Peeling potatoes near-optimally in near-linear time^{*}

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Abstract

We consider the following geometric optimization problem: find a convex polygon of maximum area contained in a given simple polygon P with n vertices. We give a randomized near-linear-time $(1 - \varepsilon)$ -approximation algorithm for this problem: in $O(n(\log^2 n + (1/\varepsilon^3) \log n + 1/\varepsilon^4))$ time we find a convex polygon contained in P that, with probability at least 2/3, has area at least $(1 - \varepsilon)$ times the area of an optimal solution. We also obtain similar results for the variant of computing a convex polygon inside P with maximum perimeter.

To achieve these results we provide new results in geometric probability. The first result is a bound relating the probability that two points chosen uniformly at random inside P are mutually visible and the area of the largest convex body inside P. The second result is a bound on the expected value of the difference between the perimeter of any planar convex body K and the perimeter of the convex hull of a uniform random sample inside K.

Keywords: geometric optimization; potato peeling; visibility graph; geometric probability; approximation algorithm.

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1 Introduction

We consider the algorithmic problem of finding a maximum-area convex set in a given simple polygon. Thus, we are interested in computing

$$A^*(P) := \sup\{\operatorname{area}(K) \mid K \subset P, K \text{ convex}\}.$$

The problem was introduced by Goodman [25], who named it the **potato peeling prob**lem. Goodman also showed that the supremum is actually achieved, so we can replace it by the maximum. Henceforth we use n to denote the number of vertices in the input polygon P.

Chang and Yap [12] showed that $A^*(P)$ can be computed in $O(n^7)$ time. Since there have been no improvements in the running time of exact algorithms, it is natural to turn the attention to faster, approximation algorithms. A step in this direction is made by Hall-Holt et al. [27], who show how to obtain a constant-factor approximation in $O(n \log n)$ time.

In this paper we present a randomized $(1 - \varepsilon)$ -approximation algorithm. Besides the simple polygon P, the algorithm takes as input a parameter $\varepsilon \in (0, 1)$ controlling the approximation. In time $O\left(n(\log^2 n + (1/\varepsilon^3)\log n + 1/\varepsilon^4)\right)$ the algorithm returns a convex polygon contained in P that, with probability at least 2/3, has area at least $(1 - \varepsilon) \cdot A^*(P)$. For any constant ε , and more generally for any $\varepsilon = \Omega\left(1/\log^{1/3} n\right)$, the running time becomes $O(n\log^2 n)$. As usual, the probability of error can be reduced to $\delta \in (0, 1)$ using $O(\log(1/\delta))$ independent repetitions of the algorithm. Note that for $\varepsilon < 1/n^{3/2}$, the exact algorithm of Chang and Yap [12] is faster as it runs in time $O(n^7) = O(n/\varepsilon^4)$.

Overview of the approach. Let R be a set of points contained in P. The *visibility* graph of R, denoted by G(P, R), has R as vertex set and, for any two points x and y in R, the edge xy is in G(P, R) whenever the segment xy is contained in P. See Figure 1.

Let us assume that the set of points R is obtained by uniform sampling in P. We note the following properties:

- For each convex polygon $K \subseteq P$, the area of the convex hull $\operatorname{conv}(K \cap R)$ is similar to the area of K, provided that $|K \cap R|$ is large enough. For this, it is convenient to have large |R|.
- For each convex polygon $K \subseteq P$, the boundary of $\operatorname{conv}(K \cap R)$ is made of edges in G(P, R).
- With dynamic programming one can find a maximum-area convex polygon defined by edges of G(P, R). For this to be efficient, it is convenient that G(P, R) has few edges.

Thus, we have a trade-off on the number of points in R that are needed. We argue that there is a suitable size for R such that G(P, R) has a near-linear expected number of edges and, with reasonable probability, the edges of G(P, R) give a good inner approximation to an optimal solution. Instead of finding the optimal solution directly in G(P, R), we make a search in a small parallelogram of area $\Theta(A^*(P))$ around each edge of G(P, R), performing a second sampling. The core of the argument is a bound relating $A^*(P)$ and the probability that two random points in P are visible. Such relation was unknown and we believe that it is of independent interest. See Theorems 9, 10 and the follow up work [5] (summarized in Theorem 11 here) for the precise relations.



Figure 1: A portion of the visibility graph of a point set. Only the edges incident to three vertices are displayed.

Perimeter. We are also interested in finding a convex polygon inside P with maximum perimeter. Let per(K) denote the perimeter of a convex body K. In the case that K is a segment, then per(K) is twice the length of K. Let

$$L^*(P) := \sup\{\operatorname{per}(K) \mid K \subset P, K \text{ convex}\}.$$

By the same compactness argument as used by Goodman [25, Proposition 1], using the Blaschke selection theorem, the supremum is achieved and so it can be replaced by the maximum.

We provide a randomized algorithm to compute a convex polygon (or segment) inside P whose perimeter is at least $(1 - \varepsilon) \cdot L^*(P)$. For every $\delta > 0$, to succeed with probability $1 - \delta$, the algorithm uses time

$$O\left(n\left[(1/\varepsilon^4)\log^2 n + \left((1/\varepsilon)\log^2 n + (1/\varepsilon^6)\log n + 1/\varepsilon^8\right)\log(1/\delta)\right]\right).$$

The main obstacle in this case is that the polygons with near-optimal perimeter may be very skinny and thus have arbitrarily small area. For that case, random sampling of points is futile, but we can use a longest segment contained in P to approximate $L^*(P)$. More precisely, if the perimeter-optimal convex polygon has aspect ratio $O(\varepsilon)$, then we can $(1 - \varepsilon)$ -approximate it via a longest segment inside P, which in turn can be $(1 - \varepsilon)$ approximated in near-linear time [27]. If the perimeter-optimal polygon has aspect ratio $\Omega(\varepsilon)$, then it has area at least $\Omega(\varepsilon \cdot A^*(P))$, and the approach based on random samples of points can be adapted, with a larger number of sample points. To bound the number of sample points we use a new theorem in geometric probability bounding the expected difference between the perimeter of any planar convex body K and the perimeter of the convex hull of a random sample inside K. See our Theorem 18 for the precise statement.

Other related work. There have been several results about finding maximum-area objects of certain type inside a given simple polygon. DePano, Ke and O'Rourke [20] consider squares and equilateral triangles, Daniels, Milenkovic and Roth [18] consider axis-parallel rectangles, Melissaratos and Souvaine [29] consider arbitrary triangles. Subquadratic algorithms to find a longest segment contained in a simple polygon were first given by Chazelle and Sharir [16] and improved by Agarwal, Sharir and Toledo [1, 2]. Hall-Holt et al. [27] present near-linear time algorithms for a $(1 - \varepsilon)$ -approximation of the longest segment.

Aronov et al. [3] consider a variation where the search is restricted to convex polygons whose edges are edges of a given triangulation (with inner points) of P. They show how

to compute a maximum-area convex polygon for this model in $O(m^2)$ time, where m is the number of edges in the triangulation.

Dumitrescu, Har-Peled and Tóth [21] consider the following problem: given a unit square Q and a set X of points inside Q, find a maximum-area convex body inside Q that does not have any point of X in its interior. This is an instance of the potato peeling problem for polygons with holes. They provide a $(1 - \varepsilon)$ -approximation in time $O(n^2/\varepsilon^6)$. For any fixed ε , the running time is quadratic. Our algorithm exploits the absence of holes in P, so it does not produce an improvement in this case.

The potato peeling problem can be understood as finding a largest set of points that are mutually visible. Rote [32] showed how to compute in polynomial time the probability that two random points inside a polygon are visible. A faster algorithm has been proposed by Buchin et al. [11]. Cheong, Efrat and Har-Peled [17] consider the problem of finding a point in a simple polygon whose visibility region is maximized. They provide a $(1 - \varepsilon)$ approximation algorithm using near-quadratic time. The approach is based on taking a random sample of points in the polygon, constructing the visibility region of each point, and taking a point lying in most visibility regions.

Roadmap. In Section 2 we provide tools related to convex bodies. In Section 3 we relate the probability of two random points being visible and $A^*(P)$. We present and analyze the algorithm to approximate $A^*(P)$ in Section 4. In Section 5 we discuss the adaptation to maximize the perimeter. We conclude in Section 6.

Assumptions. We will have to generate points uniformly at random inside a triangle. For this, we will assume that a random number in the interval [0, 1] can be generated in constant time.

2 About convexity

Here we provide tools related to convexity.

2.1 Inner approximation using random sampling

In this subsection, we provide results about the number of points that have to be sampled inside a convex body K so that the area of the convex hull of the sample is a good approximation to the area of K. We may think of K as a maximum-area convex set in P for which we aim to find a $(1 - \varepsilon)$ -approximation. In our algorithm, we sample points in a superset of K, thus we also provide extensions to this case. In particular, Lemma 1 deals with the problem of sampling points inside a given convex body K. In Lemma 3 the sample is taken from a larger polygon $\Gamma \supseteq K$ and the goal is to hit K with at least C points. These two results are then combined together in Lemma 4.

Lemma 1. Let K be a convex body in the plane and let R be a sample of points chosen uniformly at random inside K. There is some universal constant C_1 such that, if $|R| \ge C_1/\varepsilon^{3/2}$, then with probability at least 5/6 it holds that $\operatorname{area}(\operatorname{conv}(R)) \ge (1-\varepsilon) \cdot \operatorname{area}(K)$.

Proof. We use as a black box known extremal properties and bounds on the so-called missed area of a random polygon. See the lectures by Bárány [4, 2nd lecture], the survey [6] or [7] for an overview.

Let us scale K so that it has area 1. We have to show that $1 - \operatorname{area}(\operatorname{conv}(R)) \ge \varepsilon$ holds with probability at most 1/6.

Let K_m denote the convex hull of m points chosen uniformly at random in K and define $X(m) = 1 - \operatorname{area}(K_m)$. Thus X(m) is the missed area, that is, the area of $K \setminus K_m$. Groemer [26] showed that $\mathbb{E}[X(m)]$ is maximized when K is a disk of area 1. Rényi and Sulanke [31] showed that for every smooth convex set K there exists some constant C_K , depending on K, such that $\mathbb{E}[X(m)] \leq C_K \cdot m^{-2/3}$. This result also follows from a similar upper bound by Rényi and Sulanke [30] on the expected number E_m of edges of K_m and from Efron's [23] identity $\mathbb{E}[X(m)] = \mathbb{E}[E_{m+1}]/(m+1)$. Both statements together imply that

$$\mathbb{E}[X(m)] \le \frac{C'}{m^{2/3}},$$

where C' is the constant C_K when K is a unit-area disk. (From the results of [31], or subsequent works, one can explicitly compute that $C' \leq 5$, so the constant is very reasonable.)

We set $C_1 := (6C')^{3/2}$. Whenever $|R| \ge C_1 \cdot \varepsilon^{-3/2}$, we can use Markov's inequality to obtain

$$\Pr[1 - \operatorname{area}(\operatorname{conv}(R)) \ge \varepsilon] = \Pr[X(|R|) \ge \varepsilon]$$

$$\leq \frac{\mathbb{E}[X(|R|)]}{\varepsilon}$$

$$\leq \frac{C' |R|^{-2/3}}{\varepsilon}$$

$$\leq \frac{C' \left((6C')^{3/2} \cdot \varepsilon^{-3/2}\right)^{-2/3}}{\varepsilon}$$

$$= \frac{1}{6}.$$

Remark 2. For convenience we will assume that $C_1/\varepsilon^{3/2} \ge 3$ for all $\varepsilon \in (0,1)$. This is not problematic because we can replace C_1 with $\max(C_1,3)$, if needed.

Lemma 3. Let K be a convex body contained in a polygon Γ , let R be a random sample of points inside Γ , and let $C \geq 3$ be an arbitrary value. If

$$|R| \geq 4 \cdot C \cdot \frac{\operatorname{area}(\Gamma)}{\operatorname{area}(K)},$$

then with probability at least 5/6 it holds that $|R \cap K| \ge C$.

Proof. Let $X = |R \cap K|$. The random variable X is a sum of |R| independent Bernoulli random variables, each with expected value

$$p = \frac{\operatorname{area}(K)}{\operatorname{area}(\Gamma)}.$$

Standard calculations (or formulas) show that

$$\mathbb{E}[X] = |R| \cdot p \ge 4 \cdot C \cdot \frac{\operatorname{area}(\Gamma)}{\operatorname{area}(K)} \cdot \frac{\operatorname{area}(K)}{\operatorname{area}(\Gamma)} = 4 \cdot C$$

and

$$\operatorname{Var}[X] = |R| \cdot p(1-p) \leq \mathbb{E}[X].$$

We can now use Chebyshev's inequality in its form

$$\forall a > 0: \quad \Pr[|X - \mathbb{E}[X]| \ge a] \le \frac{\operatorname{Var}[X]}{a^2}$$

and the inequality $C\geq 3$ to obtain the following:

$$Pr[X \le C] \le Pr[X \le \frac{1}{4}\mathbb{E}[X]]$$

$$\le Pr[|X - \mathbb{E}[X]| \ge \frac{3}{4}\mathbb{E}[X]]$$

$$\le \frac{4^2}{3^2} \cdot \frac{Var[X]}{(\mathbb{E}[X])^2}$$

$$\le \frac{16}{9} \cdot \frac{1}{\mathbb{E}[X]}$$

$$\le \frac{16}{9} \cdot \frac{1}{4 \cdot C}$$

$$\le \frac{16}{9} \cdot \frac{1}{4 \cdot 3}$$

$$< \frac{1}{6}.$$

Lemma 4. Let K be a convex body contained in a polygon Γ , let R be a random sample of points inside Γ , and let C_1 be the constant in Lemma 1. If

$$|R| \geq 4 \cdot \frac{C_1}{\varepsilon^{3/2}} \cdot \frac{\operatorname{area}(\Gamma)}{\operatorname{area}(K)},$$

then with probability at least 2/3 it holds that $\operatorname{area}(\operatorname{conv}(R \cap K)) \ge (1 - \varepsilon) \operatorname{area}(K)$.

Proof. We define the following events:

$$\mathcal{E}: |R \cap K| \ge C_1/\varepsilon^{3/2}, \mathcal{F}: \operatorname{area}(\operatorname{conv}(R \cap K)) \ge (1-\varepsilon) \cdot \operatorname{area}(K).$$

For each event \mathcal{A} we use $\overline{\mathcal{A}}$ for its negation. Since $C_1/\varepsilon^{3/2} \geq 3$ (see Remark 2), Lemma 3 implies

$$\Pr\left[\overline{\mathcal{E}}\right] \leq \frac{1}{6},$$

and Lemma 1 implies

$$\Pr\left[\overline{\mathcal{F}} \mid \mathcal{E}\right] \leq \frac{1}{6}.$$

Therefore

$$\begin{aligned} \Pr\left[\overline{\mathcal{F}}\right] &= \Pr\left[\overline{\mathcal{F}} \mid \mathcal{E}\right] \cdot \Pr\left[\mathcal{E}\right] + \Pr\left[\overline{\mathcal{F}} \mid \overline{\mathcal{E}}\right] \cdot \Pr\left[\overline{\mathcal{E}}\right] \\ &\leq \Pr\left[\overline{\mathcal{F}} \mid \mathcal{E}\right] + \Pr\left[\overline{\mathcal{E}}\right] \\ &\leq \frac{1}{6} + \frac{1}{6} \\ &= \frac{1}{3}. \end{aligned}$$



Figure 2: Proof of Lemma 5.

2.2 Outer containment in a parallelogram

In the previous subsection we have proved that, given a superset Γ of K, for samples R of points in Γ of a certain size, area $(\operatorname{conv}(R \cap K))$ is a good approximation to the area of Kwith positive constant probability. If we set $\Gamma := P$, the size of R might turn out too big to yield a subquadratic algorithm. For this reason, we want to find a smaller superset of K to take the sample from. In this subsection we show a method to find a parallelogram Γ containing K with area proportional to the area of K, and that this parallelogram can be found with positive constant probability, using a relatively small random sample from P.

Let K be a convex body in \mathbb{R}^2 . We use y(p) to denote the y-coordinate of a point p. For each $\alpha \in (0,1)$ we define $y_{\alpha}(K)$ as the unique value satisfying

area
$$(\{p \in K \mid y(p) \le y_{\alpha}(K)\}) = \alpha \cdot \operatorname{area}(K).$$

Thus, the horizontal line at height $y_{\alpha}(K)$ breaks K into two parts and the lower one has a proportion α of the area of K. We further define

$$y_0(K) := \min\{y(p) \mid p \in K\} \text{ and } y_1(K) := \max\{y(p) \mid p \in K\}.$$

Lemma 5. For each convex body K in \mathbb{R}^2

$$y_1(K) - y_{4/5}(K) \le y_{4/5}(K) - y_{1/5}(K)$$
 and $y_{1/5}(K) - y_0(K) \le y_{4/5}(K) - y_{1/5}(K)$.

Proof. In this proof, let us drop the dependency on K in the notation and set $y_{\alpha} := y_{\alpha}(K)$ for each $\alpha \in [0, 1]$. We only show that $y_1 - y_{4/5} \leq y_{4/5} - y_{1/5}$; the other inequality is symmetric.

For $\alpha \in [0, 1]$, let ℓ_{α} be the horizontal line with *y*-coordinate y_{α} . Let p_1 be a highest point of *K*, let *cd* be the intersection of $\ell_{4/5}$ with *K*, let ℓ_c be the line through p_1 and *c*, let ℓ_d be the line through p_1 and *d*, let *c'* be the intersection of ℓ_c with $\ell_{1/5}$, and let *d'* be the intersection of ℓ_d with $\ell_{1/5}$. See Figure 2.

By the convexity of K, the triangle p_1cd is contained in the portion of K between ℓ_1 and $\ell_{4/5}$, and the portion of K between $\ell_{4/5}$ and $\ell_{1/5}$ is contained in the trapezoid cc'd'd. Thus

$$\operatorname{area}(cc'd'd) \ge 3\operatorname{area}(p_1cd)$$

and

$$\operatorname{area}(p_1c'd') = \operatorname{area}(cc'd'd) + \operatorname{area}(p_1cd) \ge 4\operatorname{area}(p_1cd).$$
(1)



Figure 3: Left: parallelogram $\Gamma(a, b, A)$. Right: proof of Lemma 6.

The triangle $p_1c'd'$ is similar to the triangle p_1cd with scale factor $(y_1-y_{1/5})/(y_1-y_{4/5})$. By (1), the scale factor is at least 2, that is

$$y_1 - y_{1/5} \ge 2(y_1 - y_{4/5})$$

and so

$$y_{4/5} - y_{1/5} \ge y_1 - y_{4/5}.$$

For any two points a and b and any value $A \ge 0$, let a' := 2a - b, b' := 2b - a, and let $\Gamma(a, b, A)$ denote the parallelogram whose vertices are the four horizontal translates of the points a' and b' by distance $\frac{2A}{|y(a)-y(b)|}$. See Figure 3, left. Note that |a'b'| = 3|ab| and area $(\Gamma(a, b, A)) = 12 \cdot A$.

Lemma 6. Let K be a convex body and assume that $A \ge \operatorname{area}(K)$. Let a and b be points in K such that

$$y(a) \ge y_{4/5}(K)$$
 and $y(b) \le y_{1/5}(K)$.

Then K is contained in $\Gamma(a, b, A)$.

Proof. In this proof, let us drop the dependency on K in the notation and set $y_{\alpha} := y_{\alpha}(K)$ for each $\alpha \in [0, 1]$.

By Lemma 5 we have

$$y(a') = 2y(a) - y(b) \ge y(a) + y_{4/5} - y_{1/5} \ge y(a) + y_1 - y_{4/5} \ge y_1$$

and similarly

$$y(b') = 2y(b) - y(a) \le y(b) + y_{1/5} - y_{4/5} \le y(b) + y_0 - y_{1/5} \le y_0.$$

Therefore K is contained between the horizontal lines y = y(a') and y = y(b'). These are the lines supporting the top and bottom side of $\Gamma(a, b, A)$.

Assume, for the sake of a contradiction, that K has some point z outside $\Gamma(a, b, A)$. Since z lies between the lines y = y(b') and y = y(a'), it must be that the horizontal distance from z to a'b' is more than $\frac{2A}{y(a)-y(b)}$. See Figure 3, right. Since the triangle abz is contained in K we would have

$$\operatorname{area}(K) \ \geq \ \operatorname{area}(abz) \ > \ \frac{1}{2} \cdot (y(a) - y(b)) \cdot \frac{2A}{y(a) - y(b)} \ = \ A \ \geq \ \operatorname{area}(K),$$

which is a contradiction. Therefore any point of K is contained in $\Gamma(a, b, A)$.

Lemma 7. Let K be a convex body contained in a polygon P, and assume that $A \ge \operatorname{area}(K)$. If R is a random sample of points inside P with

$$|R| \geq 60 \cdot \frac{\operatorname{area}(P)}{\operatorname{area}(K)},$$

then with probability at least 2/3 it holds that R contains two points a and b such that ab is an edge of G(P, R) and $\Gamma(a, b, A)$ contains K.

Proof. Define

$$K_{\leq 1/5} := \{ p \in K \mid y(p) \leq y_{1/5}(K) \}$$
 and $K_{\geq 4/5} := \{ p \in K \mid y(p) \geq y_{4/5}(K) \},\$

and consider the following events:

$$\mathcal{E}_{\leq 1/5}$$
: $K_{\leq 1/5} \cap R \neq \emptyset$ and $\mathcal{E}_{\geq 4/5}$: $K_{\geq 4/5} \cap R \neq \emptyset$.

Since

$$|R| \geq 4 \cdot 3 \cdot \frac{\operatorname{area}(P)}{\operatorname{area}(K)/5} = 4 \cdot 3 \cdot \frac{\operatorname{area}(P)}{\operatorname{area}(K_{\leq 1/5})} = 4 \cdot 3 \cdot \frac{\operatorname{area}(P)}{\operatorname{area}(K_{\geq 4/5})}$$

Lemma 3 implies

$$\Pr\left[\mathcal{E}_{\leq 1/5}\right] \geq \frac{5}{6} \text{ and } \Pr\left[\mathcal{E}_{\geq 4/5}\right] \geq \frac{5}{6}.$$

Applying the Fréchet inequality

$$\Pr[A \cap B] \ge \max\{0, \Pr[A] + \Pr[B] - 1\}$$

which does not require any independence assumption, we obtain that

$$\Pr\left[\mathcal{E}_{\leq 1/5} \cap \mathcal{E}_{\geq 4/5}
ight] \geq rac{2}{3}$$

When $\mathcal{E}_{\leq 1/5}$ and $\mathcal{E}_{\geq 4/5}$ hold, there are points $a \in K_{\leq 1/5} \cap R$ and $b \in K_{\geq 4/5} \cap R$ and Lemma 6 implies that K is contained in $\Gamma(a, b, A)$. Moreover, ab is an edge of G(P, R)because K is a convex body contained in P.

2.3 Largest convex polygon in a visibility graph.

In this subsection we give an algorithm to find a largest convex polygon whose edges are defined by a visibility graph inside a polygon. In our algorithm LARGEPOTATO, described in Section 4, the vertices of the visibility graph are points of a random sample in P, and the algorithm in the current subsection is used to find the largest convex polygon defined by that sample.

Let H be a visibility graph in some simple polygon. We denote the set of vertices and edges of H by V(H) and E(H), respectively. We assume that the coordinates of the vertices of H are known. A set of vertices U from H is a **convex clique** if: (i) there is an edge between any two vertices of U, and (ii) the points of U are in convex position. The **area of a convex clique** U is the area of conv(U).

Let s be a point of V(H). We are interested in finding a convex clique of maximum area in H, denoted by $\varphi(H, s)$, that has s as highest point. Thus we want

 $\varphi(H,s) \in \arg\max\{\operatorname{area}(U) \mid U \subseteq V(H) \text{ a convex clique, } s \text{ highest point in } U\}.$

Lemma 8. For any point s of V(H), we can compute $\varphi(H,s)$ in time $O(|V(H)|^2)$.

Proof. Pruning vertices, we can assume that all vertices of H are adjacent to s and below s. We can then use the algorithm of Bautista-Santiago et al. [8], which is an improvement over the algorithm of Fischer [24], restricted to the edges that are in H. For completeness, we provide a quick overview of the approach.

For this proof, let us denote n = |V(H)| - 1. We sort the points of $V(H) \setminus \{s\}$ counterclockwise radially from s. Let x_1, x_2, \ldots, x_n be the labeling of the points of $V(H) \setminus \{s\}$ according to that ordering. Thus, for each i < j the sequence x_i, s, x_j is a right turn.

Using a standard point-line duality and constructing the arrangement of lines dual to the points V(H), we get the circular order of the edges around each point x_i [28]. For this we spend in total $O(n^2)$ time [15, 22].

For each i < j such that $x_i x_j \in E(H)$, let OPT[i, j] be the largest-area convex clique U that has x_i, x_j , and s consecutively along the boundary of conv(U). We then have

$$\operatorname{area}(\varphi(H,s)) = \max_{i < j, x_i x_j \in E(H)} \operatorname{Opt}[i, j].$$

Taking the convention that $\max \emptyset = 0$, the values OPT[i, j] satisfy the following recursion

 $OPT[i, j] = area(sx_ix_j)$ $+ max{OPT[h, i] | h < i, x_hx_i \in E(H), x_h, x_i, x_j makes a left turn}.$

To argue the correctness of the recursion, one needs to observe that the right side of the equation does indeed correspond to the construction of a convex polygon.

For any fixed *i*, the values OPT[i, *], * > i, can be computed in O(n) time, provided that the edges incident to x_i are already radially sorted and the values OPT[h, i] are already available for all h < i. To achieve linear time, one performs a scan of the edges incident to x_i and uses the property that

$$\{x_h x_i \in E(H) \mid h < i, x_h, x_i, x_j \text{ makes a left turn}\}$$

forms a contiguous sequence in the circular ordering of edges incident to x_i .

Thus, we can fill in the whole table $OPT[\cdot, \cdot]$ in time $O(n^2)$. With this we can compute $area(\varphi(H, s))$ and construct an optimal solution $\varphi(H, s)$ by standard backtracking. See [8] for additional details.

In Section 5 we will also need to find a convex clique U whose convex hull has maximum perimeter. It is easy to modify the algorithm to compute, for a point $s \in V(H)$, the value

$$\max\{\operatorname{per}(U) \mid U \subseteq V(H) \text{ a convex clique, } s \text{ highest point in } U\}$$

and a corresponding optimal solution. Here we assume a model of computation where the length of segments can be added in constant time.

3 Probability for visibility

In this section we give a relation between $A^*(P)$ and the probability that two random points in P are visible. Such a relation is used later to bound the expected complexity of the visibility graph of a suitably sized random sample of points.

A polygon P is *weakly visible* from a segment s in P if, for each point $p \in P$, there exists some point $x \in s$ such that $xp \subset P$.



Figure 4: Situation in the proof of Theorem 9. a) Points a and b are on the same side of s. b) Points a and b are on different sides of s.

Theorem 9. Let P be a unit-area polygon weakly visible from a diagonal s. Let a and b be two points chosen uniformly at random in P. Then

- (i) $\Pr[ab \subset P] \leq 18 \cdot A^*(P) and^1$
- (*ii*) $\Pr[ab \subset P \text{ and } ab \cap s \neq \emptyset] \leq 6 \cdot A^*(P).$

Proof. Without loss of generality we assume that s is a horizontal segment on the x-axis. In this proof we use y(a) to denote the y-coordinate of a point a. Since the event that y(a)y(b) = 0 has zero probability, we may assume that $y(a) \neq 0$ and $y(b) \neq 0$. To simplify the notation, in this proof we use $A^* = A^*(P)$.

Consider first the point a fixed. We first bound the probability that a and b are visible and $|y(a)| \ge |y(b)|$ to obtain the following:

$$\Pr[ab \subset P \text{ and } |y(a)| \ge |y(b)|] \le 9 \cdot A^*.$$

This is seen showing that the set of points b satisfying $ab \subset P$ and $|y(a)| \ge |y(b)|$ is inside a region of area at most $9A^*$.

We distinguish two cases:

- 1) y(a)y(b) > 0 (a and b are on the same side of s).
- 2) y(a)y(b) < 0 (a and b are on the opposite sides of s).

Let us first consider case 1). We assume that y(a) > 0, the other case is symmetric. Refer to Figure 4a). We know that a sees some point s_a on s. We may assume that b does not lie on the segment as_a as the event that b lies on as_a has zero probability. We know that b sees some point s_b on s. We have a generalized polygon $Q = abs_bs_a$ (in which the sides as_a and bs_b may cross or some of the vertices may coincide) whose boundary is in P, and therefore the whole interior of Q is also in P. Here we use that P has no holes. If b sees a, we can choose s_b so that as_a and bs_b share a common point: indeed, if as_a and bs_b are disjoint, then the polygon Q is simple and thus b sees s_a , so we can set s_b to s_a . Let c be the common point of as_a and bs_b . By our assumptions, y(c) < y(a).

Let h be a horizontal line through b and let a' be the intersection between h and the segment as_a . The interior of Q is made of two triangles, abc and s_as_bc , both contained in P and thus each of them has area at most A^* . The triangle s_as_bc degenerates to a point if $s_a = s_b$.

For the triangle *abc*, we have area $(abc) = \frac{1}{2}|a'b| \cdot (y(a) - y(c))$, which implies that

$$|a'b| \le \frac{2A^*}{y(a) - y(c)}.$$
 (2)

¹Item (i) is not used elsewhere in this paper. However, we believe that it is an interesting fact that strengthens Theorem 10 for weakly edge-visible polygons.

If the triangle $s_a s_b c$ is not degenerate, we have $y(c) \cdot |s_a s_b| \leq 2A^*$. By the similarity of the triangles $s_a s_b c$ and a'bc, we have $|s_a s_b| = |a'b| \cdot y(c)/(y(b) - y(c))$, which implies that

$$|a'b| \le \frac{2A^*}{y(c)^2} \cdot (y(b) - y(c)).$$
(3)

Since the upper bound on |a'b| is increasing in y(c) in (2) and decreasing in y(c) in (3), the minimum of the two upper bounds is maximal when they are equal; that is, when y(c) = y(a)y(b)/(y(a) + y(b)). It follows that

$$|a'b| \le \frac{2A^*}{y(a)^2} \cdot (y(a) + y(b)).$$
(4)

The condition (4) implies that b is inside a trapezoid of height y(a) with bases of length $4A^*/y(a)$ and $8A^*/y(a)$, which has area $6A^*$. This finishes case 1).

We now consider case 2). Refer to Figure 4b). Let a'a'' be the maximum subsegment of s that is visible from a. Since the triangle aa'a'' is contained in P we have

$$\operatorname{area}(aa'a'') = \frac{1}{2}|a'a''| \cdot y(a) \leq A^*.$$

If b sees a, then the segment ab intersects the segment a'a''. Thus b is contained in a trapezoid of height y(a) with bases of length |a'a''| and $2 \cdot |a'a''|$. Such trapezoid has area

$$\frac{|a'a''| + 2|a'a''|}{2} \cdot y(a) = \frac{3}{2}|a'a''| \cdot y(a) \le 3A^*.$$

This finishes case 2).

Considering cases 1) and 2) together, for each fixed point $a \in P$ we have

 $\Pr\left[ab \subset P \text{ and } |y(a)| \ge |y(b)|\right] \le 9 \cdot A^*.$

Since this bound holds for each fixed a, it also holds when a is chosen at random. Because of symmetry we have

$$\Pr[ab \subset P] = 2 \cdot \Pr[ab \subset P \text{ and } |y(a)| \ge |y(b)|] \le 18 \cdot A^*,$$

which proves part (i) of the theorem.

Part (ii) follows by a similar consideration using case 2) only.

We can use a divide and conquer approach to obtain a bound for arbitrary polygons.

Theorem 10. Let P be an arbitrary unit-area polygon. Let a and b be two points chosen uniformly at random in P. Then

$$\Pr[ab \subset P] \leq 12 \cdot A^{*}(P) \cdot (1 + \log_{2}(1/A^{*}(P))).$$

Proof. For this proof, let us set $A^* = A^*(P)$.

For each polygon Q there exists a segment that splits Q into two polygons, each of area at most $\frac{2}{3} \operatorname{area}(Q)$ [10]. We recursively split P using such a segment in each polygon, for $h = \log_{3/2}(1/A^*)$ levels. Thus, at the bottommost level, each polygon has area bounded by A^* .

At each level ℓ of the recursion, where $\ell = 0, \ldots, h$, we have 2^{ℓ} polygons, which we denote by $Q_{\ell,1}, \ldots, Q_{\ell,2^{\ell}}$. In particular, $Q_{0,1} = P$. Since the polygons at each level ℓ are disjoint, we have

$$\sum_{i=1}^{2^{\ell}} \operatorname{area}(Q_{\ell,i}) = \operatorname{area}(P) = 1.$$

For each polygon $Q_{\ell,i}$, where $\ell < h$, let $e_{\ell,i}$ be the segment used to split $Q_{\ell,i}$. Let $\widehat{Q_{\ell,i}}$ be the portion of $Q_{\ell,i}$ that is weakly visible from $e_{\ell,i}$. At each level $\ell < h$ we have

$$\sum_{i=1}^{2^{\ell}} \operatorname{area}(\widehat{Q_{\ell,i}}) \leq \sum_{i=1}^{2^{\ell}} \operatorname{area}(Q_{\ell,i}) = 1.$$

Let $\mathcal{E}_{a,b,\ell,i}$ be the event $ab \subset \widehat{Q_{\ell,i}}$ and $ab \cap e_{\ell,i} \neq \emptyset$. Using the union bound and part (ii) of Theorem 9 we obtain

$$\Pr\left[\bigcup_{\ell=0}^{h-1}\bigcup_{i=1}^{2^{\ell}}\mathcal{E}_{a,b,\ell,i}\right] \leq \sum_{\ell=0}^{h-1}\Pr\left[\bigcup_{i=1}^{2^{\ell}}\mathcal{E}_{a,b,\ell,i}\right]$$
$$= \sum_{\ell=0}^{h-1}\sum_{i=1}^{2^{\ell}}\Pr[\mathcal{E}_{a,b,\ell,i}]$$
$$= \sum_{\ell=0}^{h-1}\sum_{i=1}^{2^{\ell}}\Pr[\mathcal{E}_{a,b,\ell,i} \mid a \in \widehat{Q_{\ell,i}}, b \in \widehat{Q_{\ell,i}}] \cdot \Pr[a \in \widehat{Q_{\ell,i}}, b \in \widehat{Q_{\ell,i}}]$$
$$\leq \sum_{\ell=0}^{h-1}\sum_{i=1}^{2^{\ell}}\left(6 \cdot \frac{A^{*}}{\operatorname{area}(\widehat{Q_{\ell,i}})} \cdot (\operatorname{area}(\widehat{Q_{\ell,i}}))^{2}\right)$$
$$= 6 \cdot A^{*} \sum_{\ell=0}^{h-1}\sum_{i=1}^{2^{\ell}}\operatorname{area}(\widehat{Q_{\ell,i}})$$
$$\leq 6 \cdot A^{*} \sum_{\ell=0}^{h-1} 1$$
$$= 6 \cdot A^{*} \cdot h.$$

At the bottommost level h, we can use that $\operatorname{area}(Q_{h,i}) \leq A^*$ for each i to obtain

$$\Pr\left[\bigcup_{i=1}^{2^{h}} [ab \subset Q_{h,i}]\right] = \sum_{i=1}^{2^{h}} \Pr\left[ab \subset Q_{h,i}\right]$$
$$\leq \sum_{i=1}^{2^{h}} \Pr\left[a \in Q_{h,i}, b \in Q_{h,i}\right]$$
$$= \sum_{i=1}^{2^{h}} (\operatorname{area}(Q_{h,i}))^{2}$$
$$\leq \sum_{i=1}^{2^{h}} A^{*} \cdot (\operatorname{area}(Q_{h,i}))$$
$$= A^{*}.$$

We then note that, if a sees b, then the event $\mathcal{E}_{a,b,\ell,i}$ occurs for some $\ell < h$ and $i \leq 2^{\ell}$, or

a and b are in the same polygon $Q_{h,i}$, where $i \leq 2^h$. Thus

$$\Pr\left[ab \subset P\right] \leq \Pr\left[\bigcup_{\ell=0}^{h-1} \bigcup_{i=1}^{2^{\ell}} [\mathcal{E}_{a,b,\ell,i}]\right] + \Pr\left[\bigcup_{i=1}^{2^{h}} [ab \subset Q_{h,i}]\right]$$
$$\leq 6 \cdot A^{*} \cdot h + A^{*}$$
$$= A^{*} + 6 \cdot A^{*} \cdot \log_{3/2}(1/A^{*})$$
$$\leq A^{*} + 12 \cdot A^{*} \cdot \log_{2}(1/A^{*}).$$

In the conference version of our paper we proved and used Theorem 10. We included the proof here for completeness and archiving purposes. Also, it is a key contribution of this paper to realize that such connection between the probability of being co-visible and the area of the largest convex body could exist. This result has been improved by Balko et al. [5]. Using their new result slightly improves the final running time of our algorithms. Thus, we will use in the rest of our paper the following theorem.

Theorem 11 (Corollary 4 in [5]). Let P be an arbitrary unit-area polygon. Let a and b be two points chosen uniformly at random in P. Then

$$\Pr\left[ab \subset P\right] \leq 180 \cdot A^*(P).$$

4 Algorithm

In this section we discuss the eventual algorithm. The input to the algorithm is a polygon P, a parameter $\varepsilon \in (0, 1)$, and a parameter $\delta \in (0, 1)$. Without loss of generality we assume that P has unit area. The algorithm, called LARGEPOTATO, is summarized in Figure 5. In the first part of the section we explain in detail each step and the notation that is still undefined. In the second part we analyze the algorithm.

4.1 Description

Sampling points. Let A(P) be a constant-factor approximation for $A^*(P)$. Thus, $A(P) \leq A^*(P) \leq C_2A(P)$ for some constant $C_2 \geq 1$. Hall-Holt et al. [27] provide an algorithm to compute such value A(P) in $O(n \log n)$ time.

algorithm to compute such value A(P) in $O(n \log n)$ time. Let us define $r := \frac{60}{A(P)}$. Since the largest triangle in any triangulation of P has area at least 1/n, we have $A^*(P) \ge 1/n$ and thus r = O(n).

Let R be a sample of r points chosen independently at random from the polygon P. The sample R can be constructed in $O(n + r \log n)$ time, as follows. By the linear-time algorithm² of Chazelle [13], we compute a triangulation of P, giving triangles T_1, \ldots, T_{n-2} . We then compute the prefix sums $S_i = \operatorname{area}(T_1) + \cdots + \operatorname{area}(T_i)$ for $i = 1, \ldots, n-2$. This is done in O(n) time. To sample a point, we select a random number x in the interval [0, 1], perform a binary search to find the smallest index j such that $x \leq S_j$, and sample a random point inside T_j . A random point inside T_j can be generated using a random point inside a parallelogram that contains two congruent copies of T_j ; such a point can be generated using two random numbers in the interval [0, 1]. In total, each point takes $O(\log n)$ time plus the time needed to generate three random numbers in the interval [0, 1]. A similar approach is described in [17].

²Computing a triangulation of P is not the bottleneck of our algorithm. Since Chazelle's algorithm is complicated, for practical purposes it would be easier to use a simpler triangulation algorithm running in $O(n \log n)$ time such as the one described in [19].

Algorithm LARGEPOTATO **Input:** Unit-area polygon $P, \varepsilon \in (0, 1)$, and $\delta \in (0, 1)$ find a value A(P) such that $A(P) \leq A^*(P) \leq C_2 \cdot A(P)$; 1. 2. $r \leftarrow 60/A(P);$ $best \leftarrow \emptyset;$ 3. **repeat** $3\log_2(1/\delta)$ times 4. $R \leftarrow \text{sample } r \text{ points uniformly at random in } P;$ 5.6. if G(P, R) has at most $C_3 \cdot n$ edges then 7. compute G(P, R); 8. for $ab \in E(G(P, R))$ do $R_{ab} \leftarrow \text{sample } 96 \cdot C_1 \cdot C_2 / (\varepsilon/2)^{3/2} \text{ points uniformly at ran-}$ 9. dom in the parallelogram $\Gamma(a, b, C_2 \cdot A(P))$; $S_{ab} \leftarrow \text{sample } 288 \cdot C_2 / \varepsilon \text{ points uniformly at random in the}$ 10. parallelogram $\Gamma(a, b, C_2 \cdot A(P));$ $G_{ab} \leftarrow G(P, (R_{ab} \cup S_{ab}) \cap P);$ 11. for $s \in S_{ab}$ do 12.13. $U \leftarrow \varphi(G_{ab}, s);$ if $\operatorname{area}(U) > \operatorname{area}(best)$ then $best \leftarrow U$; 14. 15. return conv(best);

Figure 5: Algorithm. The constant C_1 is from Lemma 1. The constant C_2 is the approximation factor from Hall-Holt et al. [27]; see Section 4.1. The constant C_3 is from Lemma 12.

Size of the visibility graph. Using the expected number of edges in the visibility graph G(P, R) and Markov's inequality lead to the following bound.

Lemma 12. There exists a constant $C_3 > 0$ such that, with probability at least 5/6, the graph G(P, R) has at most $C_3 \cdot n$ edges.

Proof. In this proof we use G := G(P, R). Using linearity of expectation, Theorem 11, the estimates $1/n \le A^*(P) \le C_2A(P)$ and the obvious fact that $n \ge 3$, we obtain

$$\mathbb{E}[|E(G)|] = \binom{r}{2} \cdot \Pr[\text{two random points are visible in } P]$$

$$\leq \frac{1}{2} \left(\frac{60}{A(P)}\right)^2 \cdot 180 \cdot A^*(P)$$

$$\leq 324000 \cdot \frac{A^*(P)}{A(P)} \cdot \frac{1}{A(P)}$$

$$\leq 324000 \cdot C_2 \cdot C_2 \cdot n.$$

Let us take $C_3 = 6 \cdot 324000 \cdot (C_2)^2$. By Markov's inequality we have

$$\Pr[|E(G)| \ge C_3 \cdot n] \le \frac{\mathbb{E}[|E(G)|]}{C_3 \cdot n} \le \frac{1}{6}.$$

Constructing the visibility graph and checking its size. We will use the following result by Ben-Moshe et al. [9].

Theorem 13 (Ben-Moshe et al. [9]). Let P be a simple polygon with n vertices and let R be a set of r points inside P. The visibility graph G(P, R) can be constructed in time $O(n + r \log r \log(rn) + k)$, where k is the number of edges in G(P, R).

In line 6 of the algorithm LARGEPOTATO, we want to check whether G(P, R) has at most $C_3 \cdot n$ edges. For this we use that the algorithm of Theorem 13 is output-sensitive and takes time $T_{[9]}(n, r, k) = O(n+r \log r \log(rn)+k)$. We run the algorithm of Theorem 13 for at most $T_{[9]}(n, r, C_3 \cdot n)$ steps. If the construction of G(P, R) is not finished, we know that $|E(G(P, R))| > C_3 \cdot n$. Otherwise the algorithm outputs whether $|E(G(P, R))| \leq C_3 \cdot n$ or not. Thus, the test in line 6 can be made in time

$$T_{[9]}(n, r, C_3 \cdot n) = O(n + r \log r \log(rn) + C_3 \cdot n)$$

= $O(n + n \log^2 n + n)$
= $O(n \log^2 n).$

The construction in line 7 takes the same time, if it is actually made.

Remark 14. For each constant ε , the bottleneck in the running time of our algorithm is here, in our use of Theorem 13 to compute the visibility graph. With the improvement of Balko et al. [5], stated in Theorem 11, all other steps can be made to run in time $O(n \log n \log(1/\delta))$ (for constant ε).

Work for each edge ab. We now discuss the work done in lines 9–14 for each edge ab of G(P, R). The parallelogram $\Gamma(a, b, C_2 \cdot A(P))$ was defined in Section 2.2. Note that $\Gamma(a, b, C_2 \cdot A(P))$ has area

$$12 \cdot C_2 \cdot A(P) \leq 12 \cdot C_2 \cdot A^*(P) = \Theta(A^*(P)).$$

Since $\Gamma(a, b, C_2 \cdot A(P))$ is a parallelogram, it is straightforward to construct the random samples R_{ab} and S_{ab} . Note that $|R_{ab}| = \Theta(\varepsilon^{-3/2})$ and $|S_{ab}| = \Theta(\varepsilon^{-1})$. We select the subset of $R_{ab} \cup S_{ab}$ contained in the polygon P and construct its visibility graph G_{ab} . We then compute a maximum-area convex clique in G_{ab} among those cliques whose highest vertex s is from S_{ab} . We make this restriction to reduce the number of candidate highest points from $\Theta(\varepsilon^{-3/2})$ to $\Theta(\varepsilon^{-1})$. This is equivalent to computing $\varphi(G_{ab}, s)$ for each $s \in S_{ab}$, which is discussed in Section 2.3. Finally, we compare the solutions U_{ab} that we obtain against the solution stored in the variable *best* and, if appropriate, update *best*.

4.2 Analysis

Lemma 15 (Time bound). For each $\varepsilon \in (0,1)$, the algorithm LARGEPOTATO can be adapted to use $O\left(n(\log^2 n + (1/\varepsilon^3)\log n + 1/\varepsilon^4)\log(1/\delta)\right)$ time.

Proof. The value A(P) can be computed in time $O(n \log n)$, as discussed before.

We first preprocess the polygon P for segment containment using the algorithm of Chazelle et al. [14]: after O(n) preprocessing time we can answer whether a query segment is contained in P in $O(\log n)$ time. In particular, we can decide in $O(\log n)$ time whether a query point is in P.

We claim that each iteration of the for-loop (lines 9–14) takes $O((1/\varepsilon^3) \log n + 1/\varepsilon^4)$ time. The samples R_{ab} and S_{ab} can be constructed in $O(|R_{ab}| + |S_{ab}|) = O((1/\varepsilon^{3/2}) \log n)$. We construct $(R_{ab} \cup S_{ab}) \cap P$ by testing each point of $R_{ab} \cup S_{ab}$ for containment in P. The graph G_{ab} is constructed by checking for each pair of points from $(R_{ab} \cup S_{ab}) \cap P$ whether the corresponding segment is contained in P. Thus G_{ab} is constructed in $O((1/\varepsilon^{3/2})^2 \log n) = O((1/\varepsilon^3) \log n)$ time. By Lemma 8, each iteration of the lines 13–14 takes time $O(|R_{ab}|^2) = O(1/\varepsilon^3)$. Thus the running time of the for loop in lines 12–14 takes time $O(|S_{ab}| \cdot (1/\varepsilon^3)) = O(1/\varepsilon^4)$. The claim follows. We next show that each iteration of the repeat-loop (lines 5–14) takes $O(n \log^2 n + (n/\varepsilon^3) \log n + n/\varepsilon^4)$ time. Since r = O(n), the sample R can be computed in $O(n \log n)$ time, as discussed in Section 4.1. As discussed before, we can make the test in line 6 in $O(n \log^2 n)$ time.

If G(P, R) has more than $C_3 \cdot n$ edges, this finishes the time spent in the iteration. Otherwise, we make $O(C_3 \cdot n) = O(n)$ iterations of the for-loop in lines 9–14. Since each iteration of the for-loop takes $O((1/\varepsilon^3) \log n + 1/\varepsilon^4)$ time, as argued earlier in this proof, the bound per iteration of the repeat-loop follows.

Lemma 16 (Correctness of one iteration). In one iteration of the repeat-loop (lines 5– 14) of the algorithm LARGEPOTATO the algorithm finds a convex polygon of area at least $(1 - \varepsilon)A^*(P)$ with probability at least 1/4.

Proof. Let K^* be a convex polygon of largest area contained in P. Therefore $\operatorname{area}(K^*) = A^*(P)$. Consider one iteration of the repeat-loop. Suppose G(P, R) passes the test on line 6. Then the following two conditions are sufficient for a successful iteration: R contains two visible points a and b such that the parallelogram $\Gamma(a, b, C_2 \cdot A(P))$ (used in lines 9–14) contains K^* , and S_{ab} contains a point s such that $\operatorname{area}(\varphi(G_{ab}, s))$ (computed in line 13) is a $(1 - \varepsilon)$ -approximation to $\operatorname{area}(K^*)$. This motivates the definition of the following events:

 $\begin{aligned} \mathcal{E}_{K^*}: & \text{ for some edge } ab \text{ of } G(P,R), \ K^* \text{ is contained in } \Gamma(a,b,C_2 \cdot A(P)), \\ \mathcal{E}_G: & |E(G(P,R))| \leq C_3 \cdot n, \\ \mathcal{E}_{\Gamma}: & \text{ for some edge } ab \text{ of } G(P,R), \text{ there is } s \in S_{ab} \text{ such that} \\ & \operatorname{area}(\varphi(G_{ab},s)) \geq (1-\varepsilon) \cdot A^*(P). \end{aligned}$

Since

$$|R| = \frac{60}{A(P)} \ge \frac{60}{\operatorname{area}(K^*)}$$

and $A^*(P) \leq C_2 \cdot A(P)$, Lemma 7 implies that

$$\Pr\left[\mathcal{E}_{K^*}\right] \geq \frac{2}{3}.$$

By Lemma 12 we have

$$\Pr[\mathcal{E}_G] \ge \frac{5}{6}$$

and therefore, since $\Pr[A \cap B] \ge \Pr[A] + \Pr[B] - 1$,

$$\Pr\left[\mathcal{E}_{K^*} \text{ and } \mathcal{E}_G\right] \ge \frac{1}{2}.$$
(5)

For the rest of the proof, we assume that \mathcal{E}_{K^*} and \mathcal{E}_G hold. Let a_0b_0 be the edge of G(P, R) such that $\Gamma_0 := \Gamma(a_0, b_0, C_2 \cdot A(P))$ contains K^* . The algorithm executes the code in lines 9–14 for $ab = a_0b_0$. Let $K^*_{\varepsilon/2}$ be the portion of K^* above $y = y_{1-\varepsilon/2}(K^*)$ and let $K^*_{1-\varepsilon/2} = K^* \setminus K^*_{\varepsilon/2}$. It holds that

$$\operatorname{area}(K^*_{\varepsilon/2}) = (\varepsilon/2) \cdot \operatorname{area}(K^*)$$
 and $\operatorname{area}(K^*_{1-\varepsilon/2}) = (1-\varepsilon/2) \cdot \operatorname{area}(K^*).$

The bound

$$|S_{a_0b_0}| = \frac{288 \cdot C_2}{\varepsilon} = 4 \cdot 3 \cdot \frac{12 \cdot C_2 \cdot A^*(P)}{(\varepsilon/2) \cdot A^*(P)} \ge 4 \cdot 3 \cdot \frac{\operatorname{area}(\Gamma_0)}{\operatorname{area}(K^*_{\varepsilon/2})}$$

and Lemma 3 (with $P = \Gamma_0$ and $K = K^*_{\varepsilon/2}$) imply that

$$\Pr\left[S_{a_0b_0} \cap K^*_{\varepsilon/2} \neq \emptyset \mid \mathcal{E}_{K^*} \text{ and } \mathcal{E}_G\right] \geq \frac{5}{6}.$$
(6)

The bound

$$|R_{a_0b_0}| = \frac{96 \cdot C_1 \cdot C_2}{(\varepsilon/2)^{3/2}} = 4 \cdot \frac{C_1}{(\varepsilon/2)^{3/2}} \cdot \frac{12 \cdot C_2 \cdot A^*(P)}{A^*(P)/2} \ge 4 \cdot \frac{C_1}{(\varepsilon/2)^{3/2}} \cdot \frac{\operatorname{area}(\Gamma_0)}{\operatorname{area}(K_{1-\varepsilon/2}^*)}$$

and Lemma 4 (with $P = \Gamma_0$ and $K = K^*_{1-\varepsilon/2}$) imply that

 $\Pr\left[\operatorname{area}(\operatorname{conv}(R_{a_0b_0} \cap K_{1-\varepsilon/2}^*)) \ge (1-\varepsilon/2) \cdot \operatorname{area}(K_{1-\varepsilon/2}^*) \mid \mathcal{E}_{K^*} \text{ and } \mathcal{E}_G\right] \ge \frac{2}{3}.$ Noting that

Noting that

$$(1 - \varepsilon/2) \cdot \operatorname{area}(K_{1 - \varepsilon/2}^*) = (1 - \varepsilon/2) \cdot (1 - \varepsilon/2) \cdot A^*(P)$$

> $(1 - \varepsilon) \cdot A^*(P),$

we have

$$\Pr\left[\operatorname{area}(\operatorname{conv}(R_{a_0b_0} \cap K^*_{1-\varepsilon/2})) \ge (1-\varepsilon) \cdot A^*(P) \mid \mathcal{E}_{K^*} \text{ and } \mathcal{E}_G\right] \ge \frac{2}{3}.$$
(7)

Joining (6) and (7) we obtain that, with probability at least 1/2, it holds

$$\operatorname{area}(\operatorname{conv}(R_{a_0b_0} \cap K^*_{1-\varepsilon/2})) \ge (1-\varepsilon) \cdot A^*(P) \text{ and } S_{a_0b_0} \cap K^*_{\varepsilon/2} \neq \emptyset.$$

If these two events occur and s is a point of $S_{a_0b_0} \cap K^*_{\varepsilon/2}$, then

$$\begin{aligned} \operatorname{area}(\varphi(G_{a_0b_0}, s)) &\geq \operatorname{area}(\operatorname{conv}((K^*_{1-\varepsilon/2} \cap R_{a_0b_0}) \cup \{s\})) \\ &\geq \operatorname{area}(\operatorname{conv}(K^*_{1-\varepsilon/2} \cap R_{a_0b_0})) \\ &\geq (1-\varepsilon) \cdot A^*(P). \end{aligned}$$

We conclude that

$$\Pr\left[\mathcal{E}_{\Gamma} \mid \mathcal{E}_{K^*} \text{ and } \mathcal{E}_G\right] \geq \frac{1}{2}$$

and using (5) obtain

 $\Pr\left[\mathcal{E}_{K^*} \text{ and } \mathcal{E}_G \text{ and } \mathcal{E}_\Gamma\right] = \Pr\left[\mathcal{E}_{\Gamma} \mid \mathcal{E}_{K^*} \text{ and } \mathcal{E}_G\right] \cdot \Pr\left[\mathcal{E}_{K^*} \text{ and } \mathcal{E}_G\right] \geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$

When \mathcal{E}_{K^*} , \mathcal{E}_G and \mathcal{E}_{Γ} occur, the test in line 6 is satisfied and in one of the iterations of the loop in lines 13–14 we will obtain a $(1 - \varepsilon)$ -approximation to $A^*(P)$.

Theorem 17. Let P be a polygon with n vertices, let ε and δ be parameters with $0 < \varepsilon < 1$ and $0 < \delta < 1$. In time $O\left(n(\log^2 n + (1/\varepsilon^3)\log n + 1/\varepsilon^4)\log(1/\delta)\right)$ we can find a convex polygon contained in P that, with probability at least $1-\delta$, has area at least $(1-\varepsilon) \cdot A^*(P)$.

Proof. We consider the output K given by LARGEPOTATO(P, ε, δ). By Lemma 15, we can assume that the output is computed in time $O\left(n(\log^2 n + (1/\varepsilon^3)\log n + 1/\varepsilon^4)\log(1/\delta)\right)$.

The polygon K returned by LARGEPOTATO (P, ε, δ) is always a convex polygon contained in P. We have $\operatorname{area}(K) < (1 - \varepsilon) \cdot A^*(P)$ if and only if all iterations of the repeat-loop (lines 5–14) fail to find such a $(1 - \varepsilon)$ -approximation. Since each such iteration fails with probability at most 3/4 due to Lemma 16, and there are $3\log_2(1/\delta)$ iterations, we have

$$\Pr[\operatorname{area}(K) < (1 - \varepsilon) \cdot A^*(P)] \leq \left(\frac{3}{4}\right)^{3\log_2(1/\delta)} < \left(\frac{1}{2}\right)^{\log_2(1/\delta)} = \delta.$$

We observe that, if we perform only four iterations of the repeat-loop, we obtain the result stated in the abstract. It is also interesting to note that, when ε is constant, then the running time of the algorithm is $O(n \log^2 n \log(1/\delta))$.

5 Convex body of maximum perimeter

In this section we present an adaptation of the previous algorithm in order to maximize the perimeter. Recall that for a convex body K in the plane, we denote its perimeter by per(K), and we have defined

 $L^*(P) = \max\{\operatorname{per}(K) \mid K \subset P, K \text{ convex}\}.$

Before presenting the actual algorithm, we first introduce a few tools.

5.1 Perimeter of the convex hull of random samples

Let K be a convex body in the plane and let R be a random sample of n points inside K. How well does per(conv(R)) approximate per(K)? There has been quite some research on this problem, often on the high-dimensional generalizations called intrinsic volumes. See [34, 35] for an overview of the known results on this problem. However, all the results we found use constants that depend on K. Since in our application the target convex body K is unknown, we are interested in bounds using universal constants, independent of K.

Theorem 18. There is some universal constant $C_4 \ge 1$ such that the following holds. For each convex body K in the plane, if K_m denotes the convex hull of m points chosen uniformly at random inside K, then $\operatorname{per}(K) - \mathbb{E}[\operatorname{per}(K_m)] \le \frac{C_4}{\sqrt{m}} \cdot \operatorname{per}(K)$.

Proof. Let us first provide some notation used only in this proof. In the notation we drop the dependency on K. Let \mathcal{U} be the set of unit vectors in the plane. For each $u \in \mathcal{U}$ and each $t \in \mathbb{R}_{\geq 0}$ we define the following values; see Figure 6.

$$\begin{split} h(u) &:= \max\{\langle p, u \rangle \mid p \in K\},\\ \mathrm{dwidth}(u) &:= h(u) + h(-u),\\ S(u,t) &:= \{p \in \mathbb{R}^2 \mid h(u) - t \leq \langle p, u \rangle \leq h(u)\},\\ \ell(u,t) &:= \{p \in \mathbb{R}^2 \mid h(u) - t = \langle p, u \rangle\},\\ v(u,t) &:= \operatorname{area}(K \cap S(u,t)). \end{split}$$

Note that dwidth(u) is the so-called *directional width* of K in direction u: the length of the orthogonal projection of K onto any line parallel to u. The line $\ell(u, t)$ is perpendicular to u and $\ell(u, 0)$ is tangent to K. Moreover, S(u, t) is an infinite slab of width t defined by $\ell(u, 0)$ and $\ell(u, t)$. When t > 0 the slab S(u, t) intersects the interior of K. The value v(u, t) tells the area of the portion of K contained in S(u, t).

In the proof we are going to use the classical Crofton's formula that tells

$$\operatorname{per}(K) = \int_{\mathcal{U}} \operatorname{dwidth}(u) \, d\omega(u) = \int_{\mathcal{U}} \int_{-\infty}^{+\infty} \mathbb{1}_{K \cap \ell(u,t) \neq \emptyset} \, dt \, d\omega(u), \tag{8}$$

where ω is the uniform Lebesgue measure on \mathcal{U} satisfying $\int_{\mathcal{U}} 1 d\omega(u) = 1/2$.

We adapt the approach of Schneider and Wieacker [33, 36], based on an observation of Efron [23].



Figure 6: Notation in the proof of Theorem 18.

By scaling, we can assume that K has unit area. Consider a line $\ell(u, t)$ that intersects K. The line $\ell(u, t)$ does not intersect K_m if and only if all points of the random sample lie in the interior of S(u, t) or all the points of the random sample lie in $K \setminus S(u, t)$. Since K has area 1 and $K \cap S(u, t)$ has area v(u, t), this means that

$$\forall u \in \mathcal{U} \text{ and } t \in [0, \operatorname{dwidth}(u)]: \quad \Pr[\ell(u, t) \cap K_m \neq \emptyset] = 1 - v(u, t)^m - (1 - v(u, t))^m.$$

Using Fubini's theorem we have

$$\mathbb{E}[\operatorname{per}(K_m)] = \int_{\mathcal{U}} \int_0^{\operatorname{dwidth}(u)} \Pr[\ell(u,t) \cap K_m \neq \emptyset] \, dt \, d\omega(u)$$

$$= \int_{\mathcal{U}} \int_0^{\operatorname{dwidth}(u)} \left[1 - v(u,t)^m - (1 - v(u,t))^m \right] \, dt \, d\omega(u)$$

$$= \int_{\mathcal{U}} \operatorname{dwidth}(u) \, d\omega(u) - \int_{\mathcal{U}} \int_0^{\operatorname{dwidth}(u)} \left[v(u,t)^m + (1 - v(u,t))^m \right] \, dt \, d\omega(u)$$

$$= \operatorname{per}(K) - \int_{\mathcal{U}} \int_0^{\operatorname{dwidth}(u)} \left[v(u,t)^m + (1 - v(u,t))^m \right] \, dt \, d\omega(u).$$

Note that for each $u \in \mathcal{U}$ and each t with $0 \le t \le dwidth(u)$ we have

$$1 - v(u, t) = v(-u, \operatorname{dwidth}(u) - t),$$

and therefore, by applying the change of variables w = -u, s = dwidth(u) - t and renaming the new variables as u, t,

$$\int_{\mathcal{U}} \int_0^{\operatorname{dwidth}(u)} \left(1 - v(u, t)\right)^m dt \, d\omega(u) = \int_{\mathcal{U}} \int_0^{\operatorname{dwidth}(u)} v(u, t)^m dt \, d\omega(u).$$

Thus, rearranging terms we get

$$\operatorname{per}(K) - \mathbb{E}[\operatorname{per}(K_m)] = 2 \int_{\mathcal{U}} \int_0^{\operatorname{dwidth}(u)} (1 - v(u, t))^m dt \, d\omega(u).$$

Now we adapt the estimate of Schneider [33] to make it independent of K. The right of Figure 6 may be helpful. For each $u \in \mathcal{U}$, let $p(u) \in K$ be a point maximizing $\langle p, u \rangle$. For each $u \in \mathcal{U}$ and each $t \in \mathbb{R}_{\geq 0}$, let K(u, t) be a copy of K scaled by $\frac{t}{\text{dwidth}(u)}$ with center p(u). Note that K(u, t) is contained in S(u, t) and, if $0 \leq t \leq \text{dwidth}(u)$, it is also contained in K. Therefore, since $\operatorname{area}(K) = 1$, we have

$$\forall u \in \mathcal{U} \text{ and } t \in [0, \operatorname{dwidth}(u)]: \quad v(u, t) \ge \operatorname{area}(K(u, t)) = \left(\frac{t}{\operatorname{dwidth}(u)}\right)^2.$$

Thus we have the estimate

$$\operatorname{per}(K) - \mathbb{E}[\operatorname{per}(K_m)] \leq 2 \int_{\mathcal{U}} \int_0^{\operatorname{dwidth}(u)} \left(1 - \left(\frac{t}{\operatorname{dwidth}(u)}\right)^2\right)^m dt \, d\omega(u).$$

For each $u \in \mathcal{U}$ we can use the change of variable $x = (t/\operatorname{dwidth}(u))^2$ and we obtain that

$$\operatorname{per}(K) - \mathbb{E}[\operatorname{per}(K_m)] \leq \int_{\mathcal{U}} \int_0^1 x^{-1/2} (1-x)^m \operatorname{dwidth}(u) \, dx \, d\omega(u)$$
$$= \int_{\mathcal{U}} \operatorname{dwidth}(u) \int_0^1 x^{-1/2} (1-x)^m \, dx \, d\omega(u).$$

Using the standard formula

$$\int_0^1 x^{a-1} (1-x)^{b-1} \, dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

for the beta function and the gamma function, and the known value $\Gamma(1/2) = \sqrt{\pi}$, we further derive

$$\operatorname{per}(K) - \mathbb{E}[\operatorname{per}(K_m)] \leq \int_{\mathcal{U}} \operatorname{dwidth}(u) \int_0^1 x^{-1/2} (1-x)^m \, dx \, d\omega(u)$$
$$= \int_{\mathcal{U}} \operatorname{dwidth}(u) \frac{\sqrt{\pi} \, \Gamma(m+1)}{\Gamma(m+3/2)} \, d\omega(u)$$
$$= \frac{\sqrt{\pi} \, \Gamma(m+1)}{\Gamma(m+3/2)} \int_{\mathcal{U}} \operatorname{dwidth}(u) \, d\omega(u)$$
$$= \frac{\sqrt{\pi} \, \Gamma(m+1)}{\Gamma(m+3/2)} \cdot \operatorname{per}(K).$$

Now we use that

$$\lim_{m \to \infty} \frac{\Gamma(m+3/2)}{\Gamma(m+1)\sqrt{m}} = 1$$

to conclude that, for some constant $C_4 \ge 1$,

$$\operatorname{per}(K) - \mathbb{E}[\operatorname{per}(K_m)] \leq \frac{C_4}{\sqrt{m}} \operatorname{per}(K).$$

Note that the bound in this theorem is optimal. When K is an equilateral triangle of unit area, to get a $(1 - \varepsilon)$ -approximation of per(K), we need to sample at least one point at distance at most $O(\varepsilon)$ from each vertex, and these regions have area $\Theta(\varepsilon^2)$.

The following lemma is the analogue to Lemma 4 for the perimeter.

Lemma 19. Let K be a convex body contained in a polygon P, let R be a random sample of points inside P, and let C_4 be the constant in Theorem 18. If

$$|R| \geq 4 \cdot (6C_4/\varepsilon)^2 \cdot \frac{\operatorname{area}(P)}{\operatorname{area}(K)},$$

then with probability at least 2/3 it holds that $\operatorname{per}(\operatorname{conv}(R \cap K)) \ge (1 - \varepsilon) \operatorname{per}(K)$.

Proof. We define the following events:

$$\begin{aligned} \mathcal{E} : & |R \cap K| \geq (6C_4/\varepsilon)^2, \\ \mathcal{F} : & \operatorname{per}(\operatorname{conv}(R \cap K)) \geq (1-\varepsilon) \cdot \operatorname{per}(K) \end{aligned}$$

For each event \mathcal{A} we use $\overline{\mathcal{A}}$ for its negation. Since $C_4 \geq 1$, then $(6C_4/\varepsilon)^2 \geq 3$ and Lemma 3 implies

$$\Pr\left[\mathcal{E}\right] \geq \frac{5}{6}$$

Assuming the event \mathcal{E} , that is, $|R \cap K| \geq (6C_4/\varepsilon)^2$, it follows from Markov's inequality and Theorem 18 that

$$\Pr\left[\operatorname{per}(K) - \operatorname{per}(\operatorname{conv}(R \cap K)) \ge \varepsilon \cdot \operatorname{per}(K)\right] \le \frac{1}{\varepsilon \cdot \operatorname{per}(K)} \cdot \mathbb{E}\left[\operatorname{per}(K) - \operatorname{per}(\operatorname{conv}(R \cap K))\right]$$
$$\le \frac{1}{\varepsilon \cdot \operatorname{per}(K)} \cdot \frac{C_4}{\sqrt{|R \cap K|}} \cdot \operatorname{per}(K)$$
$$\le \frac{C_4}{\varepsilon \cdot \operatorname{per}(K) \cdot (6C_4/\varepsilon)} \cdot \operatorname{per}(K)$$
$$= \frac{1}{6}.$$

This means that

$$\Pr\left[\overline{\mathcal{F}} \mid \mathcal{E}\right] \leq \frac{1}{6}$$

and therefore

$$\Pr\left[\overline{\mathcal{F}}\right] \leq \Pr\left[\overline{\mathcal{F}} \mid \mathcal{E}\right] + \Pr\left[\overline{\mathcal{E}}\right] \leq \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

5.2 Bounds depending on the fatness

We recall a result about approximation of convex bodies in the plane by rectangles:

Lemma 20 (Schwarzkopf et al. [37]). Given a convex body K in the plane, there exist two similar and parallel rectangles $\Pi_{in}(K)$ and $\Pi_{out}(K)$ such that $\Pi_{in}(K) \subseteq K \subseteq \Pi_{out}(K)$ and the sides of $\Pi_{out}(K)$ are at most twice as long as the sides of $\Pi_{in}(K)$.

Note that the statement does *not* guarantee that $\Pi_{in}(K)$ and $\Pi_{out}(K)$ have a common center. Given a convex body K, we denote by $d_1(K)$ and $d_2(K)$ the lengths of the sides of $\Pi_{in}(K)$, with $d_1(K) \ge d_2(K)$. Although the rectangles $\Pi_{in}(K)$ and $\Pi_{out}(K)$ are not necessarily unique, this does not affect our arguments.

We will run two algorithms and choose the best between both outputs. One of the algorithms is a $(1-26\varepsilon)$ -approximation when the optimal solution K^* satisfies $d_2(K^*)/d_1(K^*) \leq \varepsilon$, while the other covers the case $d_2(K^*)/d_1(K^*) \geq \varepsilon$. We now develop bounds for both cases.

Lemma 21. Let K^* be a convex body contained in P such that $per(K^*) = L^*(P)$. Let ℓ be the length of a longest line segment contained in P. If $d_2(K^*)/d_1(K^*) \leq \varepsilon \leq 2/5$, then $L^*(P) \leq 2 \cdot \ell \cdot (1+25 \cdot \varepsilon)$.

Proof. To simplify the notation, in this proof we set $d_1 := d_1(K^*)$ and $d_2 := d_2(K^*)$. Without loss of generality, we assume that the longer side of $\prod_{in}(K^*)$ is horizontal, and the shorter one is vertical.



Figure 7: Situation in the proof of Lemma 21. In gray, the parallelogram Ψ .

Let s be a longest line segment contained in K^* , let ℓ_{K^*} be its length, and let a and b be its two endpoints. Clearly, $\ell_{K^*} \leq \ell$. Since $\prod_{in}(K^*) \subset K^*$, we have $\ell_{K^*} \geq d_1$.

We first observe that s is not vertical. Indeed, in this case the containment $K^* \subset \Pi_{out}(K^*)$ implies $\ell_{K^*} \leq 2 \cdot d_2$. Thus, we would have $d_1 \leq 2 \cdot d_2$, which contradicts the assumption that $d_2/d_1 \leq \varepsilon \leq 2/5$.

Without loss of generality, we assume that s has non-negative slope α . Since s is a longest line segment in K^* , K^* is contained in the infinite region of the plane bounded by the lines through a and b perpendicular to s. We denote by Ψ the parallelogram resulting from the intersection of this region and the region of the plane bounded by the lines supporting the horizontal edges of $\Pi_{out}(K^*)$. See Figure 7. The horizontal sides of Ψ have length $\ell_{K^*}/\cos \alpha$, while the other sides have length at most $2 \cdot d_2/\cos \alpha$. It is wellknown, and an easy consequence of Crofton's formula (see equation (8) in Theorem 18), that if a convex body K_1 is contained in a convex body K_2 , then $\operatorname{per}(K_1) \leq \operatorname{per}(K_2)$. Since $K^* \subseteq \Psi$, it is enough to prove that $\operatorname{per}(\Psi) \leq 2 \cdot \ell \cdot (1 + 25 \cdot \varepsilon)$.

Since s is contained in $\Pi_{out}(K^*)$, the maximum slope is attained when one of the endpoints of s lies in the lower side of $\Pi_{out}(K^*)$, and the other endpoint in the upper side. Therefore,

$$\sin \alpha \ \le \ \frac{2 \cdot c_2}{\ell_{K^*}} \ \le \ \frac{2 \cdot \varepsilon \cdot d_1}{d_1} \ = \ 2 \cdot \varepsilon \,.$$

Since $\varepsilon < 1/2$,

$$\cos \alpha \geq \sqrt{1 - 4 \cdot \varepsilon^2} > 1 - 2 \cdot \varepsilon$$
.

Thus, we have

$$per(\Psi) \leq \frac{2 \cdot \ell_{K^*}}{\cos \alpha} + \frac{4 \cdot d_2}{\cos \alpha}$$
$$\leq \frac{1}{\cos \alpha} \cdot (2 \cdot \ell + 4 \cdot \varepsilon \cdot d_1)$$
$$< \frac{1}{1 - 2 \cdot \varepsilon} \cdot (2 \cdot \ell + 4 \cdot \varepsilon \cdot \ell)$$
$$= \frac{2 \cdot \ell \cdot (1 + 2 \cdot \varepsilon)}{1 - 2 \cdot \varepsilon}$$
$$< 2 \cdot \ell \cdot (1 + 25 \cdot \varepsilon),$$

where the last inequality holds because $\varepsilon \leq 2/5$.

Lemma 22. Let K^* be a convex body contained in P such that $per(K^*) = L^*(P)$. If $d_2(K^*)/d_1(K^*) \ge \varepsilon$, then $area(K^*) \ge \frac{\varepsilon}{16} \cdot A^*(P)$.

Proof. To simplify the notation, in this proof we set $d_1 := d_1(K^*)$ and $d_2 := d_2(K^*)$. Let K' be a convex shape contained in P with $\operatorname{area}(K') = A^*(P)$. We further define

 $d'_1 := d_1(K')$ and $d'_2 := d_2(K')$. We have the following obvious relations:

$$\operatorname{per}(K^*) \leq \operatorname{per}(\Pi_{out}(K^*)) \leq 4 \cdot (d_1 + d_2),$$

$$\operatorname{per}(K') \geq \operatorname{per}(\Pi_{in}(K')) = 2 \cdot (d'_1 + d'_2),$$

$$\operatorname{area}(K^*) \geq \operatorname{area}(\Pi_{in}(K^*)) = d_1 \cdot d_2,$$

$$\operatorname{area}(K') \leq \operatorname{area}(\Pi_{out}(K')) \leq 4 \cdot d'_1 \cdot d'_2.$$

Combining the first two inequalities with $per(K') \leq per(K^*)$, we obtain

$$d_1' + d_2' \le 2 \cdot (d_1 + d_2). \tag{9}$$

If $2 \cdot d_1 \geq d'_1$, then

$$\operatorname{area}(K^*) \geq d_1 \cdot d_2 \geq d_1 \cdot \varepsilon \cdot d_1 \geq (\varepsilon/4) \cdot (d_1')^2$$

$$\geq (\varepsilon/4) \cdot d_1' \cdot d_2' \geq (\varepsilon/16) \cdot \operatorname{area}(K')$$

$$= (\varepsilon/16) \cdot A^*(P) \,.$$

Let us now consider the case $2 \cdot d_1 < d'_1$. Since the inequality (9) implies

$$2 \cdot (d_1 + d_2) \ge d'_1 + d'_2 > 2 \cdot d_1 + d'_2$$

we obtain $2 \cdot d_2 > d'_2$. Adding the inequalities $d'_1 > 2 \cdot d_1$ and $2 \cdot d_2 > d'_2$ we get that

$$d_1' - d_2' > 2 \cdot (d_1 - d_2). \tag{10}$$

Combining (9) and (10) we have

$$d_1 \cdot d_2 = 1/4 \cdot ((d_1 + d_2)^2 - (d_1 - d_2)^2)$$

> 1/4 \cdot (1/4 \cdot (d_1' + d_2')^2 - 1/4 \cdot (d_1' - d_2')^2)
= (1/4) \cdot d_1' \cdot d_2'.

Thus, also in this case we get

$$\operatorname{area}(K^*) \geq d_1 \cdot d_2 > (1/4) \cdot d'_1 \cdot d'_2 \geq (\varepsilon/16) \cdot 4 \cdot d'_1 \cdot d'_2$$

$$\geq (\varepsilon/16) \cdot \operatorname{area}(K') = (\varepsilon/16) \cdot A^*(P) . \square$$

5.3 Algorithm

As already mentioned, we run two algorithms to find a $(1 - \varepsilon)$ -approximation of $L^*(P)$. In fact, to keep the computations slightly simpler, we will provide for a $(1 - 26\varepsilon)$ approximation of $L^*(P)$.

Let K^* be a convex body inside P with $per(K^*) = L^*(P)$. The first algorithm finds a $(1 - 26\varepsilon)$ -approximation of the value $L^*(P)$ when $d_2(K^*)/d_1(K^*) \leq \varepsilon$, while the second algorithm returns a $(1 - \varepsilon)$ -approximation of the value $L^*(P)$ when $d_2(K^*)/d_1(K^*) \geq \varepsilon$. Since both algorithms compute a convex polygon contained in P, taking the best of the two solutions we obtain a $(1 - 26\varepsilon)$ -approximation in any case. We can assume that $\varepsilon \leq 2/5$, as otherwise we can just take $\varepsilon = 2/5$.

Consider first the case $d_2(K^*)/d_1(K^*) \leq \varepsilon$. This means that the optimal solution K^* is "skinny". Let ℓ be the length of a longest segment contained in P. Let \bar{s} be a line segment contained in P of length at least $(1 - \varepsilon) \cdot \ell$. Lemma 21 implies that

$$\operatorname{per}(\bar{s}) \geq 2 \cdot (1-\varepsilon) \cdot \ell \geq L^*(P) \cdot \frac{1-\varepsilon}{1+25 \cdot \varepsilon} \geq L^*(P) \cdot (1-26 \cdot \varepsilon).$$

Hall-Holt et al. [27] show how to compute such a segment \bar{s} in $O((n/\varepsilon^4) \log^2 n)$ time. We conclude that, whenever $d_2(K^*)/d_1(K^*) \leq \varepsilon$, we can obtain a $(1 - 26 \cdot \varepsilon)$ -approximation to $L^*(P)$ in $O((n/\varepsilon^4) \log^2 n)$ time.

Consider now the case $d_2(K^*)/d_1(K^*) \ge \varepsilon$. This means that the optimal solution K^* is "slightly fat". By Lemma 22 we have

$$\frac{\varepsilon}{16} \cdot A^*(P) \leq \operatorname{area}(K^*) \leq A^*(P).$$

Let A(P) be the approximation computed in the algorithm LARGEPOTATO, line 1 of Figure 5. We then know that

$$\frac{\varepsilon}{16} \cdot A(P) \leq \operatorname{area}(K^*) \leq C_2 \cdot A(P).$$

We divide the interval $\left[\frac{\varepsilon}{16} \cdot A(P), C_2 \cdot A(P)\right]$ into the following $O(\log 1/\varepsilon)$ subintervals:

$$I_i := [C_2 \cdot A(P)/2^{i+1}, C_2 \cdot A(P)/2^i], \quad i \in \mathbb{Z}, \ 0 \le i \le \lceil \log_2(16C_2/\varepsilon) \rceil.$$

For each integer i we can apply a modification of the algorithm LARGEPOTATO, as follows.

Lemma 23. A modification of the algorithm LARGEPOTATO taking as an extra parameter an integer i has the following properties. It always takes time

$$O\left(n\left[2^{i}\log^{2}n+(4^{i}/\varepsilon^{4})\log n+4^{i}/\varepsilon^{6}\right]\log(1/\delta)\right).$$

If $\operatorname{area}(K^*)$ is in I_i , then the algorithm finds a convex polygon of perimeter at least $(1 - \varepsilon)L^*(P)$ with probability at least $1 - \delta$. If $\operatorname{area}(K^*)$ is not in I_i , then the algorithm returns a convex polygon inside P.

Proof. As stated in the algorithm LARGEPOTATO, we assume that P has unit area. To simplify the computation, set $A_i = C_2 \cdot A(P)/2^{i+1}$, that is, the lower endpoint of the interval I_i . The value r in line 2 of the algorithm LARGEPOTATO is set to $r = 60/A_i$. Since $A(P) = \Omega(1/n)$ we have $r = O(2^i n)$.

Consider one of the iterations of the repeat loop (lines 5–14). A slight modification of the proof of Lemma 12 gives that the expected size of the visibility graph G(P, R) is bounded by

$$\mathbb{E}[|E(G(P,R))|] = \binom{r}{2} \cdot \Pr[\text{two random points are visible in } P]$$

$$\leq \frac{1}{2} \left(\frac{60}{A_i}\right)^2 \cdot 180 \cdot A^*(P)$$

$$\leq 324000 \cdot \left(\frac{1}{C_2 \cdot A(P)/2^{i+1}}\right)^2 \cdot A^*(P)$$

$$\leq 4^{i+1} \cdot 324000 \cdot n.$$

Therefore, the condition in line 6 of LARGEPOTATO becomes "if G(P, R) has at most $C' \cdot n$ edges **then**", where $C' = 6 \cdot 4^{i+1} \cdot 324000$. This condition is satisfied in each iteration of the repeat loop with probability at least 5/6. This condition can be checked using Theorem 13 in

$$O(n + r\log r\log(rn) + 4^{i}n) = O(2^{i}n\log(2^{i}n)\log(2^{i}n^{2}) + 4^{i}n) = O(n[4^{i} + 2^{i}\log^{2}n])$$

time.

Under the assumption that $A_i \leq \operatorname{area}(K^*) \leq 2A_i$, we have $r \geq 60/\operatorname{area}(K^*)$ and Lemma 7 ensures that, with probability at least 2/3, the body K^* lies in some parallelogram $\Gamma(a_0, b_0, 2A_i)$ for some edge a_0b_0 of the visibility graph G(P, R).

It remains to discuss how to find a maximum-perimeter polygon contained in the parallelogram $\Gamma(a_0, b_0, 2A_i)$ that contains K^* (lines 9–14). For the other parallelograms $\Gamma(a, b, 2A_i), ab \in E(G(P, R))$, we just need to make sure that we find some convex polygon contained in P.

Consider an edge ab of the visibility graph G(P, R) and note that

$$\operatorname{area}(\Gamma(a, b, 2A_i)) = 12 \cdot 2A_i = 24 \cdot A_i$$

We have

$$4 \cdot (6C_4/\varepsilon)^2 \cdot \frac{\operatorname{area}(\Gamma(a, b, 2A_i))}{\operatorname{area}(K^*)} \leq \frac{144 \cdot (C_4)^2}{\varepsilon^2} \cdot \frac{24 \cdot A_i}{A_i} = \frac{C_5}{\varepsilon^2},$$

for some constant C_5 .

Using Lemma 19 we obtain the following: if we take a sample $R_{a_0b_0}$ of C_5/ε^2 points inside $\Gamma(a_0, b_0, 2A_i)$, then with probability at least 2/3, we have $\operatorname{per}(\operatorname{conv}(R_{a_0b_0} \cap K^*)) \ge (1-\varepsilon) \cdot L^*(P)$. Thus, we proceed, for each $ab \in E(G(P, R))$ as follows: take a sample R_{ab} of C_5/ε^2 points inside $\Gamma(a, b, 2A_i)$, build the visibility graph G_{ab} of R_{ab} , and find a convex clique in G_{ab} of largest perimeter. As discussed in Lemma 15, the visibility graph can be built in $O(|R_{ab}|^2 \log n) = O((1/\varepsilon)^4 \log n)$ time. For computing the convex clique of largest perimeter, we use the modification of Lemma 8 mentioned thereafter, using each point of R_{ab} as highest point. Unlike in the case of approximating the area, here we cannot afford to use an asymptotically smaller sample S_{ab} for the highest points, for the following reason. Let T be an equilateral triangle with a horizontal base at the bottom and a vertex v on top. To approximate the perimeter of T by a convex hull of a set of points inside Twith error at most ε , the highest point of the set must be in a region of area $\Omega(\varepsilon^2 \cdot \operatorname{area}(T))$ near v. Hence we would need at least $\Omega(1/\varepsilon^2)$ points in the sample S_{ab} . Thus, we need $O(|R_{ab}| \cdot |R_{ab}|^2) = O(1/\varepsilon^6)$ time to compute the convex clique of largest perimeter. We conclude that for each edge ab of G(P, R) we spend $O((1/\varepsilon^4) \log n + 1/\varepsilon^6)$ time.

Since we make $|E(G(P, R))| = O(4^i n)$ iterations of the for loop (lines 9–14), and each iteration of the for loop (lines 9–14) takes $O((1/\varepsilon^4) \log n + 1/\varepsilon^6)$ time, in the for loop of lines 8–14 we spend $O(n \cdot 4^i \cdot ((1/\varepsilon^4) \log n + 1/\varepsilon^6))$ time. To make the test in line 6 we spend $O(n[4^i + 2^i \log^2 n])$. It follows that in each iteration of the repeat loop we spend $O(n[2^i \log^2 n + (4^i/\varepsilon^4) \log n + 4^i/\varepsilon^6])$ time. Since the algorithm makes $O(\log(1/\delta))$ iterations of the repeat loop, the claimed time bound follows.

Under the assumption that area (K^*) lies in the interval I_i , the graph G(P, R) passes the test of line 6 with probability at least 5/6, one of the parallelograms $\Gamma(a_0, b_0, 2A_i)$ contains K^* with probability at least 2/3, and the sample $R_{a_0b_0}$ has the property that $\operatorname{per}(K^* \cap R_{a_0b_0}) \geq (1-\varepsilon)L^*(P)$ with probability at least 2/3. When all three events occur, the algorithm finds a $(1-\varepsilon)$ -approximation. As shown in the proof of Lemma 16, with probability at least 1/4, the three events occur simultaneously, and thus some iteration of the repeat loop is successful with probability at least $1-\delta$, as shown in the proof of Theorem 17. We conclude that, when $\operatorname{area}(K^*)$ lies in the interval I_i , the output of the algorithm is a $(1-\varepsilon)$ -approximation with probability at least $1-\delta$.

When $\operatorname{area}(K^*)$ does not lie in the interval I_i , we spend the same time and we return a convex polygon contained in P (possibly degenerated to a single point) without any guarantee. **Lemma 24.** When $d_2(K^*)/d_1(K^*) \ge \varepsilon$, we can find a convex polygon of perimeter at least $(1 - \varepsilon)L^*(P)$ with probability at least $1 - \delta$ in time

 $O\left(n\left[(1/\varepsilon)\log^2 n + (1/\varepsilon^6)\log n + 1/\varepsilon^8\right]\log(1/\delta)\right).$

Proof. We use the algorithm of Lemma 23 for each interval I_i , where $i = 0, 1, ..., \lceil \log_2(16C_2/\varepsilon) \rceil$, and return the polygon with largest perimeter we get over all iterations. The running time is

$$\sum_{i=0}^{\lceil \log_2(16C_2/\varepsilon)\rceil} O\left(n\left[2^i \log^2 n + \frac{4^i}{\varepsilon^4} \log n + \frac{4^i}{\varepsilon^6}\right] \log(1/\delta)\right).$$

Using that $\sum_i 2^i = O(1/\varepsilon)$ and $\sum_i 4^i = O(1/\varepsilon^2)$, this becomes

$$O\left(n\left[\frac{1}{\varepsilon}\log^2 n + \frac{1}{\varepsilon^6}\log n + \frac{1}{\varepsilon^8}\right]\log(1/\delta)\right).$$

The algorithm is successful in getting a $(1 - \varepsilon)$ -approximation whenever the iteration with area (K^*) in I_i is successful. Thus, the whole algorithm is a $(1 - \varepsilon)$ -approximation with probability at least $1 - \delta$, for the case $d_2(K^*)/d_1(K^*) \ge \varepsilon$.

Combining the algorithms for $d_2(K^*)/d_1(K^*) \leq \varepsilon$ and $d_2(K^*)/d_1(K^*) \geq \varepsilon$ we get a $(1 - 26\varepsilon)$ -approximation. Replacing ε with $\varepsilon/26$ in the whole discussion, we obtain the following final result for maximizing the perimeter.

Theorem 25. Let P be a polygon with n vertices, let ε and δ be parameters with $0 < \varepsilon < 1$ and $0 < \delta < 1$. In time O $\left(n\left[(1/\varepsilon^4)\log^2 n + ((1/\varepsilon)\log^2 n + (1/\varepsilon^6)\log n + 1/\varepsilon^8)\log(1/\delta)\right]\right)$ we can find a convex polygon contained in P that, with probability at least $1 - \delta$, has perimeter at least $(1 - \varepsilon) \cdot L^*(P)$.

6 Conclusions

There are several directions for future work. We explicitly mention the following:

- Finding a deterministic (1ε) -approximation using near-linear time.
- Achieving subquadratic time for polygons with an unbounded number of holes.

In the conference version of this paper (in the proceedings of SoCG 2014), we also mentioned the following two questions that have been answered affirmatively by Balko et al. [5].

- Does Theorem 9(i) hold for arbitrary simple polygons? We conjecture so, possibly with a larger constant.
- Are similar results about the probability of random points being co-visible achievable in 3-dimensions?

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