Quasi-Polynomial Time Approximation Scheme for Sparse Subsets of Polygons*

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Abstract

We describe how to approximate, in quasi-polynomial time, the largest independent set of polygons, in a given set of polygons. Our algorithm works by extending the result of Adamaszek and Wiese [AW13, AW14] to polygons of arbitrary complexity. Surprisingly, the algorithm also works for computing the largest subset of the given set of polygons that has some sparsity condition. For example, we show that one can approximate the largest subset of polygons, such that the intersection graph of the subset does not contain a cycle of length 4 (i.e., $K_{2,2}$).

1. Introduction

Let $F = \{f_1, \ldots, f_n\}$ be a set of n objects in the plane, with weights $w_1, w_2, \ldots, w_n > 0$, respectively. In this paper, we are interested in the problem of finding an independent set of maximum weight. Here a set of objects is **independent**, if no pair of objects intersect.

A natural approach to this problem is to build an *intersection graph* $G = G_I(F) = (V, E)$, where the objects form the vertices, and two objects are connected by an edge if they intersect, and weights are associated with the vertices. We want the maximum independent set in G. This is of course an NP-Complete problem, and it is known that no approximation factor is possible within $|V|^{1-\varepsilon}$ for any $\varepsilon > 0$ if $NP \neq ZPP$ [Has96]. Surprisingly, even if the maximum degree of the graph is bounded by 3, no PTAS is possible in this case [BF99].

Fat (convex) objects. In geometric settings, better results are possible. If the objects are fat (e.g., disks and squares), PTASes are known. One approach [Cha03, EJS05] relies on a hierarchical spatial subdivision, such as a quadtree, combined with dynamic programming techniques [Aro98]; it works even in the weighted case. Another approach [Cha03] relies on a recursive application of a nontrivial generalization of the planar separator theorem [LT79, SW98]; this approach is limited to the unweighted case.

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Arbitrary objects. If the objects are not fat, only weaker results are known. For the problem of finding a maximum independent set of unweighted axis-parallel rectangles, an $O(\log \log n)$ -approximation algorithm was given by Chalermsook and Chuzhoy [CC09]. For the weighted case of rectangles, Chan and Har-Peled [CH12] provided a $O(\log n/\log \log n)$ approximation. Furthermore, they provided a PTAS for independent set of pseudo-disks. Surprisingly, the algorithm is a simple local search strategy, that relies on using the planar separator theorem, to argue that if the local solution is far from the optimal, then there is a "small" beneficial exchange.

For line segments, a roughly $O(\sqrt{n_{\rm opt}})$ -approximation is known [AM06], where $n_{\rm opt}$ is the size of the optimal solution. Recently, Fox and Pach [FP11] have improved the approximation factor to n^{ε} for line segments, and also curves that intersect a constant number of times. Their argument relies on the intersection graph having a large biclique if it is dense, and a cheap separator if the intersection graph is sparse.

Recent progress. Adamaszek and Wiese [AW13, AW14] showed recently a QPTAS (i.e., Quasi-polynomial time approximation scheme¹) for independent set of weighted axis-parallel rectangles; specifically, for n axis-parallel rectangles and approximation parameter $\varepsilon > 0$, the algorithm outputs an independent set of weight $\geq (1 - \varepsilon)W_{\text{opt}}$, in $n^{\text{poly}(\log n,1/\varepsilon)}$ time, where $\text{poly}(\cdot)$ denotes some constant degree polynomial function, and W_{opt} is the weight of the optimal solution. Adamaszek and Wiese argued that there is always a closed polygonal curve, of complexity $O(\text{poly}(\log n.1/\varepsilon))$, that intersects $O(\varepsilon/\log n)$ -fraction of the optimal solution, and partition the optimal solution in a balanced way. Furthermore, one can easily enumerate over such polygons. Now, a recursive divide and conquer algorithm results in a QPTAS for the problem.

To prove the existence of this cheap curve, Adamaszek and Wiese construct a rather involved partition of the plane into regions, such that each region boundary intersects only a small fraction of the optimal solution, and then using the deus ex machina (i.e., the planar separator theorem) it follows that this "cheap" curve exists.

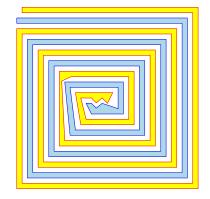
More recently, in an upcoming SODA 2014 paper, Adamaszek and Wiese [AW13, AW14] extended their results to polygons with polylog number of vertices. Furthermore, as pointed to them by the author, their approach can be dramatically simplified by using cuttings [Cha93, BS95], and their paper sketches this alternative approach.

Our results. In this paper, we extend Adamaszek and Wiese results to polygons of arbitrary complexity. Our approximation algorithm is *polynomial* in the total complexity of the input, and quasi-polynomial in the number of input polygons. In detail, we show the following.

(A) Canonical decomposition of an arrangement of polygons.

¹Not to be confused with queasy-polynomial time.

We want to apply the Clarkson-Shor technique on a set of polygons as described above. To this end, we need to come up with a decomposition of the complement the union of disjoint polygons, into "simple" canonical regions, where each such region has a constant size set of polygons that define it. In spirit, the idea is similar to computing the vertical decomposition of pseudo-disks, except that things are significantly more subtle, as we are not assuming that the polygons have constant number of extremal points in any direction. As such, two polygons might be tangled together in such a way that vertical decomposition (or any similar scheme) would have unbounded complexity, see figure on the right.



As such, we need a more topological approach to the task. To this end, we use the medial axis of the complement of the union, to define the decomposition.

We also extend this decomposition to the non-disjoint case. Specifically, we show that for t polygons with k intersections of their boundaries, one can decompose the arrangement into O(k+t) canonical cells.

The author is unaware of this decomposition being described in the literature before, and it is thus probably new.

(B) Clarkson-Shor technique, exponential decay, and cuttings.

The above decomposition now complies with the requirements of the Clarkson-Shor technique [CS89], and we can use it here to compute cuttings. Specifically, we need weighted cuttings, and while this is an easy extension of known techniques, this is not written explicitly in detail anywhere. As such, for the sake of self-containment, we reprove here the weighted version of the exponential decay lemma of Chazelle and Friedman [CF90]. Our proofs seems to be (somewhat) simpler than previous proofs, and the constants are somewhat better, and as such it might be of independent interest.

This is already sufficient to prove a weak version of cuttings. Specifically, we show that given a set of disjoint polygons of total weight W, and a parameter r, one can decompose the plane into $O(r \log r)$ canonical cells, such that the total weight of the polygons intersecting each canonical cell is at most W/r. If every input polygon intersects all lines a constant number of points, then one can prove the stronger version of cuttings, where the number of cells in the cutting is only O(r).

(C) **QPTAS** for independent set of polygons.

In Section 4.1, we describe how to use the above cutting result to argue that there is always a cheap separating curve for the optimal independent set. Our proof works by using the planar separator theorem on the cuttings computed above. Our proof is significantly simpler than the proof of Adamaszek and Wiese [AW13, AW14], and it uses a significantly weaker version of the planar separator theorem.

In Section 4.2, we plug our machinery into the algorithm of Adamaszek and Wiese [AW13, AW14], and get the desired QPTAS. Our algorithm running time is $O(m^{\text{poly}(\log m, 1/\varepsilon)} + m^{O(1)}n)$, where m is the number of input polygons, and n is their total complexity.

(D) Extensions.

For our algorithm to go through, all one needs is that the 1/r-cutting has subquadratic complexity in r. To this end, we assume that every pair of input polygons intersects a constant number of times (note, that we did not need this assumption in the independent set case). Now, cuttings have sub-quadratic complexity if the number of vertices in the original arrangement of the optimal

subset we want to compute, has subquadratic number of vertices. In particular, we get a QPTAS for the following problems.

- (i) **Pseudo-disks of bounded depth.** Given a set F of weighted pseudo-disks, and a parameter d (say a constant), we show that one can compute (1ε) -approximation to the largest subset $H \subseteq F$ of pseudo-disks, such that no point in the plane is covered by more than d regions of H.
- (ii) **Sparse subsets.** Consider a weighted set of polygons \mathcal{P} , where we want to find the heaviest subset $\mathcal{B} \subseteq \mathcal{B}$, such that the intersection graph $\mathsf{G}_I(\mathcal{B})$ does not contain the biclique $K_{s,t}$, where s and t are constants. The graph $\mathsf{G}_I(\mathcal{B})$ must be sparse in this case, and one can get a QPTAS to the largest such subset. In particular, any condition that guarantees the sparsity of $\mathsf{G}_I(\mathcal{B})$, facilities a QPTAS for finding the largest subset that has the desired property.

In particular, the above implies that the framework of Adamaszek and Wiese [AW13, AW14] can be used to approximate the largest induced sparse subgraphs of the intersection graph of well-behaved geometric regions. Here, the type of sparse subgraphs that can be approximated, are ones where the sparsity is a hereditary property that holds for any subset of vertices (similar in spirit to the independence matroid). Surprisingly, such sparse intersection graphs must have only linear number of edges, see Section 5 for details.

This is a significant strengthening of the work of Adamaszek and Wiese, and the author is unaware of any previous work that provides such guarantees (this new problem can be interpreted as a packing problem, and there are some results known about geometric packing, see [EHR12] and references therein).

Paper organization. In Section 2, we describe the canonical decomposition of the complement of the union of k disjoint polygons, and how to extend it to arbitrary intersecting polygons. In Section 3 we reprove the exponential decay lemma, and show how to build weak 1/r-cuttings of disjoint polygons of size $O(r \log r)$, and spell out the conditions enabling one to compute smaller 1/r-cuttings of size O(r). In Section 4, we sketch the QPTAS for independent set of polygons. In Section 5, we describe how the extension to a QPTAS for computing the heaviest sparse subset of polygons. We conclude in Section 6 with some comments.

2. Decomposing an arrangement of polygons into corridors

2.1. Canonical decomposition for disjoint polygons

Let $\mathcal{P} = \{P_1, \dots, P_m\}$ be a set of m disjoint simple polygons in the plane, of total complexity n. We also have a special outside square that contains all the polygons of \mathcal{P} , which we refer to as the **frame**. For the sake of simplicity of exposition we assume that all the edges involved in \mathcal{P} and the frame are neither horizontal nor vertical (this can be ensured by slightly rotating the axis system)².

We are interested in a canonical decomposition of the complement of the union of the polygons of \mathcal{P} inside the frame, that has the property that the numbers of cells is O(m), and every cell is defined by a constant number of polygons of \mathcal{P} . To this end, consider the medial axis of \mathcal{P} . To make the presentation easier³, we will use the L_{∞} -medial axis $\mathcal{M} = \mathcal{M}(\mathcal{P})$. Specifically, a point $\mathfrak{p} \in \mathbb{R}^2$ is in \mathcal{M} if there is an L_{∞} -ball (i.e., an axis-parallel square \square centered in \mathfrak{p}) that touches the polygons of \mathcal{P} in two or more

²In the example of Figure 2.1 we do not bother to do this, and the frame is axis parallel.

³Or at least making the drawing of the figures easier.

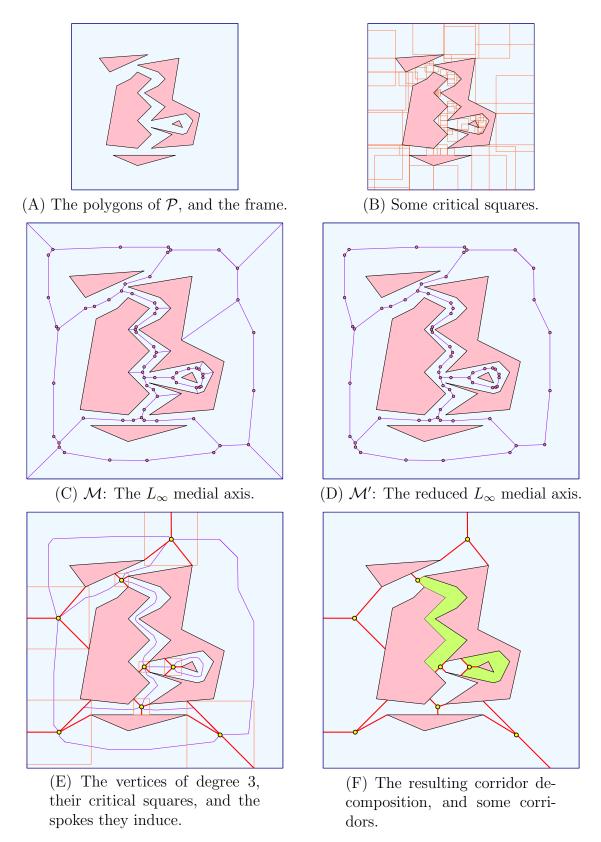


Figure 2.1: Building up the corridor decomposition.

points, and the interior of \square does not intersect any of the polygons of \mathcal{P} . We will refer to \square as a *critical square*.

The L_{∞} -medial axis is a connected collection of interior disjoint segments (i.e., it is the boundary of the Voronoi diagram of the polygons in \mathcal{P} under the L_{∞} metric together with some extra bridges involving points of the medial axis that have the same polygon on both sides). The medial axis \mathcal{M} contains some features that are of no use to us – specifically, we repeatedly remove vertices of degree one in \mathcal{M} and the segments that supports them – this process removes unnecessary tendrils. Let \mathcal{M}' be the resulting structure after this cleanup process.

Let $V = V(\mathcal{M}')$ be the set of vertices of \mathcal{M}' of degree at least three. Each such vertex $p \in V$ has a critical square \square_p associated with it. For such a square \square_p , there are $k \geq 2$ input polygons that it touches, and let p_1, \ldots, p_k be these k points of contact. We refer to the segments pp_1, pp_2, \ldots, pp_k as the **spokes** of p. Since no edge of the input polygons, or the frame is axis parallel, the spokes are uniquely defined.

Let S be the set of all spokes defined by the vertices of V. Consider the arrangement formed by the polygons of \mathcal{P} together with the segments of S. This decomposes the complement of the union of \mathcal{P} into simple polygons. Each such polygon boundary is made out of two polygonal chains that lie on two polygons of \mathcal{P} , and four spokes, see Figure 2.1 for an example. We refer to such a polygon as a **corridor**.

2.1.1. Corridor decomposition

Let $\mathcal{C}(\mathcal{P})$ denote the set of resulting polygons, which is the *corridor decomposition* of \mathcal{P} . We observe the following properties:

- (A) Consider a corridor $C \in \mathcal{C}(\mathcal{Q})$, for some $\mathcal{Q} \subseteq \mathcal{P}$. Then, there exists a subset $\mathcal{B} \subseteq \mathcal{P}$ of size at most 4 such that $C \in \mathcal{C}(\mathcal{B})$. The set $\mathcal{B} = D(C)$ is the **defining set** of C.
- (B) For such a corridor C, a polygon $\sigma \in \mathcal{P}$ conflicts with C, if C is not a corridor of $\mathcal{C}(D(C) \cup \{P\})$. This happens if σ intersects C, or alternatively, the presence of σ prevents the creation in the medial axis of the two vertices of the medial axis defining C. The set of polygons in $\mathcal{P} \setminus D(C)$ that conflict with C is the stopping set (or conflict list) of C.

Lemma 2.1. For a set Q of m disjoint simply connected polygons in the plane, we have that $|\mathcal{C}(Q)| = O(m)$.

Proof. Consider the reduced median axis \mathcal{M}' . It can be naturally be interpreted as a connected planar graph, where the vertices of degree at least three form the vertex set V, and two vertices are connected by an edge if there is a path π on \mathcal{M}' that connects them, and there is no vertex of V in the interior of π . Let G = (V, E) be the resulting graph.

Observe that the drawing of G has m+1 faces, as each face contains a single polygon of \mathcal{Q} in its interior (except for the outer one, which "contains" the frame). The graph G might contain both self loops, and parallel edges. However, every vertex of G has degree at least 3. As such, we have that $e \geq 3v/2$, where v and e are the number of vertices and edges in G, respectively.

Euler's formula in this case states that m+1-e+v=2 (the formula holds even if the graph contains loops and parallel edges), As such we have that $m+1-(3v/2)+v\geq 2$, which implies that $2m+2\geq v+4$; that is $v\leq 2m-2$. This in turn implies that $m+1-e+(2m-2)\geq 2$, which implies that $e\leq 3m-3$. Now, clearly, every corridor corresponds to one edge of G, which implies the claim.

2.1.2. Canonical decomposition for intersecting polygons

Let $Q = \{P_1, \ldots, P_m\}$ be a set of m simple polygons in the plane. We naturally assume that no three boundaries of the polygons passes through a common point.

For two polygons, we refer to an intersection point of their boundaries as an *intersection vertex*. Next, let t be the total number of intersection vertices in the arrangement $\mathcal{A}(\mathcal{Q})$. We would like to perform the same kind of canonical decomposition as above. To this end, consider any face f of the arrangement $\mathcal{A}(\mathcal{Q})$, that has t_f intersection vertices on its boundary, and has k_f distinct polygons on its boundary. Applying Lemma 2.1 to f, where the outer boundary of face replace the frame, results in a decomposition of this face into $O(k_f + t_f)$ corridors. Repeating this to all the faces in the arrangement, results in the desired decomposition. We thus get the following.

Lemma 2.2. Let Q be a set of m simply connected polygons in the plane, and let t be the total number of intersection vertices in A(Q). One can compute a decomposition C(Q) of the plane into corridors, such that no polygon boundary intersects the interior of a corridor, and each corridor is defined by at most four polygons of P. The total number of corridors in C(Q) is O(m+t).

Proof. The decomposition is described above. As for the total number of corridors, observe that every intersection vertex, can contribute to at most four faces, and one can also charge the breakup of the boundary of polygons passing through this vertex, to the vertex. Every vertex get charged O(1) times, there are $\leq t = O(t)$ intersection vertices, and the result follows.

3. Sampling, exponential decay, and cuttings

We next show that one can compute 1/r-cuttings for disjoint polygons, and sparse set of polygons. We start by reproving the exponential decay lemma.

3.1. Exponential decay

Let \mathcal{P} be a set of m disjoint polygons in the plane. Lemma 2.2 implies that the set of all possible corridors induced by any subset of \mathcal{P} is of size $O(m^4)$.

Consider any subset $S \subseteq P$. It is easy to verify that the following two conditions hold (see Section 2.1.1 for notations):

- (i) For any $C \in \mathcal{C}(\mathcal{S})$, we have $D(C) \subseteq \mathcal{S}$ and $\mathcal{S} \cap K(C) = \emptyset$.
- (ii) If $D(C) \subseteq \mathcal{S}$ and $K(C) \cap \mathcal{S} = \emptyset$, then $C \in \mathcal{C}(\mathcal{S})$.

Namely, the corridor decomposition complies with the Clarkson-Shor technique [CS89] (see also [Har11a, Chapter 8]). We prove a standard implication of this technique, for our settings, in a slightly more general settings.

Consider a set \mathcal{P} of m disjoint polygons, where every polygon $\sigma_i \in \mathcal{P}$ has weight $w_i > 0$, and $W = \sum_{i=1}^m w_i$. We prove the following version of the exponential decay lemma – this proof is an easy extension of the standard proof (if slightly simpler), and is presented here for the sake of completeness.

Definition 3.1. For a target size ρ , a ρ -sample is a random sample $S \subseteq \mathcal{P}$, where polygon σ_i is picked with probability $\rho w_i/W$.

Definition 3.2. A monotone increasing function $u(\cdot) \ge 0$ is **slowly growing**, if for any integer i > 0, we have that $u(in) \le i^{O(1)}u(n)$.

Lemma 3.3 (The weighted exponential decay lemma). Let $\mathcal{P} = \{\sigma_1, \ldots, \sigma_m\}$ be a set of m disjoint weighted polygons in the plane, $\rho \leq m$, and $1 \leq t \leq \rho/4$ be parameters. For $i = 1, \ldots, m$, let $w_i > 0$ be the weight of the ith polygon σ_i . Let $W = \sum_i w_i$. Consider two independent random ρ -samples \mathcal{S}_1 and \mathcal{S}_2 of \mathcal{P} , and let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. A corridor C is t-heavy if the total weight of the polygons in its conflict list is $\geq tW/\rho$. Let $\mathcal{C}_{\geq t}(\mathcal{S})$ be the set of all t-heavy corridors of $\mathcal{C}(\mathcal{S})$. We have that $\mathbf{E}[|\mathcal{C}_{\geq t}(\mathcal{S})|] = O(\rho \exp(-t))$.

More generally, if the polygons are not disjoint, and the corridor decomposition of any m' of them has complexity u(m'), where u(m') is a slowly growing function, then we have that $\mathbf{E}[|\mathcal{C}_{\geq t}(\mathcal{S})|] = O(u(\rho) \exp(-t))$.

Proof. The basic argument is to use double sampling. Intuitively (but outrageously wrongly), a heavy corridor of $\mathcal{C}(\mathcal{S})$ has constant probability to be presented in $\mathcal{C}(\mathcal{S}_1)$, but then it has exponentially small probability (i.e., e^{-t}) of not being "killed" by a conflicting polygon present in the second sample \mathcal{S}_2 .

For a polygon $Q \in \mathcal{P}$, we have that $\Pr[Q \in \mathcal{S}_1 \mid Q \in \mathcal{S}] = \Pr[Q \in \mathcal{S}_2 \mid Q \in \mathcal{S}] \ge 1/2$. Now, consider a corridor $C \in \mathcal{C}(\mathcal{S})$, and let $Q_1, \ldots, Q_4 \in \mathcal{P}$ be its defining set. Clearly, we have that

$$\nu = \mathbf{Pr} \Big[C \in \mathfrak{C}(\mathcal{S}_1) \ \Big| \ C \in \mathfrak{C}(\mathcal{S}) \Big] = \mathbf{Pr} \Big[Q_1, Q_2, Q_3, Q_4 \in \mathcal{S}_1 \ \Big| \ C \in \mathfrak{C}(\mathcal{S}) \Big]$$

$$= \prod_{i=1}^4 \mathbf{Pr} \Big[Q_i \in \mathcal{S}_1 \ \Big| \ C \in \mathfrak{C}(\mathcal{S}) \Big] = \prod_{i=1}^4 \mathbf{Pr} \Big[Q_i \in \mathcal{S}_1 \ \Big| \ Q_1, \dots, Q_4 \in \mathcal{S} \Big]$$

$$= \prod_{i=1}^4 \mathbf{Pr} \Big[Q_i \in \mathcal{S}_1 \ \Big| \ Q_i \in \mathcal{S} \Big] \ge \frac{1}{16}.$$

This in turn implies that

$$16 \operatorname{Pr} \left[\left(C \in \mathcal{C}(\mathcal{S}_1) \right) \cap \left(C \in \mathcal{C}(\mathcal{S}) \right) \right] \ge \operatorname{Pr} \left[C \in \mathcal{C}(\mathcal{S}) \right]. \tag{1}$$

Next, consider a corridor $C \in \mathcal{C}(S_1)$ that is t-heavy, with, say, $\{P_1, \ldots, P_k\}$ being its conflict list. Clearly, the probability that S_2 fails to pick one of the conflicting polygons in S_2 , is bounded by

$$\mathbf{Pr}\Big[C \in \mathcal{C}(\mathcal{S}) \mid C \in \mathcal{C}(\mathcal{S}_1)\Big] = \mathbf{Pr}\Big[\forall i \in \{1, \dots, k\} \quad P_i \notin \mathcal{S}_2\Big] = \prod_{i=1}^k \Big(1 - \rho \frac{w_i}{W}\Big)$$

$$\leq \prod_{i=1}^k \exp\Big(-\rho \frac{w_i}{W}\Big) = \exp\Big(-\frac{\rho}{W} \sum_i w_i\Big)$$

$$\leq \exp\Big(-\frac{\rho}{W} \cdot t \frac{W}{\rho}\Big) = e^{-t}.$$

Let \mathcal{F} be the set of possible corridors, and let $\mathcal{F}_{\geq t} \subseteq \mathcal{F}$ be the set of all t-heavy corridors. We have that

$$\mathbf{E}\Big[|\mathcal{C}_{\geq t}(\mathcal{S})|\Big] = \sum_{C \in \mathcal{F}_{\geq t}} \mathbf{Pr}\Big[C \in \mathcal{C}(\mathcal{S})\Big] \leq \sum_{C \in \mathcal{F}_{\geq t}} 16 \, \mathbf{Pr}\Big[(C \in \mathcal{C}(\mathcal{S}_{1})) \cap (C \in \mathcal{C}(\mathcal{S}))\Big]$$

$$\leq 16 \sum_{C \in \mathcal{F}_{\geq t}} \mathbf{Pr}\Big[C \in \mathcal{C}(\mathcal{S}) \mid C \in \mathcal{C}(\mathcal{S}_{1})\Big] \mathbf{Pr}[C \in \mathcal{C}(\mathcal{S}_{1})] \leq 16e^{-t} \sum_{C \in \mathcal{F}_{\geq t}} \mathbf{Pr}\Big[C \in \mathcal{C}(\mathcal{S}_{1})\Big]$$

$$\leq 16e^{-t} \sum_{C \in \mathcal{T}} \mathbf{Pr}\Big[C \in \mathcal{C}(\mathcal{S}_{1})\Big] = 16e^{-t} \, \mathbf{E}\Big[|\mathcal{C}(\mathcal{S}_{1})|\Big] = 16e^{-t} \, \mathbf{E}\Big[O(|\mathcal{S}_{1}|)\Big] = O(e^{-t}\rho),$$

by Lemma 2.1, and since $\mathbf{E}[|\mathcal{S}_1|] = \rho$.

As for the second claim, by the Chernoff inequality, and since $u(\cdot)$ is slowly growing, there are constants c and c', such that

$$\mathbf{E}\Big[|\mathcal{C}(\mathcal{S}_1)|\Big] \le u(\rho) + \sum_{i=1}^{\infty} \mathbf{Pr}\Big[|\mathcal{S}_1| \ge i\rho\Big] u\Big((i+1)\rho\Big) \le u(\rho) + \sum_{i=1}^{\infty} 2^{-i}c(i+1)^{c'}u(\rho) = O(u(\rho)).$$

Our proof of the exponential decay lemma is inspired by the work of Sharir [Sha03]. The resulting computations seems somewhat easier than the standard argumentation.

3.1.1. Weak Cuttings

For a set \mathcal{P} of polygons of total weight W, a 1/r-cutting is a decomposition \mathcal{C} of the plane into regions, such that

- (A) each region is "simple",
- (B) the total number of regions in \mathcal{C} is small (as a function of r), and
- (C) for a region $C \in \mathcal{C}$, the total weight of the polygons of \mathcal{Q} , such that their boundary intersects the interior of C is at most W/r.

See [CF90, BS95, Har00] and references therein for more information about cuttings.

Lemma 3.4. Let \mathcal{P} be a set of weighted polygons of total weight W, not necessarily disjoint, such that for any subset $\mathcal{S} \subseteq \mathcal{P}$, the complexity of $\mathcal{C}(\mathcal{S})$ is $u(|\mathcal{S}|)$, and $u(\cdot)$ is a slowly growing function. Then there exists 1/r-cutting with $O(u(r \log r))$ corridors. Furthermore, this cutting can be computed efficiently.

Proof. Let S_1 and S_2 , be ρ -samples for $\rho = cr \ln r$, where c is a sufficiently large constant, and let $S = S_1 \cup S_2$. We have that for any corridor $C \in \mathcal{C}(S)$, the total weight of the conflict list of C is of size $\leq W/r$. This holds with probability $\geq 1 - 1/r^{O(1)}$, by Lemma 3.3.

Indeed, since $u(\cdot)$ is slowly growing, it must be that $u(i) = i^{O(1)}$. Now, a corridor of $\mathcal{C}(\mathcal{S})$ that has conflict-list weight $\geq W/r$, is t-heavy for $t = c \ln r$. The number of such corridors by Lemma 3.3 is in expectation

$$\mathbf{E}\Big[|\mathcal{C}_{\geq t}(\mathcal{S})|\Big] = O\Big(u(\rho)\exp(-t)\Big) = O\Big(\rho^{O(1)}\exp(-t)\Big) < \frac{1}{r^{O(1)}},$$

for c sufficiently large. Now, the claim follows from Markov's inequality, as $\mathbf{Pr}\Big[|\mathcal{C}_{\geq t}(\mathcal{S})| \geq 1\Big] \leq \mathbf{E}\Big[|\mathcal{C}_{\geq t}(\mathcal{S})|\Big] \leq 1/r^{O(1)}$.

3.1.2. Smaller cuttings

Getting 1/r-cuttings of size O(r) (for disjoint polygons), where every cell in the cutting is "nice" is somewhat more challenging. However, for our purposes, any 1/r-cutting of size $O(r^c)$ (where c < 2 is a constant) is sufficient. Nevertheless, one way to get the smaller cuttings, is by restricting the kind of polygons under consideration.

Lemma 3.5. Let \mathcal{P} be a set of weighted polygons of total weight W, not necessarily disjoint, such that for any subset $S \subseteq \mathcal{P}$, the complexity of $\mathcal{C}(S)$ is u(|S|), and $u(\cdot)$ is a slowly growing function. In addition, assume that every polygon in \mathcal{P} have O(1) intersection points with any line, and the boundaries of every pair of polygons of \mathcal{P} have a constant number of intersections.

Then, there exists 1/r-cutting of \mathcal{P} into O(u(r)) regions, where every region is the intersection of two corridors. Furthermore, this cutting can be computed efficiently.

Proof. This follows by the standard two level sampling used in the regular cutting construction. Specifically, we first take a sample of size r, and then we fix-up any corridor that its conflict-list is too heavy by doing a second level sampling, using Lemma 3.4. In the resulting decomposition, we have to clip every corridor generated in the second level, to its parent corridor. The assumption about every polygon intersecting a line constant number of times implies the desired bound.

We omit the easy details – see de Berg and Schwarzkopf [BS95] and Chazelle and Friedman [CF90]. ■

4. **QPTAS** for independent set

4.1. Structural lemma about good separating polygon

Lemma 4.1. Consider a weighted set \mathcal{P} of m polygons, of total complexity n, not necessarily disjoint, and let \mathcal{O} be the heaviest independent set of polygons in \mathcal{P} , where $n_{\text{opt}} = |\mathcal{O}|$ and $W_{\text{opt}} = w(\mathcal{O}) = \sum_{Q \in \mathcal{P}} w(Q)$. And let r be a parameter. Then there exists a polygon σ , such that:

 (\widetilde{A}) The total weight of the polygons of \mathcal{O} completely inside (resp. outside) it is at least a constant fraction of W_{opt} .

- (B) The total weight of the polygons of \mathcal{O} that intersect the boundary of σ is $O\left(\sqrt{\frac{\log r}{r}}W_{\mathrm{opt}}\right)$.
- (C) The polygon σ can be fully encoded by a binary string having $O(\sqrt{r \log r} \log m)$ bits.

Proof. We compute an 1/r-cutting of \mathcal{O} using Lemma 3.4, which results in a decomposition with $\rho = O(r \log r)$ corridors. We now interpret this cutting as a planar map, with $O(\rho)$ faces. As such, by the planar separator theorem, it has a cycle separator of size $O(\sqrt{\rho})$.

We need to be slightly more careful. We assign every polygon of \mathcal{O} to the corridor in the cutting that contains its leftmost vertex. Thus, the weight of a corridor of the cutting is the total weight of the polygons that get assigned to it (notice, that a polygon of \mathcal{O} might intersect several corridors, but it is assigned only to one of them).

The cycle separator we need is a partition of the planar graph that balances out these weights of the faces. The existence of such a cycle separator follows readily by known results [Mil86]⁴.

The separating cycle σ has $O(\sqrt{\rho})$ edges in the planar graph. Here, an edge is either a spoke or a subchain of one of the polygons of \mathcal{P} . Now, the total weight of polygons of \mathcal{P} that intersect a spoke⁵ used in the 1/r-cutting can be at most W_{opt}/r , it follows that the total weight of polygons in \mathcal{O} intersecting σ is

$$O\left(\sqrt{\rho} \frac{W_{\text{opt}}}{r}\right) = O\left(\sqrt{r \log r} \cdot \frac{W_{\text{opt}}}{r}\right) = O\left(W_{\text{opt}} \sqrt{\frac{\log r}{r}}\right).$$

We next show how to encode each edge of σ using $O(\log m)$ bits, which implies the claim. Compute the $O(m^4)$ possible corridors induced by any subset of polygons of $\mathcal P$ that do not intersect. Let $\mathcal X$ be

⁴For the interested reader here is a quick sketch – consider the dual graph, and observe that under general position assumptions it is triangulated, and as such it has the desired cycle separator. Interpreting this cycle, as a cycle of faces in the original graph, and converting it into a cycle of edges yields the desired separator, by plugging it into a "standard" weighted cycle separator result, see for example [Har11b] for a recent proof.

⁵Note, that by disjointness of the polygons of \mathcal{O} , no polygonal chain can intersect any polygons, at least for the case of independent set.

this set of polygons. Every corridor of \mathcal{X} induces ≤ 4 vertices where its spokes touch its two adjacent polygons. In particular, let P be the set of all such vertices. Clearly, there are $O(m^4)$ such vertices.

Consider an edge e of σ . If it is a spoke we can encode it by specifying which spoke it is, which requires $O(\log m^4)$ bits, since there are $O(m^4)$ possible spokes. Otherwise, the edge is a subchain of one of the polygons of \mathcal{P} . We specify which one of the polygons it is on, which requires $O(\log m)$ bits, and then we need to specify the start and end vertex of the subchain, which can be done by specifying the two relevant vertices of P , using $O(\log |\mathsf{P}|) = O(\log m)$ bits. We also need to specify which one of the two possible polygonal subchains we refer to, which requires an extra bit. Overall, the number of bits needed to encode σ is $O(\sqrt{\rho} \log m)$, as claimed.

4.2. Computing an approximate independent set of polygons

Algorithm sketch. Let \mathcal{P} be the given set of m unweighted polygons, with total complexity n. Assume that the largest independent set \mathcal{O} has size n_{opt} . We need to set r such that

$$\sqrt{\frac{\log r}{r}} < c' \frac{\varepsilon}{\log m},$$

where c' is some fixed constant which is sufficiently large. Assuming $m > 1/\varepsilon$, this holds if we set $r = O\left(\left(\frac{\log m}{\varepsilon}\right)^2 \log \frac{\log m}{\varepsilon}\right)$. By Lemma 4.1 there exists a polygon σ that can be encoded by a binary string of length $L = O(\sqrt{r \log r} \log m) = \text{poly}(\log m, 1/\varepsilon)$. This polygon has the property that its boundary intersects at most $n_{\sigma} \leq c \frac{\varepsilon}{\log m} n_{\text{opt}}$ polygons of \mathcal{O} , and σ splits \mathcal{O} in a balanced way. In particular, assume for the time being that we knew σ . The algorithm then recurse on $\mathcal{P}_{\text{in}} = \left\{\tau \in \mathcal{P} \mid \tau \subseteq \sigma\right\}$ and $\mathcal{P}_{\text{out}} = \left\{\tau \in \mathcal{P} \mid \tau \cap \sigma = \emptyset\right\}$. Since the polygon σ partition \mathcal{O} in a balanced way, this recursion would have depth $H = O(\log m)$, before the subproblem would be of size $O(\text{poly}(\log m, 1/\varepsilon))$. At this point, the recursion bottoms out, and the algorithm tries all possibilities, to find the largest independent set.

A recursive instance is defined by the boundary of at most H polygons, each of them can be encoded by a string of length L. As such, the number of recursive subproblems is $2^{O(LH)} = 2^{\text{poly}(\log m, 1/\varepsilon)}$. As such, a dynamic programming algorithm would work in this case, as in each level of the recursion there are $2^{O(L)}$ different separating polygons to consider, and in addition, one can try to solve the given subproblem using brute force, for subsets of size up to some $O(\text{poly}(\log m, 1/\varepsilon))$. Returning the best combined solution found (on the inside and outside subproblems), among all possibilities tried, results in the desired approximation algorithm.

As for the quality of approximation, we pick c such that $c\varepsilon/\log m < \varepsilon/4H$. Clearly, at each level of the recursion, we lose $\varepsilon/4H$ fraction of the optimal solution, and as such, overall, the solution output has weight at least $(1 - H \cdot (\varepsilon/4H))n_{\text{opt}} \ge (1 - \varepsilon)n_{\text{opt}}$.

Weighted case. Observe that we can assume that $m > 1/\varepsilon$ (otherwise, a brute force algorithm would work). As such, if the maximum weight polygon in the given instance is W, then we can ignore all polygons of weight $\varepsilon W/4m \le W/m^2$. In particular, normalizing weights, the weight of every polygon is going to be an integer in the range 1 to (say) m^3 . Now, the above algorithm would work verbatim, as the depth of the recursion is going to be $O(\log m^3) = O(\log m)$ before the subproblem weight becomes zero. The only difference is that we add the weight of a polygon $\tau \in \mathcal{P}$ to the cell in the cutting that contains its leftmost endpoint when arguing about the existence of a cheap separating cycle. The rest then go through without any change.

See [AW13, AW14] for further details. The above implies the following result.

Theorem 4.2. Given a set \mathcal{P} of m simple weighted polygons in the plane, of total complexity n, one can compute an independent set of \mathcal{P} of weight $\geq (1-\varepsilon)W_{\text{opt}}$, where W_{opt} is the maximum weight of the optimal independent set. The running time of the algorithm is $O(m^{\text{poly}(\log m, 1/\varepsilon)} + m^{O(1)}n)$.

Proof. Computing the set of all possible corridors takes $O(m^4n)$ time. The rest now follows the algorithm sketched above and in [AW13, AW14].

5. Extension: **QPTAS** for sparse properties

Let \mathcal{P} be a set of polygons in the plane, no pair of them is contained inside each other. We are interested in the intersection graph $\mathsf{G} = (\mathcal{P}, \mathsf{E})$ it induces; that is, $\mathsf{E} = \left\{ \sigma \tau \ \middle| \ \sigma, \tau \in \mathcal{P}, \sigma \cap \tau \neq \emptyset \right\}$. For a subset $X \subseteq \mathcal{P}$, let $\mathsf{G}_X = (X, \mathsf{E}_X)$ denote the **induced subgraph** of G on X; that is, $\mathsf{E}_X = \left\{ \sigma \tau \ \middle| \ \sigma, \tau \in X \text{ and } \sigma \tau \in \mathsf{E} \right\}$. We refer to two subsets $X \subseteq \mathcal{P}$ and $Y \subseteq \mathcal{P}$ as **separate**, if no polygon of X intersects any polygon in Y.

Consider a property Π on graphs (e.g., a graph is planar). We can naturally define the set system of all subsets of \mathcal{P} that have this property. That is $\Pi_{\mathcal{P}} = (\mathcal{P}, \mathcal{I})$, where $\mathcal{I} = \left\{ X \subseteq \mathcal{P} \mid \mathsf{G}_X \text{ has property } \Pi \right\}$. We are interested here in **hereditary** properties. Specifically, if $X \in \Pi_{\mathcal{P}}$ then $Y \in \Pi_{\mathcal{P}}$, for all

We are interested here in **hereditary** properties. Specifically, if $X \in \Pi_{\mathcal{P}}$ then $Y \in \Pi_{\mathcal{P}}$, for all $Y \subseteq X$. We also require that the property would be **mergeable**; that is, for any two separate subsets $X, Y \subseteq \mathcal{P}$, such that $X, Y \in \Pi_{\mathcal{P}}$ we have that $X \cup Y \in \Pi_{\mathcal{P}}$. Notice, that the combinatorial structure $\Pi_{\mathcal{P}}$ is similar to a matroid, except that we do not require to have the augmentation property (this is also known as an independence system).

The **weight** of $X \in \Pi_{\mathcal{P}}$ is the total weight of the polygons of X (or the cardinality of X in the unweighted case). We are interested in computing (or approximating) the heaviest set $X \in \Pi_{\mathcal{P}}$.

As a concrete example, consider the property Π that a set $X \subseteq \mathcal{P}$ has no pair of intersecting polygons. Thus, finding the heaviest set in $\Pi_{\mathcal{P}}$ that has the desired property, in this case, corresponds to finding the heaviest independent set in \mathcal{P} .

Definition 5.1. The property $\Pi_{\mathcal{P}}$ is sparse if there are constants δ, c , such that $\delta > 0$, and for any $X \in \Pi_{\mathcal{P}}$, we have that $|\mathsf{E}(\mathsf{G}_X)| \leq c' |X|^{2-\delta}$.

Informally, sparsity implies that the number of pairs of polygons intersecting each other, in any set $X \in \Pi_{\mathcal{P}}$, is strictly subquadratic in the size of X. Surprisingly, for an intersection graph of curves, where every pair of curves intersects only a constant number of times, sparsity implies that the number of edges in the intersection graph is linear [FP08].

Lemma 5.2 ([FP08]). Let \mathcal{P} be a set of polygons such that the boundaries of every pair of polygons has a constant number of intersections. Let $\Pi_{\mathcal{P}}$ be a sparse property. Then, for any $X \in \Pi_{\mathcal{P}}$, we have that $|\mathsf{E}(\mathsf{G}_X)| = O(|X|)$.

Proof. This result is known [FP08]. We include a sketch of the proof here for the sake of completeness. We think about the boundaries of the polygons of \mathcal{P} as curves in the plane, and let m = |X|. The intersection graph G_X has subquadratic numbers of edges, and as such, the arrangement of the curves of X has at most $m' = O(m^{2-\delta})$ vertices. By the planar separator theorem, there is a set of $O(\sqrt{m'}) = O(m^{1-\delta/2})$ vertices, that their removal disconnects this arrangement into a set of m_1, m_2

curves, where $m_1, m_2 \leq (2/3)m$ and $m_1 + m_2 \leq m + O(\sqrt{m'})$ (here we break the curves passing through a vertex of the separator into two curves, sent to the respective subproblems). Applying the argument now to both sets recursively, we get that the total number of vertices is $T(m) = O(m^{1-\delta/2}) + T(m_1) + T(m_2)$, and the solution of this recurrence is T(m) = O(m).

A property $\Pi_{\mathcal{P}}$ is **exponential time checkable**, if for any subset $X \subseteq V$, one can decide if $X \in \Pi_{\mathcal{P}}$ in time $2^{|X|^{O(1)}}$.

Theorem 5.3. Let \mathcal{P} be a weighted set of m polygons in the plane, no pair of them is contained inside each other, of total complexity n, such that the boundaries of every pair of them intersects only a constant number of times. Let $\Pi_{\mathcal{P}}$ be a hereditary, sparse and mergeable property that is exponential time checkable.

Then, for a parameter $\varepsilon > 0$, one can compute in quasi-polynomial time (i.e., $2^{O(\operatorname{poly}(\log m, 1/\varepsilon))})$ a subset $X \subseteq \mathcal{P}$, such that $X \in \Pi_{\mathcal{P}}$, and $w(X) \geq (1 - \varepsilon)W_{\operatorname{opt}}$, where W_{opt} is the maximum weight of a set in $\Pi_{\mathcal{P}}$.

Proof. One need to verify that the algorithm of Section 4 works also in this case. As before, we are going to argue that there exists a cheap separating polygon.

So, let $\mathcal{O} \in \Pi_{\mathcal{P}}$ be the heaviest set. By Lemma 5.2, for any subset $\mathcal{S} \subseteq \mathcal{P}$ of size $O(r \log r)$, the arrangement $\mathcal{A}(\mathcal{S})$ has $O(r \log r)$ vertices, and as such $\mathcal{C}(\mathcal{S})$ form a planar graph with, say, $O(r \log r)$ vertices, edges and faces. In particular, Lemma 3.4 implies that we can get a 1/r-cutting with this number of cells, and let \mathcal{C} be this cutting. Distributing the polygons of \mathcal{O} to the cells of \mathcal{C} , and finding a balanced weight separator, as done in Lemma 4.1, with $L = O(\sqrt{r \log r})$ edges, results in a separating polygon of weight

$$O\left(\frac{W_{\text{opt}}}{r}\sqrt{r\log r}\right) = O\left(\frac{W_{\text{opt}}}{r^{1/3}}\right) \le \frac{\varepsilon}{c\log m}W_{\text{opt}},$$

for $r = \Omega((\log m/\varepsilon)^3)$. It is easy to verify that such a polygon can be encoded using $O(L \log m) = O(\text{poly}(\log m, 1/\varepsilon))$ bits (vertices used by the cycle are either vertices rising out of intersection of polygons, and there are $O(m^2)$ such vertices, or medial axis vertices). The rest of the algorithm now works as described in Section 4.2. Note, that because of the mergeablity assumption, we need to verify that the generated sets have the desired property only in the bottom of the recursions. But such subsets have size $O(\text{poly}(\log m, 1/\varepsilon))$, and thus they can be checked in $2^{O(\text{poly}(\log m, 1/\varepsilon))}$ time, by the exponential time checkablity assumption.

Notice, that without the assumption that no pair of input polygons is contained inside each other, we have to deal with the non-trivial technicality that the separating cycle might be fully contained inside some input polygon.

Properties that comply with our conditions, and thus one can now Theorem 5.3 to get a QPTAS for the largest subset \mathcal{O} of \mathcal{P} that have this property, include the following:

- (A) All the polygons of \mathcal{O} are independent.
- (B) The intersection graph of \mathcal{O} is planar, or has low genus.
- (C) The intersection graph of \mathcal{O} does not contain $K_{s,t}$ as a subgraph, for s and t constants.
- (D) If the boundaries of every pair of polygons of \mathcal{P} intersects at most twice, then they behave like pseudo-disks. In particular, the union complexity of m pseudo-disks is linear, and the by the Clarkson-Shor technique, the complexity of the arrangement of depth k of m pseudo-disks is O(km). This implies that if \mathcal{O} is a set pseudo-disks with bounded depth, then the intersection

graph has only $O(|\mathcal{O}|)$ edges, and as such this is a sparse property, and it follows that one can $(1-\varepsilon)$ -approximate (in quasi-polynomial time) the heaviest subset of pseudo-disks where the maximum depth is bounded. Previously, only a constant approximation was known [EHR12].

6. Conclusions

We extended the QPTAS of Adamaszek and Wiese [AW13, AW14] for polygons of arbitrary complexity, in the process showing a new interesting case where the Clarkson-Shor technique holds. We also showed that the framework of Adamaszek and Wiese [AW13, AW14] applies not only for the problem of computing the heaviest independent set, but also computing the heaviest subset that has certain sparsity conditions (e.g., the intersection graph of the subset of polygons is a planar graph, or does not contain $K_{2,2}$, etc).

The most interesting open problem is trying to get a PTAS for these problems. For example, Adamaszek and Wiese [AW13], show a PTAS for the case of "large" axis-parallel rectangles. Along these lines, the existence of a separator for sparse intersection graphs [FP08, FP10, FP13], suggest that potentially in some (unweighted) cases one should be able to use a local search strategy, as was done by Chan and Har-Peled [CH12]. The technical problem is that Chan and Har-Peled applies the separator to the intersection graph that contains the optimal \mathcal{O} and local \mathcal{L} solutions together. Of course, $G_{\mathcal{O}}$ and $G_{\mathcal{L}}$ being sparse, in no way guarantees that $G_{\mathcal{O}\cup\mathcal{L}}$ is sparse. The only case where this sparsity still holds is for the case of searching for the largest subset of pseudo-disks such that their maximum depth is bounded by a constant. In particular, we conjecture that one gets a PTAS in this case via local search.

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