} \geq T/e$. Therefore,

$$\mathcal{R}(\mathbf{p}, \mathbf{T}) \geq \frac{1/e}{1 + 1/e} \mathcal{R}_n^M \;=\; \frac{1}{1+e} \mathcal{R}_n^M \;\approx\; 0.27 \cdot \mathcal{R}_n^M.$$

2. When $T \leq r$, we upper-bound $\mathcal{R}_n^M$ as follows:

$$\begin{aligned}\mathcal{R}_n^M &= \mathbb{E}[\max \phi(v_i) \cdot \mathbb{1}\{\max \phi(v_i) \geq 0\}] \;\leq\; \mathbb{E}[\sum_i \phi(v_i) \cdot \mathbb{1}\{\phi(v_i) \geq 0\}] \\ &= n \cdot r \cdot (1 - F(r)),\end{aligned}$$

   where the final equality follows from the same argument about a single buyer as above. Let $q^M = 1 - F(r)$ be the probability that the valuation of an agent with distribution F is above the Myerson reserve price r. Because F is regular, as argued in Section 2.3, the revenue curve $R(q)$ is a concave function. By the definition of the Myerson Reserve Price as the maximizer of expected revenue, R is maximized at $q = q^M$. Because we are in the case that $T \leq r$, we get that $q^M = 1 - F(r) \leq 1 - F(T) = \frac{1}{n} < 1$. We can therefore write $\frac{1}{n}$ as a convex combination $\frac{1}{n} = \lambda \cdot q^M + (1 - \lambda) \cdot 1$, with $\lambda = \frac{1 - \frac{1}{n}}{1 - q^M}$. The concavity of R, together with $R(1) = 0$, now implies that $R(\frac{1}{n}) \geq \frac{1-\frac{1}{n}}{1-q^M} R(q^M)$. On the other hand, $R(\frac{1}{n}) = T \cdot (1 - F(T))$; by combining these, we obtain

$$T \cdot (1 - F(T)) = R(1/n) \;\geq\; \frac{1 - \frac{1}{n}}{1 - q^M} \cdot R(q^M) \;\geq\; \left(1 - \frac{1}{n}\right) \cdot r \cdot (1 - F(r)). \tag{2}$$

We can therefore bound the posted price revenue as

$$\begin{aligned} \mathcal{R}(\mathbf{p}, \mathbf{T}) = n \cdot T \cdot (1 - F(T)) \cdot F(T)^{n-1} \;&=\; n \cdot T \cdot (1 - F(T)) \cdot (1 - 1/n)^{n-1} \\ \geq\; n \cdot r \cdot (1 - F(r))(1 - 1/n)^n \;&\geq\; \frac{1}{4} \mathcal{R}_n^M. \end{aligned}$$

The first inequality follows by Equation (2); for the second inequality, we bound $(1 - 1/n)^n \geq \frac{1}{4}$, and use that the optimal revenue from selling a single item to n agents is at most n times the optimal revenue from selling a single item to one agent at the Myerson Reserve Price. ■

Having shown that the symmetric equilibrium has revenue within a constant factor of the Myerson Revenue for a single item, it remains to analyze the asymmetric equilibria. (Recall that $p = T \cdot F(T)^{n-1}$, where T is such that $F(T) = 1 - \frac{1}{n}$.)

Lemma 9 *Let $\mathbf{T} = T \cdot \mathbf{1}$ be the symmetric equilibrium with threshold T, and $\mathbf{T}' \in \mathcal{N}_{\mathbf{p}}$ be an arbitrary equilibrium. Then, $\mathcal{R}(\mathbf{p}, \mathbf{T}') \geq \Omega(1) \cdot \mathcal{R}(\mathbf{p}, \mathbf{T})$.*

Proof. We express the revenue of the symmetric equilibrium as $\mathcal{R}(\mathbf{p}, \mathbf{T}) = p \cdot \sum_i (1 - F(T)) = p$. By the Union Bound, the revenue at the equilbrium $\mathbf{T}'$ is lower-bounded by $\mathcal{R}(\mathbf{p}, \mathbf{T}') = \sum_i p \cdot (1 - F(T_i')) \geq p \cdot (1 - \prod_i F(T_i'))$. We will prove that $\prod_i F(T_i') \leq (1 - 1/n)^{n-1} \leq \frac{1}{2}$, which will imply that $\mathcal{R}(\mathbf{p}, \mathbf{T}') \geq \frac{1}{2} p = \frac{1}{2} \mathcal{R}(\mathbf{p}, \mathbf{T})$. For contradiction, assume that $\prod_i F(T_i') > (1 - 1/n)^{n-1}$.

Using that $p = T \cdot F(T)^{n-1} = T \cdot (1 - 1/n)^{n-1}$, applying the equilibrium condition (1) to $\mathbf{T}'$, and using our contradiction assumption, we get that for all j,

$$T_j' = \frac{p}{\prod_{i \neq j} F(T_i')} \;=\; \frac{T \cdot (1 - 1/n)^{n-1}}{\prod_{i \neq j} F(T_i')} \;\leq\; T \cdot \frac{(1 - 1/n)^{n-1}}{\prod_i F(T_i')} \;<\; T \cdot \frac{(1 - 1/n)^{n-1}}{(1 - 1/n)^{n-1}} \;=\; T.$$

Thus, $T_j' < T$ for all j, implying that $F(T_j') \leq F(T)$ as well. But this contradicts that $T_j' \cdot \prod_{i \neq j} F(T_i') = p = T \cdot \prod_{i \neq j} F(T)$, completing the proof. ■

4 Pricing Locally Public Goods

In this section, we turn to scenarios in which the good is not completely public. That is, the graph G is not necessarily complete; rather, G is an arbitrary network, and agents share benefits only with neighbors in G. We refer to such a good as *locally public*.

We first analyze the case when G is d-regular, for some arbitrary $d \geq 1$. For such graphs, we describe how to explicitly calculate prices that are approximately revenue-optimal, in the worst case over equilibria of agent behavior. We then consider the case of general networks. We present evidence that the pricing problem for general graphs is substantially more difficult, and that the approaches used in previous cases cannot be extended to handle the general case, even for uniform distributions. Nevertheless, we show that approximately optimal prices can be found in the special case that agent values are drawn from the uniform distribution.

4.1 d-Regular Graphs

We consider the problem of pricing locally public goods when the underlying graph is d-regular; i.e., $|N(i)| = d$ for every $i \in [n]$. As before, we assume that the value distribution F of the agents is atomless and regular.

Recall that in the case of globally public goods (Section 3), we showed how to compute a price for which the revenue at the *worst* equilibrium is a good approximation to the revenue at the *best* equilibrium for any price vector. In other words, p is such that

$$\min_{\mathbf{T} \in \mathcal{N}_p} \mathcal{R}(p \cdot \mathbf{1}, \mathbf{T}) \geq \Omega(1) \cdot \max_{\mathbf{p}} \max_{\mathbf{T} \in \mathcal{N}_{\mathbf{p}}} \mathcal{R}(\mathbf{p}, \mathbf{T}).$$

One might hope for a similar result for locally public goods. Unfortunately, we show that this is not possible even for d-regular networks: in Example 13, we give an instance of a d-regular graph for which the gap in revenue between different equilibria is linear in d. The same example also shows that for d-regular graphs, we cannot find a single price that is competitive against non-uniform price vectors. Thus, unlike for globally public goods, a constant-factor revenue approximation for d-regular graphs must specifically compare revenue-minimizing equilibria at given price vectors.

We establish the existence of a price p that depends only on the degree d and the distribution F, but not on the particular structure of G, such that when p is offered to all the agents, the seller obtains a constant fraction of the *worst-case* revenue at *any* price. In other words, we establish the existence of a price $p = p(d, F)$ such that

$$\min_{\mathbf{T} \in \mathcal{N}_p} \mathcal{R}(p \cdot \mathbf{1}, \mathbf{T}) \geq \Omega(1) \cdot \max_{p'} \min_{\mathbf{T} \in \mathcal{N}_{p'}} \mathcal{R}(p' \cdot \mathbf{1}, \mathbf{T}).$$

We emphasize that the key aspect here is that p is independent of the actual network structure of G, and that it can be computed efficiently from F and d.

Theorem 10 *In the locally public good setting with d-regular graphs, let $p = F^{-1}(1-1/d) \cdot (1-1/d)^d$. Then, if the price p is offered to all agents, the worst-case revenue among the equilibria $\mathbf{T} \in \mathcal{N}_p$ is at least a constant fraction of the revenue of the worst equilibrium for the best* network-specific *uniform price to offer the agents.*

Our approach to proving Theorem 10 is the following. We first study the symmetric equilibria of the game. We know from Section 2 that every uniform price vector $p \cdot \mathbf{1}$ admits a symmetric equilibrium. We consider a price p for which, in the corresponding symmetric equilibrium, each player buys with probability $\frac{1}{d}$. In Lemma 11, we show that the revenue of this symmetric equilibrium is a constant fraction of the revenue of any other symmetric equilibrium (across all potential prices). In particular, this implies that the worst-case revenue at any other price is at most a constant factor larger than the revenue of the symmetric equilibrium at price p. Then, in Lemma 12, we show that for this particular price p, *every* equilibrium generates at least a constant fraction of the revenue of the symmetric equilibrium.

For the remainder of this section, we fix T such that $F(T) = 1 - \frac{1}{d}$, and $p = T \cdot F(T)^d = T \cdot (1 - 1/d)^d$. Let $\mathbf{p} = p \cdot \mathbf{1}$ be the vector in which all agents are offered p.

Lemma 11 *Consider a locally public goods problem in which the underlying network is a d-regular graph and agents have valuations drawn i.i.d. from a regular distribution F. Let $\mathbf{T}$ be the symmetric equilibrium with threshold T. Then,*

$$\mathcal{R}(\mathbf{p}, \mathbf{T}) \geq \Omega(1) \cdot \mathcal{R}(p' \cdot \mathbf{1}, T' \cdot \mathbf{1}),$$

for all prices p' and threshold vectors $T' \cdot \mathbf{1}$ corresponding to the symmetric equilibrium with price p'.

Proof. Let $\mathcal{R}_d^M$ be the revenue obtained by Myerson's mechanism for selling one item to d players with i.i.d. valuations drawn according to F. The Prophet Inequality (Section 2.3) guarantees that there exists a price T^* such that a sequential posted-prices mechanism with price T^* offered to all agents guarantees at least half of the Myerson Revenue $\mathcal{R}_d^M$. On the other hand, the Myerson Revenue is optimal, and therefore clearly serves as an upper bound on the revenue that can be obtained by any price T of the sequential posted prices mechanism. In summary, there exists a T^* such that

$$\sum_{i=1}^{d} T^*(1-F(T^*))F(T^*)^{i-1} \geq \frac{1}{2}\cdot\mathcal{R}_d^M \geq \frac{1}{2}\cdot\Big(\sum_{i=1}^{d} T'(1-F(T'))F(T')^{i-1}\Big), \quad \text{for all } T'. \tag{3}$$

We distinguish between two cases, based on the relation of T^* with T.

1. If $T > T^*$, then given any price p' and corresponding symmetric equilibrium T',

$$\begin{aligned}\mathcal{R}(p'\cdot\mathbf{1}, T'\cdot\mathbf{1}) = n\cdot T'\cdot(1-F(T'))\cdot F(T')^d \;&=\; \frac{n}{d}\Big(d\cdot T'\cdot(1-F(T'))\cdot F(T')^d\Big)\\ &\leq \frac{n}{d}\Big(\sum_{i=1}^{d} T'\cdot(1-F(T'))\cdot F(T')^{i-1}\Big).\end{aligned}$$

Using both sides of Equation (3), we get:

$$\mathcal{R}(p'\cdot\mathbf{1}, T'\cdot\mathbf{1}) \leq \frac{n}{d}\cdot\mathcal{R}_d^M \;\leq\; \frac{2n}{d}\cdot\Big(\sum_{i=1}^{d} T^*(1-F(T^*))F(T^*)^{i-1}\Big) \;\leq\; \frac{2n}{d}\cdot T^*. \tag{4}$$

Next, we establish a lower bound on $\mathcal{R}(\mathbf{p},\mathbf{T})$.

$$\mathcal{R}(\mathbf{p},\mathbf{T}) = n\cdot T\cdot(1-F(T))\cdot F(T)^d = \frac{n}{d}(1-1/d)^d\cdot T \geq \frac{n}{4d}\cdot T^* \geq \frac{1}{8}\cdot\mathcal{R}(p'\cdot\mathbf{1}, T'\cdot\mathbf{1}).$$

For the first inequality, we bound $\big(1-\frac{1}{d}\big)^d \geq \frac{1}{4}$, and use that $T \geq T^*$, by the assumption of case (1); the second inequality follows from Inequality (4).

2. If $T \leq T^*$, then let $q^* = 1-F(T^*) \leq 1-F(T) = \frac{1}{d}$. Similar to the proof of Lemma 8, we use the concavity of the revenue function $R(q) = q\cdot F^{-1}(1-q)$ to derive that

$$T\cdot(1-F(T)) \geq (1-1/d)\cdot T^*\cdot(1-F(T^*)). \tag{5}$$

It follows that

$$\begin{aligned}\mathcal{R}(\mathbf{p},\mathbf{T}) &= n\cdot T\cdot(1-F(T))\cdot F(T)^d \;=\; n\cdot(1-1/d)^d\cdot T\cdot(1-F(T))\\ &\geq n(1-1/d)^{d+1}\cdot T^*\cdot(1-F(T^*))\\ &\geq (1-1/d)^{d+1}\cdot\frac{n}{d}\cdot\Big(\sum_{i=1}^{d} T^*(1-F(T^*))F(T^*)^{i-1}\Big)\\ &\geq \frac{n}{16d}\cdot\Big(\sum_{i=1}^{d} T'(1-F(T'))F(T')^{i-1}\Big)\\ &\geq \frac{n}{16}\cdot T'\cdot(1-F(T'))\cdot F(T')^d \;=\; \frac{1}{16}\cdot\mathcal{R}(p'\cdot\mathbf{1}, T'\cdot\mathbf{1}).\end{aligned}$$

The first inequality follows by Inequality (5); the second inequality holds because $F(T^*)^{i-1} \leq 1$ for every i; the third inequality follows by Inequality (3), and by bounding $(1-1/d)^{d+1} \geq \frac{1}{8}$; and for the final inequality, we use that $i \leq d$ for every i.

The assertion of the lemma follows. ■

Having shown that the symmetric equilibrium associated with price p generates a good approximation to the optimal revenue attainable at symmetric equilibria, we now show that there are no other (asymmetric) equilibria associated with price p that generate significantly less revenue.

Lemma 12 *Let $\mathbf{T} = T \cdot \mathbf{1}$ be the symmetric threshold vector associated with price p. Then, $\mathcal{R}(\mathbf{p}, \mathbf{T}') \geq \Omega(1) \cdot \mathcal{R}(\mathbf{p}, \mathbf{T})$ for any threshold vector $\mathbf{T}' \in \mathcal{N}_{\mathbf{p}}$.*

Proof. For any equilibrium $\mathbf{T}' \in \mathcal{N}_{\mathbf{p}}$,

$$\mathcal{R}(\mathbf{p}, \mathbf{T}') = \sum_i p(1 - F(T_i')) = \frac{p}{d} \sum_i \sum_{j \in N(i)} (1 - F(T_j')) \geq \frac{p}{d} \sum_i \Big(1 - \prod_{j \in N(i)} F(T_j')\Big),$$

where the last inequality follows by applying the union bound. By the equilibrium conditions, $T_i' \cdot \prod_{j \in N(i)} F(T_j') = p = T \cdot F(T)^d$ for all i; therefore, $1 - \prod_{j \in N(i)} F(T_j') = 1 - T \cdot F(T)^d / T_i'$. We get that $\mathcal{R}(\mathbf{p}, \mathbf{T}') \geq \frac{p}{d} \sum_i \left(1 - \frac{T \cdot F(T)^d}{T_i'}\right)$. From the last inequality and the equality $\mathcal{R}(\mathbf{p}, \mathbf{T}') = \sum_i p(1 - F(T_i'))$, we can bound $\mathcal{R}(\mathbf{p}, \mathbf{T}')$ as follows:

$$\mathcal{R}(\mathbf{p}, \mathbf{T}') \geq p \cdot \sum_i \frac{1}{2} \left[(1 - F(T_i')) + \frac{1}{d}\left(1 - \frac{T \cdot F(T)^d}{T_i'}\right)\right].$$

Focus on one term i of the sum. The first term in brackets is decreasing in T_i', while the second is increasing in T_i'. Consequently, we distinguish between two cases: (i) If $T_i' \leq T$, then $1 - F(T_i') \geq 1 - F(T) = \frac{1}{d}$. (ii) If $T_i' \geq T$, then

$$\frac{1}{d}\left(1 - \frac{T \cdot F(T)^d}{T_i'}\right) \geq \frac{1}{d}\left(1 - (1 - 1/d)^d\right) \geq \frac{1}{d} \cdot \left(1 - \frac{1}{e}\right).$$

Thus, summing over all i, $\mathcal{R}(\mathbf{p}, \mathbf{T}') \geq p \cdot \sum_i \frac{1}{2}\left(\frac{1}{d} \cdot \left(1 - \frac{1}{e}\right)\right)$, implying that

$$\mathcal{R}(\mathbf{p}, \mathbf{T}') \geq p \cdot \sum_i \frac{1}{2}\left(\frac{1}{d} \cdot (1 - 1/e)\right) = \Omega(1) \cdot p \cdot \frac{n}{d} = \Omega(1) \cdot \mathcal{R}(\mathbf{p}, \mathbf{T}). \;\blacksquare$$

We now show that comparing against the best worst-case revenue, rather than the best revenue in equilibrium, is a necessity rather than an artifact of our analysis.

Example 13 (Revenue gap) Consider an instance with n players whose valuations are drawn i.i.d. from the uniform distribution with support $[0, 1]$. Let the underlying network be a d-regular bipartite graph with $\frac{n}{2}$ nodes on each side. We showed in Lemma 11 that the best worst-case revenue is upper bounded by $\frac{n}{d} \cdot \mathcal{R}_d^M \leq \frac{n}{d}$ (where $\mathcal{R}_d^M$ is the revenue obtained by Myerson's mechanism for selling one item to d players).

Now, consider the following equilibrium $\mathbf{T} \in \mathcal{N}_{1/2}$ in the bipartite graph: the nodes on one side buy whenever their value exceeds the price, while the nodes on the other side always free-ride. That is, $T_i = \frac{1}{2}$ for each player i on the left, and $T_i = 2^{d-1} > 1$ for each player i on the right. This equilibrium generates a revenue of $\frac{n}{2} \cdot \frac{1}{2} \cdot (1 - \frac{1}{2}) = \frac{n}{8}$. The gap between the best worst-case revenue and the best revenue can therefore be as large as $\frac{n}{8} \cdot (\frac{n}{d})^{-1} = \frac{d}{8}$.

Notice that the same instance also shows a gap between the worst-case revenue of the best uniform price vector and the best discriminating prices. The seller can offer all nodes on the right a price of 1 and the left a price of $\frac{1}{2}$; in the unique equilibrium, the bidders on the right never buy and the bidders on the left choose a threshold of $\frac{1}{2}$.

4.2 Hardness of Bounding Revenue for General Graphs

We would like to extend the results from the previous sections beyond complete and d-regular graphs, and find a method to compute prices that approximately optimize revenue for arbitrary networks. Recall the nature of our analysis for Theorem 6 and Theorem 10: in each case, we constructed a price p and then bounded the revenue of the worst-case equilibrium $\mathbf{T} \in \mathcal{N}_p$ with respect to either an upper bound on the revenue of *any* equilibrium for *any* price vector (in the case of Theorem 6) or the worst-case revenue for any uniform price vector (for Theorem 10). Can we hope to extend these methods to general networks?

In this section, we show that there are inherent difficulties in extending these approaches to handle general networks. In appendix A, we give an instance of a network such that, for *every* price, the gap between the best-case and worst-case revenues is $\Omega(n)$. (The complete bipartite graph $K_{n/2,n/2}$, generalizing Example 13, shows the same for carefully chosen prices, but not all prices.)

One might instead hope to analyze worst-case revenues directly, as in Theorem 10. However, we again find that this poses a difficulty in general networks. Theorem 14 (whose proof is given in appendix B) shows that it is NP-hard to approximate the worst-case revenue for a given p, over all equilibria $\mathbf{T} \in \mathcal{N}_p$, to within a factor of $n^{1-\epsilon}$, even when F is the uniform distribution.

Theorem 14 *Assume that all agents' valuations are drawn i.i.d. from the uniform distribution on* $[0, 1]$*. Given a graph* G *with* n *nodes and a uniform price vector* $\mathbf{p} = p \cdot \mathbf{1}$*, it is NP-hard to approximate the worst-case revenue to within a factor* $n^{1-\epsilon}$*.*

4.3 General Graphs, Uniform Distribution

Motivated by the gap between best-case and worst-case revenue, and the approximation hardness, we explore an alternative approach. While Theorem 14 demonstrates that we cannot hope to compute the revenue generated by any given price, we show that for i.i.d. uniform distributions of agent valuations, the impact of the equilibrium and of the price choice can be decoupled, so that an approximately optimum price can be set even without knowledge of the network. It turns out that a price of $\frac{1}{2}$ gives a constant-factor approximation.

The key to our analysis is to show that, for the case of the uniform distribution, there is an underlying structure to each equilibrium that does not depend on the price chosen by the seller. Even further, it is possible to express revenue as the product of two terms, the first determined by the chosen price and the second by the structure of the equilibrium selected by the agents. This allows us to optimize the price term independently of the equilibrium structure.

Theorem 15 *Let* G *be an arbitrary network, and assume that the agents' valuations are drawn i.i.d. from the uniform distribution on* $[0, 1]$*. Then, the worst-case revenue obtained from offering a uniform price of* $\frac{1}{2}$ *is at least an* $\frac{e}{4}$ *fraction of the worst-case revenue for the optimum (network-specific) price. Formally:*

$$\min_{\mathbf{T} \in \mathcal{N}_{1/2}} \mathcal{R}(\tfrac{1}{2} \cdot \mathbf{1}, \mathbf{T}) \;\geq\; \frac{e}{4} \cdot \max_{\mathbf{p}=p \cdot \mathbf{1}} \min_{\mathbf{T} \in \mathcal{N}_{\mathbf{p}}} \mathcal{R}(\mathbf{p}, \mathbf{T}).$$

Proof. Given a price vector $\mathbf{p} = p \cdot \mathbf{1}$, an equilibrium $\mathbf{T} \in \mathcal{N}_{\mathbf{p}}$ is a vector such that $T_i \cdot \prod_{j \in N(i)} F(T_j) = p$, where $F(T_j) = \min\{1, T_j\}$ for the uniform distribution on $[0, 1]$. Note that a threshold $T_i > 1$ is "behaviorally" equivalent to a threshold $T_i = 1$, since the support of the valuations is $[0,1]$. Applying this definition of the distribution function, the equilibrium condition becomes

$$\text{(i) } T_i \in [p,1]; \qquad \text{(ii) } \prod_{j \in N(i)\cup\{i\}} T_j \le p; \qquad \text{(iii) } \prod_{j \in N(i)\cup\{i\}} T_j < p \implies T_i = 1.$$

The worst-case equilibrium for the price vector $\mathbf{p}$ can therefore be expressed as the following mathematical program:

$$\begin{array}{lll} \text{Minimize}_{\mathbf{T}\in\mathcal{N}_p} & \mathcal{R}(p\cdot\mathbf{1},\mathbf{T}) \;=\; p\cdot\sum_i(1-T_i) & \\ \text{subject to} & \prod_{j\in N(i)\cup\{i\}} T_j \le p & \text{for all } i \\ & \prod_{j\in N(i)\cup\{i\}} T_j < p \implies T_i = 1 & \text{for all } i \\ & p \le T_i \le 1 & \text{for all } i. \end{array}$$

We use the transformation $x_i = \frac{\log(1/T_i)}{\log(1/p)}$ (and thus $T_i = p^{x_i}$) in order to bring the program into a form in which the constraints carry only information about the graph and are independent of the price p. In addition, as a result, the objective function depends only on the price and not on the graph structure. This decouples the two aspects of the problem:

$$\begin{array}{lll} \text{Minimize}_{\mathbf{T}\in\mathcal{N}_p} & \mathcal{R}(p\cdot\mathbf{1},\mathbf{T}) \;=\; p\cdot\sum_i(1-\exp(-x_i\cdot\log(1/p))) & \\ \text{subject to} & \sum_{j\in N(i)\cup\{i\}} x_j \ge 1 & \text{for all } i \\ & \sum_{j\in N(i)\cup\{i\}} x_j > 1 \implies x_i = 0 & \text{for all } i \\ & 0 \le x_i \le 1 & \text{for all } i. \end{array} \tag{6}$$

For the range $y \in [0, \log(1/p)]$, elementary facts about the exponential function imply the following bounds: $\frac{(1-p)}{\log(1/p)} \cdot y \le 1 - e^{-y} \;\le\; y$. Writing X for the set of vectors $\mathbf{x}$ that are feasible for the program (6), we apply the bound on the exponential function to the program (6), obtaining that

$$p\cdot(1-p)\cdot\min_{\mathbf{x}\in X}\sum_i x_i \le \min_{\mathbf{T}\in\mathcal{N}_p}\mathcal{R}(p\cdot\mathbf{1},\mathbf{T}) \;\le\; p\cdot\log(1/p)\cdot\min_{\mathbf{x}\in X}\sum_i x_i.$$

Thus, we have upper and lower bounds on the value of the worst-case revenue for each price p. Notice that the upper bound is maximized for $p = 1/e$, so

$$\max_{\mathbf{p}=p\cdot\mathbf{1}}\min_{\mathbf{T}\in\mathcal{N}_\mathbf{p}}\mathcal{R}(\mathbf{p},\mathbf{T}) \le \frac{1}{e}\cdot\min_{\mathbf{x}\in X}\sum_i x_i.$$

On the other hand, setting $p = \frac{1}{2}$ maximizes the lower-bound, giving us that

$$\min_{\mathbf{T}\in\mathcal{N}_{1/2}}\mathcal{R}(\frac{1}{2}\cdot\mathbf{1},\mathbf{T}) \ge \frac{1}{4}\cdot\min_{\mathbf{x}\in X}\sum_i x_i \;\ge\; \frac{e}{4}\cdot\max_{\mathbf{p}=p\cdot\mathbf{1}}\min_{\mathbf{T}\in\mathcal{N}_\mathbf{p}}\mathcal{R}(\mathbf{p},\mathbf{T}). \;\blacksquare$$

The analysis above was based on choosing the Myerson Price for the uniform distribution, in order to maximize the "price component" of the product; the "equilibrium component" factored out, and contributed at most a constant-factor loss in revenue. One might suspect that the Myerson Price would provide a constant approximation for all (regular) distributions. However, in appendix A, we provide an example which shows that this is not the case, even for the complete network.

5 Concluding Remarks

In this paper, we initiated the study of revenue-maximal pricing for locally public goods. We conclude by discussing potential extensions and questions left open by our work.

Simultaneous vs. Sequential purchases In our model, all agents simultaneously observe the price of the good and decide whether to purchase. In an alternative scenario, the seller sequentially offers the good to each agent in turn, who can then decide whether to buy given the choices of those who came before. This leads to a pricing problem that is similar to the one studied here, except that the natural solution concept for the pricing game becomes the subgame perfect equilibrium rather then Nash Equilibrium. Does the increased possibility of coordination in sequential sales unambiguously help or harm revenue?

Imperfect public goods We consider scenarios where the good is a perfect public good: the benefit of owning it and having a neighbor that owns it are the same. Most goods, however, have both public and private components. The purchase of a big-screen television provides some benefit to the purchaser's friends, who can visit and watch/play, but the greatest benefit goes to the purchaser himself. One can consider a utility model where $u_i(v_i, S) = v_i - p_i$ if $i \in S$, $(1-\epsilon)v_i$ if $i \notin S$ but $N(i) \cap S \neq \emptyset$, and $u_i(v_i, S) = 0$ otherwise.

Strength of social ties We assumed that all links of the social network are homogeneous. One could also consider the case in which network links are weighted. The weights might correspond to the extent to which benefits are shared along a link. In such a case, one can assume that the network is represented by a matrix w_{ij} where $w_{ii} = 1$, $w_{ij} = 0$ for $j \notin N(i) \cup \{i\}$ and $w_{ij} \in [0,1]$ otherwise. Then, the utility of agent i for an outcome S is given by $u_i(v_i, S) = v_i \cdot \max_{j \in S} w_{ij} - p_i \cdot \mathbb{1}\{i \in S\}$. An even further generalization is to consider a submodular function $f_i : 2^{[n]} \to \mathbb{R}_+$ for each agent such that his utility is given by $u_i = v_i \cdot f_i(S) - p_i \cdot \mathbb{1}\{i \in S\}$.

Other applications and objectives We believe that our model and techniques can be useful for additional related settings. Consider, for example, the following *snow-shoveling* setting. Suppose that a landlord of an apartment building wants to make sure that snow is shoveled from the sidewalk in front of his building. Thereto, he imposes a fine on each tenant in the case that the sidewalk is not shoveled. The tenants now face a problem that is similar in spirit to purchasing a public good. Each tenant incurs a personal cost from snow shoveling, drawn from some distribution, and needs to decide whether to exert effort (and incur the associated cost), or else pay the fine if none of the other tenants shoveled. The landlord, in determining the fine, must balance between different objectives, such as getting the snow shoveled, his own revenue, and the social welfare of the tenants. This is an example of a broader class of problems in which a policy maker must decide on mechanisms that are only applicable to individuals in order to encourage group behaviors. This example illustrates the appeal of this problem with objectives other then revenue. We believe that this problem has a structure similar to public goods, and that our techniques might be useful there.

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A Revenue Gap Examples

Example 16 (Revenue Gap for Myerson Reserve) Let the underlying network be the clique $G = K_n$, and the valuations drawn i.i.d. from the exponential distribution $F(v) = 1 - e^{-v}$. First, we notice that for any price p, the only equilibrium is the symmetric one: Given any price p, each equilibrium $\mathbf{T} \in \mathcal{N}_p$ satisfies $T_i \prod_{j \neq i} F(T_j) = p$ for all i, so $\frac{F(T_i)}{T_i} = \frac{F(T_j)}{T_j}$ for any $i \neq j$. The fact that F is strictly concave implies that $\frac{F(T)}{T}$ is strictly monotone decreasing. Therefore, $\frac{F(T_i)}{T_i} = \frac{F(T_j)}{T_j}$ implies that $T_i = T_j$. So for each price, the only equilibria are symmetric ones, characterized by the equation $T \cdot F(T)^{n-1} = p$.

The exponential distribution has virtual value function $\phi(v) = v - 1$; therefore, the Myerson Reserve Price is $r = 1$. For this price, the threshold T applied by the agents satisfies $1 = T(1 -$

$e^{-T})^{n-1} =: g(T)$. Note that $g(T)$ is strictly increasing in T. We write $m = n-1$ for convenience. Let $T_1 = \log m - \log\log\log m$. Then,

$$\begin{aligned} g(T_1) &= (\log m - \log\log\log m)\left(1 - \frac{\log\log m}{m}\right)^{m-1} \\ &< (\log m)\left(1 - \frac{\log\log m}{m}\right)^{\frac{m}{\log\log m}\log\log m}\left(1 - \frac{\log\log m}{m}\right)^{-1} \\ &< (\log m)\cdot e^{-\log\log m} \;=\; 1. \end{aligned}$$

Thus, $T = g^{-1}(1)$ is greater than T_1, and hence, $F(T) > F(T_1) = 1 - \frac{\log\log(n-1)}{(n-1)}$. We therefore have $\mathcal{R}(1) = \mathcal{R}(1,T) = n \cdot (1 - F(T)) < 2\log\log n$.

On the other hand, if we set $p = \log n \cdot (1 - 1/n)^{n-1}$, the corresponding threshold is $T' = g^{-1}(p) = \log n$. We then have $\mathcal{R}(p) = \mathcal{R}(p, T') = pn \cdot (1 - F(T')) = p > \frac{1}{4}\log n$. The gap between the optimal revenue and the revenue of the Myerson reserve price can therefore be as large as $\Omega(\frac{\log n}{\log\log n})$.

Example 17 (Linear gap in revenue among equilibria) Consider a graph G composed of a cycle of size 5; in addition, for every triple of 3 out of the 5 nodes in the cycle, there is a set of N nodes, each connected to each of the nodes in the triple. The total number of nodes is $n = 10N + 5$.

We choose F to be the uniform distribution on $[0,1]$. Pick an arbitrary price $p \in (0,1)$. We now consider two equilibria from $\mathcal{N}_p$. In the first, each of the 5 buyers on the cycle chooses a threshold of $p^{1/3}$, and each of the remaining $10N$ buyers chooses a threshold of 1. Note that $T_i \cdot \prod_{j \in N(i)} F(T_j) = (p^{1/3})^3 = p$ for each buyer i, so this is indeed an equilibrium. The revenue at this equilibrium is $5p(1 - p^{1/3})$.

For the second equilibrium, two non-adjacent buyers on the cycle will choose threshold p. The remaining three buyers on the cycle, say S, will choose threshold 1. The N buyers who are adjacent to the three nodes in S will choose threshold p; the remaining $9N$ buyers will choose threshold 1. Note that $T_i \prod_{j \in N(i)} F(T_j) \le p$ for each buyer i. Moreover, for each buyer for whom $T_i \prod_{j \in N(i)} F(T_j) < p$ (i.e., each buyer with multiple neighbors who have threshold p), we have $T_i = 1$. The proposed thresholds therefore form an equilibrium. The revenue at this equilibrium is $(N+2) \cdot p \cdot (1-p)$.

Taking N to be arbitrarily large relative to p, we have that the gap between the revenue of these two equilibria is $\Omega(N) = \Omega(n)$.

B Proof of Theorem 14

In this section, we provide a proof of Theorem 14. We restate the theorem here for convenience.

Theorem 14 *Assume that all agents' valuations are drawn i.i.d. from the uniform distribution on* $[0,1]$. *Given a graph G with n nodes and a uniform price vector* $\mathbf{p} = p \cdot \mathbf{1}$, *it is NP-hard to approximate the worst-case revenue to within a factor* $n^{1-\epsilon}$.

Proof. We will use the characterization of the worst-case revenue with price p established in the proof of Theorem 15. There, we showed that the revenue can be characterized by the mathematical program (6), repeated here for convenience:

$$\begin{array}{lll} \text{Minimize} & p \cdot \sum_i (1 - \exp(-x_i \cdot \log(1/p))) & \\ \text{subject to} & \sum_{j \in N(i) \cup \{i\}} x_j \ge 1 & \text{for all } i \\ & \sum_{j \in N(i) \cup \{i\}} x_j > 1 \implies x_i = 0 & \text{for all } i \\ & 0 \le x_i \le 1 & \text{for all } i. \end{array}$$

To prove approximation hardness, we give a reduction from 3-SAT. More precisely, given a 3-SAT instance, we construct an instance of the pricing problem such that if a formula is satisfiable, the revenue is $O(1)$, whereas if it is not satisfiable, the revenue is $\Omega(n^{1-\epsilon})$.

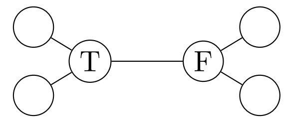

Figure 1: A variable gadget.

The reduction Given a 3-SAT formula with k clauses and m variables, construct an instance of the pricing problem as follows. For each variable in the formula, construct a "gadget," as depicted in Figure 1; the nodes T and F will correspond to true and false assignments, respectively. For each clause, we introduce k^L nodes (for some fixed large L), each connected to three other nodes, as follows. For every positive literal in the clause, each of the k^L clause nodes is connected to the T-node of the corresponding variable gadget, and for every negative literal, each of the k^L clause nodes is connected to the F-node of the corresponding variable gadget

An observation on the resulting instance We observe the following fact on the instance of the pricing problem obtained from the reduction. Let u and v be the respective T-node and F-node of a variable gadget. We claim that all feasible solutions to the program (6) have $x_u, x_v \in \{0,1\}$. Indeed, let w_1 and w_2 be the leaf nodes of the variable gadget attached to u. We know by the first constraint of the program that $x_{w_1} \geq 1 - x_u$ and $x_{w_2} \geq 1 - x_u$. Therefore, $\sum_{i \in N(u) \cup \{u\}} x_i \geq x_u + 2 \cdot (1 - x_u) = 2 - x_u$. If $x_u < 1$, then $\sum_{i \in N(u) \cup \{u\}} x_i > 1$, and the second condition in (6) implies that $x_u = 0$. The same analysis applies to x_v.

Also notice that $x_u = x_v = 1$ violates the second condition, since $\sum_{i \in N(u) \cup \{u\}} x_i \geq 2 > 1$, but $x_u \neq 0$. Therefore, the pair (x_u, x_v) must be one of the set $\{(0,1), (1,0), (0,0)\}$.

We also observe that if i is a clause node that is connected to a variable node u with $x_u = 1$, then $x_i = 0$ by the second condition.

Satisfiable formula implies low worst-case revenue If the formula is satisfiable, consider a satisfying truth assignment. If a certain variable is assigned True, then set $x_u = 1$ for its T-node and $x_v = 0$ for its F-node. Apply the opposite assignment for a variable that is assigned False. For the leaf nodes of the variable gadget, set their values to be one minus the value of their neighbor. Finally, set $x_v = 0$ for all all clause nodes v. Since the assignment is satisfiable, each clause node is connected to at least one node that is assigned 1. Therefore, the worst-case revenue is at most $p \cdot (1-p) \cdot 3 \cdot m$: each node assigned 1 produces revenue $p(1-p)$, and there are exactly $3m$ such nodes.

Unsatisfiable formula implies high worst-case revenue If the formula is not satisfiable, then for every assignment of values x_v to nodes that is feasible for (6), there exists at least one clause whose nodes are connected to three nodes with value 0. By the first condition of the program, all such clause variables must be set to 1. The revenue is therefore at least $3m + k^L$.

Since the graph has $n = 6m + k \cdot k^L$ nodes, by setting L to be sufficiently large, one cannot distinguish between a solution of revenue $n^{1-1/L}$ and a solution of revenue $O(1)$. This gives an $\Omega(n^{1-\epsilon})$ hardness for approximating the worst-case revenue, for every $\epsilon > 0$. ∎