# Does Treewidth Help in Modal Satisfiability? (Extended Abstract)

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**Abstract.** Many tractable algorithms for solving the Constraint Satisfaction Problem (CSP) have been developed using the notion of the treewidth of some graph derived from the input CSP instance. In particular, the *incidence graph* of the CSP instance is one such graph. We introduce the notion of an incidence graph for modal logic formulae in a certain normal form. We investigate the parameterized complexity of modal satisfiability with the modal depth of the formula and the treewidth of the incidence graph as parameters. For various combinations of Euclidean, reflexive, symmetric and transitive models, we show either that modal satisfiability is FPT, or that it is W[1]-hard. In particular, modal satisfiability in general models is FPT, while it is W[1]-hard in transitive models. As might be expected, modal satisfiability in transitive and Euclidean models is FPT.

# 1 Introduction

Treewidth as a parameter has been very successful in obtaining Fixed Parameter Tractable (FPT) algorithms for many classically intractable problems. One such class of problems is constraint satisfaction and closely related problems like satisfiability in propositional logic and the homomorphism problem [8, 30]. There have been recent extensions to quantified constraint satisfaction [6, 27]. In such problems, treewidth is used as a measure of modularity inherent in the given problem instance and algorithms make use of the modularity to increase their efficiency. Understanding the extent to which treewidth can be stretched in such problems is an active area of research [24, 15]. This work explores the applicability of such techniques to modal satisfiability.

Apart from having many applications (reasoning about knowledge [10], programming [28] and hardware verification [29] etc.), modal logics have nice computational properties [33, 14]. Many tools have been built for checking satisfiability of modal formulae [21, 26], despite being intractable in the classical sense (PSPACE-complete or NP-complete in most cases). Complexity of modal logic decision problems is well studied [23, 17, 16]. Another motivation for this work is to strengthen the complexity classification of modal logics through the refined analysis offered by parameterized complexity.

Our results: It is known that any modal logic formula can be effectively converted into a Conjunctive Normal Form (CNF) [9, 20]. Given a modal logic formula in CNF, we associate a graph with it. Restricted to propositional CNF formulae (which are modal formulae with modal depth 0), this graph is precisely the *incidence graph* associated with propositional CNF formulae (see [30] for details). We prove that

- 1. with the treewidth of the graph and the modal depth of the formula as parameters, satisfiability in general models is FPT,
- 2. with treewidth and modal depth as parameters, satisfiability in transitive models is W[1]-hard and
- 3. with treewidth as the parameter, satisfiability in models that are Euclidean<sup>1</sup> and any combination of reflexive, symmetric and transitive is FPT.

Since modal formulae of modal depth 0 contain all propositional formulae, bounding modal depth alone will not give FPT results (unless PTIME=NP). The main idea behind our FPT results is to express satisfiability of a modal formula in Monadic Second Order (MSO) logic over the formula's associated graph and then apply Courcelle's theorem [7]. Modal formulae with low treewidth are quite powerful, capable of encoding complex problems (see the conclusion for relevant pointers). On the other hand, modal formulae with low treewidth contain propositional CNF formulae of low treewidth, which arise naturally in many practical applications. See [12, Section 1.4] and references therein for some context on this.

Related work: In [16], Halpern considers the effect of bounding different parameters (such as the number of propositional variables, modal depth etc., but not treewidth) on complexity. In [25], Nguyen shows that satisfiability of many modal logics reduce to PTIME under the restriction of Horn fragment and bounded modal depth. In [1], Achilleos et. al. consider parameterized complexity of modal satisfiability in general models with the number of propositional variables and other structural aspects (but not treewidth) as

<sup>&</sup>lt;sup>1</sup> A binary relation  $\mapsto$  is Euclidean if  $\forall x, y, z, x \mapsto y$  and  $x \mapsto z$  implies  $y \mapsto z$ .

parameters. In [2], Adler et. al. associate treewidth with First Order (FO) formulae and use it to obtain a FPT algorithm for model checking.

The Complexity of satisfiability of modal logics follow a pattern. In [18], Halpern et. al. prove that with the addition of Euclidean property, complexity of (infinitely) many modal logics drop from PSPACE-hard to NP-complete. [19] is another work in this direction. Similar pattern is observed in graded modal logics [22]. With treewidth and modal depth as parameters, our results indicate similar behaviour in the world of parameterized complexity — satisfiability in transitive models is W[1]-hard, while satisfiability in Euclidean and transitive models is FPT, even with treewidth as the only parameter. However, more work is needed in this direction. First, the results in [18, 19] hold for infinitely many cases while we consider only a few fixed cases. Second, satisfiability in general models is PSPACE-complete and drops to NP-complete with the addition of Euclidean property. In our setting, satisfiability in general models is already FPT (but see conclusion for a discussion about why satisfiability in general models is not FPT unless PTIME=NP, when treewidth is the only parameter).

# 2 Preliminaries

Let  $\mathbb{N}$  denote the set of natural numbers. For  $k \in \mathbb{N}$ , we denote the set  $\{1, \ldots, k\}$  by [k]. We use standard notation about parameterized complexity like FPT algorithms, FPT reductions and W[1]-hardness from [13]. We will also use notation and definitions of relational structures and their tree decompositions from [13]: a *relational vocabulary*  $\tau$  is a set of relation symbols. Each relation symbol R has an arity  $arity(R) \geq 1$ . A  $\tau$ -structure S consists of a set D called the *domain* and an interpretation  $R^S \subseteq D^{arity(R)}$  of each relation symbol  $R \in \tau$ . A graph is an  $\{E\}$ -structure, where E is a binary edge relation. A *tree* is a graph without cycles. A path decomposition is a tree decomposition [13, Definition 11.23] whose underlying tree is a path. The pathwidth of a structure is the minimum of the widths of all path decompositions. It is known that computing optimal tree and path decompositions of a relational structure is FPT when parameterized by treewidth; cf. [13, Corollary 11.28] and [5].

Courcelle's theorem ([13, Theorem 11.37]) states that given a relational structure and a MSO sentence, checking whether the MSO sentence is true in the structure is FPT when parameterized by the treewidth of the structure and the length of the sentence.

We use standard notation for modal logic from [3]: well formed modal logic formulae are defined by the grammar  $\phi ::= q \in \Phi \mid \perp \mid \neg \phi \mid \phi \lor \psi \mid \Diamond \phi \mid \Box \phi$ , where  $\Phi$  is a set of propositional variables. A Kripke model for the basic modal language is a triple  $\mathcal{M} = (W, \mapsto, Vl)$ , where W is a set of worlds,  $\mapsto$  is a binary accessibility relation on W and  $V\iota: W \times \Phi \to \{\top, \bot\}$  is a valuation function. For  $w, v \in W$ , if  $w \mapsto v$ , v is said to be a successor of w. The pair  $(W, \mapsto)$  is called the frame  $\mathcal{A}$  underlying  $\mathcal{M}$ . If  $\mapsto$  is reflexive, then  $\mathcal{A}$  and  $\mathcal{M}$  are said to be a reflexive frame and a reflexive model respectively. Similar nomenclature is followed for other properties of  $\mapsto$ . The relation  $\mapsto$  is Euclidean if for all  $w_1, w_2, w_3, w_1 \mapsto w_2$  and  $w_1 \mapsto w_3$ implies  $w_2 \mapsto w_3$ . We denote the fact that a modal formula  $\phi$  is satisfied at a world w in a model  $\mathcal{M}$  by  $\mathcal{M}, w \models \phi$ . For  $q \in \Phi, \mathcal{M}, w \models q$  iff  $V_l(w, q) = \top$ . Negation  $\neg$  and disjunction  $\lor$  are treated in the standard way. For any formula  $\phi$ ,  $\mathcal{M}, w \models \Diamond \phi$  ( $\mathcal{M}, w \models \Box \phi$ ) iff some (all) successor(s) v of w satisfy  $\mathcal{M}, v \models \phi$ . A modal formula  $\phi$  is satisfiable if there is a model  $\mathcal{M}$  and a world w in  $\mathcal{M}$  such that  $\mathcal{M}, w \models \phi$ . A world w' is said to be *reachable* from w if there are worlds  $w_1, w_2, \ldots, w_m$  such that  $w \mapsto w_1 \mapsto \cdots \mapsto w_m \mapsto w'$ . It is well known that if some modal formula is satisfied at some world w in some Kripke model, discarding worlds not reachable from w does not affect satisfiability [3, Proposition 2.6]. Henceforth, if some modal formula is satisfied at some world w in some Kripke model  $\mathcal{M}$ , we will assume that  $\mathcal{M}$  consists of only those worlds reachable from w. Satisfiability in general, reflexive and transitive models are all PSPACE-complete [23], while in equivalence models, it is NP-complete [23].

The modal depth  $\operatorname{md}(\phi)$  of a modal formula  $\phi$  is inductively defined as follows.  $\operatorname{md}(q) = \operatorname{md}(\bot) = 0$ .  $\operatorname{md}(\neg \phi) = \operatorname{md}(\phi)$ .  $\operatorname{md}(\phi \lor \psi) = \max{\operatorname{md}(\phi), \operatorname{md}(\psi)}$ .  $\operatorname{md}(\Diamond \phi) = \operatorname{md}(\Box \phi) = \operatorname{md}(\phi) + 1$ . We will use the Conjunctive Normal Form (CNF) for modal logic defined in [20]:

 $\begin{array}{ll} \textit{literal} ::= q \mid \neg q \mid \Box \textit{clause} \mid \Diamond \textit{CNF} \\ \textit{clause} ::= \textit{literal} \mid \textit{clause} \lor \textit{clause} \mid \bot \\ \textit{CNF} ::= \textit{clause} \mid \textit{CNF} \land \textit{CNF} \end{array}$ 

where q ranges over  $\Phi$ . Any arbitrary modal formula  $\phi$  can be effectively transformed into CNF preserving satisfiability [9]. A *CNF* is a conjunction of clauses and a *clause* is a disjunction of literals. A *literal* is either a propositional variable, a negated propositional variable or a formula of the form  $\Box$  *clause* or  $\Diamond$  *CNF*. If one of the many literals in a clause is  $\perp$ , then  $\perp$  can be ignored without affecting satisfiability. A literal of the form  $\Diamond \perp$  can similarly be ignored. However, a clause that has  $\perp$  as the only literal cannot be ignored since  $\Box \perp$  is satisfied by a world in some Kripke model iff that world has no successors. Henceforth, we will assume that  $\perp$  occurs only inside sub-formulae of the form  $\Box \perp$ .

Suppose  $\phi$  is a modal formula in CNF. If  $\phi$  is of the form  $clause_1 \wedge clause_2 \wedge \cdots \wedge clause_m$ , then  $clause_1, clause_2, \ldots, clause_m$  and all literals appearing in these clauses are said to be at  $level \operatorname{md}(\phi)$ . If  $\Box clause_1$  is a *literal* at some level *i*, then  $clause_1$  and all literals occurring in  $clause_1$  are said to be at level i - 1. If  $\Diamond CNF$  is a literal at some level *i* and CNF is of the form  $clause_1 \wedge clause_2 \wedge \cdots \wedge clause_m'$ , then  $clause_1, clause_2, \cdots, clause_{m'}$  and all literals appearing in these clauses are said to be at level i - 1. Note that a single propositional variable can occur in the form of a *literal* at different levels. The concept of level is similar to the concept of distance defined in [26]. The process of checking satisfiability we describe in section 3 can be considered a variant of the level-based bottom-up algorithm given in [26], which is also implicitly used in [1, Theorem 5]. It requires more work and combination of other ideas to prove that this process can be formalized in MSO logic.

#### 3 Modal satisfiability in general models

In this section, we will associate a relational structure with a modal CNF formula. We show that checking satisfiability of a modal CNF formula is FPT, parameterized by modal depth and the treewidth of the associated relational structure. We begin with an example modal CNF formula.

Consider the modal CNF formula  $\{\neg q \lor \Box [r \lor \neg s]\} \land \{q \lor \Diamond \bot\} \land \{r \lor \Diamond [\neg s]\} \land \{\neg r \lor \Diamond [(t \lor \neg s) \land (r)]\}$ . Its modal depth is 1 and has 4 clauses at level 1. Figure 1 shows a graphical representation of this formula, which is very similar to the formula's syntax tree. The 4 clauses at level 1 are represented by  $e_1, e_2, e_3$  and  $e_4$ .  $e_1$  represents the clause  $\{\neg q \lor \Box [r \lor \neg s]\}$ . Since  $\neg q$  occurs as a literal in this clause, there is a dotted arrow from  $e_1$  to q.  $\Box [r \lor \neg s]$  (represented by  $e_9$ ) also occurs as a literal in clause  $e_1$  and hence there is an arrow from  $e_1$  to  $e_9$ .  $e_4$  represents the fourth clause at level 1, which contains  $\Diamond [(t \lor \neg s) \land (r)]$  as a literal. This  $\Diamond CNF$  formula is represented by  $e_{10}$ . The two clauses  $(t \lor \neg s)$  and (r) are represented by  $e_7$  and  $e_8$  respectively and are connected to  $e_{10}$  by arrows. The propositional variable r occurs as literal at 2 levels, indicated as  $Lv_0$  and  $Lv_1$ .



**Fig. 1.** Relational structure associated with the modal formula  $\{\neg q \lor \Box [r \lor \neg s]\} \land \{q \lor \Diamond \bot\} \land \{r \lor \Diamond [\neg s]\} \land \{\neg r \lor \Diamond [(t \lor \neg s) \land (r)]\}$ 

Now we will formalize the above example. The intuition behind the following definition is to represent all clauses and literals of a modal CNF formula by the domain elements of a relational structure. Binary relations are used to indicate which literals occur in which clause (and which clauses occur in which literal). Unary relations are used to indicate which elements represent literals and which elements represent clauses. This will enable us to reason about clauses, literals and their dependencies using MSO formulae over the relational structure.

**Definition 3.1.** Given a modal CNF formula  $\phi$ , we associate with it a relational structure  $S(\phi)$ . It will have one domain element for every literal of the form  $\Box$  clause or  $\Diamond$  CNF in  $\phi$ . It will also have one domain element for every propositional variable used in  $\phi$ . There are no domain elements representing the propositional constant  $\bot$ . They will be handled as special cases.

The relational structure will have two binary relations  $O_c$  (occurs) and  $\overline{O_c}$  (occurs negatively).  $\overline{O_c}(e_1, e_2)$ iff  $e_1$  represents a clause and  $e_2$  represents a propositional variable occurring negated as a literal in the clause represented by  $e_1$ . If  $e_1$  represents a clause, then  $O_c(e_1, e_2)$  iff  $e_2$  represents a literal (occurring in the clause represented by  $e_1$ ) of the form  $\Box$  clause,  $\Diamond CNF$  or a non-negated propositional variable. If  $e_1$  represents a literal of the form  $\Box$  clause, then  $O_c(e_1, e_2)$  iff  $e_2$  represents the corresponding clause. If  $e_1$  represents a literal of the form  $\Diamond CNF$ , then  $O_c(e_1, e_2)$  iff  $e_2$  represents a clause in the corresponding CNF. Finally, the following unary relations are present:

 $C_{1} : contains all domain elements representing clauses$   $L_{t} : all domain elements representing literals$   $U : all literals of the form \Box \bot$   $B_{\Box} : all literals of the form \Box clause$   $D\diamond : all literals of the form \diamond CNF$ 

 $(L_{v_i})_{0 \le i \le md(\phi)}$ : all clauses and literals at level i

For clauses and literals of the form  $\Box$  clause or  $\Diamond CNF$ , there is one domain element for every occurrence of the clause or literal. For example, if the literal  $\Diamond (q_1 \land q_2)$  occurs in two different positions of a big formula  $\phi$ , the two occurrences will be represented by two different domain elements in  $\mathcal{S}(\phi)$ . In contrast, different occurrences of a literal that is just a propositional variable will be represented by the same domain element. In the rest of the paper, whenever we refer to the treewidth of a modal CNF formula  $\phi$ , we mean the treewidth of  $\mathcal{S}(\phi)$ .

If  $e_1$  represents a *clause*,  $O_c(e_1, e_2)$  means that the clause represented by  $e_1$  can be satisfied by satisfying the literal represented by  $e_2$ .  $\overline{O_c}(e_1, e_2)$  means that the clause represented by  $e_1$  can be satisfied by setting the propositional variable represented by  $e_2$  to false.

If  $C_{\ell_0} \subseteq C_l \cap L_{v_0}$  is a subset of domain elements representing clauses at level 0, let  $CNF(C_{\ell_0})$  be the modal CNF formula that is the conjunction of clauses represented by domain elements in  $C_{\ell_0}$ . We will now see how to check satisfiability of  $CNF(\{e_7, e_8\})$  in Fig. 1 and describe the generalization of this process given in (1) below. We use  $c_\ell$  and  $l_t$  for first order variables intended to represent clauses and literals respectively. First of all, there must be a subset  $T_{r_0} \subseteq \{r, s, t\} = L_t \cap L_{v_0}$  that will be set to  $\top$ , as written in the beginning of (1). Then, we must check that this assignment satisfies each clause  $c_\ell$  in  $C_{\ell_0}$ , written as  $\forall c_\ell \in C_{\ell_0}$  in (1). To check that the clause represented by  $e_7$  is satisfied, either a positively occurring literal like t must be set to  $\top$  and hence in  $T_{r_0}$  (written as " $\exists l_t \in T_{r_0} : O_c(c_\ell, l_t)$ " in (1)) or a negatively occurring literal like s must be set to  $\perp$  and hence not in  $T_{r_0}$  (" $\exists l_t \in (L_t \cap L_{v_0}) \setminus T_{r_0} : O_c(c_\ell, l_t)$ " in (1)). A similar argument applies to  $e_8$  as well.

$$\xi[0](C_{\ell_0}) \stackrel{\Delta}{=} \exists T_{r_0} \subseteq (Lt \cap Lv_0) : \forall c_\ell \in C_{\ell_0} : \left[ (\exists lt \in T_{r_0} : O_c(c_\ell, lt)) \lor (\exists lt \in (Lt \cap Lv_0) \setminus T_{r_0} : \overline{O_c}(c_\ell, lt)) \right]$$
(1)

$$\xi[i](C_{\ell_i}) \stackrel{\simeq}{=} \exists T_{r_i} \subseteq (L_t \cap L_{v_i}) : \forall c_\ell \in C_{\ell_i} : \left[ (\exists l_t \in T_{r_i} : O_c(c_\ell, l_t)) \lor (\exists l_t \in (L_t \cap L_{v_i}) \setminus T_{r_i} : \overline{O_c}(c_\ell, l_t)) \right] \land [C_{m_{i-1}} = \{c_\ell' \in (C_l \cap L_{v_{i-1}}) \mid \exists l_t' \in T_{r_i} \cap B_{\Box}, O_c(l_t', c_\ell')\} \Rightarrow \forall l_t \in T_{r_i} \cap D_\diamond : D_{m_{i-1}} = \{c_\ell \in (C_l \cap L_{v_{i-1}}) \mid O_c(l_t, c_\ell)\} \Rightarrow \xi[i-1](D_{m_{i-1}} \cup C_{m_{i-1}})]$$
(2)

Checking satisfiability at higher levels is slightly more complicated. Suppose  $C_{\ell_i} \subseteq C_l \cap L_{v_i}$  is a subset of clauses at level i. We will take  $C_{\ell_1} = \{e_1, e_3, e_4\}$  from Fig. 1 as an example. If some world w in some Kripke model  $\mathcal{M}$  satisfies  $CNF(C_{\ell_1})$ , there must be some subset  $T_{r_1}$  of literals at level 1 satisfied at w  $(\exists T_{r_i} \subseteq (L_t \cap L_{v_i}))$ " in (2)). As before, we check that for every clause represented in  $C_{\ell_1}$  (" $\forall c_\ell \in C_{\ell_i}$ " in (2)), there is either a positively occurring literal in  $T_{r_1}$  (" $\exists l_t \in T_{r_i} : O_c(c_\ell, l_\ell)$ " in (2)) or a negatively occurring literal not in  $T_{r_1}$  (" $\exists l_t \in (L_t \cap L_{v_i}) \setminus T_{r_i} : \overline{O_c}(c_\ell, l_t)$ " in (2)). Next, we must check that the literals we have chosen to be satisfied at w (by putting them into  $T_{r_1}$ ) can actually be satisfied. Suppose  $T_{r_1}$  was  $\{e_9, q, r, e_{10}\}$ . Since  $e_9$  represents a literal of the form  $\Box$  clause (with the clause represented by domain element  $e_5$ ), we are committed to satisfy the clause represented by  $e_5$  in any world succeeding w. Let  $C_{m_0} = \{e_5\}$  be the set of clauses occurring at level 0 that we have committed to as a result of choosing corresponding  $\Box$  clause literals to be in  $T_{r_1}$  (" $C_{m_{i-1}} = \{c_i \in (C_i \cap L_{v_{i-1}}) \mid \exists l_i \in T_{r_i} \cap B_{\Box}, O_c(l_i, c_i')\}$ " in (2)). Now, since we have also chosen  $e_{10}$  to be in  $T_{r_1}$  and  $e_{10}$  represents a  $\Diamond CNF$  formula, there is a demand to create a world w' that succeeds w and satisfies the corresponding CNF formula. We have to check that every such demand in  $T_{r_1}$  can be satisfied (" $\forall lt \in T_{r_i} \cap D \diamond$ " in (2)) by creating successor worlds. In case of the demand created by  $e_{10}$ ,  $\{e_7, e_8\} = D_{m_0}$  is the set of clauses in the demanded CNF formula  $("D_{m_{i-1}} = \{c_{\ell} \in (C_{\ell} \cap L_{v_{i-1}}) \mid O_{c}(l_{\ell}, c_{\ell})\}"$  in (2)). Our aim now is to create a successor world w' in which all clauses represented in  $D_{m_0}$  are satisfied. However, w' is a successor world and we have already committed to satisfying all clauses represented in  $C_{m_0}$  in all successor worlds. Hence, we actually check if the clauses represented in  $C_{m_0} \cup D_{m_0}$  are satisfiable by inductively invoking  $\xi[0](D_{m_0} \cup C_{m_0})$  (" $\xi[i-1](D_{m_{i-1}} \cup C_{m_{i-1}})$ " in (2)).

For the sake of clarity, we have skipped handling literals of the form  $\Box \perp$  in the above discussion. They will be handled in the formal arguments that follow.

**Lemma 3.2.** The property  $\xi[i](C_{\ell_i})$  can be written in a MSO logic formula of size linear in *i*. If  $\phi$  is any modal formula in CNF and  $C_{\ell_i}$  is any subset of domain elements representing clauses at level *i*, then  $CNF(C_{\ell_i})$  is satisfiable iff  $\xi[i](C_{\ell_i})$  is true in  $S(\phi)$ .

*Proof.* We will prove the first claim by induction on *i*. Formula (3) below is same as (1) written in formal MSO syntax. (4) is a formal MSO statement of (2) and two additional conditions for handling literals of the form  $\Box \bot$  and  $\Diamond \bot$ . We will prove that the length  $|\xi[i]|$  of  $\xi[i]$  is linear in *i*. Let *c* be the length of  $\xi[i]$  without length of  $\xi[i-1]$  counted. As can be seen,  $|\xi[0]| \le c$ . Inductively assume that  $|\xi[i-1]| \le ic$ . Then,  $|\xi[i]| = c + |\xi[i-1]|$ . Hence,  $|\xi[i]| \le c + ic = c(i+1)$ .

We will now prove the second claim by induction on i.

Base case i = 0: The modal formula  $CNF(C_{\ell_0})$  is a propositional CNF formula. Suppose  $\xi[0](C_{\ell_0})$  is true in  $S(\phi)$ . Hence, there is a subset  $T_{r_0}$  of domain elements that satisfy the last four conditions of  $\xi[0]$  defined in (3). The second condition  $\forall x (T_{r_0}(x) \Rightarrow (Lt(x) \land Lv_0(x)))$  ensures that all domain elements in  $T_{r_0}$  are also in Lt and  $Lv_0$ . Hence, all domain elements in  $T_{r_0}$  represent literals at level 0. Since the only literals at level 0 are propositional variables or their negations,  $T_{r_0}$  is in fact a subset of propositional variables. Consider the Kripke model  $\mathcal{M}$  with a single world w at which, all propositional variables in  $T_{r_0}$  are set to  $\top$  and all others are set to  $\bot$ . We will now prove that all clauses represented in  $C_{\ell_0}$  are satisfied in w. Let  $c_\ell$  be some element in  $C_{\ell_0}$  representing some clause. Since  $C_{\ell_0}(c_\ell)$  is true and  $S(\phi)$  satisfies the last three conditions of  $\xi[0]$ , we have that either  $\exists lt (T_{r_0}(lt) \land O_c(c_\ell, lt))$  or  $\exists lt (Lt(lt) \land Lv_0(lt) \land \neg T_{r_0}(lt) \land \overline{O_c}(c_\ell, lt))$  is true in  $S(\phi)$ . In the first case,  $O_c(c_\ell, lt)$  means that lt is a positively occurring literal in the clause  $c_\ell$  and  $T_{r_0}(lt) \land Lt(lt) \land Lv_0(lt)$ means that lt is a literal negatively occurring in clause  $c_\ell$  and  $\neg T_{r_0}(lt)$  and  $T_{r_0}(lt) \land Lt(lt) \land Lv_0(lt)$ means that  $l_t$  is a literal negatively occurring in clause  $c_\ell$  and  $\neg T_{r_0}(lt)$  is not in  $T_{r_0}$  (and hence it is set to  $\bot$  in w, again satisfying clause  $c_\ell$ ).

Now suppose that there is a Kripke model  $\mathcal{M}$  and a world w such that  $\mathcal{M}, w \models CNF(C_{\ell_0})$ . We will prove that  $\xi[0](C_{\ell_0})$  is true in  $\mathcal{S}(\phi)$ . The first requirement is to find a suitable subset  $T_{r_0}$  of domain elements. We will set  $T_{r_0}$  to be the set of precisely those domain elements that represent propositional variables occurring at level 0 and set to  $\top$  in the world w. This will ensure that the condition  $\forall x (T_{r_0}(x) \Rightarrow (L_t(x) \land L_{v_0}(x)))$  in  $\xi[0]$  is satisfied. Now we have to prove that last three conditions of  $\xi[0]$  are satisfied. So let  $c_{\ell} \in C_{\ell_0}$  be any domain element so that it satisfies  $C_{\ell_0}(c_{\ell})$ . We have to now prove that this  $c_{\ell}$  satisfies one of the last two conditions of  $\xi[0]$ . Since  $c_{\ell} \in C_{\ell_0}$ , it represents a clause in  $\phi$  occurring at level 0. Since  $\mathcal{M}, w \models CNF(C_{\ell_0})$ , the clause represented by  $c_{\ell}$  is satisfied in w. Hence there is either a positively occurring propositional variable set to  $\top$  in w (so that it is in  $T_{r_0}$ , thus satisfying  $\exists l_t (T_{r_0}(l_t) \land O_c(c_{\ell}, l_t)))$  or a negatively occurring propositional variable set to  $\perp$  in w (so that it is not in  $T_{r_0}$ , thus satisfying  $\exists l_t (L_t(l_t) \land L_{v_0}(l_t) \land \neg T_{r_0}(l_t) \land \overline{O_c}(c_{\ell}, l_t)))$ . This completes the base case.

Induction step: Suppose  $C_{\ell_i}$  is a subset of domain elements representing clauses occurring at level i and  $\xi[i](C_{\ell_i})$  is true in  $\mathcal{S}(\phi)$ . We will build a Kripke model  $\mathcal{M}$  and prove that it has a world w such that  $\mathcal{M}, w \models CNF(C_{\ell_i})$ . We will start with a single world w. Since  $\xi[i](C_{\ell_i})$  is true in  $\mathcal{S}(\phi)$ , there must be a subset  $T_{r_i}$  of domain elements satisfying the last eleven conditions of  $\xi[i](C_{\ell_i})$ . The condition  $\forall x (T_{r_i}(x) \Rightarrow (Lt(x) \land Lv_i(x)))$  ensures that all domain elements in this  $T_{r_i}$  represent literals occurring at level i. Let  $PV(T_{r_i}) \subseteq T_{r_i}$  be those domain elements in  $T_{r_i}$  that represent propositional variables. Similarly, let  $BL(T_{r_i})$  and  $DL(T_{r_i})$  be the domain elements in  $T_{r_i}$  representing literals of the form  $\Box$  clause and  $\Diamond CNF$  respectively. In our world w,

set all propositional variables in  $PV(T_{r_i})$  to  $\top$  and set all others to  $\bot$ . Now, w satisfies all literals represented in  $PV(T_{r_i})$ . We will later prove how to satisfy literals represented in  $BL(T_{r_i})$  and  $DL(T_{r_i})$  in the world w.

Now, assuming that all literals represented in  $T_{r_i}$  are satisfied at w, we will prove that  $\mathcal{M}, w \models CNF(C_{\epsilon_i})$ . This part of the proof is similar to the base case. If  $c_{\ell}$  is any clause in  $C_{\ell_i}$ , it satisfies  $C_{\ell_i}(c_{\ell})$  and hence either  $\exists lt (T_{r_i}(lt) \land O_c(c_{\ell}, lt))$  or  $\exists lt (Lt(lt) \land Lv_i(lt) \land \neg T_{r_i}(lt) \land \overline{O_c}(c_{\ell}, lt))$  is true in  $\mathcal{S}(\phi)$ . In the first case, a positively occurring literal at level i is in  $T_{r_i}$ , and since all literals in  $T_{r_i}$  are satisfied at w, the clause represented by  $c_{\ell}$  is also satisfied at w. In the second case, a negatively occurring literal at level i is not in  $T_{r_i}$ . Since only propositional variables can occur negatively in clauses, we can in fact conclude that a negatively occurring propositional variable is not in  $T_{r_i}$ . Since all propositional variables not in  $T_{r_i}$  are set to  $\perp$  in w, the clause represented by  $c_{\ell}$  is satisfied in w.

Now we will prove that literals represented in  $BL(T_{r_i})$  and  $DL(T_{r_i})$  can be satisfied in w by adding appropriate successor worlds. First note that since  $\forall x ((T_{r_i}(x) \land D \diamond (x)) \Rightarrow \neg U(x))$  is true in  $\mathcal{S}(\phi)$ , no element x in  $T_{r_i}$  represents a literal of the form  $\Diamond \bot$  (since U is the unary relation containing all domain elements representing literals of the form  $\Box \bot$  or  $\Diamond \bot$ ). Second, note that since  $\exists x(T_{r_i}(x) \land D \diamond (x)) \Rightarrow$  $\forall y (T_{r_i}(y) \Rightarrow \neg U(y))$  is true in  $\mathcal{S}(\phi)$ , if  $DL(T_{r_i})$  is not empty, then no element in  $BL(T_{r_i})$  represents a literal of the form  $\Box \bot$ . Therefore, we can hope to add a new successor for each literal represented by some element  $l_t$  in  $DL(T_{r_i})$ , satisfying the CNF formula in the literal represented in  $l_t$  as well as all clauses in literals represented in  $BL(T_{r_i})$ . Now we will prove that this can actually be done.

Since  $\xi[i](C_{\ell_i})$  is true in  $\mathcal{S}(\phi)$ , there is a subset  $C_{m_{i-1}}$  of domain elements satisfying the last four conditions of  $\xi[i](C_{\ell_i})$ . The condition  $\forall c_\ell(C_{m_i-1}(c_\ell) \Leftrightarrow \exists l_t(T_{r_i}(l_\ell) \land B_{\Box}(l_\ell) \land O_c(l_t, c_\ell)))$  ensures that  $C_{m_i-1}$ contains exactly those domain elements representing some clause  $clause_1$  at level i-1 such that  $\Box clause_1$  is a literal in  $T_{r_i}$  (and hence  $\Box clause_1$  is in  $BL(T_{r_i})$ ). An element  $l_t$  satisfies  $T_{r_i}(l_t) \land D \diamond (l_t)$  iff  $l_t \in DL(T_{r_i})$ . Hence, the condition  $\forall lt ((T_{r_i}(lt) \land D \diamond (lt))) \Rightarrow [\cdots]$  ensures that the last two conditions of  $\xi[i](C_{\ell_i})$  is true for every element  $l_t$  in  $T_{r_i}$  representing a literal of the form  $\Diamond CNF$  occurring at level *i*. Consider any one such element lt. The condition  $\exists D_{m_{i-1}} \forall c \ell' (D_{m_{i-1}}(c\ell) \Leftrightarrow ((C_{m_{i-1}}(c\ell)) \lor O_c(lt, c\ell)))$  ensures that  $D_{m_{i-1}}$  contains exactly those elements representing some clause  $clause_1$  (occurring at level i-1) such that  $\Box$  clause<sub>1</sub> is represented in  $BL(T_{r_i})$  or clause<sub>1</sub> occurs in the CNF formula in the  $\Diamond$  CNF literal represented by lt. Since  $\xi[i-1](D_{m_{i-1}})$  is true, we can apply the induction hypothesis and conclude that there is some Kripke model  $\mathcal{M}'$  and a world w' such that  $\mathcal{M}', w' \models CNF(D_{mi-1})$ . Now, w' satisfies the CNF formula in the  $\Diamond CNF$  literal represented by lt. For every literal of the form  $\Box clause_1$  in  $BL(T_{r_i})$ , w' satisfies clause\_1. Now, we add the Kripke model  $\mathcal{M}'$  to  $\mathcal{M}$  and make w' a successor of w. We repeat this procedure for every element  $l_t$  in  $DL(T_{r_i})$ . Now, for every literal in  $T_{r_i}$  of the form  $\Diamond CNF$ , there is a successor of w that satisfies the corresponding CNF formula (we have already proved that literals of the form  $\Diamond \perp$  will not be present in  $T_{r_i}$ ). For every literal in  $T_{r_i}$  of the form  $\Box$  clause, all successors of w will satisfy the corresponding clause (we have already proved that if there is a literal of the form  $\Box \perp$  in  $T_{r_i}$ , then  $T_{r_i}$  will not have any literals of the form  $\Diamond CNF$  and hence we will not add any successor worlds to w).

Now we will prove the other direction of the induction step. Suppose  $C_{\ell_i}$  is a subset of domain elements representing clauses occurring at level *i* and that there is a Kripke model  $\mathcal{M}$  and a world *w* such that  $\mathcal{M}, w \models CNF(C_{\ell_i})$ . We will prove that  $\xi[i](C_{\ell_i})$  is true in  $\mathcal{S}(\phi)$ . To begin with, we will choose  $T_{r_i}$  to be the set of precisely those domain elements that represent literals occurring at level *i* that are satisfied at *w*. If literals of the form  $\Box \bot$  occur at level *i*, then they will also be included in  $T_{r_i}$  by definition if there are no successor worlds at *w*. Now we will prove that last eleven conditions of  $\xi[i](C_{\ell_i})$  are true in  $\mathcal{S}(\phi)$ . The condition  $\forall x (T_{r_i}(x) \Rightarrow (L_t(x) \land L_{v_i}(x)))$  is true since all elements *x* in  $T_{r_i}$  are representing literals  $(L_t(x))$ at level *i*  $(L_{v_i}(x))$ . Next we will prove that the condition  $\forall c_\ell C_{\ell_i}(c_\ell) \Rightarrow [\ldots]$  is true. Let  $c_\ell$  be some arbitrary element in  $C_{\ell_i}$ . Since  $c_\ell$  represents a clause that is satisfied at the world *w*, there must be either a positively occurring literal that is satisfied at *w* (and hence the domain element representing that literal will be in  $T_{r_i}$ , thus implying that  $\exists l_t (T_{r_i}(l_t) \land O_c(c_\ell, l_t))$  is true in  $\mathcal{S}(\phi)$ ) or there must be a negatively occurring literal that is not satisfied at *w*. In the latter case, since only propositional variables can occur negatively, we can in fact conclude that there is a negatively occurring propositional variable that is set to  $\bot$  at *w* (and hence not in  $T_{r_i}$ , which implies that  $\exists l_t (L_t(l_t) \land L_{v_i}(l_t) \land \neg T_{r_i}(l_t) \land \overline{O_c}(c_\ell, l_t))$  is true in  $\mathcal{S}(\phi)$ .

Next we will prove that the condition  $\forall x ((T_{r_i}(x) \land D \diamond (x)) \Rightarrow \neg U(x))$  is true. If any element x is in  $T_{r_i}(T_{r_i}(x))$  and represents a literal of the form  $\Diamond CNF(D \diamond (x))$ , then x will not represent  $\Diamond \bot (\neg U(x))$  since x represents a literal that is satisfied at w and  $\Diamond \bot$  cannot be satisfied.

Next we will prove that the condition  $\exists x(T_{r_i}(x) \land D \diamond (x)) \Rightarrow \forall y(T_{r_i}(y) \Rightarrow \neg U(y))$  is true. Suppose there is some element x in  $T_{r_i}$  that represents a literal of the form  $\Diamond CNF$  ( $\exists x(T_{r_i}(x) \land D \diamond (x))$ ). Since the literal represented by x is satisfied at w, there is a successor world in which the corresponding CNF formula is satisfied. Since w has successor worlds, it cannot satisfy  $\Box \bot$  and hence none of the elements in  $T_{r_i}$  represent literals of the form  $\Box \bot$  ( $\forall y(T_{r_i}(y) \Rightarrow \neg U(y)$ )).

Finally, we will prove that the condition  $\exists C_{m_{i-1}}[\ldots]$  is true. Let us first construct the set  $C_{m_{i-1}}$ . For any element  $c_{\ell}$  ( $\forall c_{\ell}$ ), we will put  $c_{\ell}$  in  $C_{m_{i-1}}$  iff  $(C_{m_{i-1}}(c_{\ell}) \Leftrightarrow)$  there is some element  $l_{\ell}$  ( $\exists l_{\ell}$ ) in  $T_{r_{i}}$  $(T_{r_i}(l_t))$  representing a literal of the form  $\Box$  clause  $(B \Box(l_t))$  such that  $c_\ell$  represents the corresponding clause  $(O_c(lt, c_\ell))$ . The condition  $\forall c_\ell (C_{m_{i-1}}(c_\ell) \Leftrightarrow \exists lt (T_{r_i}(lt) \land B_{\Box}(lt) \land O_c(lt, c_\ell)))$  is true in  $\mathcal{S}(\phi)$  by construction. Next we will prove that the condition  $\forall lt ((T_{r_i}(l_t) \land D \diamond (l_t))) \Rightarrow [\ldots]$  is true. Suppose  $l_t$  is any element in  $T_{r_i}$  representing a literal of the form  $\Diamond CNF_1$   $(D \diamond (l_t))$ . Since  $\Diamond CNF_1$  is satisfied at the world w, there is a successor world w' that satisfies the corresponding  $CNF_1$  formula. Let  $D_{m_{i-1}}$  be the set that includes some element  $c_{\ell}$  iff  $(\forall c_{\ell}' D_{m_{i-1}}(c_{\ell}') \Leftrightarrow) c_{\ell}'$  is in the set  $C_{m_{i-1}}$  constructed above  $(C_{m_{i-1}}(c_{\ell}'))$  or it represents a clause occurring in the  $CNF_1$  formula contained in the  $\Diamond CNF_1$  literal represented by  $lt (O_c(lt, c_{\ell'}))$ .  $D_{m_{i-1}}$ satisfies the condition  $\forall c\ell' (D_{m_i-1}(c\ell) \Leftrightarrow ((C_{m_i-1}(c\ell)) \lor O_c(lt, c\ell)))$  by construction. If  $c\ell'$  is any element in  $D_{m_{i-1}}$ , then it represents some  $clause_1$  at level i-1 such that  $clause_1$  occurs in the  $CNF_1$  formula contained in the  $\Diamond CNF_1$  literal represented by lt or  $\Box clause_1$  appears in  $T_{r_i}$ . Hence, all clauses represented in  $D_{m_{i-1}}$ are satisfied at w' (since w' is a successor of w that satisfies  $CNF_1$  and all literals of the form  $\Box clause$ represented in  $T_{r_i}$  are satisfied at w). By the induction hypothesis, we conclude that  $\xi[i-1](D_{m_{i-1}})$  is true in  $\mathcal{S}(\phi)$ . П

**Theorem 3.3.** Given a modal CNF formula  $\phi$ , there is a FPT algorithm that checks if  $\phi$  is satisfiable in general models, with treewidth of  $S(\phi)$  and modal depth of  $\phi$  as parameters.

Proof. Given  $\phi$ ,  $S(\phi)$  can be constructed in polynomial time. To check that all clauses of  $\phi$  at level md( $\phi$ ) are satisfiable in some world w of some Kripke model  $\mathcal{M}$ , we check whether the formula  $\exists C_{\ell_{\mathrm{md}}(\phi)} \forall c_{\ell}(C_{\ell_{\mathrm{md}}(\phi)}(c_{\ell}) \Leftrightarrow (C_{\ell_{\mathrm{rd}}(\phi)}(c_{\ell}))) \land \xi[\mathrm{md}(\phi)](C_{\ell_{\mathrm{md}}(\phi)})$  is true in  $S(\phi)$ . By Lemma 3.2, this is possible iff  $\phi$  is satisfiable and length of the above formula is linear in md( $\phi$ ). An application of Courcelle's theorem will give us the FPT algorithm.

#### 3.1 On the relevance of treewidth for modal logic

Informally, treewidth is a measure of how close a graph is to being a tree. Given a modal logic formula  $\phi$ , the associated structure  $\mathcal{S}(\phi)$  is very similar to the syntax tree of  $\phi$ . The structure  $\mathcal{S}(\phi)$  is not a tree (i.e., it has cycles) because a single propositional variable may be shared by many clauses of the formula. Thus, if very few variables are shared across clauses,  $\mathcal{S}(\phi)$  is very close to a tree, i.e.,  $\mathcal{S}(\phi)$  will have small treewidth. In the example of Fig. 1, if we replace q and s by r, the number of shared variables will increase. As can be seen in Fig. 2, the number of cycles will also increase. For example,  $e_1$  was not part of any cycle in Fig. 1 but forms a cycle with  $e_9, e_5$  and r in Fig. 2.



**Fig. 2.** Relational structure associated with the modal formula  $\{\neg r \lor \Box [r]\} \land \{r \lor \Diamond \bot\} \land \{r \lor \Diamond [\neg r]\} \land \{\neg r \lor \Diamond [(t \lor \neg r) \land (r)]\}$ 

Treewidth is a very fundamental concept and naturally arises in many contexts, even in industrial applications like software verification [32]. Applications of treewidth related techniques to propositional logic is extensively studied — see [12, Section 1.4] and references therein. Modal logic being a natural and very useful extension of propositional logic, we might expect some benefit by exploring applicability of treewidth related techniques to modal logic.

The set of modal formulas with small treewidth is powerful enough to encode complex formulas. In [1, Lemma 1], there is a translation of propositional CNF formulae into equivalent modal formulae. We can verify that the resulting modal formula always has a small constant treewidth (the resulting modal formula uses only one propositional variable). Hence, the restriction of bounded treewidth is not a severe one. Given a formula  $\phi$ ,  $S(\phi)$  can be computed in PTIME. Though computing treewidth of  $S(\phi)$  is NP-complete, it is FPT when parameterized by treewidth.

### 4 Models with Euclidean property

In this section, we will investigate the parameterized complexity of satisfiability in Euclidean models. The main observation leading to the FPT algorithm is the fact that if a modal formula is satisfied in a Euclidean model, then it is satisfied in a rather simple model. As proved in [22], if a modal formula is satisfied at some world  $w_0$  in some Euclidean model  $\mathcal{M}$ , then it is satisfied in a model whose underlying frame is of the form  $(W \cup \{w_0\}, \mapsto)$  where  $W \times W \subseteq \mapsto$ . Therefore, almost all worlds are successors of almost all other worlds. If one world satisfies a formula  $\Box clause_1$ , then almost all worlds satisfy the formula  $clause_1$  (and hence satisfy  $\Box clause_1$  as well). If one world satisfies a formula  $\Diamond CNF_1$ , then almost all worlds satisfy  $\Diamond CNF_1$  as well. Thus, most of the worlds are very similar to each other and we can reason about them using small MSO formulae. This holds even if we add more properties like reflexivity, transitivity etc. The rest of this section is devoted to proving the following theorem.

**Theorem 4.1.** Let  $\phi$  be a modal CNF formula. With treewidth of  $S(\phi)$  as parameter, there is a FPT algorithm for checking whether  $\phi$  is satisfiable in a Kripke model that satisfies Euclidean property and any combination of reflexivity, symmetry and transitivity.

We will drop all unary relations  $(Lv_i)_{0 \le i \le md(\phi)}$ . Instead, we will have one unary relation  $P_v$  containing all domain elements representing propositional variables and one unary relation  $H_i$  containing all other domain elements. This will not change the treewidth of  $S(\phi)$ . To make the presentation easier to follow, we will use informal description of MSO formulae. Let  $C_{\ell_1}$  and  $GC_{\ell}$  be sets of clauses (we will see later that clauses in  $GC_{\ell}$  will be satisfied in almost all worlds of a model). The following MSO formula checks if all clauses in  $C_{\ell_1}$  are satisfiable in a model in which, all worlds satisfy all clauses in  $GC_{\ell}$ .

$$\chi(C_{\ell_1}, GC_{\ell}) \stackrel{\simeq}{=} \exists T_r \subseteq (L_t \cap D\diamond) : \exists T_{r_0} \subseteq P_v :$$

$$G_{\ell_t} = \{l_t \in (L_t \cap B_{\Box}) \mid \exists c_\ell \in GC_\ell \wedge O_c(l_t, c_\ell)\}$$

$$\land \forall c_\ell \in (C_{\ell_1} \cup GC_\ell) :$$

$$\exists l_t \in (G_{\ell_t} \cup T_r \cup T_{r_0}) : O_c(c_\ell, l_t)$$

$$\lor \exists l_t \in P_v \setminus T_{r_0} : \overline{O_c}(c_\ell, l_t)$$

$$\land \forall l_t \in T_r : \exists T_{r_1} \subseteq P_v :$$

$$D_m = \{c_\ell \in Cl \mid O_c(l_t, c_\ell')\} \Rightarrow$$

$$\forall c_\ell \in (D_m \cup GC_\ell) :$$

$$\exists l_\ell' \in (T_r \cup G_{\ell_t} \cup T_{r_1}) : O_c(c_\ell, l_\ell')$$

$$\lor \exists l_\ell' \in P_v \setminus T_{r_1} : \overline{O_c}(c_\ell, l_\ell')$$

**Lemma 4.2.** Let  $C_{\ell_1}$  and  $GC_{\ell}$  be sets of clauses occurring in a modal CNF formula  $\phi$ . If  $\chi(C_{\ell_1}, GC_{\ell})$  is true in  $S(\phi)$ , then there is a Kripke model  $\mathcal{M}$  and a world w in it such that:

- 1. w satisfies all clauses in  $C_{\ell_1}$ ,
- 2. all worlds in  $\mathcal{M}$  satisfy all clauses in  $GC_{\ell}$ ,
- 3. the accessibility relation  $\mapsto$  in  $\mathcal{M}$  is the equivalence relation on the set of all worlds in  $\mathcal{M}$  and
- if a new world satisfying all clauses in GCℓ is added to M (making it accessible from all existing worlds of M), w will still satisfy all clauses in Cℓ1.

Proof. Suppose  $\chi(C_{\ell_1}, GC_{\ell})$  is true in  $\mathcal{S}(\phi)$ . We will build a Kripke model  $\mathcal{M}$  satisfying the required properties. To begin with, there must be a set  $T_r$  of literals of the form  $\Diamond CNF$  and a set  $T_{r_0}$  of propositional variables such that the rest of the formula  $\chi(C_{\ell_1}, GC_{\ell})$  is true in  $\mathcal{S}(\phi)$ . Let  $G_{\ell t} = \{lt \in (Lt \cap B_{\Box}) \mid \exists c_{\ell} \in GC_{\ell} \land O_c(lt, c_{\ell})\}$ be the set of literals of the form  $\Box clause$  such that the corresponding clause is in  $GC_{\ell}$ . We will start with one world w in our model in which precisely those propositional variables are set to  $\top$  that are in the set  $T_{r_0}$ . We will then add exactly one world  $w_i$  for each literal  $l_{t_i}$  in  $T_r$ . For any literal  $l_{t_i}$  in  $T_r$ , there will be a subset  $T_{r_1}$  of propositional variables such that last four conditions of  $\chi(C_{\ell_1}, GC_{\ell})$  are true in  $\mathcal{S}(\phi)$ . In the world  $w_i$ , we will set precisely those propositional variables to  $\top$  that are in the set  $T_{r_1}$  corresponding to  $l_{t_i}$ . Our model consists of the above worlds and the accessibility relation  $\mapsto$  is the equivalence relation on the set of all worlds. For any literal  $l_{t_i} \in T_r$  (which is of the form  $\Diamond CNF$ ), let  $D_{m_i} = \{c_{\ell'} \in Cl \mid O_c(lt_i, c_{\ell'})\}$  be the set of clauses that make up the corresponding CNF formula. By induction on modal depth of any clause  $c_{\ell}$ , we will prove that if  $c_{\ell} \in C_{\ell_1} \cup GC_{\ell}$ , then  $c_{\ell}$  is satisfied in w and if  $c_{\ell} \in D_{m_i} \cup GC_{\ell}$ , then  $c_{\ell}$  is satisfied in  $w_i$ .

In the base case, modal depth of  $c_{\ell}$  is 0. We will first prove that if  $c_{\ell} \in C_{\ell_1} \cup GC_{\ell}$ , then  $c_{\ell}$  is satisfied in w. Suppose  $\exists lt \in (G_{\ell t} \cup T_r \cup T_{r_0}) : O_c(c_{\ell}, lt)$  is true in  $\mathcal{S}(\phi)$ . If  $lt \in G_{\ell t} \cup T_r$ , then lt will have modal depth at least 1 (because it is of the form  $\Box clause$  or  $\Diamond CNF$ ) and hence modal depth of  $c_{\ell}$  (which contains lt as a sub-formula) will also be more than 1. Hence,  $lt \in T_{r_0}$ . This means that lt is a propositional variable set to  $\top$  in w that occurs positively in  $c_{\ell}$  and hence  $c_{\ell}$  is satisfied in w. If  $\exists lt \in P_v \setminus T_{r_0} : \overline{O_c}(c_{\ell}, lt)$  is true in  $\mathcal{S}(\phi)$ , then there is a propositional variable set to  $\bot$  in w that occurs negatively in  $c_{\ell}$  and hence,  $c_{\ell}$  is satisfied in w. Now, we will take up the case of  $c_{\ell} \in D_{m_i} \cup GC_{\ell}$ . In the formula  $\chi(C_{\ell_1}, GC_{\ell})$ , suppose  $\exists lt' \in (T_r \cup G_{\ell t} \cup T_r) : O_c(c_{\ell}, lt')$  is true in  $\mathcal{S}(\phi)$ . As before, lt' has be to in  $T_{r_1}$ , which means that there is a propositional variable set to  $\top$  at  $w_i$  that occurs positively in  $c_{\ell}$  lence,  $c_{\ell}$  is satisfied in  $w_i$ . If on the other hand,  $\exists lt' \in P_v \setminus T_{r_1} : \overline{O_c}(c_{\ell}, lt')$  is true, then there is a propositional variable set to  $\bot$  at  $w_i$  that occurs negatively in  $c_{\ell}$  is satisfied in  $w_i$ .

For the induction step, suppose  $c_{\ell} \in (C_{\ell_1} \cup GC_{\ell})$ . Suppose that in  $\mathcal{S}(\phi)$ , the formula  $\exists l_t \in (G_{\ell t} \cup T_r \cup T_{r_0})$ :  $O_c(c_{\ell}, l_t)$  is true. If  $l_t \in G_{\ell t}$ , then it is of the form  $\Box clause$  such that the corresponding clause (of modal depth lower than  $c_{\ell}$ ) is in  $GC_{\ell}$ . By the induction hypothesis, all worlds in  $\mathcal{M}$  will satisfy all clauses in  $GC_{\ell}$  of modal depth less than  $c_{\ell}$  (and by condition 4 of the lemma, all new worlds added will also satisfy all clauses in  $GC_{\ell}$ ) and hence  $\Box clause$  is satisfied in w and hence  $c_{\ell}$  is satisfied in w. If  $l_t \in T_r$ , then  $l_t$  is a literal of the form  $\Diamond CNF$  such that there is a world  $w_i$  in  $\mathcal{M}$  added to satisfy the corresponding CNF formula. All clauses in this CNF formula (which form the set  $Dm_i$ ) have modal depth less than  $c_{\ell}$  and by the induction hypothesis, they are all satisfied at  $w_i$ . Hence,  $w_i$  satisfies the corresponding CNF formula, and hence  $w_i$ satisfies the corresponding  $\Diamond CNF$  formula (since  $w_i$  is a successor of w) and hence  $c_{\ell}$  is satisfied at w. If  $l_t \in T_{r_0}$ , then  $l_t$  is a propositional variable set to  $\top$  in w and that occurs positively in  $c_{\ell}$ . Hence,  $c_{\ell}$  is satisfied in w. On the other hand, if  $\exists l_t \in P_v \setminus T_{r_0} : \overline{O_c}(c_{\ell}, l_t)$  is true in  $\mathcal{S}(\phi)$ , then  $l_t$  is a propositional variable set to  $\bot$  in w and that occurs negatively in  $c_{\ell}$ . Hence, in this case also,  $c_{\ell}$  is satisfied in w.

Finally, for the induction step, suppose  $c_{\ell} \in D_{m_i} \cup GC_{\ell}$ . Suppose that in  $\mathcal{S}(\phi)$ , the formula  $\exists lt' \in (T_r \cup G_{\ell \ell} \cup T_{r_1}) : O_c(c_{\ell}, lt')$  is true. If  $lt' \in G_{\ell \ell}$ , then it is of the form  $\Box clause$  such that the corresponding clause (of modal depth lower than  $c_{\ell}$ ) is in  $GC_{\ell}$ . By the induction hypothesis, all worlds in  $\mathcal{M}$  will satisfy all clauses in  $GC_{\ell}$  of modal depth less than  $c_{\ell}$  (and by condition 4 of the lemma, all new worlds added will also satisfy all clauses in  $GC_{\ell}$ ) and hence  $\Box clause$  is satisfied in  $w_i$  and hence  $c_{\ell}$  is satisfied in  $w_i$ . If  $lt' \in T_r$ , then lt' is a literal of the form  $\Diamond CNF$  such that there is a world  $w_{i'}$  in  $\mathcal{M}$  added to satisfy the corresponding CNF formula. All clauses in this CNF formula (which form the set  $D_{m_{i'}}$ ) have modal depth less than  $c_{\ell}$  and by the induction hypothesis, they are all satisfied at  $w_{i'}$ . Hence,  $w_{i'}$  satisfies the corresponding CNF formula, and hence  $w_i$  satisfies the corresponding  $\Diamond CNF$  formula (since  $w_{i'}$  is a successor of  $w_i$ ) and hence  $c_{\ell}$  is satisfied at  $w_i$ . If  $lt' \in T_{r_1}$ , then lt' is a propositional variable set to  $\top$  in  $w_i$  and that occurs positively in  $c_{\ell}$ . Hence,  $c_{\ell}$  is satisfied in  $w_i$ . On the other hand, if  $\exists lt' \in P_v \setminus T_{r_1} : \overline{O_c}(c_{\ell}, lt')$  is true in  $\mathcal{S}(\phi)$ , then lt' is a propositional variable set to  $\bot$  in this case also,  $c_{\ell}$  is satisfied in  $w_i$ . This completes the induction step and hence the proof.

The following formula makes use of  $\chi(C_{\ell_1}, GC_{\ell})$  to check if a set of clauses  $C_{\ell_0}$  is satisfiable in an Euclidean model.

$$\chi(C_{\ell_0}) \stackrel{\bigtriangleup}{=} \exists T_{r_0} \subseteq Lt : \forall lt \in T_{r_0} : \exists c_\ell \in C_{\ell_0} : O_c(c_\ell, lt) \\ \land \forall c_\ell \in C_{\ell_0} : \\ \exists lt \in T_{r_0} : O_c(c_\ell, lt) \\ \lor \exists lt \in P_v \setminus T_{r_0} : \overline{O_c}(c_\ell, lt) \\ \land C_{m_0} = \{c_\ell \in C_l \mid \exists lt \in (T_{r_0} \cap B_{\Box}) \land O_c(lt, c_\ell)\} \Rightarrow \\ \exists GC_\ell \subseteq (C_l) : \forall c_\ell \in GC_\ell : \exists lt \in (L_\ell \cap B_{\Box}) : O_c(l_\ell, c_\ell) \\ \land \forall lt \in (T_{r_0} \cap D \diamond) : \\ D_{m_0} = \{c_\ell \in C_l \mid O_c(l_\ell, c_\ell)\} \Rightarrow \\ \chi(D_{m_0} \cup C_{m_0}, GC_\ell) \end{cases}$$
(6)

**Lemma 4.3.** Let  $C_{\ell_0}$  be a set of clauses occurring in a modal CNF formula  $\phi$ .  $CNF(C_{\ell_0})$  is satisfiable at a world w in an Euclidean model  $\mathcal{M}$  in which w is not its own successor iff  $\chi(C_{\ell_0})$  is true in  $\mathcal{S}(\phi)$ .

Proof. Suppose  $\chi(C_{\ell_0})$  is true in  $S(\phi)$ . We will build an Euclidean Kripke model  $\mathcal{M}$  satisfying  $CNF(C_{\ell_0})$ . We will begin with a single world w. In w, precisely those propositional variables are set to  $\top$  that appear in the set  $T_{r_0}$  that witnesses truth of  $\chi(C_{\ell_0})$  in  $S(\phi)$ . For now, we will assume that literals of the form  $\Diamond CNF$ and  $\Box clause$  in  $T_{r_0}$  will be satisfied in w by addition of suitable worlds. If  $c_\ell$  is any clause in  $C_{\ell_0}$ , then either  $\exists l_t \in T_{r_0} : O_c(c_\ell, l_t)$  or  $\exists l_t \in P_v \setminus T_{r_0} : \overline{O_c}(c_\ell, l_t)$  is true. In the former case, a propositional variable that is set to  $\top$  at w or a literal of the form  $\Diamond CNF$  or  $\Box clause$  that is satisfied in w occurs in  $c_\ell$  and hence  $c_\ell$  is satisfied in w. In the later case, a propositional variable set to  $\bot$  in w occurs negatively in  $c_\ell$  and hence  $c_\ell$  is satisfied in w.

As promised, we will now add suitable successors such that all literals of the form  $\Diamond CNF$  and  $\Box clause$ in  $T_{r_0}$  are satisfied at w. Let  $C_{m_0} = \{c_\ell \in Cl \mid \exists lt \in (T_{r_0} \cap B_{\Box}) \land O_c(lt, c_\ell)\}$  be the set of clauses we have committed to satisfy in all successors of w by choosing the corresponding  $\Box clause$  to be in  $T_{r_0}$ . For any literal lt of the form  $\Diamond CNF$  in  $T_{r_0} \cap D\diamond$ , let  $D_{m_0} = \{c_\ell \in Cl \mid O_c(lt, c_\ell)\}$  be the set of clauses in the corresponding CNF formula. Since  $\chi(D_{m_0} \cup C_{m_0}, GC_\ell)$  is true in  $\mathcal{S}(\phi)$ , there is a model  $\mathcal{M}_1$  and a world  $w_1$  in it as specified in Lemma 4.2, such that  $w_1$  satisfies all clauses in  $D_{m_0} \cup C_{m_0}$  and all worlds in  $\mathcal{M}_1$ satisfy all clauses in  $GC_\ell$ . Let  $(\mathcal{M}_1, w_1), (\mathcal{M}_2, w_2), \ldots$  be the models given by Lemma 4.2 for the demand sets created by each of the literals of the form  $\Diamond CNF$  in  $T_{r_0} \cap D\diamond$ . Adding all worlds of  $\mathcal{M}_1, \mathcal{M}_2, \ldots$  to  $\mathcal{M}$ and making  $w_1, w_2, \ldots$  successors of w will result in all literals of the form  $\Diamond CNF$  in  $T_{r_0} \cap D\diamond$  being satisfied at w in  $\mathcal{M}$ . Since  $w_1, w_2, \ldots$  all satisfy all clauses in  $C_{m_0}$ , all literals of the form  $\Box clause$  in  $T_{r_0} \cap B_{\Box}$  are also satisfied at w in  $\mathcal{M}$ . Making all worlds other than w successors of all worlds other than w will ensure that  $\mathcal{M}$  is based on an Euclidean frame. Condition 4 of Lemma 4.2 will ensure that due to the additional accessibility relation pairs created, the worlds  $w_1, w_2, \ldots$  will not stop satisfying clauses required to satisfy the demands created by literals in  $T_{r_0}$ .

Now, suppose that  $CNF(C_{\ell_0})$  is satisfied in an Euclidean model. We will prove that  $\chi(C_{\ell_0})$  is true in  $\mathcal{S}(\phi)$ . As proved in [22],  $CNF(C_{\ell_0})$  is satisfied in a model  $\mathcal{M}$  at a world w such that the underlying frame of  $\mathcal{M}$  is of the form  $(W \cup \{w\}, \mapsto)$  such that  $W \times W \subseteq \mapsto$ . As stated in the lemma, w is not its own successor. If w was the successor of any other world, then Euclidean property will force w to be its own successor, hence w can not be the successor of any other world. To prove that  $\chi(C_{\ell_0})$  is true in  $\mathcal{S}(\phi)$ , we will first construct a set  $T_{r_0}$  of literals. Since  $CNF(C_{\ell_0})$  is satisfied at w, for every clause in  $C_{\ell_0}$ , there must be a literal occurring in that clause satisfied at w. Let  $T_{r_0}$  be the set of such literals of the form  $\Box clause$  or  $\Diamond CNF$  and the propositional variables set to  $\top$  in w and occurring positively in some clause in  $C_{\ell_0}$ . The condition  $\forall lt \in T_{r_0} : \exists c_\ell \in C_{\ell_0} : O_c(c_\ell, lt)$  is true by construction of  $T_{r_0}$ . If  $c_\ell$  is any clause in  $C_{\ell_0}$ , then either

- 1. there is some literal of the form  $\Box clause$  or  $\Diamond CNF$  or a positively occurring propositional variable that occurs in  $c_{\ell}$  and present in  $T_{r_0}$  (in which case  $\exists l_t \in T_{r_0} : O_c(c_{\ell}, l_t)$  is true) or
- 2. there is a negatively occurring propositional variable that is set to  $\perp$  in w (in which case  $\exists lt \in P_v \setminus T_{r_0}$ :  $\overline{O_c}(c_\ell, l_t)$  is true).

Let  $C_{m_0} = \{c_{\ell} \in Cl \mid \exists lt \in (T_{r_0} \cap B_{\Box}) \land O_c(lt, c_{\ell})\}$  be the set of clauses that are satisfied in all successors of w. Let  $GC_{\ell} = \{c_{\ell} \in Cl \mid \exists lt \in (Lt \cap B_{\Box}) \land O_c(lt, c_{\ell}) \land \mathcal{M}, w' \models lt, w' \neq w\}$  be the set of all clauses such that corresponding  $\Box clause$  formula is satisfied at some world w' other than w. Since all worlds other than w are successors of w', all worlds other than w satisfy all clauses in  $GC_{\ell}$ . The condition  $\forall c_{\ell} \in GC_{\ell} : \exists lt \in (Lt \cap B_{\Box}) : O_c(lt, c_{\ell})$  is true by construction of  $GC_{\ell}$ . Note that  $GC_{\ell}$  and  $T_{r_0} \cap D \diamond$  will be empty if w has no successors, hence the rest of  $\chi(C_{\ell 0})$  is vacuously true. For any literal  $lt \in T_{r0} \cap D\diamond$ , let  $D_{m0} = \{c_{\ell} \in Cl \mid O_c(lt, c_{\ell})\}$  be the set of clauses in the *CNF* formula contained in lt. In  $\mathcal{M}$ , there is a successor  $w_1$  of w that satisfies all clauses in  $D_{m0} \cup C_{m0}$ . We will prove that  $\chi(D_{m0} \cup C_{m0}, GC_{\ell})$  is true in  $\mathcal{S}(\phi)$ .

We first select a subset  $T_r \subseteq (Lt \cap D\diamond)$  so that the rest of the formula  $\chi(D_{m_0} \cup C_{m_0}, GC_\ell)$  can be satisfied. Let  $T_r = \{lt \in (Lt \cap D\diamond) \mid \mathcal{M}, w' \models lt, w' \neq w\}$  be the set of literals of the form  $\diamond CNF$  such that some world w' other than w satisfies the  $\diamond CNF$  formula (since w is not a successor of w', some other world w'' succeeding w' will satisfy the corresponding CNF formula). Let  $T_{r_0}$  be the set of propositional variables set to  $\top$  in the world  $w_1$  mentioned above. Let  $G_{\ell t} = \{lt \in (Lt \cap B_{\Box}) \mid \exists c_\ell \in GC_\ell \land Oc(lt, c_\ell)\}$  be the set of literals of the form  $\Box clause$  such that the corresponding clause is in  $GC_\ell$ . Let  $c_\ell$  be any clause in  $D_{m_0} \cup C_{m_0} \cup GC_\ell$ . Since  $w_1$  satisfies  $c_\ell$ , there must be a literal lt occurring in  $c_\ell$  such that lt is satisfied in  $w_1$ .

- 1. If  $l_t$  is of the form  $\Box clause$ , then it is in  $G_{\ell t}$  and hence  $\exists l_t \in G_{\ell t} : O_c(c_\ell, l_t)$  is true.
- 2. If lt is of the form  $\Diamond CNF$ , then it is in  $T_r$  and hence  $\exists lt \in T_r : O_c(c_\ell, lt)$  is true.
- 3. If lt is a positively occurring propositional variable, then  $\exists lt \in T_{r_0} : O_c(c_{\ell}, lt)$  is true.
- 4. If lt is a negatively occurring propositional variable, then  $\exists lt \in P_v \setminus T_{r_0} : \overline{O_c}(c_\ell, lt)$  is true.

Let  $lt \in T_r$  be any literal of the form  $\Diamond CNF$  in  $T_r$ . By definition of  $T_r$ , there is some world w' other than w such that w' satisfies the corresponding CNF formula. Let  $T_{r_1}$  be the set of propositional variables set to  $\top$  in w' and let  $D_m = \{c_\ell \in Cl \mid O_c(lt, c_\ell')\}$  be the set of clauses in the CNF formula in lt. Let  $c_\ell$  be any clause in  $D_m \cup GC_\ell$ . Since w' satisfies  $c_\ell$ , there must be a literal lt' occurring in  $c_\ell$  such that lt' is satisfied in w'.

- 1. If lt' is of the form  $\Box clause$ , then it is in  $G_{\ell t}$  and hence  $\exists lt' \in G_{\ell t} : O_c(c_{\ell}, lt')$  is true.
- 2. If lt' is of the form  $\Diamond CNF$ , then it is in  $T_r$  and hence  $\exists lt' \in T_r : O_c(c_\ell, lt')$  is true.
- 3. If lt' is a positively occurring propositional variable, then  $\exists lt' \in T_{r_1} : O_c(c_\ell, lt')$  is true.
- 4. If lt' is a negatively occurring propositional variable, then  $\exists lt' \in Pv \setminus Tr_1 : \overline{Oc}(c_\ell, lt')$  is true.

Suppose a modal formula is satisfied at a world w in an Euclidean model where w is its own successor. Then Euclidean property will force the accessibility relation  $\mapsto$  to be the equivalence relation on the set of all worlds. Hence, any  $\Box$  clause literal chosen to be satisfied in w will result in all worlds (including w) satisfying the corresponding clause. This can be easily handled by modifying  $\chi(C_{\ell_0})$  as follows.

$$\chi'(C_{\ell_0}) \stackrel{\triangle}{=} \exists T_{r_0} \subseteq Lt :$$

$$\exists GC_{\ell} \subseteq (Cl) : \forall c_{\ell} \in GC_{\ell} : \exists lt \in (Lt \cap B_{\Box}) : O_c(lt, c_{\ell})$$

$$\land \forall c_{\ell} \in (C_{\ell_0} \cup GC_{\ell}) :$$

$$\exists lt \in T_{r_0} : O_c(c_{\ell}, lt)$$

$$\lor \exists lt \in Pv \setminus T_{r_0} : \overline{O_c}(c_{\ell}, lt)$$

$$\land \forall lt \in (T_{r_0} \cap D\diamond) :$$

$$Dm_0 = \{c_{\ell} \in Cl \mid O_c(lt, c_{\ell})\} \Rightarrow$$

$$\chi(Dm_0, GC_{\ell})$$

$$(7)$$

Now to check if a modal formula  $\phi$  is satisfiable in an Euclidean model, we just have to check if  $\chi(C_{\ell_0}) \lor \chi'(C_{\ell_0})$  is true in  $\mathcal{S}(\phi)$ , where  $C_{\ell_0}$  is the set of clauses at the highest level. An application of Courcelle's theorem will give us the FPT algorithm. Note that in this case, the size of the MSO formula we need to check is independent of modal depth.

To check if a modal formula  $\phi$  is satisfiable in a reflexive and Euclidean model, we just check if  $\chi'(C_{\ell_0})$  is true in  $\mathcal{S}(\phi)$ .

Suppose a modal formula  $\phi$  is satisfied at some world w in an Euclidean and symmetric model. If w has any other successors, then Euclidean property will force all worlds reachable from w to be successors of wand w to be a successor of all worlds reachable from w. This is same as a reflexive and Euclidean model and can be handled by  $\chi'(C_{\ell_0})$ . If w has no other successors but is its own successor it can again be handled by  $\chi'(C_{\ell_0})$ . If w has no successors and is not its own successor, then all clauses of  $\phi$  at the highest level are satisfied at w by literals of the form  $\Box$  clause or propositional variables. This can be easily checked by a small MSO formula.

#### 4.1 Euclidean and transitive models

Suppose we want to check satisfiability of a modal CNF formula in models that are both Euclidean and transitive. As seen above, the modal CNF formula is satisfied in a model with an underlying frame of the

form  $(W \cup \{w\}, \mapsto)$  where  $W \times W \subseteq \mapsto$ . In addition, all other worlds are successors of w. Hence any literal of the form  $\Box$  clause satisfied at w will result in all other worlds satisfying the corresponding clause. This can be handled by modifying  $\chi(C_{\ell})$  as follows.

$$\chi''(C_{\ell_0}) \stackrel{\Delta}{=} \exists T_{r_0} \subseteq Lt : \forall lt \in T_{r_0} : \exists c_\ell \in C_{\ell_0} : O_c(c_\ell, lt) \\ \land \forall c_\ell \in C_{\ell_0} : \\ \exists lt \in T_{r_0} : O_c(c_\ell, lt) \\ \lor \exists lt \in P_v \setminus T_{r_0} : \overline{O_c}(c_\ell, lt) \\ \land C_{m_0} = \{c_\ell \in C_l \mid \exists lt \in (T_{r_0} \cap B_{\Box}) \land O_c(lt, c_\ell)\} \Rightarrow \\ \exists GC_\ell \subseteq (C_l) : \forall c_\ell \in GC_\ell : \exists lt \in (Lt \cap B_{\Box}) : O_c(lt, c_\ell) \\ \land \forall lt \in (T_{r_0} \cap D \diamond) : \\ D_{m_0} = \{c_\ell \in C_l \mid O_c(lt, c_\ell)\} \Rightarrow \\ \chi(D_{m_0} \cup C_{m_0}, GC_\ell \cup C_{m_0}) \end{cases}$$
(8)

The Euclidean property is very strong in the sense that it makes the complexity of infinitely many modal logics drop from PSPACE-hard to NP-complete [18]. One might hope for extending the results of this section to any modal logic whose frames is a subset of Euclidean frames. The results in [18] use semantic characterizations while our MSO formulae can only reason about syntax of modal logic formulae. Even though there is a close relation between the syntax and semantics of modal logic of Euclidean frames (which have been used to obtain the results of this section), it seems difficult to exploit this relation to obtain FPT algorithms for arbitrary extensions of modal logic of Euclidean frames. It remains to be seen if other tools from the theory of MSO logic on graphs can be used to achieve this.

#### 5 Reflexive models

As an example of how the basic technique described in section 3 can be extended to satisfiability in models satisfying some other properties, we will show satisfiability in reflexive models. We will need the following MSO formula to define the set of vertices reachable from a given vertex in a finite directed acyclic graph.

$$\mathcal{R}(x,X) \stackrel{\Delta}{=} \forall y \quad (y \in X) \Leftrightarrow \\ \begin{bmatrix} & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & \forall \exists z \in X : O_c(z,y) \end{bmatrix}$$
(9)

**Lemma 5.1.** Let G be a finite directed acyclic graph in which,  $O_c$  is the binary relation represented by the directed edges. Let x be a vertex and X be a subset of vertices in G. Then, X is the set of precisely those vertices reachable from x by a directed path of length 1 or more iff  $\mathcal{R}(x, X)$  defined in (9) is true in G.

*Proof.* Suppose X is the set of precisely those vertices reachable from x by a directed path of length 1 or more. If some vertex y is in X (i.e., there is a directed path of length 1 or more from x to y), we will prove that  $O_c(x, y) \vee \exists z \in X : O_c(z, y)$  is true in G. If the length of the path from x to y is 1, there is an edge from x to y, making  $O_c(x, y)$  true in G. If the directed path between x and y is at least 2 and is of the form  $x \to y' \to \cdots \to y'' \to y$ , then we can take y'' as witness for z in  $\exists z \in X : O_c(z, y)$ , making  $\exists z \in X : O_c(z, y)$  true in G. On the other hand, suppose  $O_c(x, y) \vee \exists z \in X : O_c(z, y)$  is true in G for some vertex y. We will prove that y is in X (i.e., there is a directed path of length 1 or more from x to y). Suppose  $O_c(x, y)$  is true in G. Then there is an edge from x to y, which is a directed path of length 1. Suppose  $\exists z \in X : O_c(z, y)$  is true G, then there is a directed path of length 1 or more from x to z (since z is in X). Appending the edge from z to y to this path gives us a path of length 2 or more from x to y.

Now, suppose that  $\forall y \quad (y \in X) \Leftrightarrow [O_c(x, y) \lor (\exists z \in X : O_c(z, y))]$  is true in *G*. We will prove that *X* is the set of precisely those vertices reachable from *x* by a directed path of length 1 or more. We will first prove that *X* does not contain any vertex not reachable from *x*. Suppose to the contrary that there is a vertex *y* in *X* not reachable from *x*. Since  $y \in X$ ,  $O_c(x, y) \lor (\exists z \in X : O_c(z, y))$  is true in *G*. Since  $O_c(x, y)$  cannot be true (as that would mean *y* is reachable from *x*), there is some  $z_1 \in X$  with  $O_c(z_1, y)$ . Since  $z_1$  is in *X*,  $O_c(x, z_1) \lor (\exists z \in X : O_c(z, z_1))$  is true. Since  $O_c(x, z_1)$  cannot be true (as that would mean *y* is reachable from *x*), there is some  $z_2 \in X$  with  $O_c(z_2, z_1)$ . The vertex  $z_2$  has to be distinct from *y* and  $z_1$  since

otherwise, the fact that G is devoid of directed cycles is violated. Continuing this type of reasoning leads us to an infinite sequence  $y, z_1, z_2, \ldots$  of distinct vertices, contradicting the fact that G is a finite graph. Hence, X does not contain any vertex not reachable from x. Next we will prove that every vertex y reachable from x is in X, by induction on the length i of the shortest directed path from x to y. In the base case i = 1, there is an edge from x to y, which means that  $O_c(x, y)$  is true in G, and  $\mathcal{R}(x, X)$  forces y to be in X. Suppose there is a directed path of length i + 1 from x to y. Let z be the vertex preceding y in this path. Since there is a directed path of length at most i from x to z, we can use induction hypothesis conclude that  $z \in X$ . Since there is an edge from z to y,  $\exists z \in X : O_c(z, y)$  is true in G, and again  $\mathcal{R}(x, X)$  forces y to be in X.

Let  $C_{\ell i}$  be some set of domain elements representing clauses at level at most *i*. The property  $\zeta[i](C_{\ell i})$  defined below checks if there is a reflexive Kripke model  $\mathcal{M}$  and a world *w* in it that satisfies all clauses in  $C_{\ell i}$ .

$$\zeta[0](C_{\ell_0}) \stackrel{\triangle}{=} \exists T_{r_0} \subseteq (L_t \cap L_{v_0}) : \forall c_\ell \in C_{\ell_0} : \\ \left[ (\exists l_t \in T_{r_0} : O_c(c_\ell, l_t)) \lor (\exists l_t \in (L_t \cap L_{v_0}) \setminus T_{r_0} : \overline{O_c}(c_\ell, l_t)) \right]$$
(10)

$$\begin{aligned}
\zeta[i](C_{\ell_i}) &\stackrel{\triangle}{=} \exists T_{r_i} \subseteq Lt: \\
\forall lt \in T_{r_i} \quad \exists c_{\ell} \in C_{\ell_i} : \exists X : (\mathcal{R}(c_{\ell}, X) \land lt \in X) \\
\land C_{m_{i-1}} = \{c_{\ell'} \in Cl \mid \exists lt' \in T_{r_i} \cap B_{\Box}, Oc(lt', c_{\ell'})\} \Rightarrow \\
\forall c_{\ell} \in C_{\ell_i} \cup C_{m_{i-1}}: \\
[(\exists lt \in T_{r_i} : Oc(c_{\ell}, lt)) \lor (\exists lt \in Lt \setminus T_{r_i} : \overline{Oc}(c_{\ell}, lt))] \\
\land \forall lt \in T_{r_i} \cap D \diamond : D_{m_{i-1}} = \{c_{\ell} \in Cl \mid Oc(lt, c_{\ell})\} \Rightarrow \\
\zeta[i-1](D_{m_{i-1}} \cup C_{m_{i-1}})
\end{aligned}$$
(11)

**Lemma 5.2.** The property  $\zeta[i](C_{\ell_i})$  can be written in a MSO logic formula of size linear in i. If  $\phi$  is any modal formula in CNF and  $C_{\ell_i}$  is any subset of domain elements representing clauses at level at most i, then  $CNF(C_{\ell_i})$  is satisfiable in a reflexive model iff  $\zeta[i](C_{\ell_i})$  is true in  $S(\phi)$ .

*Proof.* We will prove the first claim by induction on *i*. We will prove that the length  $|\zeta[i]|$  of  $\zeta[i]$  is linear in *i*. Let *c* be the length of  $\zeta[i]$  without the length of  $\zeta[i-1]$  counted. As can be seen,  $|\zeta[0]| \leq c$ . Inductively assume that  $|\zeta[i-1]| \leq ic$ . Then,  $|\zeta[i]| = c + |\zeta[i-1]|$ . Hence,  $|\zeta[i]| \leq c + ic = c(i+1)$ .

We will now prove the second claim by induction on i.

Base case i = 0: Suppose  $\zeta[0](C_{\ell_0})$  is true in  $\mathcal{S}(\phi)$ . Hence, there is a subset  $T_{r_0}$  of domain elements that satisfy all the conditions of  $\zeta[0]$  defined in (10). Since all domain elements in  $T_{r_0}$  represent literals at level 0 and the only literals at level 0 are propositional variables or their negations,  $T_{r_0}$  is in fact a subset of propositional variables. Consider the reflexive Kripke model  $\mathcal{M}$  with a single world w at which, all propositional variables in  $T_{r_0}$  are set to  $\top$  and all others are set to  $\bot$ . We will now prove that all clauses represented in  $C_{\ell_0}$  are satisfied in w. Let  $c_{\ell}$  be some element in  $C_{\ell_0}$  representing some clause. We have that either  $\exists lt \in T_{r_0} : O_c(c_{\ell}, lt)$  or  $\exists lt \in (Lt \cap L_{v_0}) \setminus T_{r_0} : \overline{O_c}(c_{\ell}, lt)$  is true in  $\mathcal{S}(\phi)$ . In the first case, a positively occurring propositional variable is set to  $\top$  in w and in the second case, a negatively occurring propositional variable is set to  $\bot$  in w.

Now suppose that there is a reflexive Kripke model  $\mathcal{M}$  and a world w such that  $\mathcal{M}, w \models CNF(C_{\ell_0})$ . We will prove that  $\zeta[0](C_{\ell_0})$  is true in  $\mathcal{S}(\phi)$ . The first requirement is to find a suitable subset  $T_{r_0}$  of domain elements. We will set  $T_{r_0}$  to be the set of precisely those domain elements that represent propositional variables occurring at level 0 and set to  $\top$  in the world w. Since every clause  $c_\ell$  in  $C_{\ell_0}$  is satisfied in w, either there is a positively occurring propositional variable set to  $\top$  in w or there is a negatively occurring propositional variable set to  $\top$  in w or there is a negatively occurring  $\exists t \in (L_t \cap L_{v_0}) \setminus T_{r_0} : \overline{O_c}(c_\ell, t_t)$  is true in  $\mathcal{S}(\phi)$ . This completes the base case.

Induction step: Suppose  $C_{\ell_i}$  is a subset of domain elements representing clauses occurring at level at most *i* and  $\zeta[i](C_{\ell_i})$  is true in  $\mathcal{S}(\phi)$ . We will build a reflexive Kripke model  $\mathcal{M}$  and prove that it has a world *w* such that  $\mathcal{M}, w \models CNF(C_{\ell_i})$ . We will start with a single world *w*. Since  $\zeta[i](C_{\ell_i})$  is true in  $\mathcal{S}(\phi)$ , there must be a subset  $T_{r_i}$  of domain elements satisfying all the conditions of  $\zeta[i](C_{\ell_i})$ . Since  $\forall l \in T_{r_i} \quad \exists c_\ell \in$  $C_{\ell_i} : \exists X : (\mathcal{R}(c_\ell, X) \land l_\ell \in X)$  is true in  $\mathcal{S}(\phi)$ , all literals in  $T_{r_i}$  are reachable from some clause in  $C_{\ell_i}$  in  $\mathcal{S}(\phi)$ . Hence, all literals in  $T_{r_i}$  are at level at most *i*. Let  $C_{m_{i-1}} = \{c_\ell \in C_l \mid \exists l_\ell \in T_{r_i} \cap B_{\Box}, O_c(l_\ell, c_\ell')\}$  be the set of clauses that we are committed to satisfy in all successors of *w* (that includes *w* as well) as a result of chosing the corresponding  $\Box clause$  to be in  $T_{r_i}$ . All clauses in  $C_{m_{i-1}}$  are at level at most *i* – 1. For each literal  $l_{t_1}$  of the form  $\Diamond CNF$  in  $T_{r_i}$ , let  $D_{m_{i-1}} = \{c_\ell \in C_l \mid O_c(l_{t_1}, c_\ell)\}$  be the set of clauses occurring in  $l_{t_1}$ . Since all clauses in  $D_{m_{i-1}} \cup C_{m_{i-1}}$  are at level at most i - 1 and  $\zeta[i - 1](D_{m_{i-1}} \cup C_{m_{i-1}})$  is true in  $S(\phi)$ , we can apply induction hypothesis to conclude that there is a reflexive Kripke model  $\mathcal{M}_1$  and a world  $w_1$ in it that satisfies all clauses in  $D_{m_{i-1}} \cup C_{m_{i-1}}$ . Add all such models  $\mathcal{M}_1, \mathcal{M}_2, \ldots$  to our Kripke model  $\mathcal{M}$ we are constructing and make the worlds  $w_1, w_2, \ldots$  successors of w. In w, set precisely those propositional variables to  $\top$  that occur in  $T_{r_i}$ . Let  $c_\ell$  be any clause in  $C_{\ell_i} \cup C_{m_{i-1}}$ . Now, we will prove by induction on modal depth of  $c_\ell$  that in  $\mathcal{M}$ , the world w satisfies  $c_\ell$ . If  $\exists lt \in Lt \setminus Tr : \overline{O_c}(c_\ell, lt)$  is true, then a propositional variable not in  $T_{r_i}$  occurs negatively in  $c_\ell$ . Since this propositional variable is set to  $\perp$  in w,  $c_\ell$  is satisfied at w. If  $\exists lt \in T_{r_i} : O_c(c_\ell, lt)$  is true and lt is a propositional variable, then it is set to  $\top$  in w and occurs positively in  $c_\ell$ . If  $\exists lt \in T_{r_i} : O_c(c_\ell, lt)$  is true and  $l_t$  is of the form  $\Box clause$ , then the corresponding clause is in  $C_{m_{i-1}}$  and hence true in all successors of w (including w itself, by induction on modal depth of  $c_\ell$ ). If  $\exists lt \in T_{r_i} : O_c(c_\ell, lt)$  is true and  $l_t$  is of the form  $\Diamond CNF$ , then we would have added a world to satisfy the corresponding CNF formula.

Now we will prove the other direction of the induction step. Suppose  $C_{\ell_i}$  is a subset of domain elements representing clauses occurring at level at most i and that there is a reflexive Kripke model  $\mathcal{M}$  and a world w such that  $\mathcal{M}, w \models CNF(C_{\ell_i})$ . We will prove that  $\zeta[i](C_{\ell_i})$  is true in  $\mathcal{S}(\phi)$ . To begin with, we will choose  $T_{r_i}$  to be the set of precisely those literals occurring at level i or below that are satisfied at w and occur as subformulas of some clause in  $C_{\ell_i}$ . This will ensure that  $\forall lt \in T_{r_i} \quad \exists c_\ell \in C_{\ell_i} : \exists X : (\mathcal{R}(c_\ell, X) \land lt \in X)$ is true in  $\mathcal{S}(\phi)$ . Let  $C_{m_{i-1}} = \{c_\ell' \in C_l \mid \exists lt' \in T_{r_i} \cap B_{\Box}, O_c(lt', c_\ell')\}$  be the set of clauses such that the corresponding  $\Box clause$  is in  $T_{r_i}$ . The world w satisfies all clauses in  $C_{\ell_i}$  and since w is its own successor, it also satisfies all clauses in  $C_{m_{i-1}}$ . Hence, if  $c_\ell$  is any clause in  $C_{\ell_i} \cup C_{m_{i-1}}$ , some literal occurring in  $c_\ell$  must be satisfied in w. Therefore,  $\forall c_\ell \in C_{\ell_i} \cup C_{m_{i-1}} : [(\exists lt \in T_{r_i} : O_c(c_\ell, lt)) \lor (\exists lt \in Lt \setminus T_{r_i} : \overline{O_c}(c_\ell, lt))]$  is true in  $\mathcal{S}(\phi)$ .

Let lt be any literal of the form  $\Diamond CNF$  in  $T_{r_i}$  and let  $D_{m_{i-1}} = \{c_i \in C_i \mid O_c(lt, c_i)\}$  be the set of clauses in the corresponding CNF formula. Since w satisfies  $l_t$ , there must be a successor w' of w that satisfies all clauses in  $D_{m_{i-1}}$  and also all clauses in  $C_{m_{i-1}}$  since w' is a successor of w. Since all clauses in  $D_{m_{i-1}} \cup C_{m_{i-1}}$  are at level at most i-1 and w' is a world in a reflexive Kripke model that satisfies all clauses in  $D_{m_{i-1}} \cup C_{m_{i-1}}$ , we can apply induction hypothesis to conclude that  $\zeta[i-1](D_{m_{i-1}} \cup C_{m_{i-1}})$  is true in  $\mathcal{S}(\phi)$ .

**Theorem 5.3.** Given a modal CNF formula  $\phi$ , there is a FPT algorithm that checks if  $\phi$  is satisfiable in reflexive models, with the treewidth of  $S(\phi)$  and the modal depth of  $\phi$  as parameters.

*Proof.* Given  $\phi$ ,  $\mathcal{S}(\phi)$  can be constructed in polynomial time. To check that all clauses of  $\phi$  at level  $\mathrm{md}(\phi)$  are satisfiable in some world w of some reflexive Kripke model  $\mathcal{M}$ , we check whether the formula  $\exists C_{\ell_{\mathrm{md}}(\phi)} \forall c_{\ell}(C_{\ell_{\mathrm{md}}(\phi)}(c_{\ell}) \Leftrightarrow (C_{\ell}(c_{\ell}) \land L_{v_{\mathrm{md}}(\phi)}(c_{\ell}))) \land \zeta[\mathrm{md}(\phi)](C_{\ell_{\mathrm{md}}(\phi)})$  is true in  $\mathcal{S}(\phi)$ . By Lemma 5.2, this is possible iff  $\phi$  is satisfiable in a reflexive model. The length of the above formula is linear in  $\mathrm{md}(\phi)$ . An application of Courcelle's theorem will give us the FPT algorithm.

## 6 Transitive models

In transitive models, formulae with small modal depth can check properties of all worlds reachable from a given world. To formalize this into a W[1]-hardness proof, we introduce the parameterized Partitioned Weighted Satisfiability (p-Pw-SAT) problem. An instance of p-Pw-SAT problem is a triple ( $\mathcal{F}, part : \Phi \rightarrow [k], tg : [k] \rightarrow \mathbb{N}$ ), where  $\mathcal{F}$  is a propositional CNF formula, *part* partitions the set of propositional variables into k parts and we need to check if there is a satisfying assignment that sets exactly tg(p) variables to  $\top$ in each part p. Parameters are k and pathwidth of the primal graph of  $\mathcal{F}$  (one vertex for each propositional variable, an edge between two variables iff they occur together in a clause). The following lemma can be proved by a FPT reduction from the Number List Coloring Problem [11].

**Lemma 6.1.** The p-Pw-SAT problem is W[1]-hard when parameterized by the number of parts k and the pathwidth of the primal graph.

Proof. We will give a FPT reduction from the Number List Coloring Problem (NLCP). An instance of NLCP is a graph G = (V, E), a set of colors  $S_v$  for each vertex  $v \in V$  and a target function  $tg : \bigcup_{v \in V} S_v \to \mathbb{N}$ . We need to check if G can be properly colored (every adjacent pair of vertices get different colors) such that every vertex v is colored from its set  $S_v$  and there are exactly  $tg(\ell)$  vertices colored with  $\ell$  for every  $\ell \in \bigcup_{v \in V} S_v$ . In [11], it is proved that even for graphs of pathwidth 2, NLCP is W[1]-hard when parameterized by total number of colors in  $\bigcup_{v \in V} S_v$ . Given an instance of NLCP with a graph of pathwidth 2, we associate with it an instance of p-Pw-SAT with the set of propositional variables  $\{q_v^{\ell} \mid v \in V, \ell \in S_v\}$ . Every color  $\ell \in \bigcup_{v \in V} S_v$  is a partition of the set of propositional variables and contains the variables  $\{q_v^{\ell} \mid \ell \in S_v\}$ . Target function is the same as target function of the NLCP instance. The CNF formula is the conjunction of the following formulae:

$$atLeast \stackrel{\Delta}{=} \bigwedge_{v \in V} \left(\bigvee_{\ell \in S_v} q_v^\ell\right)$$
$$atMost \stackrel{\Delta}{=} \bigwedge_{v \in V} \bigwedge_{\ell \neq \ell' \in S_v} \left(\neg q_v^\ell \lor \neg q_v^{\ell'}\right)$$
$$proper \stackrel{\Delta}{=} \bigwedge_{(v,u) \in E} \bigwedge_{\ell \in S_v \cap S_u} \left(\neg q_v^\ell \lor \neg q_u^\ell\right)$$

Suppose the given NLCP instance is a YES instance. In the associated p-PW-SAT instance, set  $q_v^{\ell}$  to  $\top$  iff the vertex v receives color  $\ell$  in the witnessing coloring. Since every vertex gets a color from its set, the formula *atLeast* above is satisfied. Since every vertex gets at most one color, the formula *atMost* is satisfied. If (v, u) is any edge in the graph, then since v and u get different colors in the witnessing coloring, the formula *proper* above is also satisfied. Since target function of the p-PW-SAT instance is same as the target function of the NLCP instance, the target function of p-PW-SAT is also satisfied.

On the other hand, suppose that the instance of p-Pw-SAT is a YES instance. Color a vertex v with the color  $\ell$  iff the propositional variable  $q_v^{\ell}$  is set to  $\top$  in the witnessing satisfying assignment. The formula *atLeast* ensures that every vertex gets at least one color from its set, while the formula *atMost* ensures that every vertex gets at most one color. If (v, u) is an edge in G and  $\ell$  is a common color between  $S_v$  and  $S_u$ , then the formula *proper* above ensures that at least one of the vertices v, u do not get the color  $\ell$ . Hence, the coloring given to the graph G is proper. Again since target function of the p-Pw-SAT instance is same as the target function of the NLCP instance, the target function of NLCP is also satisfied.

Now, it is left to prove that parameters of the p-Pw-SAT instance is bounded by some functions of the parameters of the NLCP instance. First parameter of the p-Pw-SAT instance is the number of partitions, which is same as the total number of colors in the NLCP instance (and later is a parameter of the NLCP instance). Second parameter is the pathwidth of the primal graph of the CNF formula. Consider any path decomposition of width 2 of the graph G in the NLCP instance. For every bag B and every vertex v in the bag, replace v by the set  $\{q_v^{\ell} \mid \ell \in S_v\}$ . We claim that the resulting decomposition is a path decomposition of the CNF formula, there is a bag containing all propositional variables occurring as literals in that clause. For any clause in the formula atLeast or atMost associated with a vertex v, any bag that contained the vertex v before replacement will meet the above criteria. For a clause in the formula proper associated with an edge (v, u), any bag that contained the vertices v and u before replacement will suffice. In the new path decomposition, number of elements in any bag is at most 3 times the total number of colors in the NLCP instance.

# **Theorem 6.2.** With treewidth and modal depth as parameters, modal satisfiability in transitive models is W[1]-hard.

The rest of this section is devoted to a proof of the above theorem, which is by a FPT reduction from p-Pw-SAT to satisfiability of modal CNF formulae in transitive models. Given an instance  $(\mathcal{F}, part : \Phi \to [k], tg : [k] \to \mathbb{N})$  of p-Pw-SAT problem with the pathwidth of the primal graph of  $\mathcal{F}$  being  $p_w$ , we construct a modal CNF formula  $\phi_{\mathcal{F}}$  of modal depth 2 in FPT time such that the pathwidth (and hence the treewidth) of  $\mathcal{S}(\phi_{\mathcal{F}})$ is bounded by a function of  $p_w$  and k and p-Pw-SAT is a YES instance iff  $\phi_{\mathcal{F}}$  is satisfiable in a transitive model. Suppose the propositional variables used in  $\mathcal{F}$  are  $q_1, q_2, \ldots, q_n$ . The idea is that if  $\phi_{\mathcal{F}}$  is satisfied at some world  $w_0$  in some transitive model  $\mathcal{M}$ , then  $\mathcal{M}, w_0 \models \mathcal{F}$ . To check that the required targets of the number of variables set to true in each partition are met,  $\phi_{\mathcal{F}}$  will force the existence of worlds  $w_1, w_2, \ldots, w_n$ arranged as  $w_0 \mapsto w_1 \mapsto w_2 \mapsto \cdots \mapsto w_n$ . In the formula  $\phi_{\mathcal{F}}$ , we will maintain a counter for each partition of the propositional variables. At each world  $w_i$ , if  $q_i$  is true, we will force the counter corresponding to  $part(q_i)$ to increment. At the world  $w_n$ , the counters will have the number of variables set to  $\top$  in each partition. We will then verify in the formula  $\phi_{\mathcal{F}}$  that these counts meet the given target. Such counting tricks have come under standard usage in complexity theoretic arguments of modal logic. The challenge here is to implement the counting in a modal formula of small pathwidth. In a p-Pw-SAT instance containing n propositional variables and k partitions, we will denote the number of variables in partition p by n[p]. We first construct an optimal path decomposition of the primal graph of  $\mathcal{F}$  in FPT time. We will name the variables occurring in the first bag as  $q_1, \ldots, q_i$ . We will name the variables newly introduced in the second bag as  $q_{i+1}, \ldots, q_{i'}$  and so on. In the rest of the construction, we will use this same ordering  $q_1, \ldots, q_n$  of the propositional variables. This will be important to maintain the pathwidth of the resulting modal formula low. The modal CNF formula  $\phi_{\mathcal{F}}$  will use all the propositional variables  $q_1, \ldots, q_n$  used by  $\mathcal{F}$  and also use the following additional variables:

- $-t_{\uparrow_1},\ldots,t_{\uparrow_k},f_{\uparrow_1},\ldots,f_{\uparrow_k}$ : partition indicators.
- For each partition  $p, tr_p^0, \ldots, tr_p^{n[p]}, fl_p^0, \ldots, fl_p^{n[p]}$ : counters to count the number of variables set to  $\top$  and  $\perp$  in partition p.
- $d_0, \ldots, d_{n+1}$ : depth indicators.

The modal CNF formula  $\phi_{\mathcal{F}}$  is the conjunction of the formulae described below. For clarity, we have used the shorthand notation  $\Rightarrow$  but they can be easily converted to CNF. Also for notational convenience, we will use part(i) instead of  $part(q_i)$ .  $\Phi(p)$  is the set of variables among  $\{q_1, \ldots, q_n\}$  in partition p. The formula determined ensures that all successors of  $w_0$  preserve the assignment of  $q_1, \ldots, q_n$ . The formula depth ensures that for all  $i, d_i \wedge \neg d_{i+1}$  holds in the world  $w_i$ .

In  $w_{i-1}$ , if  $q_i$  is set to  $\top$ , we want to indicate that in  $w_i$ , the counter for partition part(i) should be incremented. We will indicate this in the formula *setCounter* by setting the variable  $t_{\uparrow part(i)}$  to  $\top$ . Similar indication is done for the counter keeping track of variables set to  $\bot$  in partition p.

$$\begin{split} determined &\stackrel{\triangle}{=} \bigwedge_{i=1}^{n} q_{i} \Rightarrow \Box q_{i} \land \bigwedge_{i=1}^{n} \neg q_{i} \Rightarrow \Box \neg q_{i} \\ depth &\stackrel{\triangle}{=} \Diamond (d_{1} \land \neg d_{2}) \land \bigwedge_{i=1}^{n-1} \Box \left[ (d_{i} \land \neg d_{i+1}) \Rightarrow \Diamond (d_{i+1} \land \neg d_{i+2}) \right] \\ setCounter &\stackrel{\triangle}{=} (q_{1} \Rightarrow t \uparrow_{part(1)}) \land (\neg q_{1} \Rightarrow f \uparrow_{part(1)}) \\ \land \bigwedge_{i=2}^{n} \Box \left\{ [d_{i-1} \land \neg d_{i}] \Rightarrow \left[ (q_{i} \Rightarrow t \uparrow_{part(i)}) \land (\neg q_{i} \Rightarrow f \uparrow_{part(i)}) \right] \right\} \\ incCounter &\stackrel{\triangle}{=} (t \uparrow_{part(1)} \Rightarrow \Box tr_{part(1)}^{1}) \land (f \uparrow_{part(1)} \Rightarrow \Box fl_{part(1)}^{1}) \\ \land \bigwedge_{p=1}^{k} \bigwedge_{j=0}^{n} \Box \left[ t \uparrow_{p} \Rightarrow (tr_{p}^{j} \Rightarrow \Box tr_{p}^{j+1}) \right] \land \Box [f \uparrow_{p} \Rightarrow (fl_{p}^{j} \Rightarrow \Box fl_{p}^{j+1})] \\ targetMet &\stackrel{\triangle}{=} \bigwedge_{p=1}^{k} \Box [d_{n} \Rightarrow (tr_{p}^{tg(p)} \land \neg tr_{p}^{tg(p)+1})] \\ \land \bigwedge_{p=1}^{k} \Box [d_{n} \Rightarrow (fl_{p}^{n[p]-tg(p)} \land \neg fl_{p}^{n[p]-tg(p)+1})] \end{split}$$

Variables  $tr_p^0, \ldots, tr_p^{n[p]}$  implement the counter keeping track of variables set to  $\top$  in partition p. If j variables in  $\Phi(p) \cap \{q_1, \ldots, q_i\}$  are set to  $\top$ , then we want  $tr_p^j$  to be set to  $\top$  in  $w_i$ . To maintain this, in  $w_{i-1}$ , if it is indicated that a counter is to be incremented (by setting  $t\uparrow_p$  to  $\top$ ), we will force all successors of  $w_{i-1}$  to increment the  $tr_p$  counter in the formula *incCounter*. Finally, we check that at  $w_n$ , all the targets are met in the formula *targetMet*.

The modal CNF formula  $\phi_{\mathcal{F}}$  we need is the conjunction of  $\mathcal{F}$ , the formulae defined above and the miscellaneous formulae below (which ensure that counters are initiated properly and are monotonically non-

decreasing).

$$determined' \stackrel{\Delta}{=} \bigwedge_{p=1}^{k} tr_{p}^{0} \Rightarrow \Box tr_{p}^{0} \land \bigwedge_{p=1}^{k} fl_{p}^{0} \Rightarrow \Box fl_{p}^{0}$$

$$countInit \stackrel{\Delta}{=} d_{0} \land \neg d_{1} \land \bigwedge_{p=1}^{k} (\neg tr_{p}^{1} \land \neg fl_{p}^{1} \land tr_{p}^{0} \land fl_{p}^{0})$$

$$depth' \stackrel{\Delta}{=} \bigwedge_{p=1}^{k} \bigwedge_{j=0}^{n[p]} [\Box(tr_{p}^{j} \Rightarrow \Box tr_{p}^{j}) \land \Box(fl_{p}^{j} \Rightarrow \Box fl_{p}^{j})]$$

$$countMonotone \stackrel{\Delta}{=} \bigwedge_{i=1}^{n} \Box(d_{i} \Rightarrow d_{i-1}) \land \bigwedge_{p=1}^{k} \bigwedge_{j=2}^{n[p]} [\Box(tr_{p}^{j} \Rightarrow tr_{p}^{j-1}) \land \Box(fl_{p}^{j} \Rightarrow fl_{p}^{j-1})]$$

**Lemma 6.3.** If a p-Pw-SAT instance is a YES instance, then the modal formula constructed above is satisfied in a transitive Kripke model.

Proof. We will construct a transitive Kripke model using the satisfying assignment f that satisfies  $\mathcal{F}$  while meeting the given target. The model  $\mathcal{M}$  consists of worlds  $w_0, w_1, \ldots, w_n$  arranged as  $w_0 \mapsto w_1 \mapsto w_2 \mapsto \cdots \mapsto w_n$ . In all worlds,  $q_i$  is set to  $f(q_i)$  for all i, thus ensuring that  $\mathcal{M}, w_0 \models \mathcal{F} \land determined$ . In  $w_i, \{d_0, \ldots, d_i\}$  are set to  $\top$  and  $\{d_{i+1}, \ldots, d_{n+1}\}$  are set to  $\bot$  for all i between 0 and n, thus ensuring that  $\mathcal{M}, w_0 \models depth \land d_0 \land \neg d_1$ , the last two clauses coming from the formula depth'. It also ensures that  $\mathcal{M}, w_0 \models \bigwedge_{i=1}^n \Box(d_i \Rightarrow d_{i-1})$ , which is part of *countMonotone*. We will set  $tr_p^0$  and  $f\iota_p^0$  to  $\top$  in all worlds and  $tr_p^1$  and  $f\iota_p^1$  to  $\bot$  in  $w_0$  for all partitions p, thus ensuring  $\mathcal{M}, w_0 \models countInit \land determined'$ . At  $w_{i-1}$ , we will set  $t_{\uparrow part(i)}$  to  $q_i$ 's value in the same world and  $f_{\uparrow part(i)}$  to  $\neg q_i$ 's value. This will ensure that  $\mathcal{M}, w_0 \models setCounter$ .

At  $w_i$ , for any partition p, if j variables in  $\Phi(p) \cap \{q_1, \ldots, q_i\}$  are set to  $\top$ , then we will set  $\{tr_p^0, \ldots, tr_p^j\}$ to  $\top$  and  $\{tr_p^{j+1}, \ldots, tr_p^{n[p]}\}$  to  $\bot$ . If j' variables in  $\Phi(p) \cap \{q_1, \ldots, q_i\}$  are set to  $\bot$ , we will set  $\{fl_p^0, \ldots, fl_p^{j'}\}$ to true and  $\{fl_p^{j'+1}, \ldots, fl_p^{n[p]}\}$  to  $\bot$ . For any  $p \neq part(i+1)$ , we will set  $t\uparrow_p$  and  $f\uparrow_p$  to  $\bot$  at  $w_i$ . These will ensure that  $\mathcal{M}, w_0 \models incCounter \land depth' \land countMonotone$ .

Combined with the above settings of all propositional variables in  $\mathcal{M}$ , it is easy to check that the fact that f meets the target for each partition implies  $\mathcal{M}, w_0 \models targetMet$ .

**Lemma 6.4.** Suppose the modal CNF formula  $\phi_{\mathcal{F}}$  constructed above is satisfied at some world  $w_0$  of some transitive Kripke model  $\mathcal{M}$ . Then  $\mathcal{M}$  contains distinct worlds  $w_1, \ldots, w_n$  such that for each i between 1 and  $n, w_i$  is a successor of  $w_{i-1}$ . Moreover,  $\{d_0, \ldots, d_i\}$  are set to  $\top$  and  $\{d_{i+1}, \ldots, d_{n+1}\}$  are set to  $\perp$  in  $w_i$ . For any partition p, if j variables in  $\Phi(p) \cap \{q_1, \ldots, q_i\}$  are set to  $\top$  in  $w_0$ , then  $\{tr_p^0, \ldots, tr_p^j\}$  are all set to  $\top$  in  $w_i$ . If j' variables in  $\Phi(p) \cap \{q_1, \ldots, q_i\}$  are set to  $\perp$  in  $w_0$ , then  $\{f_{l_p}^0, \ldots, f_{l_p}^{j'}\}$  are all set to  $\top$  in  $w_i$ .

*Proof.* We will first prove the existence of worlds  $w_1, \ldots, w_i$  by induction on *i*.

Base case i = 1: Since  $\mathcal{M}, w_0 \models depth$ , there must be a successor  $w_1$  of  $w_0$  that satisfies  $d_1 \wedge \neg d_2$ . Since  $\mathcal{M}, w_0 \models countInit, w_0$  satisfies  $d_0 \wedge \neg d_1$  and hence  $w_1$  can not be same as  $w_0$ . Since  $\mathcal{M}, w_0 \models \Box(d_3 \Rightarrow d_2)$  (part of *countMonotone*) and  $w_1$  is a successor of  $w_0$ , we get  $\mathcal{M}, w_1 \models d_3 \Rightarrow d_2$ . Since  $d_2$  is set to  $\bot$  in  $w_1$ , this means that  $d_3$  is also set to  $\bot$  in  $w_1$ . Similar reasoning can be used to prove that all of  $\{d_2, \ldots, d_{n+1}\}$  are set to  $\bot$  in  $w_1$ . The fact that  $\mathcal{M}, w_1 \models d_1 \Rightarrow d_0$  means that  $d_0$  is set to  $\top$  in  $w_1$  (since  $d_1$  is set to  $\top$  in  $w_1$ ).

Induction step: Assume that worlds  $w_1, \ldots, w_i$  exist in  $\mathcal{M}$  with the stated properties. Hence,  $w_i$  satisfies  $d_i \wedge \neg d_{i+1}$ . Since  $w_0$  satisfies depth and  $w_i$  is a successor of  $w_0$  (by transitivity), there must be a successor  $w_{i+1}$  of  $w_i$  that satisfies  $d_{i+1} \wedge \neg d_{i+2}$ . Since all worlds  $w_0, \ldots, w_i$  satisfy  $\neg d_{i+1}, w_{i+1}$  is distinct from all of them. The fact that  $w_{i+1}$  satisfies  $d_{i'} \Rightarrow d_{i'-1}$  for all i' (these formulae are part of *countMonotone* formula satisfied by  $w_0$ ) can be used to show that all of  $d_0, \ldots, d_{i+1}$  are set to  $\top$  in  $w_{i+1}$  and all of  $d_{i+2}, \ldots, d_{n+1}$  are set to  $\bot$  in  $w_{i+1}$ .

We will now prove the second claim of the lemma, which is about values of  $\{tr_p^0, \ldots, tr_p^j\}$  in  $w_i$ . We will first prove that  $tr_p^j$  is set to  $\top$  by induction on *i*.

Base case i = 1: If  $q_1$  is not in part p, there is nothing to prove  $(tr_{part(1)}^0 \text{ is set to } \top \text{ in all worlds})$ . If  $q_1$  is in part p and  $q_1$  is set to  $\bot$ , there is nothing to prove. If  $q_1$  is in part p and  $q_1$  is set to  $\top$ , then since  $w_0$ 

satisfies setCounter, we get  $\mathcal{M}, w_0 \models q_1 \Rightarrow t_{\uparrow part(1)}$ . Since,  $q_1$  is set to  $\top$  and part(1) = p, we get that  $t_{\uparrow p}$  is set to  $\top$  in  $w_0$ . Since  $w_0$  satisfies *incCounter*, we get  $\mathcal{M}, w_0 \models t_{\uparrow p} \Rightarrow \Box t_r_p^1$  and hence  $\mathcal{M}, w_0 \models \Box t_r_p^1$ . Since  $w_1$  is a successor of  $w_0$ , we conclude that in  $w_1, t_r_p^1$  is set to  $\top$ .

Induction step: Case 1:  $q_i$  is not in part p and none of the variables in  $\Phi(p) \cap \{q_1, \ldots, q_i\}$  are set to to  $\top$  in  $w_i$ . In this case, there is nothing to prove.

Case 2:  $q_i$  is not in part p and some  $1 \leq j < i$  variables in  $\Phi(p) \cap \{q_1, \ldots, q_i\}$  are set to  $\top$ . By the induction hypothesis,  $tr_p^j$  is set to  $\top$  in  $w_{i-1}$ . Now  $\mathcal{M}, w_0 \models depth'$ . Hence  $\mathcal{M}, w_0 \models \Box(tr_p^j \Rightarrow \Box tr_p^j)$ , and hence  $\mathcal{M}, w_{i-1} \models tr_p^j \Rightarrow \Box tr_p^j$  (since  $w_{i-1}$  is a successor of  $w_0$ ), and hence  $\mathcal{M}, w_{i-1} \models \Box tr_p^j$  (since  $tr_p^j$  is set to  $\top$  in  $w_{i-1}$ ), and hence  $\mathcal{M}, w_i \models tr_p^j$  (since  $w_i$  is a successor of  $w_{i-1}$ ).

Case 3:  $q_i$  is in part p and  $q_i$  is set to  $\perp$ . If none of the variables in  $\Phi(p) \cap \{q_1, \ldots, q_i\}$  are set to  $\top$ , then the argument is similar to case 1. If some  $1 \leq j < i$  variables in  $\Phi(p) \cap \{q_1, \ldots, q_i\}$  are set to  $\top$ , then the argument is similar to case 2.

Case 4:  $q_i$  is in part p and  $q_i$  is set to  $\top$ . We know that  $w_{i-1}$  satisfies  $d_{i-1} \wedge \neg d_i$ . Since  $w_0$  satisfies setCounter, we have  $\mathcal{M}, w_0 \models \Box \{ [d_{i-1} \wedge \neg d_i] \Rightarrow [q_i \Rightarrow t\uparrow_{part(i)}] \}$ , and hence  $\mathcal{M}, w_{i-1} \models [d_{i-1} \wedge \neg d_i] \Rightarrow [q_i \Rightarrow t\uparrow_{part(i)}] \}$  and hence  $\mathcal{M}, w_{i-1} \models [d_{i-1} \wedge \neg d_i] \Rightarrow [q_i \Rightarrow t\uparrow_{part(i)}] \}$  (since  $w_{i-1}$  is a successor of  $w_0$ ), and hence  $\mathcal{M}, w_{i-1} \models q_i \Rightarrow t\uparrow_p$  (since  $\mathcal{M}, w_{i-1} \models d_{i-1} \wedge \neg d_i$ ), and hence  $\mathcal{M}, w_{i-1} \models t\uparrow_p$  (since  $\mathcal{M}, w_{i-1} \models q_i$ ). Since  $w_0$  satisfies incCounter and  $w_{i-1}$  is a successor of  $w_0$ , we get  $\mathcal{M}, w_{i-1} \models t\uparrow_p \Rightarrow (tr_p^{j-1} \Rightarrow \Box tr_p^j)$ . We have already seen that  $t\uparrow_p$  is set to  $\top$  in  $w_{i-1}$  and  $tr_p^{j-1}$  is set to  $\top$  in  $w_{i-1}$  by the induction hypothesis (j is at least 1 since  $q_i$  is in part p and is set to  $\top$ ). Hence, we get  $\mathcal{M}, w_{i-1} \models \Box tr_p^j$ . Since  $w_i$  is a successor of  $w_{i-1}$ , we conclude that  $tr_p^j$  is set to  $\top$  in  $w_i$ .

Now, since  $w_0$  satisfies  $\Box(tr_p^j \Rightarrow tr_p^{j-1})$  (this is part of *countMonotone*) and  $w_i$  is a successor of  $w_0$ , we get  $\mathcal{M}, w_i \models tr_p^j \Rightarrow tr_p^{j-1}$ . Since  $tr_p^j$  is set to  $\top$  in  $w_i$ , it follows that  $tr_p^{j-1}$  is also set to  $\top$  in  $w_i$ . Similarly,  $tr_p^0, \ldots, tr_p^j$  are all set to  $\top$  in  $w_i$ .

The proof for values of  $\{f_{l_{p}}^{0}, \ldots, f_{l_{p}}^{j'}\}$  is symmetric to the proof of values of  $\{t_{r_{p}}^{0}, \ldots, t_{r_{p}}^{j}\}$ .

# **Theorem 6.5.** If $\phi_{\mathcal{F}}$ constructed above is satisfied in a transitive model, then the p-Pw-SAT instance is a YES instance.

Proof. Suppose  $\phi_{\mathcal{F}}$  is satisfied in some world  $w_0$  of a transitive model. Since  $\mathcal{F}$  is part of  $\phi_{\mathcal{F}}$ , the assignment to  $\{q_1, \ldots, q_n\}$  induced by  $w_0$  satisfies  $\mathcal{F}$ . We claim that this assignment also meets the targets. If not, we will derive a contradiction. For some partition p, suppose there are more than tg(p) variables set to  $\top$ . Then by Lemma 6.4,  $tr_p^{tg(p)+1}$  will be set to  $\top$  in  $w_n$ , contradicting the fact that  $w_0$  satisfies targetMet. For some partition p, if there are less than tg(p) variables set to  $\top$ , then there will be more than n[p] - tg(p) variables set to  $\bot$ . By Lemma 6.4,  $tr_p^{n[p]-tg(p)+1}$  will be set to  $\top$  in  $w_n$ , again contradicting the fact that  $w_0$  satisfies targetMet.

Given an instance of p-Pw-SAT problem, the formula  $\phi_{\mathcal{F}}$  described above can be constructed in FPT time. To complete the proof of Theorem 6.2, we will prove that the pathwidth of  $\phi_{\mathcal{F}}$  is bounded by some function of k and  $p_w$ .  $\phi_{\mathcal{F}}$  has been carefully constructed to keep pathwidth low.

#### **Lemma 6.6.** Pathwidth of $S(\phi_{\mathcal{F}})$ is at most 4pw + 2k + 5.

*Proof.* Given an optimal path decomposition of the primal graph of  $\mathcal{F}$ , depth counters can be added to the bags without increasing their size much since the order of depth counters is same as the order of  $q_1, \ldots, q_n$ . There are only 2k partition indicators  $t_{\uparrow 1}, \ldots, t_{\uparrow k}, f_{\uparrow 1}, \ldots, f_{\uparrow k}$ , so they can also be added to the bags without increasing their size very much. However, the set of 2n partition counters (of the form  $tr_p^j$  or  $f\iota_p^j$ ) has to be added carefully to maintain the size of the bags. Formulas of  $\phi_{\mathcal{F}}$  have been carefully designed to enable this. The key observation is that the only "link" between  $q_1, \ldots, q_n$  and partition counters are partition indicators and there are only 2k of them. The following proof relies on this observation.

Consider an optimal path decomposition of the primal graph of  $\mathcal{F}$  with each bag containing at most  $p_w$  elements. Ensure that for all i with  $1 \leq i < n$ , there is a bag containing both  $q_i$  and  $q_{i+1}$  or there is a bag with  $q_i$  such that the next bag contains  $q_{i+1}$  (call this the *continuity* property). If this is not the case for some i, consider the last bag B containing  $q_i$  and the first bag B' containing  $q_{i+1}$ . No bag that is between B and B' will introduce any new variable (if it did, that new variable would have been  $q_{i+1}$  according to our order). Hence, all the bags in between B and B' are subsets of B. Hence, they can all be removed and B' can become the bag immediately after B. The resulting decomposition is still a path decomposition of the primal graph of  $\mathcal{F}$  with each bag containing at most  $p_w$  elements. Moreover, the order of variables  $q_1, \ldots, q_n$  does not change due to the change we have made in the path decomposition. This new decomposition has a

bag containing  $q_i$  such that the next bag contains  $q_{i+1}$ . Now, we can repeat the above process until we get a path decomposition with the continuity property.

For any *i* with  $1 \leq i \leq n$ , let  $B_i$  be a bag containing the propositional variable  $q_i$ . We will expand this path decomposition by adding variables used in  $\phi_{\mathcal{F}}$  such that for every *clause* that appears in  $\phi_{\mathcal{F}}$ , there is a bag that contains all propositional variables appearing in that clause. Each of these expanded bags will have at most  $4p_w + 2k$  elements. We will then show how to expand this into a path decomposition of  $\mathcal{S}(\phi_{\mathcal{F}})$ , by adding at most 6 elements to each bag (creating duplicate copies of existing bags if required). This will prove that the pathwidth of  $\mathcal{S}(\phi_{\mathcal{F}})$  is at most  $4p_w + 2k + 5$ .

First, in each bag *B* and each element  $q_i$  in it, add  $d_{i-1}, d_i$  and  $d_{i+1}$ . Note that due to continuity property of the decomposition we started with, the expanded decomposition still retains the property that all bags containing an element forms a connected component, even after adding depth counters  $d_0, \ldots, d_{n+1}$ . Next, add  $t_{\uparrow 1}, \ldots, t_{\uparrow k}, f_{\uparrow 1}, \ldots, f_{\uparrow k}$  to all the bags. We will refer to the bag containing  $q_i, d_{i-1}, d_i$  and  $d_{i+1}$  as  $B_i$ . Now, we have a decomposition with each bag containing at most  $4p_w + 2k$  elements, and the last bag contains  $d_n$ . To this bag, we will append 2k paths serially. For  $1 \leq p \leq k$ ,  $(2p-1)^{\text{th}}$  path will be as follows:  $\{d_n, t_{\uparrow 1}, \ldots, t_{\uparrow k}, f_{\uparrow 1}, \ldots, f_{\uparrow k}, t_r_p^{1}, t_r_p^{1}\} - \{d_n, t_{\uparrow 1}, \ldots, t_{\uparrow k}, f_{\uparrow 1}, \ldots, f_{\uparrow k}, t_r_p^{n[p]-1}, t_r_p^{n[p]}\}$ . We will refer to these bags as  $B_p^1, \ldots, B_p^{n[p]}$ .  $2p^{\text{th}}$  path is similar, with  $f_{\iota_p}$  variables replacing  $t_{\iota_p}$  variables. We will refer to these bags in  $2p^{\text{th}}$  path as  $B_p^{1'}, \ldots, B_p^{n[p]'}$ . Each of these new bags has at most 2k + 3 elements, and the whole decomposition still retains the property that for any element, the set of bags containing that element forms a connected component.

Now we will show how to expand the above decomposition into a path decomposition of  $S(\phi_{\mathcal{F}})$ . We have to add clauses and literals occurring in  $\phi_{\mathcal{F}}$  and ensure that for any pair of elements  $O_c(e_1, e_2)$  or  $\overline{O_c}(e_1, e_2)$ , there is a bag containing both  $e_1$  and  $e_2$ . To achieve this, we may have to "augment" an existing bag with new elements. If  $B_i$  is a bag in the path decomposition  $\cdots - B_i - \ldots$ , augmenting  $B_i$  with elements  $e_1$  and  $e_2$  means that we add another bag  $\cdots - B'_i - B_i - \ldots$  with  $B'_i$  containing all elements of  $B_i$  and in addition containing  $e_1$  and  $e_2$ . If we ensure that these new elements introduced during augmentation is never added to any other bag in the decomposition, augmentation will not violate the path decomposition's property that for any element, the set of bags containing that element forms a connected component. Now, we will go through each sub-formula of  $\phi_{\mathcal{F}}$  and prove that all its clauses, literals and  $O_c$  pairs are already represented in the path decomposition we have constructed above or that the decomposition can be augmented to represent them.

- Clauses in  $\mathcal{F}$ : For each clause in  $\mathcal{F}$ , the propositional variables in that clause form a clique in the primal graph of  $\mathcal{F}$ . Hence, there is a bag B in the new decomposition that contains all propositional variables occurring in that clause. Augment B with a new domain element representing the clause.
- determined: Here, the clauses are of the form  $\neg q_i \lor \Box q_i$  and  $q_i \lor \Box \neg q_i$ . Augment the bag  $B_i$  containing  $q_i$  with 3 domain elements, one for the clause  $\neg q_i \lor \Box q_i$  itself, one for the literal  $\Box q_i$  and one for the clause in this literal that contains  $q_i$  as its only literal. Perform similar augmentation for the clause  $q_i \lor \Box \neg q_i$ .
- depth: For  $\Diamond(d_1 \land \neg d_2)$ , augment the bag  $B_1$  containing  $d_1$  and  $d_2$  with 4 domain elements representing literals and clauses of  $\Diamond(d_1 \land \neg d_2)$ . Augment the bag  $B_{i+1}$  containing  $d_i, d_{i+1}$  and  $d_{i+2}$  with 6 elements representing literals and clauses of  $\Box[\neg d_i \lor d_{i+1} \lor \Diamond(d_{i+1} \land \neg d_{i+2})]$ .
- setCounter: Augment the bag  $B_1$  containing  $q_1$  and  $t_{\uparrow part(1)}$  with one element representing the clause  $\neg q_1 \lor t_{\uparrow part(1)}$ . Do a similar augmentation for the clause  $q_1 \lor f_{\uparrow part(1)}$ .  $\Box(q \land r)$  is equivalent to  $\Box q \land$  $\Box r$ . Hence, the latter part of setCounter can be split into clauses  $\Box(\neg d_{i-1} \lor d_i \lor \neg q_i \lor t_{\uparrow part(i)})$  and  $\Box(\neg d_{i-1} \lor d_i \lor q_i \lor f_{\uparrow part(i)})$ . Augment the bag  $B_i$  containing  $d_{i-1}, d_i, q_i, t_{\uparrow part(i)}$  and  $f_{\uparrow part(i)}$  with 6 elements representing clauses and literals of these two clauses.
- *incCounter*: Augment the bag  $B_{part(1)}^1$  containing  $t_{\uparrow part(1)}$  and  $tr_{part(1)}^1$  with 3 elements representing clauses and literals of  $(\neg t_{\uparrow part(1)} \lor \Box t_{part(1)}^1)$ . Similarly augment the bag  $B_{part(1)}^{1'}$  for  $(\neg f_{\uparrow part(1)} \lor \Box f_{part(1)}^1)$ . Augment the bag  $B_p^{j+1}$  containing  $t_{\uparrow p}, tr_p^j$  and  $tr_p^{j+1}$  with 6 elements representing literals and clauses of  $\Box(\neg t_{\uparrow p} \lor \neg tr_p^j \lor \Box tr_p^{j+1})$ . Similarly augment  $B_p^{j+1'}$  for  $\Box(\neg f_{\uparrow p} \lor \neg f_p^j \lor \Box f_p^{j+1})$ .
- targetMet: Augment the bag  $B_p^{tg(p)+1}$  containing  $d_n, tr_p^{tg(p)}$  and  $tr_p^{tg(p)+1}$  with 6 elements for the literals and clauses in  $\Box(\neg d_n \lor tr_p^{tg(p)})$  and  $\Box(\neg d_n \lor \neg tr_p^{tg(p)+1})$ . Similarly augment  $B_p^{n[p]-tg(p)+1'}$  for  $\Box(\neg d_n \lor f_p^{n[p]-tg(p)+1})$
- determined': Augment the bag  $B_p^1$  containing  $tr_p^0$  with 3 elements representing literals and clauses of  $\neg tr_p^0 \lor \Box tr_p^0$ . Similarly augment  $B_p^{1'}$  for  $\neg fl_p^0 \lor \Box fl_p^0$ .

- countInit: Augment the bag  $B_1$  containing  $d_0$  and  $d_1$  with 2 elements representing the clauses in  $d_0 \wedge \neg d_1$ . Augment the bag  $B_p^1$  containing  $tr_p^0$  and  $tr_p^1$  with 2 elements representing the clauses in  $\neg tr_p^1 \wedge tr_p^0$ . Similarly augment  $B_p^{1'}$  for  $\neg fl_p^1 \wedge fl_p^0$ .
- depth': Augment the bag  $B_p^j$  containing  $tr_p^j$  with 6 elements representing literals and clauses of  $\Box(\neg tr_p^j \lor \Box tr_p^j)$ . Similarly augment  $B_p^{j'}$  for  $\Box(\neg fl_p^j \lor \Box fl_p^j)$ .
- count Monotone: Augment the bag  $B_i$  containing  $d_i$  and  $d_{i-1}$  with 3 elements representing literals and clauses of  $\Box(\neg d_i \lor d_{i-1})$ . Augment the bag  $B_p^j$  containing  $tr_p^j$  and  $tr_p^{j-1}$  with 3 elements representing literals and clauses of  $\Box(\neg tr_p^j \lor tr_p^{j-1})$ . Similarly augment  $B_p^{j'}$  for  $\Box(\neg fl_p^j \lor fl_p^{j-1})$ .

In the absence of transitivity, the above reduction would require a formula of modal depth that depends on n (and hence it would no longer be a FPT reduction). The above hardness proof will however go through for any class of transitive frames that has paths of unbounded length of the form  $w_1 \mapsto w_2 \mapsto \cdots \mapsto w_n$  without any reverse paths<sup>2</sup>. See [31] for some context on such classes of transitive frames of unbounded depth. Readers familiar with frames with no branching to the right (axiom .3) may infer that the above hardness proof will also go through for K4.3 frames.

# 7 Conclusions and Future Work

By expressing satisfiability of modal formulae as a MSO property, we obtained a FPT algorithm for modal satisfiability in general models with treewidth and modal depth as parameters. Due to the dependence of the constructed MSO sentence on modal depth, the FPT algorithm obtained in section 3 has a running time with a tower of 2's whose height is  $\mathcal{O}(\text{md}(\phi))$ . Unless, PTIME=NP, such dependence on modal depth cannot be avoided due to the following observation. In [1, Lemma 1], it is shown how to encode an arbitrary propositional CNF formula into an equivalent modal formula (the propositional formula is satisfiable in a general model). This modal formula has some very low modal depth h such that any function growing slower than a tower of 2's of height h - 5 is a polynomial in the size of the propositional formula. The treewidth of this modal formula can be verified to be a constant. This also proves that unless PTIME=NP, modal satisfiability in general models is not FPT when treewidth is the only parameter.

We can work out a composition algorithm [4], and hence conclude that with treewidth and modal depth as parameters, there is no polynomial kernel for modal satisfiability in general models.

One direction for future research is towards meta classification as done in [19], instead of the case by case analysis of this work. We can also consider variations in treewidth, such as having different domain elements representing same propositional variable at different levels in  $S(\phi)$ . Other variations are modal circuits instead of modal formulae and generalizations of primal/dual graphs instead of incidence graphs.

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# References

- A. Achilleos, M. Lampis, and V. Mitsou. Parameterized modal satisfiability. In *ICALP*, volume 6199 of *LNCS*, pages 369–380, 2010.
- [2] I. Adler and M. Weyer. Tree-width for first order formulae. In CSL, volume 5771 of LNCS, pages 71–85, 2009.
- [3] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. CUP, 2001.
- [4] H. L. Bodlaender, R. G. Downey, M. R. Fellows, and D. Hermelin. On problems without polynomial kernels. J. Comput. Syst. Sci., 75(8):423–434, 2009.
- [5] H. L. Bodlaender and T. Kloks. Efficient and constructive algorithms for the pathwidth and treewidth of graphs. J. Alg., 21(2):358-402, 1996.
- [6] H. Chen. Quantified constraint satisfaction and bounded treewidth. In R. L. de Mántaras and L. Saitta, editors, ECAI, pages 161–165. IOS Press, 2004.
- [7] B. Courcelle. The monadic second-order logic of graphs III: tree-decompositions, minors and complexity issues. ITA, 26:257–286, 1992.
- [8] V. Dalmau, P. G. Kolaitis, and M. Y. Vardi. Constraint satisfaction, bounded treewidth, and finite-variable logics. In CP '02, pages 310–326. Springer, 2002.

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- [9] P. Enjalbert and L. F. del Cerro. Modal resolution in clausal form. Theor. Comp. Sc., 65(1):1–33, 1989.
- [10] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. Reasoning About Knowledge. MIT Press, 1995.
- [11] M. Fellows, F. V. Fomin, D. Lokshtanov, F. Rosamond, S. Saurabh, S. Szeider, and C. Thomassen. On the complexity of some colorful problems parameterized by treewidth. In *Comb. Opt. and Appl.*, volume 4616 of *LNCS*, pages 366–377.
- [12] E. Fischer, J.A. Makowsky, and E.V. Ravve. Counting truth assignments of formulas of bounded tree-width or clique-width. Disc. App. Math., 156(4):511–529, 2008.
- [13] J. Flum and M. Grohe. Parameterized Complexity Theory. Springer, 2006.
- [14] E. Grädel. Why are modal logics so robustly decidable? In Current Trends in Theor. Comp. Sc., pages 393–408.
   2001.
- [15] M. Grohe. The structure of tractable constraint satisfaction problems. In MFCS, volume 4162 of LNCS, pages 58–72, 2006.
- [16] J. Y. Halpern. The effect of bounding the number of primitive propositions and the depth of nesting on the complexity of modal logic. Artif. Intell., 75(2):361–372, 1995.
- [17] J. Y. Halpern and Y. O. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. Artif. Intell., 54(3):319–379, 1992.
- [18] J. Y. Halpern and L. C. Rêgo. Characterizing the np-pspace gap in the satisfiability problem for modal logic. In *IJCAI*, pages 2306–2311, 2007.
- [19] E. Hemaspaandra and H. Schnoor. On the complexity of elementary modal logics. In STACS, pages 349–360, 2008.
- [20] A. Herzig and J. Mengin. Uniform interpolation by resolution in modal logic. In JELIA '08, volume 5293 of LNCS, pages 219–231, 2008.
- [21] U. Hustadt and R. A. Schmidt. An empirical analysis of modal theorem provers. J. Applied Non-Classical Logics, 9(4), 1999.
- [22] Y. Kazakov and I. Pratt-Hartmann. A note on the complexity of the satisfiability problem for graded modal logics. In *LICS*, pages 407–416, 2009.
- [23] R. E. Ladner. The computational complexity of provability in systems of modal propositional logic. SIAM J. Comput., 6(3):467–480, 1977.
- [24] D. Marx. Can you beat treewidth? In FOCS, pages 169–179, 2007.
- [25] L. A. Nguyen. On the complexity of fragments of modal logics, volume 5 of Advances in Modal logic, pages 249–268. 2005.
- [26] G. Pan and M. Y. Vardi. Optimizing a BDD-based modal solver. In CADE-19, volume 2741 of LNCS, pages 75–89. 2003.
- [27] G. Pan and M. Y. Vardi. Fixed-parameter hierarchies inside pspace. In LICS '06, pages 27–36. IEEE Computer Society, 2006.
- [28] V. R. Pratt. Application of modal logic to programming. Studia Logica, 39(2-3):257-274, 1980.
- [29] J. H. Reif and A. P. Sistla. A multiprocess network logic with temporal and spatial modalities. In ICALP, volume 154 of LNCS, pages 629–639, 1983.
- [30] M. Samer and S. Szeider. Constraint satisfaction with bounded treewidth revisited. J. Comput. Syst. Sci., 76(2):103–114, 2010.
- [31] B. ten Cate. A note on the expressibility problem for modal logics and star-free regular expressions. *Inf. Process. Lett.*, 109(10):509–513, 2009.
- [32] M. Thorup. All structured programs have small tree width and good register allocation. Inf. and Comput., 142(2):159 – 181, 1998.
- [33] M. Y. Vardi. Why is modal logic so robustly decidable? In Descriptive Complexity and Finite Models, pages 149–184. AMS, 1996.