Multiple-Size Divide-and-Conquer Recurrences*

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1 Introduction

This note reports a tight asymptotic solution to the following recurrence on all positive integers n:

$$T(n) = c \cdot n^{\alpha} \cdot \log^{\beta} n + \sum_{i=1}^{k} a_i \cdot T(\lceil b_i \cdot n \rceil) \quad \text{for } n \geq n_0, \tag{1}$$

$$0 < T(n) \leq d \quad \text{for } n < n_0, \tag{2}$$

where

- $\alpha \geq 0, \beta \geq 0, c > 0, d > 0$,
- k is a positive integer,
- $a_i > 0$ and $1 > b_i > 0$ for $i = 1, \ldots, k$,
- $n_0 \geq \max_{i=1}^{k} \frac{1}{1-b_i}$.

Since $n_0 \geq \max_{i=1}^{k} \frac{1}{1-b_i}$, $\lceil b_i \cdot n \rceil \leq n - 1$ for all b_i and $n \geq n_0$. Thus, the $T(n)$ term on the left-hand side of (1) is defined on T-terms with smaller n, and (2) properly specifies the initial values of T.

A special case of this recurrence, namely, $k = 1$, is discussed in [2, 5] and standard textbooks on algorithms and is used extensively to analyze divide-and-conquer strategies [1, 4]. A specific recurrence with $k = 2$ is used to analyze a divide-and-conquer algorithm for selecting a key with a given rank [1, 3, 4].

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Let $g(x) = \sum_{i=1}^{k} a_i \cdot b_i^x$. The *characteristic equation* of the general recurrence is the equation $g(x) = 1$. Our solution to the general recurrence is summarized in the following theorem.

Theorem 1 *If r is the solution to the characteristic equation of the general recurrence, then*

$$T(n) = \begin{cases} \Theta(n^r) & \textit{if } r > \alpha; \\ \Theta(n^\alpha \log^{1+\beta} n) & \textit{if } r = \alpha; \\ \Theta(n^\alpha \log^\beta n) & \textit{if } r < \alpha. \end{cases}$$

The key ingredient of our proof for this theorem is the notion of a characteristic equation. With this new notion, our proof is essentially the same as that of the case with $k = 1$ [1, 2, 4, 5]. This note concentrates on elaborating the characteristic equation's role in our proof by detailing an upper bound proof for a certain case. Once this example is understood, it is straightforward to reconstruct a general proof for Theorem 1. Consequently, we omit the general proof for the sake of brevity and clarity.

2 An Example

This section discusses the general recurrence with $k = 3$. To further focus our attention on the characteristic equation's role, we assume that $\beta = 0$, r is a positive integer, and $r > \alpha$. Then, according to Theorem 1, $T(n) = \Theta(n^r)$. We will only prove $T(n) = O(n^r)$. The lower bound proof is similar.

Let $S(n) = f_1 \cdot n^r - f_2 \cdot n^{r-\frac{1}{2}} - f_3 \cdot n^\alpha$. It suffices to show that there exist some positive constants f_1, f_2, f_3 such that $T(n) = O(S(n))$. These constants and some others are chosen as follows. Without loss of generality, we assume $b_1 < b_2 < b_3$.

$$\begin{aligned} f_3 &= \frac{c}{g(\alpha) - 1}; \\ f_2 &= \text{any positive constant}; \\ f_1 &= f_2 + f_3 + 1; \\ m_0 &= \max\{n_0, \frac{1}{b_1}, \left(\frac{f_1 \cdot 2^r \cdot \frac{1}{b_1}}{f_2 \cdot \left(g(r - \frac{1}{2}) - 1\right)}\right)^2\}; \\ M &= \max_{n < m_0}\{1, T(n)\}. \end{aligned}$$

Note that since $0 < b_i < 1$ for all b_i, g is a decreasing function. Then since $r > \alpha$ and $r > r - \frac{1}{2}$, $g(\alpha) > 1$ and $g(r - \frac{1}{2}) > 1$. Thus, the above constants are all positive. We next consider the following new recurrence:

$$\begin{aligned} R(n) &= c \cdot n^\alpha + a_1 \cdot R(\lceil b_1 \cdot n \rceil) + a_2 \cdot R(\lceil b_2 \cdot n \rceil) + a_3 \cdot R(\lceil b_3 \cdot n \rceil) && \text{for } n \geq m_0, \\ R(n) &= 1 && \text{for } n < m_0, \end{aligned} \tag{3}$$

It can be shown by induction that $T(n) \leq M \cdot R(n)$ for all n. Thus, to prove $T(n) = O(S(n))$, it suffices to show $R(n) \leq S(n)$ for all n.

Base Case: $R(m) \leq S(m)$ for all $m < m_0$. This follows from the choice of f_1. Given some $n \geq m_0$, we need to show $R(n) \leq S(n)$.
Induction Hypothesis: $R(m) \leq S(m)$ for all integers m where $m_0 \leq m < n$.
Induction Step:

$$\begin{aligned} R(n) &\leq c{\cdot}n^\alpha + a_1{\cdot}S(\lceil b_1{\cdot}n\rceil) + a_2{\cdot}S(\lceil b_2{\cdot}n\rceil) + a_3{\cdot}S(\lceil b_3{\cdot}n\rceil) && (4) \\ &\leq c{\cdot}n^\alpha + f_1{\cdot}g(r){\cdot}(n+\frac{1}{b_1})^r - f_2{\cdot}g(r-\frac{1}{2}){\cdot}n^{r-\frac{1}{2}} - f_3{\cdot}g(\alpha){\cdot}n^\alpha && (5) \\ &\leq c{\cdot}n^\alpha + f_1{\cdot}g(r){\cdot}n^r + f_1{\cdot}2^r{\cdot}n^{r-1}{\cdot}\frac{1}{b_1} - f_2{\cdot}g(r-\frac{1}{2}){\cdot}n^{r-\frac{1}{2}} - f_3{\cdot}g(\alpha){\cdot}n^\alpha && (6) \end{aligned}$$

In this above derivation,

- (4) follows from (3), the inequality $m_0 \geq n_0$, the base step and the induction hypothesis;
- (5) follows from the fact that $\lceil b_i{\cdot}n\rceil \leq b_i{\cdot}(n+\frac{1}{b_1})$;
- (6) follows from the fact that $(n+\frac{1}{b_1})^r \leq n^r + 2^r{\cdot}n^{r-1}{\cdot}\frac{1}{b_1}$ because r is a positive integer and $m_0 \geq \frac{1}{b_1}$.

To finish the induction step, note that the right-hand side of (6) is at most $S(n)$ as desired for the following reasons.

- By the choice of f_3, $c{\cdot}n^\alpha + f_3{\cdot}g(\alpha){\cdot}n^\alpha \leq -f_3{\cdot}n^\alpha$.
- Since $m_0 \geq \left(\frac{f_1\cdot 2^r\cdot\frac{1}{b_1}}{f_2\cdot(g(r-\frac{1}{2})-1)}\right)^2$, $f_1{\cdot}2^r{\cdot}n^{r-1}{\cdot}\frac{1}{b_1} - f_2{\cdot}g(r-\frac{1}{2}){\cdot}n^{r-\frac{1}{2}} \leq -f_2{\cdot}n^{r-\frac{1}{2}}$.

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