# THE COMPLEXITY OF APPROXIMATELY COUNTING TREE HOMOMORPHISMS 

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#### Abstract

We study two computational problems, parameterised by a fixed tree $H$. \#HomsTo $(H)$ is the problem of counting homomorphisms from an input graph $G$ to $H$. \#WHomsTo $(H)$ is the problem of counting weighted homomorphisms to $H$, given an input graph $G$ and a weight function for each vertex $v$ of $G$. Even though $H$ is a tree, these problems turn out to be sufficiently rich to capture all of the known approximation behaviour in $\# \mathrm{P}$. We give a complete trichotomy for \#WHomsTo $(H)$. If $H$ is a star then \#WHomsTo $(H)$ is in FP. If $H$ is not a star but it does not contain a certain induced subgraph $J_{3}$ then \#WHomsTo $(H)$ is equivalent under approximation-preserving (AP) reductions to \#BIS, the problem of counting independent sets in a bipartite graph. This problem is complete for the class $\# \mathrm{RH} \Pi_{1}$ under AP-reductions. Finally, if $H$ contains an induced $J_{3}$ then \#WHomsTo $(H)$ is equivalent under AP-reductions to \#SAT, the problem of counting satisfying assignments to a CNF Boolean formula. Thus, \#WHOMSTO $(H)$ is complete for \#P under AP-reductions. The results are similar for \#HomsTo $(H)$ except that a rich structure emerges if $H$ contains an induced $J_{3}$. We show that there are trees $H$ for which \#HomsTo $(H)$ is \#Sat-equivalent (disproving a plausible conjecture of Kelk). However, it is still not known whether \#HomsTo $(H)$ is \#SAT-hard for every tree $H$ which contains an induced $J_{3}$. It turns out that there is an interesting connection between these homomorphism-counting problems and the problem of approximating the partition function of the ferromagnetic Potts model. In particular, we show that for a family of graphs $J_{q}$, parameterised by a positive integer $q$, the problem \#HomsTo $\left(J_{q}\right)$ is AP-interreducible with the problem of approximating the partition function of the $q$-state Potts model. It was not previously known that the Potts model had a homomorphismcounting interpretation. We use this connection to obtain some additional upper bounds for the approximation complexity of \#HomsTo $\left(J_{q}\right)$.


## 1. Introduction

A homomorphism from a graph $G$ to a graph $H$ is a mapping $\sigma: V(G) \rightarrow V(H)$ such that the image $(\sigma(u), \sigma(v))$ of every edge $(u, v) \in E(G)$ is in $E(H)$. Let $\operatorname{Hom}(G, H)$ denote the set of homomorphisms from $G$ to $H$ and let $Z_{H}(G)=|\operatorname{Hom}(G, H)|$. For each fixed $H$, we consider the following computational problem.

Problem: \#HomsTo $(H)$.
Instance: Graph $G$.
Output: $Z_{H}(G)$.

[^0]The vertices of $H$ are often referred to as "colours" and a homomorphism from $G$ to $H$ can be thought of as an assignment of colours to the vertices of $G$ which satisfies certain constraints along each edge of $G$. The constraints guarantee that adjacent vertices in $G$ are assigned colours which are adjacent in $H$. A homomorphism in $\operatorname{Hom}(G, H)$ is therefore often called an " $H$-colouring" of $G$. When $H=K_{q}$, the complete graph with $q$ vertices, the elements of $\operatorname{Hom}\left(G, K_{q}\right)$ are proper $q$-colourings of $G$.

There has been much work on determining the complexity of the $H$-colouring decision problem, which is the problem of determining whether $Z_{H}(G)=0$, given input $G$. This work will be described in Section 1.1, but at this point it is worth mentioning the dichotomy result of Hell and Nešetril [21], which shows that the decision problem is solvable in polynomial time if $H$ is bipartite and that it is NP-hard otherwise. There has also been work [12, 27] on determining the complexity of exactly or approximately solving the related counting problem \#HomsTo $(H)$. This paper is concerned with the computational difficulty of \#HomsTo $(H)$ when $H$ is bipartite, and particularly when $H$ is a tree.

As an example, consider the case where $H$ is the four-vertex path $P_{4}$ (of length three). Label the vertices (or colours) $1,2,3,4$, in sequence. If $G$ is not bipartite then $\operatorname{Hom}(G, H)=\emptyset$, so the interesting case is when $G$ is bipartite. Suppose for simplicity that $G$ is connected. Then one side of the vertex bipartition of $G$ must be assigned even colours and the other side must be assigned odd colours. It is easy to see that the vertices assigned colours 1 and 4 form an independent set of $G$, and that every independent set arises in exactly two ways as a homomorphism. Thus, $Z_{P_{4}}(G)$ is equal to twice the number of independent sets in the bipartite graph $G$. We will return to this example presently.

It will sometimes be useful to consider a weighted generalisation of the homomorphismcounting problem. Suppose, for each $v \in V(G)$, that $w_{v}: V(H) \rightarrow \mathbb{Q} \geq 0$ is a weight function, assigning a non-negative rational weight to each colour. Let $W(G, H)$ be an indexed set of weight functions, containing one weight function for each vertex $v \in V(G)$, Thus,

$$
W(G, H)=\left\{w_{v} \mid v \in V(G)\right\}
$$

Our goal is to compute the weighted sum of homomorphisms from $G$ to $H$, which is expressed as the partition function

$$
Z_{H}(G, W(G, H))=\sum_{\sigma \in \operatorname{Hom}(G, H)} \prod_{v \in V(G)} w_{v}(\sigma(v)) .
$$

Given a fixed $H$, each weight function $w_{v} \in W(G, H)$ can be represented succinctly as a list of $|V(H)|$ rational numbers. This representation is used in the following computational problem.

Problem: \#WHomsTo $(H)$.
Instance: A graph $G$ and an indexed set of weight functions $W(G, H)$.
Output: $Z_{H}(G, W(G, H))$.
The complexity of exactly solving \#HomsTo $(H)$ and \#WHomsTo $(H)$ is already understood. Dyer and Greenhill have observed [12, Lemma 4.1] that \#HomsTo $(H)$ is in FP if $H$ is a complete bipartite graph. It is easy to see (see Observation (1) that the same is true of \#WHomsTo $(H)$. On the other hand, Dyer and Greenhill showed that \#HomsTo $(H)$ is \#Pcomplete for every bipartite graph $H$ that is not complete. Since \#HomsTo $(H)$ is a special case
of the more general problem \#WHomsTo $(H)$, we conclude that both problems are in FP if $H$ is a star (a tree in which some "centre" vertex is an endpoint of every edge), and that both problems are \#P-complete for every other tree $H$.

This paper maps the complexity of approximately solving \#HomsTo $(H)$ and \#WHomsTo $(H)$ when $H$ is a tree. Dyer, Goldberg, Greenhill and Jerrum [10] introduced the concept of "APreduction" for studying the complexity of approximate counting problems. Informally, an APreduction is an efficient reduction from one counting problem to another, which preserves closeness of approximation; two counting problems that are interreducible using this kind of reduction have the same complexity when it comes to finding good approximate solutions. We have already encountered an extremely simple example of two AP-interreducible problems, namely \#HomsTo $\left(P_{4}\right)$ and \#BIS, the problem of counting independent sets in a bipartite graph. Using less trivial reductions, Dyer et al. showed ([10, Theorem 5]) that several natural counting problems in addition to \#HomsTo $\left(P_{4}\right)$ are interreducible with \#BIS, and moreover that they are all complete for the complexity class $\# \mathrm{RH} \Pi_{1}$ with respect to AP-reductions. The class $\# \mathrm{RH} \Pi_{1}$ is conjectured to contain problems that do not have an FPRAS; however it is not believed to contain \#SAT, the classical hard problem of computing the number of satisfying assignments to a CNF Boolean formula. Refer to Section 2 for more detail on the technical concepts mentioned here and elsewhere in the introduction.

Steven Kelk's PhD thesis [27] examined the approximation complexity of the problem \#HomsTo $(H)$ for general $H$. He identified certain families of graphs $H$ for which \#HomsTo $(H)$ is APinterreducible with \#BIS and other large families for which \#HomsTo $(H)$ is AP-interreducible with \#Sat. He noted [27, Section 5.7.1] that, during the study, he did not encounter any bipartite graphs $H$ for which \#SAT $\leq_{\text {AP }}$ \#HomsTo $(H)$, and that he suspected [27, Section 7.3] that there were "structural barriers" which would prevent homomorphism-counting problems to bipartite graphs from being \#Sat-hard. An interesting test case is the tree $J_{3}$ which is depicted in Figure 1] Kelk referred to this tree [27, Section 7.4] as "the junction", and conjectured that \#HomsTo $\left(J_{3}\right)$ is neither \#BIS-easy nor \#Sat-hard. Thus, he conjectured that unlike the setting of Boolean constraint satisfaction, where every parameter leads to a computational problem which is FPRASable, \#BIS-equivalent, or \#SAT-equivalent [11], the complexity landscape for approximate $H$-colouring may be more nuanced, in the sense that there might be graphs $H$ for which none of these hold.

The purpose of this paper is to describe the interesting complexity landscape of the approximation problems \#HomsTo $(H)$ and \#WHomsTo $(H)$ when $H$ is a tree. It turns out that even the case in which $H$ is a tree is sufficiently rich to include all of the known approximation complexity behaviour in \#P.

First, consider the weighted problem \#WHomsTo $(H)$. For this problem, we show that there is a complexity trichotomy, and the trichotomy depends upon the induced subgraphs of $H$. We say that $H$ contains an induced $H^{\prime}$ if $H$ has an induced subgraph that is isomorphic to $H^{\prime}$. Here is the result. If $H$ contains no induced $P_{4}$ then it is a star, so \#WHomsTo $(H)$ is in FP (Observation 11). If $H$ contains an induced $P_{4}$ but it does not contain an induced $J_{3}$ then it turns out that \#WHomsTo $(H)$ is AP-interreducible with \#BIS (Lemma 4). Finally, if $H$ contains an induced $J_{3}$, then \#SAT $\leq_{\text {AP }}$ \#WHOMSTO $(H)$ (Lemma6.) Thus, the complexity of
\#WHomsTo $(H)$ is completely determined by the induced subgraphs of the tree $H$, and there are no possibilities other than those that arise in the Boolean constraint satisfaction trichotomy [11].

Now consider the problem \#HomsTo $(H)$. Like its weighted counterpart, the unweighted problem \#HomsTo $(H)$ is in FP if $H$ is a star, and it is \#BIS-equivalent if $H$ contains an induced $P_{4}$ but it does not contain an induced $J_{3}$. However, it is not known whether \#HomsTo $(H)$ is \#Sat-hard for every $H$ which contains an induced $J_{3}$. The structure that has emerged is already quite rich. First, we have discovered (Theorem 11) that there are trees $H$ for which \#HomsTo $(H)$ is \#Sat-hard. This result is surprising - it disproves the plausible conjecture of Kelk that \#HomsTo $(H)$ is not \#Sat-hard for any bipartite graph $H$. We don't know whether \#HomsTo $(H)$ is \#Sat-hard for every tree $H$ which contains an induced $J_{3}$. In fact, we have discovered an interesting connection between these homomorphism-counting problems and the problem of approximating the partition function of the ferromagnetic Potts model. In particular, Theorem 10 shows that for a family of graphs $J_{q}$, parameterised by a positive integer $q$, the problem \#HomsTo $\left(J_{q}\right)$ is AP-interreducible with the problem of approximating the partition function of the $q$-state Potts model. This is surprising because it was not known that the Potts model had a homomorphism-counting interpretation.

The Potts-model connection allows us to give a non-trivial upper bound for the complexity of \#HomsTo $\left(J_{q}\right)$. In particular, Corollary 12 shows that this problem is AP-reducible to the problem of counting proper $q$-colourings of bipartite graphs.

We are not aware of any complexity relationships between the problems \#HomsTo $\left(J_{q}\right)$, for $q>2$. At one extreme, they might all be AP-interreducible; at the other, they might all be incomparable. Another conceivable situation is that \#HomsTo $\left(J_{q}\right)$ is AP-reducible to \#HomsTo $\left(J_{q^{\prime}}\right)$ exactly when $q \leq q^{\prime}$. There is no real evidence for or against any of these or other possibilities. However, in the final section we exhibit a natural problem that provides an upper bound on the complexity of infinite families of problems of the form \#HomsTo $\left(J_{q}\right)$ where $q$ is a prime power. Specifically, we show (Corollary 15) that \#HomsTo $\left(J_{p^{k}}\right)$ is AP-reducible to the weight enumerator of a linear code over the field $\mathbb{F}_{p}$.
1.1. Previous Work. We have already mentioned Hell and Nešetril's classic work [21] on the complexity of the $H$-colouring decision problem. They showed that this problem is solvable in polynomial time if $H$ is bipartite, and that it is NP-complete otherwise. Our paper is concerned with the situation in which $H$ is an undirected graph (specifically, an undirected tree) but it is worth noting that the decision problem becomes much more complicated if $H$ is allowed to be a directed graph. Indeed, Feder and Vardi showed [13] that every constraint satisfaction problem (CSP) is equivalent to some digraph homomorphism problem. Despite much research, a complete dichotomy theorem for the digraph homomorphism decision problem is not known. Bang-Jensen and Hell [2] had conjectured a dichotomy for the special case in which the digraph $H$ has no sources and no sinks. This conjecture was proved in important recent work of Barto, Kozik and Niven [3]. Given the conjecture, Hell, Nešetřil, and Zhu [20] stated that "digraphs with sources and sinks, and in particular oriented trees, seem to be the hard part of the problem." Gutjahr, Woeginger and Welzl [19] constructed a directed tree $H$ such that determining whether a digraph $G$ has a homomorphism to $H$ is NP-complete. Of course, for some other trees, this problem is solvable in polynomial time. For example, they showed that it is solvable
in polynomial time whenever $H$ is an oriented path (a path in which edges may go in either direction). Hell, Nešetřil and Zhu [20] construct a whole family of directed trees for which the homomorphism decision problem is NP-hard, and study the problem of characterising NP-hard trees by forbidden subtrees. The reader is referred to Hell and Nešetriil's book [22] and to their survey paper [23] for more details about these decision problems.

As mentioned in the introduction, there is already some existing work [12, 27] on determining the complexity of exactly or approximately counting homomorphisms. This work is discussed in more detail elsewhere in this paper. The problem of sampling homomorphisms uniformly at random (or, in the weighed case, of sampling homomorphisms with probability proportional to their contributions to the partition function) is closely related to the approximate counting problem. We will later discuss some existing work [18] on the complexity of the homomorphism-sampling problem. First, we describe some related results on a particular approach to this problem namely, the application of the Markov chain Monte Carlo (MCMC) method. Here the idea is to simulate a Markov chain whose states correspond to homomorphisms from $G$ to $H$. The chain will be constructed so that the probability of a particular homomorphism $\sigma$ in the stationary distribution of the chain is proportional to the contribution of $\sigma$ to the partition function. If the Markov chain is rapidly mixing then it is possible to efficiently sample homomorphisms from a distribution that is very close to the appropriate distribution. This, in turn, leads to a good approximate counting algorithm [9]. First, Cooper, Dyer and Frieze [6] considered the unweighted problem. They showed that, for any non-trivial $H$, any Markov chain on $H$-colourings that changes the colours of up to some constant fraction of the vertices of $G$ in a single step will have exponential mixing time (so will not lead to an efficient approximate counting algorithm). When $H$ is a tree with a self-loop on every vertex, they construct a weight function $w_{H}: V(H) \rightarrow \mathbb{Q}_{\geq 0}$ so that rapid mixing does occur for the special case of the weighted homomorphism problem in which every vertex $v$ of $G$ has weight function $w_{v}=w_{H}$. Thus, their result gives an FPRAS for this special case of \#WHomsTo $(H)$. The slow-mixing results of [6] have been extended in [1] and in [4]. In particular, Borgs et al. [4] considered the case in which $H$ is a rectangular subset of the hypercubic lattice, and constructed a weight function $w_{H}$ for which quasi-local Markov chains (which change the colours of up to some constant fraction of the vertices in a small sublattice at each step) have slow mixing.

## 2. Preliminaries

This section brings together the main complexity-theoretic notions that are specific to the study of approximate counting problems. A more detailed account can be found in [10].

A randomised approximation scheme is an algorithm for approximately computing the value of a function $f: \Sigma^{*} \rightarrow \mathbb{R}_{\geq 0}$. The approximation scheme has a parameter $\varepsilon \in(0,1)$ which specifies the error tolerance. A randomised approximation scheme for $f$ is a randomised algorithm that takes as input an instance $x \in \Sigma^{*}$ (e.g., in the case of \#HomsTo $(H)$, the input would be an encoding of a graph $G$ ) and a rational error tolerance $\varepsilon \in(0,1)$, and outputs a rational number $z$ (a random variable depending on the "coin tosses" made by the algorithm) such that, for every instance $x$, $\operatorname{Pr}\left[e^{-\epsilon} f(x) \leq z \leq e^{\epsilon} f(x)\right] \geq \frac{3}{4}$. We adopt the convention that $z$ is represented as a pair of integers representing the numerator and the denominator. The randomised approximation
scheme is said to be a fully polynomial randomised approximation scheme, or FPRAS, if it runs in time bounded by a polynomial in $|x|$ and $\epsilon^{-1}$. As in [16], we say that a real number $z$ is efficiently approximable if there is an FPRAS for the constant function $f(x)=z$.

Our main tool for understanding the relative difficulty of approximation counting problems is approximation-preserving reductions. We use the notion of approximation-preserving reduction from Dyer et al. [10]. Suppose that $f$ and $g$ are functions from $\Sigma^{*}$ to $\mathbb{R}_{\geq 0}$. An AP-reduction from $f$ to $g$ gives a way to turn an FPRAS for $g$ into an FPRAS for $f$. The actual definition in [10] applies to functions whose outputs are natural numbers. The generalisation that we use here follows McQuillan [28]. An approximation-preserving reduction (AP-reduction) from $f$ to $g$ is a randomised algorithm $\mathcal{A}$ for computing $f$ using an oracle for $g$. The algorithm $\mathcal{A}$ takes as input a pair $(x, \varepsilon) \in \Sigma^{*} \times(0,1)$, and satisfies the following three conditions: (i) every oracle call made by $\mathcal{A}$ is of the form $(w, \delta)$, where $w \in \Sigma^{*}$ is an instance of $g$, and $\delta \in(0,1)$ is an error bound satisfying $\delta^{-1} \leq \operatorname{poly}\left(|x|, \varepsilon^{-1}\right)$; (ii) the algorithm $\mathcal{A}$ meets the specification for being a randomised approximation scheme for $f$ (as described above) whenever the oracle meets the specification for being a randomised approximation scheme for $g$; and (iii) the run-time of $\mathcal{A}$ is polynomial in $|x|$ and $\varepsilon^{-1}$ and the bit-size of the values returned by the oracle.

If an approximation-preserving reduction from $f$ to $g$ exists we write $f \leq_{\text {AP }} g$, and say that $f$ is AP-reducible to $g$. Note that if $f \leq_{\text {AP }} g$ and $g$ has an FPRAS then $f$ has an FPRAS. (The definition of AP-reduction was chosen to make this true.) If $f \leq_{\text {AP }} g$ and $g \leq_{\text {AP }} f$ then we say that $f$ and $g$ are AP-interreducible, and write $f \equiv_{\mathrm{AP}} g$. A word of warning about terminology: the notation $\leq_{\text {AP }}$ has been used (see, e.g., [7]) to denote a different type of approximationpreserving reduction which applies to optimisation problems. We will not study optimisation problems in this paper, so hopefully this will not cause confusion.

Dyer et al. [10] studied counting problems in \#P and identified three classes of counting problems that are interreducible under approximation-preserving reductions. The first class, containing the problems that have an FPRAS, are trivially AP-interreducible since all the work can be embedded into the reduction (which declines to use the oracle). The second class is the set of problems that are AP-interreducible with \#SAT, the problem of counting satisfying assignments to a Boolean formula in CNF. Zuckerman [31] has shown that \#SAT cannot have an FPRAS unless RP = NP. The same is obviously true of any problem to which \#SAT is AP-reducible.

The third class appears to be of intermediate complexity. It contains all of the counting problems expressible in a certain logically-defined complexity class, $\# \mathrm{RH} \Pi_{1}$. Typical complete problems include counting the downsets in a partially ordered set [10], computing the partition function of the ferromagnetic Ising model with local external magnetic fields [15], and counting the independent sets in a bipartite graph, which is defined as follows.

Problem: \#BIS.
Instance: A bipartite graph $G$.
Output: The number of independent sets in $G$.
In [10] it was shown that \#BIS is complete for the logically-defined complexity class \#RH ${ }_{1}$ with respect to approximation-preserving reductions. We conjecture [16] that there is no FPRAS for \#BIS.


Figure 1. The tree $J_{3}$.
A problem that is closely related to approximate counting is the problem of sampling configurations almost uniformly at random. The analogue of an FPRAS in the context of sampling problems is the PAUS, or Polynomial Almost Uniform Sampler.

Goldberg, Kelk, and Paterson [18] have studied the problem of sampling $H$-colourings almost uniformly at random. They gave a hardness result for every fixed tree $H$ that is not a star. In particular, their theorem [18, Theorem 2] shows that there is no PAUS for sampling $H$-colourings unless \#BIS has an FPRAS.

In general, there is a close connection between approximate counting and almost-uniform sampling. Indeed, in the presence of a technical condition called "self-reducibility", the counting and sampling variants of two problems are interreducible [26]. The weighted problem \#WHomsTo $(H)$ is self-reducible, so the result of [18] immediately gives an AP-reduction from \#BIS to \#WHomsTo $(H)$ for every tree $H$ that is not a star. However, it is not known whether the unweighted problem \#HomsTo $(H)$ is self-reducible.

As mentioned in Section 1.1] the paper [9] shows how to turn a PAUS for $H$-colourings into an FPRAS for \#HomsTo $(H)$, but it is not known whether there is a reduction in the other direction. Thus, we cannot directly apply the hardness result of [18] to reduce \#BIS to \#HomsTo $(H)$. However, we will see in the next section that the complexity gap between problems with an FPRAS and those that are \#BIS-equivalent still holds for \#HomsTo $(H)$ in the special case when $H$ is a tree, which is the focus of this paper.

## 3. Weighted tree homomorphisms

First, we introduce some notation and a few graphs that are of special interest.
In this paper, the graphs that we consider are undirected and simple - they do not have selfloops or multiple edges between vertices. For every positive integer $n$, let $[n]$ denote $\{1,2, \ldots, n\}$. We use $\Gamma_{H}(v)$ to denote the set of neighbours of vertex $v$ in graph $H$ and we use $d_{H}(v)$ to denote the degree of $v$, which is $\left|\Gamma_{H}(v)\right|$.

Let $P_{n}$ be the $n$-vertex path (with $n-1$ edges). An $n$-leaf star is the complete bipartite graph $K_{1, n}$. Let $J_{q}$ be the graph with vertex set

$$
V\left(J_{q}\right)=\{w\} \cup\left\{c_{i} \mid i \in[q]\right\} \cup\left\{c_{i}^{\prime} \mid i \in[q]\right\},
$$

and edge set

$$
E\left(J_{q}\right)=\left\{\left(c_{i}, c_{i}^{\prime}\right) \mid i \in[q]\right\} \cup\left\{\left(c_{i}^{\prime}, w\right) \mid i \in[q]\right\} .
$$

$J_{3}$ is depicted in Figure 1
3.1. Stars. As Dyer and Greenhill observed [12, Lemma 4.1], \#HomsTo $(H)$ is in FP if $H$ is a complete bipartite graph. We now show that \#WHomsTo $(H)$ is also in FP in this case. Suppose that $H$ is a complete bipartite graph with bipartition $\left(U, U^{\prime}\right)$ where $U=\left\{u_{1}, \ldots, u_{h}\right\}$ and $U^{\prime}=\left\{u_{1}^{\prime}, \ldots, u_{h^{\prime}}^{\prime}\right\}$. Let $G$ be an input to \#WHOMsTo $(H)$ with connected components $G^{1}, \ldots, G^{\kappa}$. Clearly, $Z_{H}(G)=\prod_{i=1}^{\kappa} Z_{H}\left(G^{i}\right)$. Also, if $G^{i}$ is non-bipartite then $Z_{H}\left(G^{i}\right)=0$. Suppose that $G^{i}$ is a connected bipartite graph with bipartition $\left(V, V^{\prime}\right)$ where $V=\left\{v_{1}, \ldots, v_{n}\right\}$ and $V^{\prime}=\left\{v_{1}^{\prime}, \ldots, v_{n^{\prime}}^{\prime}\right\}$. Then

$$
Z_{H}\left(G^{i}\right)=\prod_{j=1}^{n} \sum_{c=1}^{h} w_{v_{j}}\left(u_{c}\right) \prod_{j^{\prime}=1}^{n^{\prime}} \sum_{c^{\prime}=1}^{h^{\prime}} w_{v_{j^{\prime}}^{\prime}}\left(u_{c^{\prime}}^{\prime}\right)+\prod_{j=1}^{n^{\prime}} \sum_{c=1}^{h} w_{v_{j}^{\prime}}\left(u_{c}\right) \prod_{j^{\prime}=1}^{n} \sum_{c^{\prime}=1}^{h^{\prime}} w_{v_{j^{\prime}}}\left(u_{c^{\prime}}^{\prime}\right) .
$$

In the context of this paper, where $H$ is a tree, we can draw the following concluson.
Observation 1. Suppose that $H$ is a star. Then \#WHomsTo $(H)$ is in FP.
3.2. Trees with intermediate complexity. The purpose of this section is to prove Lemma 4 which says that if $H$ is a tree that is not a star and has no induced $J_{3}$ then \#BIS $\equiv_{\text {AP }}$ \#HomsTo $(H)$ and \#BIS $\equiv_{\text {AP }}$ \#WHOMSTo $(H)$. The main work of the section is in the proof of Lemma4, but first we need some existing results. In particular, Lemma 2 below is due to Kelk, and Lemma 3 is an easy consequence of earlier work by the authors and their coauthors on counting CSPs. We have chosen to include a proof sketch of the former because the work of Kelk is unpublished [27] and a proof of the latter because we did not state or prove it explicitly in earlier work, and it might be rather difficult for the reader to see why it is implied by that work.

If $H$ is a tree with no induced $P_{4}$ then it is a star, so, by $\operatorname{Observation~} 1$ \#WHomsTo $(H)$ is in FP. On the other hand, the following lemma shows that if $H$ contains an induced $P_{4}$ then even the unweighted problem \#HomsTo $(H)$ is \#BIS-hard. To motivate the lemma, suppose that $H$ contains an induced $P_{4}$. Then it is a bipartite graph which is not complete, so by Goldberg at al. [18, Theorem 2] the (uniform) sampling problem for $H$-colourings of a graph is as hard as the sampling problem for independent sets in a bipartite graph. This is not quite the result we are seeking, but it is close in spirit, given the close connection between sampling and approximate counting. The following lemma, which is a special case of [27, Lemma 2.19], is exactly what we need.

Lemma 2 (Kelk). Let $H$ be a tree containing an induced $P_{4}$. Then

$$
\text { \#BIS } \leq_{\mathrm{AP}} \# \operatorname{HoMSTo}(H)
$$

Proof. (Proof sketch) We will not give a complete proof of Lemma 2 since it is a special case of a lemma of Kelk, but here is a sketch to give the reader a high-level idea of the construction. Let $\Delta$ be the maximum degree of vertices of $H$ and let $\Delta^{\prime} \leq \Delta$ be the maximum degree taken by a neighbour of a degree- $\Delta$ vertex in $H$. Note that $\Delta^{\prime} \geq 2$ since $H$ cannot be a star. Let $\left(c, c^{\prime}\right)$ be any edge in $H$ with $d_{H}(c)=\Delta$ and $d_{H}\left(c^{\prime}\right)=\Delta^{\prime}$. Let $N_{c}$ be the set $\Gamma_{H}(c)-\left\{c^{\prime}\right\}$ and let $N_{c^{\prime}}=\Gamma_{H}\left(c^{\prime}\right)-\{c\}$. Since $H$ is a tree, there are no edges in $H$ between $N_{c}$ and $N_{c^{\prime}}$. Now consider a connected instance $G$ of \#BIS with bipartition $V(G)=\left(V, V^{\prime}\right)$. Let $G^{\prime}$ be the bipartite graph with vertex set $V(G) \cup\left\{C, C^{\prime}\right\}$ (where $C$ and $C^{\prime}$ are new vertices that are not in $V(G)$ ) and edge set $E(G) \cup\left\{\left(C, C^{\prime}\right)\right\} \cup\{C\} \times V^{\prime} \cup\left\{C^{\prime}\right\} \times V$. Consider an $H$-colouring $\sigma$ of $G$
with $\sigma(C)=c$ and $\sigma\left(C^{\prime}\right)=c^{\prime}$. (Standard constructions can be used to augment $G^{\prime}$ so that almost all homomorphisms to $H$ have this property.) For every vertex $v \in V, \sigma(v) \in N_{c^{\prime}} \cup\{c\}$ and for every vertex $v^{\prime} \in V^{\prime}, \sigma\left(v^{\prime}\right) \in N_{c} \cup\{c\}$. Also, $\left\{v \in V \mid \sigma(v) \in N_{c^{\prime}}\right\} \cup\left\{v^{\prime} \in V^{\prime} \mid \sigma\left(v^{\prime}\right) \in N_{c}\right\}$ is an independent set of $G$. Thus, there is an injection from independent sets of $G$ into these $H$-colourings of $G^{\prime}$. Standard tricks can be used to adjust the construction so that almost all of the homomorphisms correspond to maximum independent sets of $G$ and so that all maximum independent sets correspond to approximately the same number of homomorphisms. The proof follows from the fact that counting maximum independent sets in a bipartite graph is equivalent to \#BIS [10].

As mentioned above, the main result of this section is Lemma 4 which will be presented below. Its proof relies on earlier work on counting constraint satisfaction problems (CSPs). Suppose that $x$ and $x^{\prime}$ are Boolean variables. An assignment $\sigma:\left\{x, x^{\prime}\right\} \rightarrow\{0,1\}$ is said to satisfy the implication constraint $\operatorname{IMP}\left(x, x^{\prime}\right)$ if $\left(\sigma(x), \sigma\left(x^{\prime}\right)\right)$ is in $\{(0,0),(0,1),(1,1)\}$. The idea is that " $\sigma(x)=1$ " implies " $\sigma\left(x^{\prime}\right)=1$ ". The assignment $\sigma$ is said to satisfy the "pinning" constraint $\delta_{0}(x)$ if $\sigma(x)=0$ and the pinning constraint $\delta_{1}(x)$ if $\sigma(x)=1$. If $X$ is a set of Boolean variables then a set $C$ of $\left\{\right.$ IMP, $\left.\delta_{0}, \delta_{1}\right\}$ constraints on $X$ is a set of constraints of the form $\delta_{0}(x)$, $\delta_{1}(x)$ and $\operatorname{IMP}\left(x, x^{\prime}\right)$ for $x$ and $x^{\prime}$ in $X$. The set $S(X, C)$ of satisfying assignments is the set of all assignments $\sigma: X \rightarrow\{0,1\}$ which simultaneously satisfy all of the constraints in $C$. We will consider the following computational problem.

Problem: \#CSP (IMP, $\left.\delta_{0}, \delta_{1}\right)$.
Instance: A set $X$ of Boolean variables and a set $C$ of $\left\{\right.$ IMP, $\left.\delta_{0}, \delta_{1}\right\}$ constraints on $X$.
Output: $|S(X, C)|$.
We will also consider the following weighted version of \#CSP(IMP). Suppose, for each $x \in X$, that $\gamma_{x}:\{0,1\} \rightarrow \mathbb{Q}_{>0}$ is a weight function. For an indexed set $\gamma(X)=\left\{\gamma_{x} \mid x \in X\right\}$ of weight functions, let

$$
Z(X, C, \gamma)=\sum_{\sigma \in S(X, C)} \prod_{x \in X} \gamma_{x}(\sigma(x))
$$

Problem: \#CSP ${ }^{*}\left(\operatorname{IMP}, \delta_{0}, \delta_{1}\right)$.
Instance: A set $X$ of Boolean variables, a set $C$ of $\left\{\right.$ IMP, $\left.\delta_{0}, \delta_{1}\right\}$ constraints on $X$, and an indexed set $\gamma(X)$ of weight functions.
Output: $Z(X, C, \gamma)$.
We will use the following lemma, which follows from earlier work on counting CSPs.
Lemma 3. \# $\operatorname{CSP}^{*}\left(\operatorname{IMP}, \delta_{0}, \delta_{1}\right) \equiv{ }_{\mathrm{AP}}$ \#BIS.
Proof. Dyer, Goldberg, and Jerrum [11, Theorem 3] shows that \#CSP(IMP, $\left.\delta_{0}, \delta_{1}\right) \equiv_{\text {AP }}$ \#BIS. \#CSP (IMP, $\left.\delta_{0}, \delta_{1}\right)$ trivially reduces to \#CSP ${ }^{*}\left(\operatorname{IMP}, \delta_{0}, \delta_{1}\right)$ since it is a special case. Thus, it suffices to give an AP-reduction from \#CSP* $\left(\operatorname{IMP}, \delta_{0}, \delta_{1}\right)$ to \#CSP (IMP, $\left.\delta_{0}, \delta_{1}\right)$. The idea behind the construction that we use comes from Bulatov et al. [5, Lemma 36, Item (i)]. We give the details in order to translate the construction into the current context.

Let $(X, C, \gamma)$ be an instance of \#CSP ${ }^{*}\left(\mathrm{IMP}, \delta_{0}, \delta_{1}\right)$. We can assume without loss of generality that all of the weights $\gamma_{x}(b)$ are positive integers by multiplying all of the weights by the product
of the denominators. The construction that follows is not difficult but the details are a little bit complicated, so we use the following running example to illustrate. Let $X=\{y, z\}, C=$ $\operatorname{IMP}(y, z), \gamma_{y}(0)=5, \gamma_{y}(1)=2, \gamma_{z}(0)=1$ and $\gamma_{z}(1)=1$.

For every variable $x \in X$, consider the weight function $\gamma_{x}$. Let $k_{x}=\max \left(\left\lceil\lg \gamma_{x}(0)\right\rceil,\left\lceil\lg \gamma_{x}(1)\right\rceil\right)$. For every $b \in\{0,1\}$, write the bit-expansion of $\gamma_{x}(1 \oplus b)$ as

$$
\gamma_{x}(1 \oplus b)=a_{x, b, 0}+a_{x, b, 1} 2^{1}+\cdots+a_{x, b, k_{x}} 2^{k_{x}}
$$

where each $a_{x, b, i} \in\{0,1\}$. Note that $\gamma_{x}(1 \oplus b)>0$ so there is at least one $i$ with $a_{x, b, i}=1$. Let $\min _{x, b}=\min \left\{i \mid a_{x, b, i}=1\right\}$ and $\max _{x, b}=\max \left\{i \mid a_{x, b, i}=1\right\}$. If $i<\max _{x, b}$ and $a_{x, b, i}=1$ then let $\operatorname{next}_{x, b, i}=\min \left\{j>i \mid a_{x, b, j}=1\right\}$. If $i>\min _{x, b}$ and $a_{x, b, i}=1$ then let $\operatorname{prev}_{x, b, i}=\max \left\{j<i \mid a_{x, b, j}=1\right\}$. For the running example,

- $k_{y}=\lceil\lg 5\rceil=3$ and $k_{z}=\lceil\lg 1\rceil=0$.
- For the variable $y$, taking $b=0$ we have $\gamma_{y}(1 \oplus 0)=2^{1}$ so $a_{y, 0,0}=0, a_{y, 0,1}=1$, and $a_{y, 0,2}=a_{y, 0,3}=0$. Also, $\min _{y, 0}=1=\max _{y, 0}$.
- Similarly, taking $b=1$ gives $\gamma_{y}(1 \oplus 1)=2^{0}+2^{2}$ so $a_{y, 1,0}=1, a_{y, 1,1}=0, a_{y, 1,2}=1$ and $a_{y, 1,3}=0$. Thus $\min _{y, 1}=0$ and $\max _{y, 1}=2$. Then next ${ }_{y, 1,0}=2$ and $\operatorname{prev}_{y, 1,2}=0$.
- Finally, for the variable $z$ and $b \in\{0,1\}$, we have $\gamma_{z}(1 \oplus b)=2^{0}$ so $a_{z, b, 0}=1$ and $\min _{z, b}=0=\max _{z, b}$.
Now for every $x \in X$, for every $i \in\left\{1, \ldots, k_{x}\right\}$ and every $b \in\{0,1\}$ with $a_{x, b, i}=1$ let $A_{x, b, i}$ be the set of $i+2$ variables $\left\{x_{b, i, 1}, \ldots, x_{b, i, i}\right\} \cup\left\{L_{x, b, i}, R_{x, b, i}\right\}$. Let $C_{x, b, i}$ be the set of implication constraints $\bigcup_{j \in[i]}\left\{\operatorname{IMP}\left(L_{x, b, i}, x_{b, i, j}\right), \operatorname{IMP}\left(x_{b, i, j}, R_{x, b, i}\right)\right\}$. Note that there are $2^{i}+2$ satisfying assignments to the \#CSP instance $\left(A_{x, b, i}, C_{x, b, i}\right)$ : one with $\sigma\left(L_{x, b, i}\right)=\sigma\left(R_{x, b, i}\right)=0$, one with $\sigma\left(L_{x, b, i}\right)=\sigma\left(R_{x, b, i}\right)=1$, and $2^{i}$ with $\sigma\left(L_{x, b, i}\right)=0$ and $\sigma\left(R_{x, b, i}\right)=1$. The point here is that the sets $A_{x, b, i}$ will be combined for different values of $i$. The satisfying assignments with $\sigma\left(L_{x, b, i}\right)=\sigma\left(R_{x, b, i}\right)=0$ will correspond to contributions from a different index $i^{\prime}>i$ and the satisfying assignments with $\sigma\left(L_{x, b, i}\right)=\sigma\left(R_{x, b, i}\right)=1$ will correspond to contributions from a different index $i^{\prime}<i$. There are exactly $2^{i}$ satisfying assignments with $\sigma\left(L_{x, b, i}\right)=0$ and $\sigma\left(R_{x, b, i}\right)=1$ and these will correspond to the $a_{x, b, i} 2^{i}$ summand in the bit-expansion of $\gamma_{x}(1 \oplus b)$. For the running example,
- for the variable $y$ and for $b=0$ and $i=1$ we have $A_{y, 0,1}=\left\{y_{0,1,1}\right\} \cup\left\{L_{y, 0,1}, R_{y, 0,1}\right\}$. Then $C_{y, 0,1}$ contains $\left\{\operatorname{IMP}\left(L_{y, 0,1}, y_{0,1,1}\right), \operatorname{IMP}\left(y_{0,1,1}, R_{y, 0,1}\right)\right\}$ and there are $2+2^{1}=4$ solutions.
- For the variable $y$ and for $b=1$ and $i=2$ we have $A_{y, 1,2}=\left\{y_{1,2,1}, y_{1,2,2}\right\} \cup\left\{L_{y, 1,2}, R_{y, 1,2}\right\}$. Then $C_{y, 1,2}$ contains the constraints $\operatorname{IMP}\left(L_{y, 1,2}, y_{1,2,1}\right), \operatorname{IMP}\left(y_{1,2,1}, R_{y, 1,2}\right), \operatorname{IMP}\left(L_{y, 1,2}, y_{1,2,2}\right)$, and $\operatorname{IMP}\left(y_{1,2,2}, R_{y, 1,2}\right)$ and there are $2+2^{2}=6$ solutions.
We now add some constraints corresponding to the $i=0$ case above. For every $x \in X$ and every $b \in\{0,1\}$ with $a_{x, b, 0}=1$ let $A_{x, b, 0}$ be the set of variables $\left\{L_{x, b, 0}, R_{x, b, 0}\right\}$. Let $C_{x, b, 0}$ be the set containing the constraint $\operatorname{IMP}\left(L_{x, b, 0}, R_{x, b, 0}\right)$. Note that there are $2^{0}+2=3$ satisfying assignments to the \#CSP instance $\left(A_{x, b, 0}, C_{x, b, 0}\right)$ : one with $\sigma\left(L_{x, b, 0}\right)=\sigma\left(R_{x, b, 0}\right)=0$, one with $\sigma\left(L_{x, b, 0}\right)=\sigma\left(R_{x, b, 0}\right)=1$, and $2^{0}=1$ with $\sigma\left(L_{x, b, 0}\right)=0$ and $\sigma\left(R_{x, b, 0}\right)=1$. For the running example,
- $A_{y, 1,0}=\left\{L_{y, 1,0}, R_{y, 1,0}\right\}$ and $C_{y, 1,0}=\left\{\operatorname{IMP}\left(L_{y, 1,0}, R_{y, 1,0}\right)\right\}$.
- For $b \in\{0,1\}, A_{z, b, 0}=\left\{L_{z, b, 0}, R_{z, b, 0}\right\}$ and $C_{z, b, 0}=\left\{\operatorname{IMP}\left(L_{z, b, 0}, R_{z, b, 0}\right)\right\}$.

Now for every $x \in X$ and $b \in\{0,1\}$ let $C_{x, b}^{\prime}$ be the set of constraints forcing equality of $\sigma\left(R_{x, b, i}\right)$ and $\sigma\left(L_{x, b, j}\right)$ when $i$ and $j$ are adjacent one-bits in the bit-expansion of $\gamma_{x}(1 \oplus b)$. In particular,

$$
C_{x, b}^{\prime}=\bigcup_{\text {next }_{x, b, i}=j, \operatorname{prev}_{x, b, j}=i}\left\{\operatorname{IMP}\left(R_{x, b, i}, L_{x, b, j}\right), \operatorname{IMP}\left(L_{x, b, j}, R_{x, b, i}\right)\right\}
$$

For the running example,

- $C_{y, 0}^{\prime}=C_{z, 0}^{\prime}=C_{z, 1}^{\prime}=\emptyset$ since these variables have only one positive coefficient in the bit expansion.
- For the variable $y$ and $b=1$ the relevant non-zero coefficients are $i=0$ and $j=2$ so we get

$$
C_{y, 1}^{\prime}=\left\{\operatorname{IMP}\left(R_{y, 1,0}, L_{y, 1,2}\right), \operatorname{IMP}\left(L_{y, 1,2}, R_{y, 1,0}\right)\right\}
$$

Now consider $x \in X$. Let $C_{x, 0}^{\prime \prime}=C_{x, 0}^{\prime} \cup\left\{\delta_{0}\left(L_{x, 0, \min _{x, 0}}\right)\right\}$ and let $C_{x, 1}^{\prime \prime}=C_{x, 1}^{\prime} \cup\left\{\delta_{1}\left(R_{x, 1, \text { max }_{x, 1}}\right)\right\}$. For $x \in X$ and $b \in\{0,1\}$ let

$$
A_{x, b}=\bigcup_{i \in\left\{0, \ldots, k_{x}\right\}, a_{x, b, i}=1} A_{x, b, i}
$$

and let

$$
C_{x, b}=C_{x, b}^{\prime \prime} \cup \bigcup_{i \in\left\{0, \ldots, k_{x}\right\}, a_{x, b, i}=1} C_{x, b, i}
$$

Now will show that there are $\gamma_{x}(1)$ satisfying assignments to the \#CSP instance ( $A_{x, 0}, C_{x, 0}$ ) which have the property that $\sigma\left(R_{x, 0, \max _{x, 0}}\right)=1$ and one satisfying assignment in which $\sigma\left(R_{x, 0, \max _{x, 0}}\right)=$ 0 . To see this, note that the constraint $\delta_{0}\left(L_{x, 0, \min _{x, 0}}\right)$ forces $\sigma\left(L_{x, 0, \min _{x, 0}}\right)=0$. If $\sigma\left(R_{x, 0, \max _{x, 0}}\right)=$ 0 then all of the variables in $A_{x, 0}$ are assigned spin 0 by $\sigma$. Otherwise, there is exactly one $i$ with $a_{x, 0, i}=1$ and $\sigma\left(L_{x, 0, i}\right)=0$ and $\sigma\left(R_{x, 0, i}\right)=1$. As we noted above, there are $2^{i}$ assignments to the variables in $A_{x, b, i}$. But $\sum_{i: a_{x, 0,1}=i} 2^{i}=\gamma_{x}(1)$, as required. Similarly, there are $\gamma_{x}(0)$ satisfying assignments to the \#CSP instance $\left(A_{x, 1}, C_{x, 1}\right)$ in which $\sigma\left(L_{x, 1, \min _{x, 1}}\right)=0$ and there is one satisfying assignment in which $\sigma\left(L_{x, 1, \min _{x, 1}}\right)=1$. Let us quickly apply this to the running example.

- Taking variable $y$ and $b=0$ we have $A_{y, 0}=A_{y, 0,1}$ and $C_{y, 0}^{\prime \prime}=\left\{\delta_{0}\left(L_{y, 0,1}\right)\right\} \cup C_{y, 0,1}$. Then $\max _{y, 0}=1$. From above, there is one solution $\sigma$ with $\sigma\left(R_{y, 0, \max _{y, 0}}\right)=0$ and there are $2^{1}=\gamma_{y}(1)$ solutions $\sigma$ with $\sigma\left(R_{y, 0, \max _{y, 0}}\right)=1$.
- Taking variable $y$ and $b=1$ we have

$$
A_{y, 1}=A_{y, 1,0} \cup A_{y, 1,2}
$$

and

$$
C_{y, 1}^{\prime \prime}=\left\{\delta_{1}\left(R_{y, 1,2}\right), \operatorname{IMP}\left(R_{y, 1,0}, L_{y, 1,2}\right), \operatorname{IMP}\left(L_{y, 1,2}, R_{y, 1,0}\right)\right\} \cup C_{y, 1,0} \cup C_{y, 1,2}
$$

There is one solution $\sigma$ with $\sigma\left(L_{y, 1,0}\right)=1$. There are $2^{0}+2^{2}=\gamma_{y}(0)$ solutions $\sigma$ with $\sigma\left(L_{y, 1,0}\right)=0$.

- Taking variable $z$ we have $A_{z, b}=A_{z, b, 0}=\left\{L_{z, b, 0}, R_{z, b, 0}\right\}$. Then, taking $b=0, C_{z, 0}=$ $\left\{\delta_{0}\left(L_{z, 0,0}\right), \operatorname{IMP}\left(L_{z, 0,0}, R_{z, 0,0}\right)\right\}$. so there is $2^{0}=1=\gamma_{z}(1)$ assignment with $\sigma\left(R_{z, 0,0}\right)=$ 1 and one with $\sigma\left(R_{z, 0,0}\right)=0$. Taking $b=1, C_{z, 1}=\left\{\delta_{1}\left(R_{z, 1,0}\right), \operatorname{IMP}\left(L_{z, 1,0}, R_{z, 1,0}\right)\right\}$ so there is $2^{0}=1=\gamma_{z}(0)$ assignment with $\sigma\left(L_{z, 1,0}\right)=0$ and one with $\sigma\left(L_{z, 1,0}\right)=1$.
Finally, consider $x \in X$. Let $C_{x}$ be the set of constraints containing the four implications $\operatorname{IMP}\left(x, R_{x, 0, \max _{x, 0}}\right), \operatorname{IMP}\left(R_{x, 0, \max _{x, 0}}, x\right), \operatorname{IMP}\left(x, L_{x, 1, \min _{x, 1}}\right)$, and $\operatorname{IMP}\left(L_{x, 1, \min _{x, 1}}, x\right)$. Now there are $\gamma_{x}(1)$ solutions to $\left(A_{x, 0} \cup A_{x, 1} \cup\{x\}, C_{x, 0} \cup C_{x, 1} \cup C_{x}\right)$ with $\sigma(x)=1$ and $\gamma_{x}(0)$ solutions with $\sigma(x)=0$. Thus, we have simulated the weight function $w_{x}$ with $\left\{\right.$ IMP, $\left.\delta_{0}, \delta_{1}\right\}$ constraints. For the running example,
- first consider the variable $y$.
- With $\sigma(y)=1$ the constraints in $C_{y}$ force $\sigma\left(R_{y, 0, \max _{y, 0}}\right)=1$ which, from above, gives $\gamma_{y}(1)$ solutions to $\left(A_{y, 0}, C_{y, 0}\right)$. The constraints in $C_{y}$ also force $\sigma\left(L_{y, 1, \min (y, 1)}\right)=$ 1, which, from above, gives one solution to $\left(A_{y, 1}, C_{y, 1}\right)$.
- With $\sigma(y)=0$ the constraints in $C_{y}$ force $\sigma\left(R_{y, 0, \max _{y, 0}}\right)=0$ so there is only one solution to $\left(A_{y, 0}, C_{y, 0}\right)$. The constraints in $C_{y}$ also force $\sigma\left(L_{y, 1, \min (y, 1)}\right)=0$ so there are $\gamma_{y}(0)$ solutions to $\left(A_{y, 1}, C_{y, 1}\right)$.
- The argument for variable $z$ is similar.

Thus, the correct output for the \#CSP ${ }^{*}\left(\operatorname{IMP}, \delta_{0}, \delta_{1}\right)$ instance $(X, C, \gamma)$ is same as the correct output for the \#CSP (IMP, $\delta_{0}, \delta_{1}$ ) instance obtained from $(X, C, \gamma)$ by adding new variables and constraints to simulate each weight function $\gamma_{x}$.

We can now prove the main lemma of this section.
Lemma 4. Suppose that $H$ is a tree which is not a star and which has no induced $J_{3}$. Then

$$
\text { \#BIS } \equiv_{\mathrm{AP}} \text { \#HomsTo }(H) \text { and \#BIS } \equiv_{\mathrm{AP}} \text { \#WHOMSTo }(H) .
$$

Proof. \#HomsTo $(H)$ is a special case of \#WHomsTo $(H)$ so it is certainly AP-reducible to \#WHomsTo $(H)$. By Lemma2 \#BIS is AP-reducible to \#HomsTo $(H)$ and therefore it is APreducible to \#WHomsTo $(H)$. So it suffices to give an AP-reduction from \#WHomsTo $(H)$ to \#BIS. Applying Lemma 3, it suffices to give an AP-reduction from \#WHomsTo $(H)$ to \#CSP ${ }^{*}\left(\mathrm{IMP}, \delta_{0}, \delta_{1}\right)$.

In order to do the reduction, we will order the vertices of $H$ using the fact that it has no induced $J_{3}$. (This ordering is similar the one arising from the "crossing property" of the authors that is mentioned in [27, Section 7.3.3].) A "convex ordering" of a connected bipartite graph with bipartition $\left(U, U^{\prime}\right)$ with $|U|=h$ and $\left|U^{\prime}\right|=h^{\prime}$ and edge set $E \subseteq U \times U^{\prime}$ is a pair of bijections $\pi: U \rightarrow[h]$ and $\pi^{\prime}: U^{\prime} \rightarrow\left[h^{\prime}\right]$ such that there are monotonically non-decreasing functions functions $m:[h] \rightarrow\left[h^{\prime}\right], M:[h] \rightarrow\left[h^{\prime}\right], m^{\prime}:\left[h^{\prime}\right] \rightarrow[h]$ and $M^{\prime}:\left[h^{\prime}\right] \rightarrow[h]$ satisfying the following conditions.

- If $\pi(u)=i$ then $\left\{\pi^{\prime}\left(u^{\prime}\right) \mid\left(u, u^{\prime}\right) \in E\right\}=\left\{\ell \in\left[h^{\prime}\right] \mid m(i) \leq \ell \leq M(i)\right\}$.
- If $\pi^{\prime}\left(u^{\prime}\right)=i$ then $\left\{\pi(u) \mid\left(u, u^{\prime}\right) \in E\right\}=\left\{\ell \in[h] \mid m^{\prime}(i) \leq \ell \leq M^{\prime}(i)\right\}$.

The purpose of $\pi$ and $\pi^{\prime}$ is just to put the vertices in the correct order. For example, in Figure 2 , $\pi$ is the identity map on the set $U=\{1,2,3,4\}$ and $\pi^{\prime}$ is the identity map on the set $U^{\prime}=$ $\{1,2,3\}$. Vertex 3 in $U$ is connected to the sequence containing vertices 1,2 and 3 in $U^{\prime}$, so


Figure 2. An example of a convex ordering
$m(3)=1$ and $M(3)=3$. Every other vertex in $U$ has degree 1 and in particular $m(1)=$ $M(1)=1, m(2)=M(2)=1$ and $m(4)=M(4)=3$. Similarly, vertex 1 in $U^{\prime}$ is attached to the sequence containing vertices 1,2 and 3 in $U$ so $m^{\prime}(1)=1$ and $M^{\prime}(1)=3$ but $m^{\prime}(2)=$ $M^{\prime}(2)=3$ and $m^{\prime}(3)=M^{\prime}(3)=4$.

To see that a convex ordering of $H$ always exists, consider the following algorithm. The input is a tree $H$ with no induced $J_{3}$, a bipartition $\left(U, U^{\prime}\right)$ of the vertices of $H$, and a distinguished leaf $u \in U$ whose parent $u^{\prime}$ is adjacent to at most one non-leaf. (Note that such a leaf $u$ always exists since $H$ is a tree.) The output is a convex ordering of $H$ in which $\pi(u)=h$ and $\pi^{\prime}\left(u^{\prime}\right)=h^{\prime}$. Here is what the algorithm does. If all of the neighbours of $u^{\prime}$ are leaves, then $h^{\prime}=1$ so take any bijection $\pi$ from $U-\{u\}$ to $[h-1]$ and set $\pi(u)=h$ and $\pi^{\prime}\left(u^{\prime}\right)=h^{\prime}$. Return this output. Otherwise, let $u^{\prime \prime}$ be the neighbour of $u^{\prime}$ that is not a leaf. Let $H^{\prime}$ be the graph formed from $H$ by removing all of the $d_{H}\left(u^{\prime}\right)-1$ neighbours of $u^{\prime}$ other than $u^{\prime \prime}$. Since $H$ has no induced $J_{3}$, the graph $H^{\prime}$ has the following property: $u^{\prime}$ is a leaf whose parent, $u^{\prime \prime}$, is adjacent to at most one non-leaf. Recursively, construct a convex ordering for $H^{\prime}$ in which $\pi\left(u^{\prime}\right)=h^{\prime}$ and $\pi\left(u^{\prime \prime}\right)=h-\left(d_{H}\left(u^{\prime}\right)-1\right)$. Extend $\pi$ by assigning values to the leaf-neighbours of $u^{\prime}$, ensuring that $\pi(u)=h$.

We will now show how to reduce \#WHomsTo $(H)$ to \#CSP* $\left(\operatorname{IMP}, \delta_{0}, \delta_{1}\right)$. Let $G$ be a connected bipartite graph with bipartition $\left(V, V^{\prime}\right)$ and let $W(G, H)$ be an indexed set of weight functions. Let

$$
Z_{H}^{\prime}(G, W(G, H))=\sum_{\sigma \in \operatorname{Hom}(G, H) \text { with } \sigma(V) \subseteq U} \prod_{v \in V(G)} w_{v}(\sigma(v))
$$

and let

$$
Z_{H}^{\prime \prime}(G, W(G, H))=\sum_{\sigma \in \operatorname{Hom}(G, H) \text { with } \sigma(V) \subseteq U^{\prime}} \prod_{v \in V(G)} w_{v}(\sigma(v)) .
$$

Clearly, $Z_{H}(G, W(G, H))=Z_{H}^{\prime}(G, W(G, H))+Z_{H}^{\prime \prime}(G, W(G, H))$. We will show how to reduce the computation of $Z_{H}^{\prime}(G, W(G, H))$, given the input $(G, W(G, H))$, to the problem \# $\mathrm{CSP}^{*}\left(\mathrm{IMP}, \delta_{0}, \delta_{1}\right)$. In the same way, we can reduce the computation of $Z_{H}^{\prime \prime}(G, W(G, H))$ to \#CSP* ${ }^{*}\left(\mathrm{IMP}, \delta_{0}, \delta_{1}\right)$.

Since we are considering assignments which map $V$ to $U$ and $V^{\prime}$ to $U^{\prime}$, the vertices in $U$ will not get mixed up with the vertices in $U^{\prime}$. We can simplify the notation by relabelling the vertices so that $\pi$ and $\pi^{\prime}$ are the identity permutations. Then, given the convex ordering property, we can assume that $U=[h]$ and that $U^{\prime}=\left[h^{\prime}\right]$ and that we have monotonically non-decreasing functions functions $m:[h] \rightarrow\left[h^{\prime}\right], M:[h] \rightarrow\left[h^{\prime}\right], m^{\prime}:\left[h^{\prime}\right] \rightarrow[h]$ and $M^{\prime}:\left[h^{\prime}\right] \rightarrow[h]$ such that

- for $i \in U, \Gamma_{H}(i)=\left\{\ell \in\left[h^{\prime}\right] \mid m(i) \leq \ell \leq M(i)\right\}$, and
- for $i \in U^{\prime}, \Gamma_{H}(i)=\left\{\ell \in[h] \mid m^{\prime}(i) \leq \ell \leq M^{\prime}(i)\right\}$.

A configuration $\sigma$ contributing to $Z_{H}^{\prime}(G, W(G, H))$ is a map from $V$ to $[h]$ together with a map from $V^{\prime}$ to $\left[h^{\prime}\right]$ such that the following is true for every edge $\left(v, v^{\prime}\right) \in V \times V^{\prime}$.
(1) $m(\sigma(v)) \leq \sigma\left(v^{\prime}\right) \leq M(\sigma(v))$, and
(2) $m^{\prime}\left(\sigma\left(v^{\prime}\right)\right) \leq \sigma(v) \leq M^{\prime}\left(\sigma\left(v^{\prime}\right)\right)$.

Since $m, M, m^{\prime}$ and $M^{\prime}$ are monotonically non-decreasing, we can re-write the conditions in a less natural way which will be straightforward to apply below.
(1') $\sigma(v) \leq i$ implies $\sigma\left(v^{\prime}\right) \leq M(i)$,
(2') $\sigma\left(v^{\prime}\right) \leq i^{\prime}$ implies $\sigma(v) \leq M^{\prime}\left(i^{\prime}\right)$,
(3') $\sigma\left(v^{\prime}\right) \leq m(i)-1$ implies $\sigma(v) \leq i-1$, and
(4) $\sigma(v) \leq m^{\prime}\left(i^{\prime}\right)-1$ implies $\sigma\left(v^{\prime}\right) \leq i^{\prime}-1$.

Using monotonicity, (1) and (21) follow from the right-hand side of (11) and (2). Suppose that $\sigma\left(v^{\prime}\right)<m(i)$. Then the left-hand side of (1) gives $m(\sigma(v))<m(i)$, so by monotonicity, $\sigma(v)<i$. Equation (3) follows. In the same way, Equation (4) follows from the left-hand side of (2). Going the other direction, the right-hand sides of (11) and (2) follow from (1) and (2).To derive the left-hand side of (1), take the contrapositive of (3), which says $\sigma(v) \geq i$ implies $\sigma\left(v^{\prime}\right) \geq m(i)$ then plug in $i=\sigma(v)$. The derivation of the left-hand side of (2) is similar.

We now construct an instance of $\# \operatorname{CSP}^{*}\left(\operatorname{IMP}, \delta_{0}, \delta_{1}\right)$. For each vertex $v \in V$ introduce Boolean variables $v_{0}, \ldots, v_{h}$. Introduce constraints $\delta_{0}\left(v_{0}\right)$ and $\delta_{1}\left(v_{h}\right)$ and, for every $i \in[h]$, $\operatorname{IMP}\left(v_{i-1}, v_{i}\right)$. For each vertex $v^{\prime} \in V^{\prime}$ introduces Boolean variables $v_{0}^{\prime}, \ldots, v_{h^{\prime}}^{\prime}$. Introduce constraints $\delta_{0}\left(v_{0}^{\prime}\right)$ and $\delta_{1}\left(v_{h^{\prime}}^{\prime}\right)$ and, for every $i^{\prime} \in\left[h^{\prime}\right], \operatorname{IMP}\left(v_{i^{\prime}-1}^{\prime}, v_{i^{\prime}}^{\prime}\right)$.

Now there is a one-to-one correspondence between assignments $\sigma$ mapping $V$ to $U$ and $V^{\prime}$ to $U^{\prime}$, and assignments $\tau$ to the Boolean variables that satisfy the above constraints. In particular, $\sigma(v)=\min \left\{i \mid \tau\left(v_{i}\right)=1\right\}$. Similarly, $\sigma\left(v^{\prime}\right)=\min \left\{i^{\prime} \mid \tau\left(v_{i}^{\prime}\right)=1\right\}$.

Now, $\sigma(v) \leq i$ is exactly equivalent to $\tau\left(v_{i}\right)=1$. Thus, we can add the following further constraints to rule out assignments $\sigma$ that do not satisfy (1), (2), (3) and (4). Add all of the following constraints where $v \in V, v^{\prime} \in V^{\prime}, i \in[h]$ and $i^{\prime} \in\left[h^{\prime}\right]: \operatorname{IMP}\left(v_{i}, v_{M(i)}^{\prime}\right)$, $\operatorname{IMP}\left(v_{i^{\prime}}^{\prime}, v_{M^{\prime}\left(i^{\prime}\right)}\right)$, $\operatorname{IMP}\left(v_{m(i)-1}^{\prime}, v_{i-1}\right)$, and $\operatorname{IMP}\left(v_{m^{\prime}\left(i^{\prime}\right)-1}, v_{i^{\prime}-1}^{\prime}\right)$. Now the assignments $\tau$ of Boolean values to the variables satisfy all of the constraints if and only if they correspond to assignments $\sigma$ which satisfy (1), (2), (3) (4), and so should contribute to

$$
Z_{H}^{\prime}(G, W(G, H))=\sum_{\sigma \in \operatorname{Hom}(G, H) \text { with } \sigma(V) \subseteq U} \prod_{v \in V(G)} w_{v}(\sigma(v)) .
$$

We will next construct weight functions for the instance of \#CSP ${ }^{*}\left(\operatorname{IMP}, \delta_{0}, \delta_{1}\right)$ in order to reproduce the effect of the weight functions in $W(G, H)$.

In order to avoid division by 0 , we first modify the construction. Suppose that for some variable $v \in V$ and some $i \in[h], w_{v}(i)=0$. Configurations $\sigma$ with $\sigma(v)=i$ make no contribution to $Z_{H}^{\prime}(G, W(G, H))$. Thus, it does no harm to rule out such configurations by modifying the \# $\mathrm{CSP}^{*}\left(\mathrm{IMP}, \delta_{0}, \delta_{1}\right)$ instance to ensure that $\tau\left(v_{i}\right)=1$ implies $\tau\left(v_{i-1}\right)=1$. We do this by adding the constraint $\operatorname{IMP}\left(v_{i}, v_{i-1}\right)$. Similarly, if $w_{v^{\prime}}\left(i^{\prime}\right)=0$ for $v^{\prime} \in V$ and $i^{\prime} \in\left[h^{\prime}\right]$ then we add the constraint $\operatorname{IMP}\left(v_{i^{\prime}}^{\prime}, v_{i^{\prime}-1}^{\prime}\right)$.

Once we've made this change, we can replace $W(G, H)$ with an equivalent indexed set of weight functions $W^{\prime}(G, H)$ where $w_{v}^{\prime}(i)=w_{v}(i)$ if $w_{v}(i)>0$ and $w_{v}^{\prime}(i)=1$, otherwise.

The weight functions for the $\# \operatorname{CSP}^{*}\left(\right.$ IMP $\left., \delta_{0}, \delta_{1}\right)$ instance are then constructed as follows, for each $v \in V$. For each $i \in[h]$, let $\gamma_{v_{i-1}}(0)=1$. Let $\gamma_{v_{h}}(1)=w_{v}^{\prime}(h)$. For each $i \in[h-1]$, let $\gamma_{v_{i}}(1)=w_{v}^{\prime}(i) / w_{v}^{\prime}(i+1)$. Note that $\gamma_{v_{h}}(0)$ and $\gamma_{v_{0}}(1)$ have not yet been defined - these values can be chosen arbitrarily. They will not be relevant given the constraints $\delta_{0}\left(v_{0}\right)$ and $\delta_{1}\left(v_{h}\right)$.

Now if $\sigma(v)=i$ we have $\tau\left(v_{0}\right)=\cdots=\tau\left(v_{i-1}\right)=0$ and $\tau\left(v_{i}\right)=\cdots=\tau\left(v_{h}\right)=1$ so $\prod_{j} \gamma_{v_{j}}\left(\tau\left(v_{j}\right)\right)=w_{v}^{\prime}(i)$, as required. Similarly, for each $v^{\prime} \in V^{\prime}$, define the weight functions as follows. For each $i \in\left[h^{\prime}\right]$, let $\gamma_{v_{i-1}^{\prime}}(0)=1$. Let $\gamma_{v_{h^{\prime}}^{\prime}}(1)=w_{v^{\prime}}^{\prime}\left(h^{\prime}\right)$. For each $i \in\left[h^{\prime}-1\right]$, let $\gamma_{v_{i}^{\prime}}(1)=w_{v^{\prime}}^{\prime}(i) / w_{v^{\prime}}^{\prime}(i+1)$. Using these weight functions, we obtain the desired reduction from the computation of $Z_{H}^{\prime}(G, W(G, H))$ to \#CSP ${ }^{*}$ (IMP, $\left.\delta_{0}, \delta_{1}\right)$.
3.3. Intractable trees. Lemma 4 shows that if $H$ has no induced $J_{3}$ then \#WHomsTo $(H)$ is AP-reducible to \#BIS. The purpose of this section is to prove Lemma 6, below, which shows, by contrast, that if $H$ does have an induced $J_{3}$, then \#WHomsTo $(H)$ is \#SAT-hard.

In order to prepare for the proof of Lemma 6, we introduce the notion of a multiterminal cut. Given a graph $G=(V, E)$ with distinguished vertices $\alpha, \beta$ and $\gamma$, which we refer to as "terminals", a multiterminal cut is a set $E^{\prime} \subseteq E$ whose removal disconnects the terminals in the sense that the graph $\left(V, E \backslash E^{\prime}\right)$ does not contain a path between any two distinct terminals. The size of the multiterminal cut is the number of edges in $E^{\prime}$. Consider the following computational problem.

## Problem: \#MultiterminalCut(3).

Instance: A positive integer $b$, a connected graph $G=(V, E)$ and 3 distinct vertices $\alpha, \beta$ and $\gamma$ from $V$. The input has the property that every multiterminal cut has size at least $b$.
Output: The number of size- $b$ multiterminal cuts for $G$ with terminals $\alpha$, $\beta$, and $\gamma$.
We will use the following technical lemma, which we used before in [15] (without stating it formally).
Lemma 5. \#MultiterminalCut $(3) \equiv_{\text {AP }}$ \#Sat.
Proof. This follows essentially from the proof of Dalhaus et al. [8] that the decision version of \#MultiterminalCut(3) is NP-hard and from the fact [10, Theorem 1] that the NP-hardness of a decision problem implies that the corresponding counting problem is AP-interreducible with \#Sat. The details are given in [15, Section 4].

Lemma 6. Suppose that $H$ is a tree with an induced $J_{3}$. Then


Figure 3. The tree $J$.
Proof. We will prove the lemma by giving an AP-reduction from \#MultiterminalCut(3) to \#WHomsTo $(H)$. The lemma will then follow from Lemma 5 .

Suppose that $H$ has an induced subgraph which is isomorphic to $J_{3}$. To simplify the notation, label the vertices and edges of $H$ in such a way that the induced subgraph is (identically) the graph $J$ depicted in Figure 3 ,

Let $b, G=(V, E), \alpha, \beta$ and $\gamma$ be an input to \#MultiterminalCut(3). Let $s=2+$ $|E(G)|+2|V(G)|$. (The exact size of $s$ is not important, but it has to be at least this big to make the calculation work, and it has to be at most a polynomial in the size of $G$.) Let $G^{\prime}$ be the graph defined as follows. First, let $V^{\prime}(G)=\{(e, i) \mid e \in E, i \in[s]\}$. Thus, $V^{\prime}(G)$ contains $s$ vertices for each edge $e$ of $G$. Then let $G^{\prime}$ be the graph with vertex set $V\left(G^{\prime}\right)=V(G) \cup V^{\prime}(G)$ and edge set

$$
E\left(G^{\prime}\right)=\left\{(u,(e, i)) \mid u \in V(G),(e, i) \in V^{\prime}(G), \text { and } u \text { is an endpoint of } e\right\} .
$$

We will define weight functions $w_{v}$ for $v \in V\left(G^{\prime}\right)$ so that an approximation to the number of size- $b$ multi-terminal cuts for $G$ with terminals $\alpha, \beta$ and $\gamma$ can be obtained from an approximation to $Z_{H}\left(G^{\prime}, W\left(G^{\prime}, H\right)\right)$. We start by defining the set of pairs $(v, c) \in V\left(G^{\prime}\right) \times V(H)$ for which we will specify $w_{v}(c)>0$. In particular, define the set $\Omega$ as follows.

$$
\Omega=\left\{\left(\alpha, x_{0}\right),\left(\beta, y_{0}\right),\left(\gamma, z_{0}\right)\right\} \cup\left((V(G)-\{\alpha, \beta, \gamma\}) \times\left\{x_{0}, y_{0}, z_{0}\right\}\right) \cup\left(V^{\prime}(G) \times\left\{w, x_{1}, y_{1}, z_{1}\right\}\right)
$$

Let $w_{v}(c)=1$ if $(v, c) \in \Omega$. Otherwise, let $w_{v}(c)=0$.
Thus, $Z_{H}\left(G^{\prime}, W\left(G^{\prime}, H\right)\right)$ is the number of homomorphisms $\sigma$ from $G^{\prime}$ to $H$ with $\sigma(V(G))=$ $\left\{x_{0}, y_{0}, z_{0}\right\}, \sigma\left(V^{\prime}(G)\right) \subseteq\left\{w, x_{1}, y_{1}, z_{1}\right\}, \sigma(\alpha)=x_{0}, \sigma(\beta)=y_{0}$ and $\sigma(\gamma)=z_{0}$. We will refer to these as "valid" homomorphisms.

If $\sigma$ is a valid homomorphism, then let

$$
\operatorname{bi}(\sigma)=\{e \in E(G) \mid \quad \text { the vertices of } V(G) \text { corresponding to }
$$ the endpoints of $e$ are mapped to different colours by $\sigma\}$.

Note that, for every valid homomorphism $\sigma, \operatorname{bi}(\sigma)$ is a multiterminal cut for the graph $G$ with terminals $\alpha, \beta$ and $\gamma$.

For every multiterminal cut $E^{\prime}$, let $\kappa\left(E^{\prime}\right)$ denote the number of components in the graph $\left(V, E \backslash E^{\prime}\right)$. For each multiterminal cut $E^{\prime}$, let $Z_{E^{\prime}}$ denote the number of valid homomorphisms $\sigma$ from $G^{\prime}$ to $H$ such that $\operatorname{bi}(\sigma)=E^{\prime}$. From the definition of multiterminal cut, $\kappa\left(E^{\prime}\right) \geq 3$. If $\kappa\left(E^{\prime}\right)=3$ then

$$
Z_{E^{\prime}}=2^{s\left(E(G)-E^{\prime}\right)}
$$

since there are two choices for the colours of each vertex $(e, i)$ with $e \in E(G)-E^{\prime}$. (Since the endpoints of each such edge $e$ are assigned the same colour by $\sigma$, the vertex $(e, i)$ can either be coloured $w$, or it can be coloured with one other colour.) Also,

$$
Z_{E^{\prime}} \leq 2^{s\left(E(G)-E^{\prime}\right)} 3^{\kappa\left(E^{\prime}\right)-3}
$$

since the component of $\alpha$ is mapped to $x_{0}$ by $\sigma$, the component of $\beta$ is mapped to $y_{0}$, the component of $\gamma$ is mapped to $z_{0}$, and each remaining component is mapped to a colour in $\left\{x_{0}, y_{0}, z_{0}\right\}$.

Let $Z^{*}=2^{s(E(G)-b)}$. If $E^{\prime}$ has size $b$ then $\kappa\left(E^{\prime}\right)=3$. (Otherwise, there would be a smaller multiterminal cut, contrary to the definition of \#MultiterminalCut(3).) So, in this case,

$$
\begin{equation*}
Z_{E^{\prime}}=Z^{*} \tag{1}
\end{equation*}
$$

If $E^{\prime}$ has size $b^{\prime}>b$ then

$$
Z_{E^{\prime}} \leq 2^{s\left(E(G)-b^{\prime}\right)} 3^{\kappa\left(E^{\prime}\right)-3}=2^{-s\left(b^{\prime}-b\right)} 3^{\kappa\left(E^{\prime}\right)-3} Z^{*} \leq 2^{-s} 3^{|V(G)|} Z^{*}
$$

Clearly, there are at most $2^{|E(G)|}$ multiterminal cuts $E^{\prime}$. So, using the definition of $s$,

$$
\begin{equation*}
\sum_{E^{\prime}:\left|E^{\prime}\right|>b} Z_{E^{\prime}} \leq \frac{Z^{*}}{4} \tag{2}
\end{equation*}
$$

From Equation (1), we find that, if there are $N$ size- $b$ multiterminal cuts then

$$
Z_{H}\left(G^{\prime}, W\left(G^{\prime}, H\right)\right)=N Z^{*}+\sum_{E^{\prime}:\left|E^{\prime}\right|>b} Z_{E^{\prime}}
$$

So applying Equation (2), we get

$$
N \leq \frac{Z_{H}\left(G^{\prime}, W\left(G^{\prime}, H\right)\right)}{Z^{*}} \leq N+\frac{1}{4}
$$

Thus, we have an AP-reduction from \#MultiterminalCut(3) to \#HomsTo $(H)$. To determine the accuracy with which $Z(G)$ should be approximated in order to achieve a given accuracy in the approximation to $N$, see the proof of Theorem 3 of [10].

## 4. Tree homomorphisms capture the ferromagnetic Potts model.

The problem \#HomsTo $(H)$ counts colourings of a graph satisfying "hard" constraints: two colours (corresponding to vertices of $H$ ) are either allowed on adjacent vertices of the instance or disallowed. By contrast, the Potts model (to be described presently) is "permissive": every pair of colours is allowed on adjacent vertices, but some pairs are favoured relative to others. The strength of interactions between colours is controlled by a real parameter $\gamma$. In this section, we will show that approximating the number of homomorphisms to $J_{q}$ is equivalent in difficulty to the problem of approximating the partition function of the ferromagnetic $q$-state Potts model. Since the latter problem is not known to be \#BIS-easy for any $q>2$, we might speculate that approximating \#HomsTo $\left(J_{q}\right)$ is not \#BIS-easy for any $q>2$. If so, $J_{3}$ would be the smallest tree with this property.

It is interesting that, for fixed $q$, a continuously parameterised class of permissive problems can be shown to be computationally equivalent to a single counting problem with hard constraints. Suppose, for example, that we wanted to investigate the possibility that computing the partition
function of the $q$-state ferromagnetic Potts model formed a hierarchy of problems of increasing complexity with increasing $q$. We could equivalently investigate the sequence of problems \#HomsTo $\left(J_{q}\right)$, which seems intuitively to be an easier proposition.

We start with some definitions. Let $q$ be a positive integer. The $q$-state Potts model is a statistical mechanical model of Potts [29] which generalises the classical Ising model from two to $q$ spins. In this model, spins interact along edges of a graph $G=(V, E)$. The strength of each interaction is governed by a parameter $\gamma$ (a real number which is always at least -1 , and is greater than 0 in the ferromagnetic case which we study, where like spins attract each other). The $q$-state Potts partition function is defined as follows.

$$
\begin{equation*}
Z_{\text {Potts }}(G ; q, \gamma)=\sum_{\sigma: V \rightarrow[q]} \prod_{e=\{u, v\} \in E}(1+\gamma \delta(\sigma(u), \sigma(v))), \tag{3}
\end{equation*}
$$

where $\delta\left(s, s^{\prime}\right)$ is 1 if $s=s^{\prime}$, and is 0 otherwise.
The Potts partition function is well-studied. In addition to the complexity-theory literature mentioned below, we refer the reader to Sokal's survey [30].

In order to state our results in the strongest possible form, we use the notion of "efficiently approximable real number" from Section 2, Recall that a real number $\gamma$ is efficiently approximable if there is an FPRAS for the problem of computing it. The notion of "efficiently approximable" is not important to the constructions below - the reader who prefers to assume that the parameters are rational will still appreciate the essence of the reductions.

Let $q$ be a positive integer and let $\gamma$ be a positive efficiently approximable real. Consider the following computational problem, which is parameterised by $q$ and $\gamma$.

```
Problem: Potts (q,\gamma).
Instance: Graph G=(V,E).
Output: }\mp@subsup{Z}{\mathrm{ Potts }}{}(G;q,\gamma)\mathrm{ .
```

This problem may be defined more generally for non-integers $q$ via the Tutte polynomial. We will use some results from [16] which are more general, but we do not need the generality here.

In an important paper, Jaeger, Vertigan and Welsh [24] examined the problem of evaluating the Tutte polynomial. Their result gave a complete classification of the computational complexity of $\operatorname{Potts}(q, \gamma)$. For every fixed positive integer $q$, apart from the trivial $q=1$, and for every fixed $\gamma$, they showed that this computational problem is \#P-hard. When $q=1$ and $\gamma$ is rational, $Z_{\text {Potts }}(G ; q, \gamma)$ can easily be exactly evaluated in polynomial time. The complexity of the approximation problem has also been partially resolved. In the positive direction, Jerrum and Sinclair [25] gave an FPRAS for the case $q=2$. In the negative direction, Goldberg and Jerrum [16] showed that approximation is \#BIS-hard for every fixed $q>2$. They left open the question of whether approximating $Z_{\text {Potts }}(G ; q, \gamma)$ is as easy as \#BIS (or whether it might be even harder).

In this paper, we show that the approximation problem is equivalent in complexity to a tree homomorphism problem. In particular, we show that POTTS $(q, \gamma)$ is AP-equivalent to the problem of approximately counting homomorphisms to the tree $J_{q}$.

We first give an AP-reduction from $\operatorname{Potts}(q, 1)$ to \#HomsTo $\left(J_{q}\right)$.

Lemma 7. Let $q>2$ be a positive integer.

$$
\operatorname{PotTs}(q, 1) \leq_{\mathrm{AP}} \# \operatorname{HOMSTO}\left(J_{q}\right)
$$

Proof. Let $G$ be an instance of $\operatorname{Potts}(q, 1)$. We can assume without loss of generality that $G$ is connected, since it is clear from (3) that a graph $G$ with connected components $G_{1}, \ldots, G_{\kappa}$ satisfies $Z_{\text {Potts }}(G ; q, \gamma)=\prod_{i=1}^{\kappa} Z_{\text {Potts }}\left(G_{i} ; q, \gamma\right)$.

Let $G^{\prime}$ be the graph with

$$
V\left(G^{\prime}\right)=V(G) \cup E(G)
$$

and

$$
E\left(G^{\prime}\right)=\{(u, e) \mid u \in V(G), e \in E(G), \text { and } u \text { is an endpoint of } e\}
$$

$G^{\prime}$ is sometimes referred to as the " 2 -stretch" of $G$. For clarity, when we consider an element $e \in E(G)$ as a vertex of $G^{\prime}$ (rather than an edge of $G$ ), we shall refer to it as the "midpoint vertex corresponding to edge $e$ ".

Let $s$ be an integer satisfying

$$
\begin{equation*}
8 q(q+1)^{|V(G)|+|E(G)|} \leq\left(\frac{q}{2}\right)^{s} \tag{4}
\end{equation*}
$$

For concreteness, take $s$ to be the smallest integer satisfying (4). The exact size of $s$ is not so important. The calculation below relies on the fact that $s$ is large enough to satisfy (4). On the other hand, $s$ must be at most a polynomial in the size of $G$, to make the reduction feasible.

We will construct an instance $G^{\prime \prime}$ of \#HomsTo $\left(J_{q}\right)$ by adding some gadgets to $G^{\prime}$. Fix a vertex $v \in V(G)$. Let $G^{\prime \prime}$ be the graph with $V\left(G^{\prime \prime}\right)=V(G) \cup E(G) \cup\left\{v_{0}, \ldots, v_{s}\right\}$ and $E\left(G^{\prime \prime}\right)=$ $E\left(G^{\prime}\right) \cup\left\{\left(v, v_{0}\right)\right\} \cup\left\{\left(v_{0}, v_{i}\right) \mid i \in[s]\right\}$. See Figure 4 .

We say that a homomorphism $\sigma$ from $G^{\prime \prime}$ to $J_{q}$ is typical if $\sigma\left(v_{0}\right)=w$. Note that, in a typical homomorphism, every vertex in $V(G)$ is mapped by $\sigma$ to one of the colours from $\left\{c_{1}^{\prime}, \ldots, c_{q}^{\prime}\right\}$. Let $Z_{J_{q}}^{t}\left(G^{\prime \prime}\right)$ denote the number of typical homomorphisms from $G^{\prime \prime}$ to $J_{q}$.

Given a mapping $\sigma: V(G) \rightarrow\left\{c_{1}^{\prime}, \ldots, c_{q}^{\prime}\right\}$, the number of typical homomorphisms which induce this mapping is $2^{\operatorname{mono}(\sigma)} q^{s}$, where $\operatorname{mono}(\sigma)$ is the number of edges $e \in E(G)$ whose endpoints in $V(G)$ are mapped to the same colour by $\sigma$. (To see this, note that there are two possible colours for the midpoint vertices corresponding to such edges, whereas the other midpoint vertices have to be mapped to $w$ by $\sigma$. Also, there are $q$ possible colours for each vertex in $\left\{v_{1}, \ldots, v_{s}\right\}$.) Thus, using the definition (3), we conclude that

$$
Z_{J_{q}}^{t}\left(G^{\prime \prime}\right)=\sum_{\sigma: V(G) \rightarrow\left\{c_{1}^{\prime}, \ldots, c_{q}^{\prime}\right\}} 2^{\operatorname{mono}(\sigma)} q^{s}=q^{s} Z_{\mathrm{Potts}}(G ; q, 1)
$$

The number of atypical homomorphisms from $G^{\prime \prime}$ to $J_{q}$, which we denote by $Z_{J_{q}}^{a}\left(G^{\prime \prime}\right)$, is at most $2 q 2^{s}(q+1)^{|V(G)|+|E(G)|}$. (To see this, note, that there are $2 q$ alternative colours for $v_{0}$. For each of these, there are at most 2 colours for each vertex in $\left\{v_{1}, \ldots, v_{s}\right\}$ and at most $q+1$ colours for each vertex in $V(G) \cup E(G)$.) Using Equation (4), we conclude that $Z_{J_{q}}^{a}\left(G^{\prime \prime}\right) \leq q^{s} / 4$. Since $Z_{J_{q}}\left(G^{\prime \prime}\right)=Z_{J_{q}}^{t}\left(G^{\prime \prime}\right)+Z_{J_{q}}^{a}\left(G^{\prime \prime}\right)$, we have


Figure 4. The instance $G^{\prime \prime}$. The thick curved line between $V(G)$ and $E(G)$ indicates that the edges in $E\left(G^{\prime}\right)$ go between elements of $V(G)$ and elements of $E(G)$, but these are not shown.

$$
\begin{equation*}
Z_{\text {Potts }}(G ; q, 1) \leq \frac{Z_{J_{q}}\left(G^{\prime \prime}\right)}{q^{s}} \leq Z_{\text {Potts }}(G ; q, 1)+\frac{1}{4} \tag{5}
\end{equation*}
$$

Equation (5) guarantees that the construction is an AP-reduction from $\operatorname{PotTS}(q, 1)$ to the problem \#HomsTo $\left(J_{q}\right)$. To determine the accuracy with which $Z_{J_{q}}\left(G^{\prime \prime}\right)$ should be approximated in order to achieve a given desired accuracy in the approximation to $Z_{\text {Potts }}(G ; q, 1)$, see the proof of Theorem 3 of [10].

In order to get a reduction going the other direction, we need to generalise the Potts partition function to a hypergraph version. Let $\mathcal{H}=(\mathcal{V}, \mathcal{E})$ be a hypergraph with vertex set $\mathcal{V}$ and hyperedge (multi)set $\mathcal{E}$. Let $q$ be a positive integer. The $q$-state Potts partition function of $\mathcal{H}$ is defined as follows:

$$
Z_{\text {Potts }}(\mathcal{H} ; q, \gamma)=\sum_{\sigma: \mathcal{V} \rightarrow[q]} \prod_{f \in \mathcal{E}}(1+\gamma \delta(\{\sigma(v) \mid v \in f\})),
$$

where $\delta(S)$ is 1 if its argument is a singleton and 0 otherwise. Let $q$ be a positive integer and let $\gamma$ be a positive efficiently approximable real. We consider the following computational problem, which is parameterised by $q$ and $\gamma$.

Problem: HyperPotts $(q, \gamma)$.
Instance: A hypergraph $\mathcal{H}=(\mathcal{V}, \mathcal{E})$.

Output: $Z_{\text {Potts }}(\mathcal{H} ; q, \gamma)$.
We start by reducing \#HomsTo $\left(J_{q}\right)$ to the problem of approximating the Potts partition function of a hypergraph with parameters $q$ and 1 .

Lemma 8. Let $q$ be a positive integer.

$$
\text { \#HomsTo }\left(J_{q}\right) \leq_{\text {AP }} \operatorname{HyperPotts}(q, 1)
$$

Proof. We can assume without loss of generality that the instance to \#HomsTo $\left(J_{q}\right)$ is bipartite, since otherwise the output is zero. We can also assume that it is connected since a graph $G$ with connected components $G_{1}, \ldots, G_{\kappa}$ satisfies $Z_{J_{q}}(G)=\prod_{i=1}^{\kappa} Z_{J_{q}}\left(G_{i}\right)$. Finally, it is easy to find a bipartition of a connected bipartite graph in polynomial time, so we can assume without loss of generality that this is provided as part of the input.

Let $B=(U, V, E)$ be a connected instance of \#HomsTo $\left(J_{q}\right)$ consisting of vertex sets $U$ and $V$ and edge set $E$ (a subset of $U \times V$ ). Let $Z_{J_{q}}^{U}(B)$ be the number of homomorphisms from $B$ to $J_{q}$ in which vertices in $U$ are coloured with colours in $\left\{c_{1}^{\prime}, \ldots, c_{q}^{\prime}\right\}$. Similarly, let $Z_{J_{q}}^{V}(B)$ be the number of homomorphisms from $B$ to $J_{q}$ in which vertices in $V$ are coloured with colours in $\left\{c_{1}^{\prime}, \ldots, c_{q}^{\prime}\right\}$. Clearly, $Z_{J_{q}}(B)=Z_{J_{q}}^{U}(B)+Z_{J_{q}}^{V}(B)$. We will show how to approximate $Z_{J_{q}}^{U}(B)$ using an approximation oracle for $\operatorname{HYPERPOTTS}(q, 1)$. The approximation of $Z_{J_{q}}^{V}(B)$ is similar.

The construction is straightforward. For every $v \in V$, let $\Gamma(v)$ denote the set of neighbours of vertex $v$ in $B$. Let $F=\{\Gamma(v), \mid v \in V\}$. Let $H=(U, F)$ be an instance of $\operatorname{HyperPotts}(q, 1)$.

The reduction is immediate, because $Z_{J_{q}}^{U}(B)=Z_{\text {Potts }}(H ; q, 1)$. To see this, note that every configuration $\sigma: U \rightarrow\left\{c_{1}^{\prime}, \ldots, c_{q}^{\prime}\right\}$ contributes weight $2^{\operatorname{mono}(\sigma)}$ to $Z_{\text {Potts }}(H ; q, 1)$, where $\operatorname{mono}(\sigma)$ is the number of hyperedges in $F$ that are monochromatic in $\sigma$. Also, the configuration $\sigma$ can be extended in exactly $2^{\operatorname{mono}(\sigma)}$ ways to homomorphisms from $B$ to $J_{q}$.

The next step is to reduce the problem of approximating the Potts partition function of a hypergraph to the problem of approximating the Potts partition function of a uniform hypergraph, which is a hypergraph in which all hyperedges have the same size. The reason for this step is that the paper [16] shows how to reduce the latter to the approximation of the Potts partition function of a graph, which is the desired target of our reduction.

Let $q$ be a positive integer and let $\gamma$ be a positive efficiently approximable real. We consider the following computational problem, which, like $\operatorname{HyPERPOtTS}(q, \gamma)$, is parameterised by $q$ and $\gamma$.

```
Problem: UniformHyperPotts \((q, \gamma)\).
Instance: A uniform hypergraph \(\mathcal{H}=(\mathcal{V}, \mathcal{E})\).
Output: \(Z_{\text {Potts }}(\mathcal{H} ; q, \gamma)\).
```

We will actually only use the following lemma with $\gamma=1$ but we state, and prove, the more general lemma, since it is no more difficult to prove.

Lemma 9. Let $q$ be a positive integer and let $\gamma$ be a positive efficiently approximable real. Then

$$
\operatorname{HYPERPOTTS}(q, \gamma) \leq_{\text {AP }} \operatorname{UNIFORMHYPERPOTTS}(q, \gamma)
$$

Proof. Let $\mathcal{H}=(\mathcal{V}, \mathcal{E})$ be an instance to $\operatorname{HyperPotts}(q, \gamma)$ with $|\mathcal{V}|=n$ and $|\mathcal{E}|=m$ and $\max (|f| \mid f \in \mathcal{E})=t$. Let $s$ be any positive integer that is at least

$$
\frac{\log \left(4 q^{n+m(t-1)}(1+\gamma)^{m}\right)}{\log (1+\gamma)}
$$

As with our other reductions, the exact value of $s$ is not important, as long as it satisfies the above inequality, it is bounded from above by a polynomial in $n$ and $m$, and its can be computed in polynomial time (as a function of $n$ and $m$ ). An appropriate $s$ can be readily computed by computing crude upper and lower bounds for $\gamma$ and evaluating different values of $s$ one-by-one to find one that is sufficiently large, in terms of these bounds.

For every hyperedge $f \in \mathcal{E}$, fix some vertex $v_{f} \in f$. Introduce new vertices $\left\{u_{f, i} \mid f \in \mathcal{E}, i \in\right.$ $[t-1]\}$, and let $\mathcal{V}^{\prime}=\mathcal{V} \cup\left\{u_{f, i} \mid f \in \mathcal{E}, i \in[t-1]\right\}$. Let

$$
\mathcal{E}^{\prime}=\left\{f \cup\left\{u_{f, i} \mid i \in[t-|f|]\right\} \mid f \in \mathcal{E}\right\} \cup\left\{\left\{v_{f}, u_{f, 1}, \ldots, u_{f, t-1}\right\} \times[s] \mid f \in \mathcal{E}\right\} .
$$

That is, the multi-set $\mathcal{E}^{\prime}$ has $s$ copies of the edge $\left\{v_{f}, u_{f, 1}, \ldots, u_{f, t-1}\right\}$ and one copy of the edge $f \cup\left\{u_{f, i} \mid i \in[t-|f|]\right\}$ for each hyperedge $f \in \mathcal{E}$. Let $\mathcal{H}^{\prime}=\left(\mathcal{V}^{\prime}, \mathcal{E}^{\prime}\right)$. Note that $\mathcal{H}^{\prime}$ is $t$-uniform.

Now, the total contribution to $Z_{\text {Potts }}\left(\mathcal{H}^{\prime} ; q, \gamma\right)$ from configurations $\sigma$ which are monochromatic on every edge $\left\{v_{f}, u_{f, 1}, \ldots, u_{f, t-1}\right\}$ is exactly $Z_{\text {Potts }}(\mathcal{H} ; q, \gamma)(1+\gamma)^{s m}$. Also, the total contribution to $Z_{\text {Potts }}\left(\mathcal{H}^{\prime} ; q, \gamma\right)$ from any other configurations $\sigma$ is at most $q^{n+m(t-1)}(1+\gamma)^{m}(1+\gamma)^{s(m-1)}$ since there are at most $q^{n+m(t-1)}$ such configurations and $\gamma>0$.

So

$$
\begin{aligned}
Z_{\mathrm{Potts}}(\mathcal{H} ; q, \gamma) \leq \frac{Z_{\mathrm{Potts}}\left(\mathcal{H}^{\prime} ; q, \gamma\right)}{(1+\gamma)^{s m}} & \leq Z_{\mathrm{Potts}}(\mathcal{H} ; q, \gamma)+\frac{q^{n+m(t-1)}(1+\gamma)^{m}}{(1+\gamma)^{s}} \\
& \leq Z_{\mathrm{Potts}}(\mathcal{H} ; q, \gamma)+\frac{1}{4}
\end{aligned}
$$

which completes the reduction.
Finally, we are ready to put together the pieces to show that, for every integer $q>2$, the problem of approximating the Potts partition function is equivalent to a tree homomorphism problem.

Theorem 10. Let $q>2$ be a positive integer and let $\gamma$ be a positive efficiently approximable real. Then $\operatorname{Potts}(q, \gamma) \equiv_{\mathrm{AP}}$ \#HomsTo $\left(J_{q}\right)$.

Proof. We start by establishing the reduction from \#HomsTo $\left(J_{q}\right)$ to $\operatorname{Potts}(q, \gamma)$. By Lemmas 8 and 9 .

$$
\text { \#Homsto }\left(J_{q}\right) \leq_{\mathrm{AP}} \operatorname{HyperPotts}(q, 1) \leq_{\mathrm{AP}} \operatorname{UNIFORMHYpERPotts}(q, 1) .
$$

To complete the sequence of reductions we need to know that the last problem is reducible to $\operatorname{Potts}(q, \gamma)$. Fortunately, this step already appears in the literature in a slightly different guise, so we just need to explain how to translate the terminology from the earlier result to the current setting. For every positive integer $q$, the partition function $Z_{\text {Potts }}(\mathcal{H} ; q, \gamma)$ of the Potts model on hypergraphs is equal to the Tutte polynomial $Z_{\text {Tutte }}(\mathcal{H} ; q, \gamma)$ (whose definition we will not need here). This equality is proved in [16, Observation 2.1], using the same basic line of argument that

Fortuin and Kasteleyn [14] used in the graph case. Furthermore, for $q>2$, Lemmas 9.1 and 10.1 of [16] reduce the problem of approximating the Tutte partition function $Z_{\text {Tutte }}(\mathcal{H} ; q, 1)$, where $\mathcal{H}$ is a uniform hypergraph, to that of approximating the Tutte partition function $Z_{\text {Tutte }}(G ; q, \gamma)$, where $G$ is a graph. Given the equivalence between $Z_{\text {Tutte }}(G ; q, \gamma)$ and $Z_{\text {Potts }}(G ; q, \gamma)$ mentioned earlier, we see that

## $\operatorname{UNIFORMHYPERPOTTS}(q, 1) \leq$ AP $^{\operatorname{Potts}}(q, \gamma)$,

completing the chain of reductions.
For the other direction, we will establish an AP-reduction from $\operatorname{Potts}(q, \gamma)$ to the problem \#HomsTo $\left(J_{q}\right)$. To start, we note that since a graph is a special case of a uniform hypergraph, Lemmas 9.1 and 10.1 of [16] give an AP-reduction from $\operatorname{Potts}(q, \gamma)$ to $\operatorname{Potts}(q, 1)$. (It is definitely not necessary to go via hypergraphs for this reduction, but here it is easier to use the stated result than to repeat the work.) Finally, Lemma 7 shows that $\operatorname{PoTts}(q, 1) \leq_{\text {AP }}$ \#HomsTo $\left(J_{q}\right)$.

## 5. INAPPROXIMABILITY OF COUNTING TREE HOMOMORPHISMS

Until now, it was not known whether or not a bipartite graph $H$ exists for which approximating \#HomsTo $(H)$ is \#Sat-hard. It is perhaps surprising, then, to discover that \#HomsTo $(H)$ may be \#Sat-hard even when $H$ is a tree. However, the hardness result from Section 3 provides a clue. There it was shown that the weighted version \#WHomsTo $(H)$ is \#SAt-hard whenever $H$ is a tree containing $J_{3}$ as an induced subgraph. If we were able to construct a tree $H$, containing $J_{3}$, that is able, at least in some limited sense, to simulate vertex weights, then we might obtain a reduction from \#WHomsTo $\left(J_{3}\right)$ to \#HomsTo $(H)$. That is roughly how we proceed in this section. We will obtain our hard tree $H$ by "decorating" the leaves of $J_{3}$. These decorations will match certain structures in the instance $G$, so that particular distinguished vertices in $G$ will preferentially be coloured with particular colours. Carrying through this idea requires $H$ to have a certain level of complexity, and the tree $J_{3}^{*}$ that we actually use (see Figure 5) is about the smallest for which this approach works. Presumably the same approach could also be applied starting at $J_{q}$, for $q>3$. It is possible that there are trees $H$ that are much smaller than $J_{3}^{*}$ for which \#HomsTo $(H)$ is \#Sat-hard. It is even possible that \#HomsTo $\left(J_{3}\right)$ is \#Sat-hard. But demonstrating this would require new ideas.

Define vertex sets

$$
\begin{aligned}
X & =\left\{x_{0}, x_{1}\right\} \cup\left\{x_{2, i} \mid i \in[5]\right\} \\
Y & =\left\{y_{0}, y_{1}\right\} \cup\left\{y_{2, i} \mid i \in[4]\right\} \cup\left\{y_{3, i, j} \mid i \in[4], j \in[3]\right\} \\
Z & =\left\{z_{0}, z_{1}\right\} \cup\left\{z_{2, i} \mid i \in[3]\right\} \cup\left\{z_{3, i, j} \mid i \in[3], j \in[3]\right\} \cup\left\{z_{4, i, j, k} \mid i \in[3], j \in[3], k \in[2]\right\},
\end{aligned}
$$

and edge sets

$$
\begin{aligned}
E_{X}= & \left\{\left(x_{0}, x_{1}\right)\right\} \cup\left\{\left(x_{1}, x_{2, i}\right) \mid i \in[5]\right\}, \\
E_{Y}= & \left\{\left(y_{0}, y_{1}\right)\right\} \cup\left\{\left(y_{1}, y_{2, i}\right) \mid i \in[4]\right\} \cup\left\{\left(y_{2, i}, y_{3, i, j}\right) \mid i \in[4], j \in[3]\right\}, \\
E_{Z}= & \left\{\left(z_{0}, z_{1}\right)\right\} \cup\left\{\left(z_{1}, z_{2, i}\right) \mid i \in[3]\right\} \cup\left\{\left(z_{2, i}, z_{3, i, j}\right) \mid i \in[3], j \in[3]\right\} \\
& \cup\left\{\left(z_{3, i, j}, z_{4, i, j, k}\right) \mid i \in[3], j \in[3], k \in[2]\right\} .
\end{aligned}
$$



Figure 5. The tree $J_{3}^{*}$.

Let $J_{3}^{*}$ be the tree with vertex set $V\left(J_{3}^{*}\right)=\{w\} \cup X \cup Y \cup Z$ and edge set

$$
E\left(J_{3}^{*}\right)=\left\{\left(w, x_{0}\right),\left(w, y_{0}\right),\left(w, z_{0}\right)\right\} \cup E_{X} \cup E_{Y} \cup E_{Z}
$$

See Figure 5. Consider the equivalence relation on $V\left(J_{3}^{*}\right)$ defined by graph isomorphism - two vertices of $J_{3}^{*}$ are in the same equivalence class if there is an isomorphism of $J_{3}^{*}$ mapping one to the other. The canonical representatives of the equivalence classes are the vertices $w, x_{0}, x_{1}$, $x_{2,1}, y_{0}, y_{1}, y_{2,1}, y_{3,1,1}, z_{0}, z_{1}, z_{2,1}, z_{3,1,1}$ and $z_{4,1,1,1}$. These are shown in the figure.

In this section, we will show that \#SAT is AP-reducible to \#HomsTo $\left(J_{3}^{*}\right)$. We start by identifying relevant structure in $J_{3}^{*}$.

A simple path in a graph is a path in which no vertices are repeated. For every vertex $h$ of $J_{3}^{*}$, and every positive integer $k$, let $d_{k}(h)$ be the number of simple length- $k$ paths from $h$. The values $d_{1}(h), d_{2}(h)$ and $d_{3}(h)$ can be calculated for each canonical representative $h \in V\left(J_{3}^{*}\right)$ by inspecting the definition of $J_{3}^{*}$ (or its drawing in Figure 5). These values are recorded in the first four columns of the table in Figure 6 .

Now let $w_{k}(h)$ denote the number of length- $k$ walks from $h$ in $J_{3}^{*}$. Clearly, $w_{1}(h)=d_{1}(h)$ since $J_{3}^{*}$ has no self-loops, so all length- 1 walks are simple paths. Next, note that $w_{2}(h)=$ $d_{1}(h)+d_{2}(h)$. To see this, note that every length- 2 walk from $h$ is either a simple length- 2 path

| $h$ | $d_{1}(h)$ | $d_{2}(h)$ | $d_{3}(h)$ | $w_{1}(h)$ | $w_{2}(h)$ | $w_{3}(h)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $w$ | 3 | 3 | 12 | 3 | 6 | 24 |
| $x_{0}$ | 2 | 7 | 2 | 2 | 9 | 13 |
| $x_{1}$ | 6 | 1 | 2 | $\mathbf{6}$ | 7 | 39 |
| $x_{2,1}$ | 1 | 5 | 1 | 1 | 6 | 7 |
| $y_{0}$ | 2 | 6 | 14 | 2 | 8 | 24 |
| $y_{1}$ | 5 | 13 | 2 | 5 | $\mathbf{1 8}$ | 40 |
| $y_{2,1}$ | 4 | 4 | 10 | 4 | 8 | 30 |
| $y_{3,1,1}$ | 1 | 3 | 4 | 1 | 4 | 8 |
| $z_{0}$ | 2 | 5 | 11 | 2 | 7 | 20 |
| $z_{1}$ | 4 | 10 | 20 | 4 | 14 | $\mathbf{4 6}$ |
| $z_{2,1}$ | 4 | 9 | 7 | 4 | 13 | 32 |
| $z_{3,1,1}$ | 3 | 3 | 7 | 3 | 6 | 19 |
| $z_{4,1,1,1}$ | 1 | 2 | 3 | 1 | 3 | 6 |

Figure 6. For each canonical representative $h \in V\left(J_{3}^{*}\right)$, we record the values of $w_{1}(h)=d_{1}(h), w_{2}(h)=d_{1}(h)+d_{2}(h)$ and $w_{3}(h)=d_{1}^{2}(h)+d_{2}(h)+d_{3}(h)$.
from $J_{3}^{*}$, or it is a walk obtained by taking an edge from $h$, and then going back to $h$. Finally, $w_{3}(h)=d_{1}(h)^{2}+d_{2}(h)+d_{3}(h)$ since every length- 3 walk from $h$ is one of the following:

- a simple length-3 path from $h$,
- a simple length-2 path from $h$, with the last edge repeated in reverse, or
- a simple length- 1 path from $h$ with the last edge repeated in reverse, followed by another simple length-1 path from $h$.
These values are recorded, for each canonical representative $h \in V\left(J_{3}^{*}\right)$, in the last three columns of the table in Figure 6. The important fact that we will use is that $w_{1}(h)$ is uniquely maximised at $h=x_{1}, w_{2}(h)$ is uniquely maximised at $h=y_{1}$, and $w_{3}(h)$ is uniquely maximised at $h=z_{1}$. (These are shown in boldface in the table.)

We are now ready to prove the following theorem.
Theorem 11. \#SAT $\leq_{\text {AP }} \# \operatorname{HOMSTO}\left(J_{3}^{*}\right)$.
Proof. By Lemma 5] it suffices to give an AP-reduction from \#MultiterminalCut(3) to \#HomsTo $\left(J_{3}^{*}\right)$. The basic construction follows the outline of the reduction developed in the proof of Lemma6. However, unlike the situation of Lemma6, the target problem \#HomsTo $\left(J_{3}^{*}\right)$ does not include weights, so we must develop gadgetry to simulate the role of these.

Let $b, G=(V, E), \alpha, \beta$ and $\gamma$ be an input to \#MultiterminalCut(3). Let $s=3+|E(G)|+$ $2|V(G)|$. (As before, the exact size of $s$ is not important, but it has to be at least this big to make the calculation work, and it has to be at most a polynomial in the size of $G$.)

Let $G^{\prime}$ be the graph defined in the proof of Lemma 6. In particular, let $V^{\prime}(G)=\{(e, i) \mid e \in$ $E(G), i \in[s]\}$. Then let $G^{\prime}$ be the graph with vertex set $V\left(G^{\prime}\right)=V(G) \cup V^{\prime}(G)$ and edge set

$$
E\left(G^{\prime}\right)=\left\{(u,(e, i)) \mid u \in V(G),(e, i) \in V^{\prime}(G), \text { and } u \text { is an endpoint of } e\right\} .
$$

Now let $r$ be any positive integer such that

$$
\begin{equation*}
\left(\frac{46}{40}\right)^{r} \geq 8\left|V\left(J_{3}^{*}\right)\right|^{|V(G)|+s|E(G)|+7} \tag{6}
\end{equation*}
$$

For concreteness, take $r$ to be the smallest integer satisfying (6). Once again, the exact value of $r$ is not so important. Any $r$ would work as long as it is at most a polynomial in the size of $G$, and it satisfies (6).

We will construct an instance $G^{\prime \prime}$ of \#HomsTo $\left(J_{3}^{*}\right)$ by adding some gadgets to $G^{\prime}$. First, we define the gadgets.

- Let $\Gamma_{x}$ be a graph with vertex set $V\left(\Gamma_{x}\right)=\left\{v_{x_{1}}\right\} \cup \bigcup_{i \in[r]}\left\{v_{x, i}\right\}$ and edge set $E\left(\Gamma_{x}\right)=$ $\bigcup_{i \in[r]}\left\{\left(v_{x_{1}}, v_{x, i}\right)\right\}$.
- Let $\Gamma_{y}$ be a graph with vertex set $V\left(\Gamma_{y}\right)=\left\{v_{y_{1}}\right\} \cup \bigcup_{i \in[r]}\left\{v_{y, i}, v_{y, i}^{\prime}\right\}$ and edge set $E\left(\Gamma_{y}\right)=$ $\bigcup_{i \in[r]}\left\{\left(v_{y_{1}}, v_{y, i}\right),\left(v_{y, i}, v_{y, i}^{\prime}\right)\right\}$.
- Let $\Gamma_{z}$ be a graph with vertex set $V\left(\Gamma_{z}\right)=\left\{v_{z_{1}}\right\} \cup \bigcup_{i \in[r]}\left\{v_{z, i}, v_{z, i}^{\prime}, v_{z, i}^{\prime \prime}\right\}$ and edge set $E\left(\Gamma_{x}\right)=\bigcup_{i \in[r]}\left\{\left(v_{z_{1}}, v_{z, i}\right),\left(v_{z, i}, v_{z, i}^{\prime}\right),\left(v_{z, i}^{\prime}, v_{z, i}^{\prime \prime}\right)\right\}$.
Finally, let

$$
V\left(G^{\prime \prime}\right)=V\left(G^{\prime}\right) \cup\left\{v_{w}, v_{x_{0}}, v_{y_{0}}, v_{z_{0}}\right\} \cup V\left(\Gamma_{x}\right) \cup V\left(\Gamma_{y}\right) \cup V\left(\Gamma_{z}\right)
$$

and

$$
\begin{aligned}
E\left(G^{\prime \prime}\right) & =\left\{\left(v_{w}, v_{x_{0}}\right),\left(v_{w}, v_{y_{0}}\right),\left(v_{w}, v_{z_{0}}\right),\left(v_{x_{0}}, v_{x_{1}}\right),\left(v_{y_{0}}, v_{y_{1}}\right),\left(v_{z_{0}}, v_{z_{1}}\right),\left(v_{x_{1}}, \alpha\right),\left(v_{y_{1}}, \beta\right),\left(v_{z_{1}}, \gamma\right)\right\} \\
& \cup E\left(G^{\prime}\right) \cup\left\{\left(v_{w}, v\right) \mid v \in V(G)\right\} \cup E\left(\Gamma_{x}\right) \cup E\left(\Gamma_{y}\right) \cup E\left(\Gamma_{z}\right) .
\end{aligned}
$$

A picture of the instance $G^{\prime \prime}$ is shown in Figure 7 .
We say that a homomorphism $\sigma$ from $G^{\prime \prime}$ to $J_{3}^{*}$ is typical if $\sigma\left(v_{x_{1}}\right)=x_{1}, \sigma\left(v_{y_{1}}\right)=y_{1}$, and $\sigma\left(v_{z_{1}}\right)=z_{1}$. Note that, in a typical homomorphism, $\sigma\left(v_{w}\right)=w$, so $\sigma(V(G))=\left\{x_{0}, y_{0}, z_{0}\right\}$ and $\sigma\left(V^{\prime}(G)\right) \subseteq\left\{w, x_{1}, y_{1}, z_{1}\right\}$. Also, $\sigma(\alpha)=x_{0}, \sigma(\beta)=y_{0}$, and $\sigma(\gamma)=z_{0}$.

If $\sigma$ is a typical homomorphism, then let

$$
\operatorname{bi}(\sigma)=\{e \in E(G) \mid \quad \text { the vertices of } V(G) \text { corresponding to }
$$

$$
\text { the endpoints of } e \text { are mapped to different colours by } \sigma\} \text {. }
$$

Note that, for every typical homomorphism $\sigma, \operatorname{bi}(\sigma)$ is a multiterminal cut for the graph $G$ with terminals $\alpha, \beta$ and $\gamma$.

For every multiterminal cut $E^{\prime}$ of $G$, let $\kappa\left(E^{\prime}\right)$ denote the number of components in the graph $\left(V, E \backslash E^{\prime}\right)$. For each multiterminal cut $E^{\prime}$, let $Z_{E^{\prime}}$ denote the number of typical homomorphisms $\sigma$ from $G^{\prime \prime}$ to $J_{3}^{*}$ such that $\operatorname{bi}(\sigma)=E^{\prime}$.

As in the proof of Lemma6, $\kappa\left(E^{\prime}\right) \geq 3$. If $\kappa\left(E^{\prime}\right)=3$ then

$$
Z_{E^{\prime}}=2^{s\left|E(G)-E^{\prime}\right|} 6^{r} 18^{r} 46^{r}=2^{s\left|E(G)-E^{\prime}\right|} 4968^{r} .
$$

The $2^{s\left|E(G)-E^{\prime}\right|}$ comes from the two choices for the colour of each vertex $(e, i)$ with $e \in E(G)-$ $E^{\prime}$, as before. The $6^{r}$ comes from the choices for the vertices in $V\left(\Gamma_{x}\right) \backslash\left\{x_{1}\right\}$ according to column 5 of the table in Figure6. The $18^{r}$ comes from the choices for the vertices in $V\left(\Gamma_{y}\right) \backslash\left\{y_{1}\right\}$ (in column 6) and the $46^{r}$ comes from the choices for the vertices in $V\left(\Gamma_{z}\right) \backslash\left\{z_{1}\right\}$ (in column 7).


Figure 7. The instance $G^{\prime \prime}$. The thick curved line between $V(G)$ and $V^{\prime}(G)$ indicates that the edges in $E\left(G^{\prime}\right)$ go between vertices in $V(G)$ and vertices in $V^{\prime}(G)$, but these are not shown. Vertex $v_{w}$ is connected to each vertex in $V(G)$.

Also, for any multiterminal cut $E^{\prime}$ of $G$,

$$
Z_{E^{\prime}} \leq 2^{s\left|E(G)-E^{\prime}\right|} 3^{\kappa\left(E^{\prime}\right)-3} 4968^{r}
$$

since in any typical homomorphism $\sigma$, the component of $\alpha$ is mapped to $x_{0}$ by $\sigma$, the component of $\beta$ is mapped to $y_{0}$, the component of $\gamma$ is mapped to $z_{0}$, and each remaining component is mapped to a colour in $\left\{x_{0}, y_{0}, z_{0}\right\}$.

Let $Z^{*}=2^{s|E(G)-b|} 4968^{r}$. If $E^{\prime}$ has size $b$ then $\kappa\left(E^{\prime}\right)=3$. (Otherwise, there would be a smaller multiterminal cut, contrary to the definition of \#MultiterminalCut(3).) So, in this
case,

$$
\begin{equation*}
Z_{E^{\prime}}=Z^{*} \tag{7}
\end{equation*}
$$

If $E^{\prime}$ has size $b^{\prime}>b$ then

$$
Z_{E^{\prime}} \leq 2^{s\left|E(G)-b^{\prime}\right|} 3^{\kappa\left(E^{\prime}\right)-3} 4968^{r}=2^{-s\left(b^{\prime}-b\right)} 3^{\kappa\left(E^{\prime}\right)-3} Z^{*} \leq 2^{-s} 3^{|V(G)|} Z^{*}
$$

Clearly, there are at most $2^{|E(G)|}$ multiterminal cuts $E^{\prime}$. So, using the definition of $s$,

$$
\begin{equation*}
\sum_{E^{\prime}:\left|E^{\prime}\right|>b} Z_{E^{\prime}} \leq \frac{Z^{*}}{8} \tag{8}
\end{equation*}
$$

Now let $Z^{-}$denote the number of homomorphisms from $G^{\prime \prime}$ to $J_{3}^{*}$ that are not typical. Now

$$
Z^{-} \leq\left|V\left(J_{3}^{*}\right)\right|^{|V(G)|+\left|V^{\prime}(G)\right|+7}(40 / 46)^{r} 4968^{r},
$$

since there are at most $\left|V\left(J_{3}^{*}\right)\right|$ colours for each of the vertices in

$$
V(G) \cup V^{\prime}(G) \cup\left\{v_{w}, v_{x_{0}}, v_{y_{0}}, v_{z_{0}}, v_{x_{1}}, v_{y_{1}}, v_{z_{1}}\right\} .
$$

Also, given that the assignment to $v_{x_{1}}, v_{y_{1}}$ and $v_{z_{1}}$ is not precisely $x_{1}, y_{1}$ and $z_{1}$, respectively, it can be seen from the table in Figure 6 that the number of possibilities for the remaining vertices is at most $(40 / 46)^{r}$ times as large as it would otherwise have been. (For example, from the last column of the table, colouring $v_{z_{1}}$ with $y_{1}$ instead of with $z_{1}$ would give exactly $40^{r}$ choices for the colours of the vertices in $\Gamma_{z} \backslash\left\{v_{z_{1}}\right\}$ instead of $46^{r}$ choices. The differences in the other columns are more substantial than this.) Since $\left|V^{\prime}(G)\right|=s|E(G)|$,

$$
Z^{-} \leq\left|V\left(J_{3}^{*}\right)\right|^{|V(G)|+s|E(G)|+7}(40 / 46)^{r} 4968^{r} .
$$

We can assume that $b \leq|E(G)|$ (otherwise, the number of size- $b$ multiterminal cuts is trivially 0 ) so from the definition of $Z^{*}$,

$$
Z^{-} \leq\left|V\left(J_{3}^{*}\right)\right|^{|V(G)|+s|E(G)|+7}(40 / 46)^{r} Z^{*}
$$

Using Equation (6), we get

$$
\begin{equation*}
Z^{-} \leq \frac{Z^{*}}{8} \tag{9}
\end{equation*}
$$

From Equation (7), we find that, if there are $N$ size- $b$ multiterminal cuts then

$$
Z_{J_{3}^{*}}(G)=N Z^{*}+\sum_{E^{\prime}:\left|E^{\prime}\right|>b} Z_{E^{\prime}}+Z^{-}
$$

So applying Equations (8) and (9), we get

$$
N \leq \frac{Z_{J_{3}^{*}}(G)}{Z^{*}} \leq N+\frac{1}{4} .
$$

Thus, we have an AP-reduction from \#MultiterminalCut(3) to \#HomsTo $\left(J_{3}^{*}\right)$. To determine the accuracy with which $Z(G)$ should be approximated in order to achieve a given accuracy in the approximation to $N$, see the proof of Theorem 3 of [10].

## 6. The Potts partition function and proper colourings of bipartite graphs

Let $q$ be any integer greater than 2 . Consider the following computational problem.
Problem: \#Bipartite $q$-Col.
Instance: A bipartite graph $G$.
Output: The number of proper $q$-colourings of $G$.
Dyer et al. [10, Theorem 13] showed that \#BIS $\leq_{\text {ap }}$ \#Bipartite $q$-Col. However, it may be the case that \#Bipartite $q$-COL is easier to approximate than \#Sat. Certainly, no AP-reduction from \#Sat to \#Bipartite $q$-Col has been discovered (despite some effort!). Therefore, it seems worth recording the following upper bound on the complexity of \#HOMSTo $\left(J_{q}\right)$, which is an easy consequence of Theorem 10 .
Corollary 12. Let $q>2$ be a positive integer. Then \#HOMSTO $\left(J_{q}\right) \leq_{\text {AP }}$ \#Bipartite $q$-CoL.
Corollary 12 follows immediately from Lemma 13 below by applying Theorem 10 with $\gamma=$ $1 /(q-2)$.
Lemma 13. Let $q>2$ be a positive integer. Then $\operatorname{POTTS}(q, 1 /(q-2)) \leq_{\text {AP }}$ \#BIPARTITE $q$-COL.
Proof. Let $G=(V, E)$ be an input to $\operatorname{Potts}(q, 1 /(q-2))$. Let $G^{\prime}$ be the two-stretch of $G$ constructed as in the proof of Lemma77. In particular, $G^{\prime}$ is the bipartite graph with

$$
V\left(G^{\prime}\right)=V(G) \cup E(G)
$$

and

$$
E\left(G^{\prime}\right)=\{(u, e) \mid u \in V(G), e \in E(G), \text { and } u \text { is an endpoint of } e\} .
$$

Consider an assignment $\sigma: V(G) \rightarrow[q]$ and an edge $e=(u, v)$ of $G$. If $\sigma(u) \neq \sigma(v)$ then there are $q-2$ ways to colour the midpoint vertex corresponding to $e$ so that it receives a different colour from $\sigma(u)$ and $\sigma(v)$. However, if $\sigma(u)=\sigma(v)$ then there are $q-1$ possible colours for the midpoint vertex.

Let $N$ denote the number of proper $q$-colourings of $G^{\prime}$. Then since $(q-1) /(q-2)-1=$ $1 /(q-2)$, we have

$$
N=(q-2)^{|E|} \sum_{\sigma: V \rightarrow[q]}\left(\frac{q-1}{q-2}\right)^{\operatorname{mono}(\sigma)}=(q-2)^{|E|} Z_{\mathrm{Potts}}(G ; q, 1 /(q-2)),
$$

where $\operatorname{mono}(\sigma)$ is the number of edges $e \in E(G)$ whose endpoints in $V(G)$ are mapped to the same colour by $\sigma$.

## 7. The Potts partition function and the weight enumerator of a code

A linear code $C$ of length $N$ over a finite field $\mathbb{F}_{q}$ is a linear subspace of $\mathbb{F}_{q}^{N}$. If the subspace has dimension $r$ then the code may be specified by an $r \times N$ generating matrix $M$ over $\mathbb{F}_{q}$ whose rows form a basis for the code. For any real number $\lambda$, the weight enumerator of the code is given by $W_{M}(\lambda)=\sum_{w \in C} \lambda^{\|w\|}$ where $\|w\|$ is the number of non-zero entries in $w$. ( $\|w\|$ is usually called the Hamming weight of $w$.) We consider the following computational problem, parameterised by $q$ and $\lambda$.

Problem: $\mathrm{WE}(q, \lambda)$.
Instance: A generating matrix $M$ over $\mathbb{F}_{q}$.
Output: $W_{M}(\lambda)$.
In [17], the authors considered the special case $q=2$ and obtained various results on the complexity of $\operatorname{WE}(2, \lambda)$, depending on $\lambda$. Here we show that, for any prime $p, \operatorname{WE}(p, \lambda)$ provides an upper bound on the complexity of $\operatorname{Potts}\left(p^{k}, \gamma\right)$.
Theorem 14. Suppose that $p$ is a prime, $k$ is a positive integer satisfying $p^{k}>2$ and $\lambda \in(0,1)$ is an efficiently computable real. Then

$$
\operatorname{PotTS}\left(p^{k}, 1\right) \leq_{\mathrm{AP}} \mathrm{WE}(p, \lambda)
$$

The following corollary follows immediately from Theorem 14 and Theorem 10 ,
Corollary 15. Suppose that $p$ is a prime, $k$ is a positive integer satisfying $p^{k}>2$ and $\lambda \in(0,1)$ is an efficiently computable real. Then \#HOMSTO $\left(J_{p^{k}}\right) \leq_{\mathrm{AP}} \mathrm{WE}(p, \lambda)$.

The condition $p^{k}>2$ can in fact be removed from Corollary 15 even though the result does not follow from Theorem 14 in this situation. For the missing case where $p=2$ and $k=1$, Lemma 4 gives \#HomsTo $\left(J_{2}\right) \leq_{\text {AP }}$ \#BIS and [17, Cor. 7, Part (4)] show \#BIS $\leq_{\text {AP }} \mathrm{WE}(2, \lambda)$. A striking feature of Corollary 15 is that it provides a uniform upper bound on the complexity of the infinite sequence of problems \#HomsTo $\left(J_{p^{k}}\right)$, with $p$ fixed and $k$ varying. This uniform upper bound is interesting if (as we suspect) $\mathrm{WE}(p, \lambda)$ is not itself equivalent to \#SAT via APreducibility.
Proof of Theorem 14 Let $q=p^{k}$ and let $\gamma=\lambda^{-q(p-1) / p}-1>0$. Since Theorem 10 shows $\operatorname{Potts}\left(p^{k}, 1\right) \equiv_{\mathrm{AP}} \# \operatorname{HomsTo}\left(J_{p^{k}}\right) \equiv_{\mathrm{AP}} \operatorname{Potts}\left(p^{k}, \gamma\right)$, it is enough to given an AP-reduction from $\operatorname{Potts}\left(p^{k}, \gamma\right)$ to $\mathrm{WE}(p, \lambda)$. So suppose $G=(V, E)$ is a graph with $n$ vertices and $m$ edges. We wish to evaluate

$$
\begin{equation*}
Z_{\mathrm{Potts}}(G ; q, \gamma)=\sum_{\sigma: V \rightarrow[q]}(1+\gamma)^{\operatorname{mono}(\sigma)} . \tag{10}
\end{equation*}
$$

Our aim is to construct an instance of the weight enumerator problem whose solution is the above expression, modulo an easily computable factor. Introduce a collection of variables $X=\left\{x_{i}^{v} \mid\right.$ $v \in V$ and $i \in[k]\}$. To each assignment $\sigma: V \rightarrow[q]$ we define an associated assignment $\hat{\sigma}: X \rightarrow \mathbb{F}_{p}$ as follows: for all $v \in V$,

$$
\left(\hat{\sigma}\left(x_{1}^{v}\right), \hat{\sigma}\left(x_{2}^{v}\right), \ldots, \hat{\sigma}\left(x_{k}^{v}\right)\right)=\varphi(\sigma(v)),
$$

where $\varphi$ is any fixed bijection $[q] \rightarrow \mathbb{F}_{p}^{k}$. Note that $\sigma \mapsto \hat{\sigma}$ is a bijection from assignments $V \rightarrow[q]$ to assignments $X \rightarrow \mathbb{F}_{p}$. (Informally, we have coded the spin at each vertex as a $k$-tuple of variables taking values in $\mathbb{F}_{p}$.)

Let $\ell_{1}\left(z_{1}, \ldots, z_{k}\right), \ldots, \ell_{q}\left(z_{1}, \ldots, z_{k}\right)$ be an enumeration of all linear forms $\alpha_{1} z_{1}+\alpha_{2} z_{2}+$ $\cdots+\alpha_{k} z_{k}$ over $\mathbb{F}_{p}$, where $\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{k}\right)$ ranges over $\mathbb{F}_{p}^{k}$. This collection of linear forms has the following property:

$$
\text { If } z_{1}=z_{2}=\cdots z_{k}=0 \text {, then all of } \ell_{1}\left(z_{1}, \ldots, z_{k}\right), \ldots, \ell_{q}\left(z_{1}, \ldots, z_{k}\right) \text { are zero; }
$$

$$
\begin{equation*}
\text { otherwise, precisely } q / p=p^{k-1} \text { of } \ell_{1}\left(z_{1}, \ldots, z_{k}\right), \ldots, \ell_{q}\left(z_{1}, \ldots, z_{k}\right) \text { are zero. } \tag{11}
\end{equation*}
$$

The first claim in (11) is trivial. To see the second, assume without loss of generality that $z_{1} \neq 0$. Then, for any choice of $\left(\alpha_{2}, \ldots, \alpha_{k}\right) \in \mathbb{F}_{p}^{k-1}$, there is precisely one choice for $\alpha_{1} \in \mathbb{F}_{p}$ that makes $\alpha_{1} z_{1}+\cdots+\alpha_{k} z_{k}=0$.

Now give an arbitrary direction to each edge $(u, v) \in E$ and consider the system $\Lambda$ of linear equations

$$
\left\{\ell_{j}\left(\hat{\sigma}\left(x_{1}^{v}\right)-\hat{\sigma}\left(x_{1}^{u}\right), \hat{\sigma}\left(x_{2}^{v}\right)-\hat{\sigma}\left(x_{2}^{u}\right), \ldots, \hat{\sigma}\left(x_{k}^{v}\right)-\hat{\sigma}\left(x_{k}^{u}\right)\right)=0: j \in[q] \text { and }(u, v) \in E\right\} .
$$

(We view $\Lambda$ as a multiset, so the trivial equation $0=0$ arising from the linear form $\ell_{j}$ with $\alpha_{1}=\alpha_{2}=\cdots=\alpha_{k}=0$ occurs $m$ times, a convention that makes the following calculation simpler.) Denote by $\operatorname{sat}(\hat{\sigma})$ the number of satisfied equations in $\Lambda$. Then, from (11),

$$
\operatorname{sat}(\hat{\sigma})=q \operatorname{mono}(\sigma)+\frac{q}{p}(m-\operatorname{mono}(\sigma))
$$

and hence

$$
\operatorname{mono}(\sigma)=\frac{p}{(p-1) q} \operatorname{sat}(\hat{\sigma})-\frac{m}{p-1}
$$

Noting that $1+\gamma=\lambda^{-q(p-1) / p}$,

$$
\begin{align*}
\sum_{\sigma: V \rightarrow[q]}(1+\gamma)^{\operatorname{mono}(\sigma)} & =\sum_{\hat{\sigma}: X \rightarrow \mathbb{F}_{p}}(1+\gamma)^{(p /(p-1) q) \operatorname{sat}(\hat{\sigma})-m /(p-1)} \\
& =\lambda^{q m / p} \sum_{\hat{\sigma}: X \rightarrow \mathbb{F}_{p}} \lambda^{-\operatorname{sat}(\hat{\sigma})} \\
& =\lambda^{-(1-1 / p) q m} \sum_{\hat{\sigma}: X \rightarrow \mathbb{F}_{p}} \lambda^{\mathrm{unsat}(\hat{\sigma})} \tag{12}
\end{align*}
$$

where $\operatorname{unsat}(\hat{\sigma})=q m-\operatorname{sat}(\hat{\sigma})$ is the number of unsatisfied equations in $\Lambda$.
The system $\Lambda$ has $q m$ equations in $k n$ variables, so we may write it in matrix form $A \hat{\boldsymbol{\sigma}}=\mathbf{0}$, where $A$ is a $(q m \times k n)$-matrix, and $\hat{\boldsymbol{\sigma}}$ is a $k n$-vector over $\mathbb{F}_{p}$. The columns of $A$ and the components of $\hat{\boldsymbol{\sigma}}$ are indexed by pairs $(i, v) \in[k] \times V$, and the $(i, v)$-component of $\hat{\boldsymbol{\sigma}}$ is $\hat{\sigma}\left(x_{i}^{v}\right)$. Enumerating the columns of $A$ as $\mathbf{a}_{i}^{v} \in \mathbb{F}_{p}^{q m}$ for $(i, v) \in[k] \times V$, we may re-express $\Lambda$ in the form

$$
\sum_{i \in[k], v \in V} \hat{\sigma}\left(x_{i}^{v}\right) \mathbf{a}_{i}^{v}=\mathbf{0}
$$

where $\mathbf{0}$ is the length- $q m$ zero vector. Then unsat $(\hat{\sigma})$ is the Hamming weight of the length- $q m$ vector $\mathbf{b}(\hat{\sigma})=\sum_{i, v} \hat{\sigma}\left(x_{i}^{v}\right) \mathbf{a}_{i}^{v}$. As $\hat{\sigma}$ ranges over all assignments $X \rightarrow \mathbb{F}_{p}$, so $\mathbf{b}(\hat{\sigma})$ ranges over the vector space (or code)

$$
C=\left\{\sum_{i, v} \hat{\sigma}\left(x_{i}^{v}\right) \mathbf{a}_{i}^{v} \mid \hat{\sigma}: X \rightarrow \mathbb{F}_{p}\right\}=\left\langle\mathbf{a}_{i}^{v} \mid i \in[k], v \in V\right\rangle
$$

generated by the vectors $\left\{\mathbf{a}_{i}^{v}\right\}$.

We will argue that the mapping sending $\hat{\sigma}$ to $\mathbf{b}(\hat{\sigma})$ is $q$ to 1 , from which it follows that $\sum_{\hat{\sigma}} \lambda^{\mathrm{unsat}(\hat{\sigma})}$ is $q$ times the weight enumerator of the code $C$. Then, from (10) and (12), letting $M$ be any generating matrix for $C$,

$$
Z_{\mathrm{Potts}}(G ; q, \gamma)=q \lambda^{-(1-1 / p) q m} W_{M}(\lambda) .
$$

To see where the factor $q$ comes from, consider the assignments $\hat{\sigma}$ satisfying

$$
\begin{equation*}
\sum_{i \in[k], v \in V} \hat{\sigma}\left(x_{i}^{v}\right) \mathbf{a}_{i}^{v}=\mathbf{b}, \tag{13}
\end{equation*}
$$

for some $\mathbf{b} \in \mathbb{F}_{p}^{q m}$. For every $i \in[k]$ and every edge $(u, v) \in E$, there is an equation in $\Lambda$ specifying the value of $\hat{\sigma}\left(x_{i}^{v}\right)-\hat{\sigma}\left(x_{i}^{u}\right)$. Thus, since $G$ is connected, the vector $\mathbf{b}$ determines $\hat{\sigma}$ once the partial assigment $\left(\hat{\sigma}\left(x_{1}^{r}\right), \ldots, \hat{\sigma}\left(x_{k}^{r}\right)\right)$ is specified for some distinguished vertex $r \in V$. Conversely, each of the $q$ partial assignments $\left(\hat{\sigma}\left(x_{1}^{r}\right), \ldots, \hat{\sigma}\left(x_{k}^{r}\right)\right)$ extends to a total assignment satisfying (13).

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