## Necessary and Sufficient Conditions on Partial Orders for Modeling Concurrent Computations

Himanshu Chauhan The University of Texas at Austin himanshu@utexas.edu

#### ABSTRACT

Partial orders are used extensively for modeling and analyzing concurrent computations. In this paper, we define two properties of partially ordered sets: width-extensibility and *interleaving-consistency*, and show that a partial order can be a valid state based model: (1) of some synchronous concurrent computation iff it is width-extensible, and (2) of some asynchronous concurrent computation iff it is widthextensible and interleaving-consistent. We also show a duality between the event based and state based models of concurrent computations, and give algorithms to convert models between the two domains. When applied to the problem of checkpointing, our theory leads to a better understanding of some existing results and algorithms in the field. It also leads to efficient detection algorithms for predicates whose evaluation requires knowledge of states from all the processes in the system.

#### 1. INTRODUCTION

The 'happened-before' relation introduced by Lamport [14] is a prevalent technique for modeling executions of distributed as well shared memory concurrent programs. The relation models causality and imposes a partial order on the set of events that occur in a computation. For a large number of applications, models based on events of the computation provide adequate basis for analysis. But for many applications such as global predicate detection [10] and checkpointing [15], it is beneficial to model a distributed computation as a partial order on states of the involved processes. Events and states, however, are fundamentally different concepts. Events are instantaneous and states have duration. A state captures values of all the variables (including program counter) at a process, whereas an event captures the transition of the system from one state to the other<sup>1</sup>. Although, there are multiple papers [10, 11, 7] that

Vijay K. Garg The University of Texas at Austin garg@ece.utexas.edu

model computations as partially ordered sets (posets), there is no clear theory that brings out the distinction between posets used for modeling event based executions and those used for modeling state based executions. This paper's first contribution is in establishing such a theory. For example, consider the posets in Fig. 3. Are they valid event based (or state based) models for some computation? What is the class of posets that characterize event based and state based models — specifically, can every poset be a model for some computation or there exist some restrictions on posets that model the computations in event based or state based models? Additionally, any model of a concurrent computation must define the notion of a consistent global state. Are the definitions different in state based and event based models? One of the main goals of this paper is to establish results that form a basis to answer all these questions in a definitive manner. We study the relationship between the event based models and state based models, and characterize the exact class of posets that can be used to model computations in either framework. We show a duality between the two models that allows easy translation of algorithms from one model to the other. In short, the key contributions of this paper are the following:

- we define two properties on posets: *width-extensibility*, and *interleaving-consistency*, and show that they are necessary and sufficient conditions for posets modeling states of concurrent computations.
- we give algorithms to translate event based models to state based models and vice-versa. We establish the correspondence between the notions of the consistent global states in these two models.
- we show applications of our theory to the areas of checkpointing and predicate detection (in Section 6).

The rest of this paper is organized as follows. Section 2 covers the background concepts about modeling the concurrent computations as posets, and well-established concepts of event based models of computations. Section 3 defines the state based models, and shows how to generate them from event based models. Sections 4 and 5 give complete characterization of state based models for synchronous and asynchronous concurrent computations. We conclude in Section 6 by discussing the applications of our theory to the fields of checkpointing and predicate detection.

<sup>&</sup>lt;sup>1</sup>Alternatively, one may model states as instantaneous and events with duration. The point is that either the state or the event must be modeled with duration.

### 2. BACKGROUND & TERMINOLOGY

We use the term *program* to represent a finite set of instructions, and *computation* to represent an execution of a program. In this paper, we restrict our focus to finite computations — computations that terminate within bounded time. An event (of a computation) is a term that denotes depending on the context of the problem — the execution of a single instruction or a collection of instructions together. A concurrent computation is a computation involving more than one processes/threads — it is possible that the instructions executed by different processes/threads are different. Hence, a distributed computation is a concurrent computation without shared memory processes in which inter-process communication is possible only through message-passing. For modeling concurrent computations, the happened-before relation  $(\rightarrow)$  is defined as follows. The relation  $\rightarrow$  on the set of events of a computation is the smallest relation that satisfies the following three conditions: (1)If a and b are events in the same process and a occurs before b, then  $a \to b$ . (2) For a distributed system, if a is the sending of a message and b is the receipt of the same message, then  $a \rightarrow b$ . For a shared memory system, if a is the release of a lock by some thread and b is the subsequent acquisition of that lock by any thread then  $a \to b$ . (3) If  $a \to b$  and  $b \to c$  then  $a \to c$ .

Formally, a finite partially ordered set (*poset* in short) is a pair  $P = (E, \rightarrow)$  where E is a finite set and  $\rightarrow$  is an irreflexive, antisymmetric, and transitive binary relation on E [5]. We obtain a poset when we apply the happenedbefore ( $\rightarrow$ ) on the set of events of a finite computation. Let E be the set of events. Consider two events  $a, b \in E$ . If either  $a \rightarrow b$  or  $b \rightarrow a$ , we say that a and b are comparable; otherwise, we say a and b are incomparable or concurrent (in the context of concurrent computations), and denote this relation by  $a \parallel b$ . Observe that  $a \parallel b \wedge b \parallel c \not\Rightarrow a \parallel c$ .

It is important to note that multiple computations could have the identical posets as their model.

#### **2.1** Concepts on Posets

Let  $P = (E, \rightarrow)$  be a finite poset as defined above. A subset  $Y \subseteq E$  is called an *chain (antichain)*, if every pair of distinct points from Y is comparable (incomparable) in P. The *height* of a poset is defined to be the size of a largest chain in the poset. The *width* of a poset is defined to be the size of a largest antichain in the poset. All antichains of size equal to the width of the poset are called *width-antichains* in this paper. Let  $\mathcal{A}(P)$  denote the set of all width-antichains of P. Order  $\leq$  is defined over  $\mathcal{A}(P)$  as:

 $A \leq B \ (A,B \in \mathcal{A}(P)) \text{ iff } \forall a \in A, \exists b \in B : a \leq b \text{ in } P.$ 

We model processes/threads as chains of posets, and thus events/states of every process/thread form a totally ordered chain. A family  $\pi = (C_i \mid i = 1, 2..., n)$  of chains of P is called a chain partition of P if  $\bigcup C_i \mid i = 1, 2..., n = P$ .

Given a subset  $Y \subseteq E$ , the *meet* of Y, if it exists, is the greatest lower bound of Y and the *join* of Y is the least upper bound. A poset  $P = (X, \leq)$  is a *lattice* if joins and meets exist for all finite subsets of X. Let P be a poset with a given chain partition of width w. In a concurrent computation, P is the set of events executed under the happened-before

partial order. Each chain would correspond to a total order of events executed on a single process. In such a poset, every element e can be identified with a tuple (i, k) which represents the kth event in the *i*th process;  $1 \le i \le w$ .

A subset Q is a *downset* (also called *order ideal*), of P if it satisfies the constraint that if f is in Q and e is less than or equal to f, then e is also in Q. When a computation is modeled as a poset of events, the downsets are called *consistent cuts*, or *consistent global states* [4]. Throughout this paper, we use the term consistent cut. The set of downsets is closed under both union and intersection and therefore forms a lattice under the set containment order [5].

## 2.2 Event based Model of Concurrent Computations

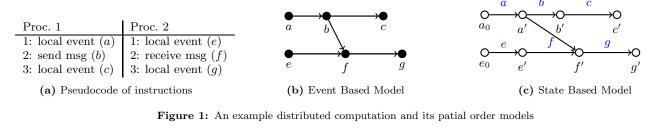
As discussed earlier, a concurrent computation is usually modeled as a set of events, E, together with a partial order happened-before [14], denoted by  $\rightarrow$ . Implicit in this model is the partition of E into chains corresponding to the processes on which the events are executed. This partition is called a *chain partition*. We make this partition explicit in our model because the translation of the event based model into the state based model depends upon it.

**Definition 1** (Event based model of computation). A concurrent computation on n processes is modeled by  $\hat{E} = (E, \rightarrow \pi)$ , where E is the set of events,  $\rightarrow$  is the happened-before relation on E, and  $\pi$  maps every event to a subset of processes from  $\{1..n\}$  such that for all  $i \in \{1..n\}$  :  $E_i = \{e \in E \mid i \in \pi(e)\}$  is totally ordered under  $\rightarrow$ .

Here,  $\pi$  is a chain partition of poset defined by  $(E, \rightarrow)$ . Intuitively, in the context of concurrent computations,  $\pi$  maps events executed on a single process to a total order such that  $E_i$  is the totally ordered set of events executed on process/thread  $C_i$ . Note that an event, such as execution of a barrier, could be assigned to multiple processes. If an event  $e \in E_i \cap E_i$ , then e is a 'shared' event for processes  $C_i$  and Fig. 1b shows the event based model  $C_j$ . of a distributed computation resulting from the execution of the pseudocode instructions listed in Fig. 1a. Fig. 2b shows the event based model of a concurrent computation on two processes that synchronize using a barrier (as per the instructions listed in Fig. 2a). Note that the model of Fig. 2b allows us to represent synchronous messages where the sender blocks for the receiver to be ready. Such synchronous messages are represented by a single event e such that  $\pi(e)$  includes the sender as well as the receiver. The model also allows us to represent barriers which require multiple processes to wait until all the processes participating in the barrier execute it. It can also model behavior of finite communicating sequential processes [3].

**Note:** In all the figures throughout this paper, events are depicted with dark filled circles, and states are depicted with empty circles.

Generally, the analysis of concurrent computations requires reasoning over the valid states of the system that could occur in these computations. These states are commonly called *consistent global states* or *consistent cuts*.



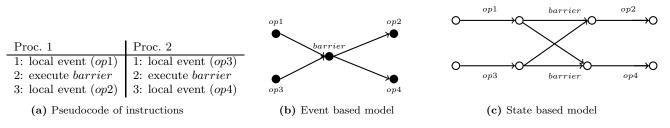


Figure 2: A computation with a barrier and its partial order models

**Definition 2** (Consistent cut in event based model). Given an event based model  $(E, \rightarrow, \pi)$  of a computation,  $G \subseteq E$ is a consistent cut of the computation if  $\forall e, f \in E : (f \in G) \land (e \rightarrow f) \Rightarrow (e \in G)$ .

Note that this definition is independent of  $\pi$  and coincides with the definition of a *down-set* of a poset [5]. It is well known that the set of downsets forms a distributive lattice. Conversely, Birkhoff showed that every finite distributive lattice can be generated as the set of downsets of a poset [2]. Thus, finite distributive lattices completely characterize the set of consistent cuts in the event based model.

The consistent cuts of the event based model in Fig. 1b are:  $\{\}, \{a\}, \{e\}, \{a, b\}, \{a, e\}, \{a, b, c\}, \{a, b, e\}, \{a, b, c, e\}, \{a, b, e, f\}, \{a, b, c, e, f\}, \{a, b, e, f, g\}, \{a, b, c, e, f, g\}.$ 

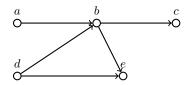
## 3. MODELING COMPUTATIONS USING STATES

For many applications in concurrent debugging [16], and predicate detection in distributed systems it is more natural to model a computation using states rather than events. For example, we may be interested in the cut (global state) in which all processes have taken their local checkpoint. We first give an intuition for state based model of concurrent computations. An event is always executed in some state, and the state before the event's execution 'existed-before' the state resulting from the execution. The existed-before relation between states is denoted using "<". The diagram (denoting the happened-before relation) of the model based on events in Fig. 1b corresponds to the state based model shown in Fig. 1c. In this figure, the execution of event *a* gets translated into an edge between two states: initial state  $a_0$ (that existed before a was executed), and state a' (the state immediately after a's execution). Thus, we have  $a_0 < a'$  in the state based model.

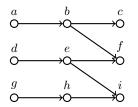
Although some concepts carry over from events to states, there are some important differences. For example, any poset of events in which all events on a single process are totally ordered can be a model of some concurrent computation in the happened-before model. But, not every poset of states is a valid concurrent computation. Consider the poset in Fig. 3a. If this poset were to be used as a state based model of a computation, the model would be incorrect — because even if the modeled states form a poset, the equivalent event based model would have a cycle (as shown in Fig. 5b)<sup>2</sup>. Thus, we can allow only those partial orders on states that do not induce cycles on the order on events.

We claim that a poset can only be a valid state based model of a concurrent computation if it satisfies a notion called *width-extensibility*.

**Definition 3** (Width-extensible Poset). A poset (X, <) is width-extensible if and only if for every antichain  $A \subseteq X$ , there exists a width-antichain W containing A.



(a) Not width-extensible: no width-antichain for  $\{b\}$ 



(b) Not width-extensible: no width-antichain for  $\{b, i\}$ 

Figure 3: Invalid posets under the state based model

Informally, when states of a concurrent computation are modeled as a poset, this property requires that for any set

 $<sup>^2{\</sup>rm The}$  techniques involved in generating event based model from state based model are in the next section.

of incomparable local states there is a possible consistent cut that includes these local states. We will show later that in the state based model, the consistent cuts correspond to width-antichains (and not down-sets). The poset in Fig. 3a is not width-extensible because there is no width-antichain that contains b.

In the above definition of width-extensible posets, we can not substitute "for all antichains" by "for all antichains of size 1". In the example of Fig. 3b, there is a width-antichain for every individual element a to i. This can be easily verified as  $\{a, d, g\}$ ,  $\{b, e, h\}$ , and  $\{c, f, i\}$  are all width-antichains. But there is no width-antichain that contains  $\{b, i\}$ . Hence, the poset is not width-extensible.

We now show a surprising result: it is sufficient to restrict our attention to antichains of size two for checking widthextensibility.

**Theorem 1.** A poset (X, <) is width-extensible if and only if for every antichain A of size at most two, there exists a width-antichain W containing A.

Proof. The necessity is obvious — the definition of widthextensibility demands that every antichain is contained in some width-antichain. Hence, for (X, <) to be width-extensible, antichains of size at most two must also be contained in a width-antichain. We now prove sufficiency. We want to prove that if every antichain of size at most two is contained in a width-antichain, then every antichain (of any size) is also contained in a width-antichain. Let w be the width of the poset (X, <) and  $\{C_1, C_2, ..., C_w\}$  be a chain partition of size w. Consider an antichain A of size  $k, 3 \le k \le w$ . If w = k, then A itself is a width-antichain, and we have the result. Suppose w > k, and A is not contained in any widthantichain. Hence, there is some chain  $C_i$  such that A does not have any elements from  $C_i$ . We know that for any pair of elements  $a, b \in A$ , with  $a \neq b$ , the antichain  $\{a, b\}$  is widthextensible. Let  $I_i(a, b)$  denote the maximal interval on  $C_i$ that contains all the elements that are incomparable to both a and b. As  $\{a, b\}$  is width-extensible, we know that  $I_i(a, b)$ is non-empty. Now consider  $a, b, c \in A$ , where all three are distinct. The width-extensibility of size two antichains guarantees that  $I_i(a, b)$ ,  $I_i(b, c)$ , and  $I_i(a, c)$  are all non-empty. Since every pair of these intervals have non-empty intersection, and all intervals are sets of one or more consecutive states in  $C_i$ , we get that  $I_i(a,b) \cap I_i(b,c) \cap I_i(a,c) \neq \phi$ . This means that  $\exists d \in C_i : (d \parallel a) \land (d \parallel b) \land (d \parallel c)$ , i.e. d is concurrent to a, b, and c. Hence, d can be added to A. By repeating this argument for all chains that do not have any element in A, we can extend A to a width-antichain. 

We can now define the state based model of a concurrent computation as follows:

**Definition 4** (State based model of concurrent computations). A concurrent computation on n processes is modeled by  $\hat{S}$ : a tuple  $(S, <, \tau)$ , where S is the set of local states, (S, <) is a width-extensible poset, and  $\tau$  is a map from S to  $\{1..n\}$  such that for all distinct states  $s, t \in S$  for all  $i \in \{1..n\}, S_i = \{s \in S \mid i \in \tau(s)\}$  is totally ordered under <. i.e.,  $\tau(s) = \tau(t) \Rightarrow (s < t) \lor (t < s)$ . Thus,  $\tau$  partitions S such that every block of the partition  $S_i$  is totally ordered. The relation < between states captures the 'existed-before' notion discussed in the first para of Section 3. Fig. 1c and 2c, are corresponding state based models of event based models shown in Fig. 1b and 2b. Note that in these figures (of state based models), the events are shown as edge labels above the edges that capture < (existed-before) relation on the states.

We now show the difference in the definitions of consistent cuts in the state based and event based model.

**Definition 5** (Consistent cut in state based model). Under the state based model,  $(S, <, \tau)$ , of a concurrent computation , a subset  $T \subseteq S$  of size equal to the width of poset (S, <) is a consistent cut if  $\forall s, t \in T : s \parallel t$ .

The order "<" over consistent cuts is defined using the " $\leq$ " relation defined over width-antichains in Section 2. Under the state based model, for any two consistent cuts A, B we have: A < B iff  $A \leq B \land A \neq B$ . Hence,  $A < B \Rightarrow \exists a \in A, \exists b \in B : a < b$  in (S, <). It is clear that the consistent cuts in state based model correspond to width-antichains of the poset.

The consistent cuts of the state based model of Fig. 1c are:  $\{a_0, e_0\}, \{a', e_0\}, \{a_0, e'\}, \{b', e_0\}, \{a', e'\}, \{c', e_0\}, \{b', e'\}, \{c', e'\}, \{b', f'\}, \{c', f'\}, \{b', g'\}, \{c', g'\}.$ 

At this point we have two notions of a consistent cut of a concurrent computation: one in the event based model (Defn. 2) and the other in the state based model (Defn. 5). Dilworth [6] proved that the set of all width-antichains also forms a distributive lattice, and Koh [13] showed that every finite distributive lattice can be generated as the set of width-antichains of a poset. The lattice of width-antichains is in general a sublattice of the lattice of downsets. Thus, the notion of consistent global states is different in event based and state based models, a distinction that has not been explored in distributed computing literature. It is also important to question that what is the relationship between these two definitions? In the next section, we show that there is a 'one-to-one' correspondence between consistent cuts in the event based and the state based models.

# 3.1 Translation between event based and state based models

Let  $\hat{E} = (E, \rightarrow, \pi)$  be an event based model of a computation on *n* processes/threads. Let  $\pi$  partition *E* into *n* chains:  $(E_i \mid i = 1, 2, ..., n)$ . For each i = 1, 2, ..., n, let  $|E_i| = n_i (\geq 1)$ . Suppose the elements of  $E_i$  are named as follows:  $E_i : (i, 1) \rightarrow (i, 2) \rightarrow ... \rightarrow (i, n_i - 1) \rightarrow (i, n_i)$ . Note that if an event is 'shared' between two processes *i* and *j*, then it will have two labels (i, x) and (j, y), with  $1 \leq x \leq n_i$ , and  $1 \leq y \leq n_j$ .<sup>3</sup> We generate a state based model  $\hat{S} = (S, <, \tau)$  from  $\hat{E}$  using the following function.

<sup>&</sup>lt;sup>3</sup>By extension of this rule, an event that is 'shared' between k processes would have k labels.

Function ES Transform: For each i = 1, 2, ..., n, let  $S_i$  be an  $|n_i + 1|$  element chain where  $n_i = |E_i|$  as above. Define the elements in  $S_i$  as follows:

$$S_i: [i,0] < [i,1] < \dots < [i,n_i-1] < [i,n_i].$$

Let  $S = \bigcup_{i=1}^{n} S_i$  and define a binary relation "<" on S by putting [i, r] < [j, s] in S  $(i, j = 1, 2, ..., n; 0 \le r \le n_i, 0 \le s \le n_j)$  iff:

- r < s, if i = j
- (i, r+1) and (j, s) are both present in E
- and (i, r+1) < (j, s) in E if  $i \neq j$ .

A special case of this transform, on disjoint chain partitions, was used by Koh in [13] to prove properties of lattice of width-antichains.

Fig. 4 gives illustrations of the application of this transform.

In the generated  $\hat{S}$ ,  $\tau$  is dependent on the chain partition  $\pi$  in  $\hat{E}$ . Intuitively, every state chain  $S_i$  contains the states of process i, such that event (i, k) in  $E_i$ , here  $1 \leq k \leq n_i$ , causes a transition from state [i, k-1] to [i, k]. On chain  $S_i$ , the state [i, 0] represents the initial state of the process i, and  $[i, n_i]$  represents the final state of the process i. The worst-case complexity of the ES transform is  $\mathcal{O}(|E|^2)$ .

We show that this  $\hat{S}$ , generated by applying the ES transform on  $\hat{E}$ , is a valid state based model of the concurrent computation, i.e., it is a width-extensible poset. We first show that it is a poset.

**Lemma 1.** If  $\hat{S}$  is the result of applying ES transform on an event based model  $\hat{E} = (E, \rightarrow, \pi)$  of a concurrent computation then  $\hat{S}$  is a poset under the "<" relation.

*Proof.* We show that the relation "<" on S is transitive and antisymmetric, and thus irreflexive.

- Claim (i) The relation "<" is asymmetric. Proof: Let  $[i, r], [j, s] \in S$  such that [i, r] < [j, s]. Clearly,  $[j, s] \not< [i, r]$  if i = j; otherwise we would get  $s \to r$  in E. Assume  $i \neq j$  and [j, s] < [i, r]. Then by definition, we have  $(i, r + 1) \to (j, s) \to (j, s + 1) \to$  (i, r) in E, which is impossible as it violates the asymmetry of  $\to$  in E.
- Claim (ii) The relation "<" is transitive. Proof: Let  $[i, r], [j, s], [k, t] \in \hat{S}$  such that [i, r] < [j, s]and [j, s] < [k, t]. Assume  $i \neq j$  and  $k \neq j$ . Then we have  $(i, r + 1) \rightarrow (j, s) \rightarrow (j, s + 1) \rightarrow (k, t)$  and hence  $(i, r + 1) \rightarrow (k, t)$  in E, which implies that [i, r] < [k, t]whether i = k or  $i \neq k$ . The cases for i = j or j = kcan be proved similarly.

Hence 
$$\hat{S}$$
 forms a poset under the "<" relation.

The following lemma proves the 'one-to-one' relation between consistent cuts of event based and state based models of a concurrent computation. **Lemma 2.** Let  $\hat{E} = (E, \rightarrow, \pi)$  and  $\hat{S} = (S, <, \tau)$  be event and state based models of a concurrent computation. Then there is a bijection between consistent cuts of  $\hat{E}$  and  $\hat{S}$ .

Proof. In Appendix A. 
$$\Box$$

Let us now study the properties of the posets that model concurrent computations using states.

## 4. CHARACTERISTICS OF STATE BASED MODELS OF SYNCHRONOUS CONCUR-RENT COMPUTATIONS

The event based model of Defn. 1 accepts chain partitions that allow 'shared' events, which in turn allows modeling synchronous executions. We will show that posets that model such synchronous concurrent computations must be width-extensible. We start by showing that  $\hat{S} = (S, <)$  constructed from any  $(E, \rightarrow, \pi)$  by applying the *ES* transform is width-extensible. First, we define the three properties  $\omega_1, \omega_2$ , and  $\omega_3$  of  $\hat{S}$ .

For  $1 \le i, j, k \le n, \hat{S} = (S, <, \tau)$ :

- $(\omega_1) \forall i, j: [i, 0] || [j, 0]$ . All initial states are concurrent.
- $(\omega_2) \forall i, j: [i, n_i] \mid\mid [j, n_j]$ . All final states are concurrent.
- $(\omega_3) \forall i, j, k$ , such that for  $i \neq j \land j \neq k$ :  $[i, s] < [j, t] \land [j, t-1] < [k, u] \Rightarrow [i, s] < [k, u].$

We now prove that these properties are observed in  $\hat{S}$ .

**Lemma 3.**  $\hat{S}$  satisfies  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$ .

*Proof.*  $\omega_1$  follows immediately from the construction of  $\hat{S}$  because there is no state [i, s] such that [i, s] < [i, 0] for any i. That is, on any state chain  $S_i$  there does not exist a state that is a precursor to the initial state of  $S_i$ . Hence, all the initial states must be concurrent.

Similarly,  $\omega_2$  follows when applied to the last states of  $S_i$  in a similar manner because there is no state on any state chain  $S_i$  that is a successor of the final state of  $S_i$ .

$$\begin{split} & \omega_3 \colon [i,s] < [j,t] \land [j,t-1] < [k,u]. \text{ Using the construction} \\ & \text{rules, we can infer that } (i,s+1) \rightarrow (j,t) \land (j,t) \rightarrow (k,u) \text{ in} \\ & E. \text{ Which by transitivity means } (i,s+1) \rightarrow (k,u). \text{ Hence,} \\ & [i,s] < [k,u] \text{ in } S. \end{split}$$

The first condition,  $\omega_1$ , ensures that all n initial states are pairwise concurrent. This is a valid requirement as all the processes would start in some default (individual) state, and at the start of the computation these states would not have any dependency amongst them. The second condition, given by  $\omega_2$ , ensures that all n final states are pairwise concurrent. This is also a valid requirement because irrespective of the events/commands executed, all the n processes end up in some individual final state at the end of the computation. Hence, when the computation is finished all the final states

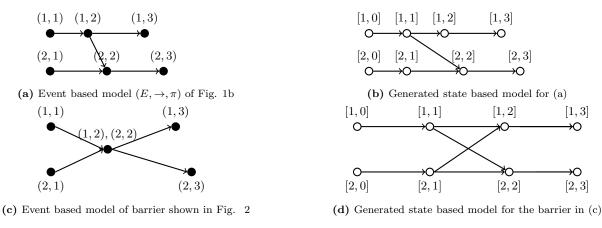


Figure 4: Event to State transform for computations of earlier examples

would not have any dependency amongst them, and thus be concurrent to each other.

The third condition,  $\omega_3$ , guarantees that causal dependency between events under the event based model translates to causal dependency between corresponding states under the state based model. Note that the labels of states in the dependency relation are different from those of events. Suppose that for two events e and f, we have  $e \to f$  under the event based model,  $\hat{E}$ . Then  $\omega_3$  translates that dependency from  $\hat{E}$  to  $\hat{S}$  such that the state preceding the execution of e is guaranteed to have existed <u>before</u> the state that is generated <u>after</u> the execution of f.

We now show that any state based model that is generated by applying the ES transform on an event based model is a valid state based model. To be a valid state based model, it is sufficient that the generated poset be width-extensible.

**Theorem 2.** Let  $\hat{S} = (S, <, \tau)$  be a state based model for some concurrent computation. If  $\hat{S}$  satisfies  $\omega_1, \omega_2$  and  $\omega_3$ , then the poset (S, <) is width-extensible.

*Proof.* We show that any antichain  $A \subset S$  can be extended to a width-antichain. It is sufficient to show that when |A| < n, there exists an antichain  $A \subset B$  such that |B| = |A| + 1. Consider any process  $C_i$  that does not contribute a state to A. We will show that there exists a state in  $S_i$  that is concurrent with all states in A. Let s and s' be two distinct states in A.

We first claim that for any state s and any process  $C_i$ , there exists a nonempty sequence of consecutive states called the "*interval* concurrent to s on  $C_i$ " and denoted by  $I_i(s)$  such that:

- 1.  $I_i(s) \subseteq S_i$  i.e., the interval consists of only states from process  $C_i$ , and
- 2.  $\forall t \in I_i(s) : t \mid | s \text{i.e.}, \text{ all states in the interval are concurrent with } s.$

For a state  $v \in S_i$ , let index(v) denote the index of state v on  $S_i$ . Thus  $0 \leq index(v) \leq n_i$ . Define  $I_i(s).lo = \min\{v \mid v \in S_i \land v \neq s\}$ . This is well-defined since

 $[i, n_i] \not\leq s$  due to  $\omega_2$ . Similarly, on account of  $\omega_1$ , we can define  $I_i(s).hi = \max\{v \mid v \in S_i \land s \not\leq v\}$ . We show that  $I_i(s).lo \leq I_i(s).hi$  by the following case analysis.

**Case 1**: There exists  $v : I_i(s).hi < v < I_i(s).lo$ .

Since  $v < I_i(s).lo$  implies v < s and  $I_i(s).hi < v$  implies s < v, we get a contradiction (v < s < v).

**Case 2**:  $index(I_i(s).hi) + 1 = index(I_i(s).lo)$ .

Let  $I_i(s).lo$  be the  $r^{th}$  state on  $S_i$ , i.e.,  $I_i(s).lo = [i, r]$ . Then,  $I_i(s).hi = [i, r-1]$ . Let s correspond to state [j, t]. From the definition of  $I_i(s).lo$ , [i, r-1] < [j, t]. From the definition of  $I_i(s).hi$ , [j, t] < [i, r]. We now have, [j, t] < [i, r]and [i, r-1] < [j, t]. From  $\omega_3$ , we get [j, t] < [j, t] which contradicts irreflexivity of <.

From the above discussion it follows that  $I_i(s).lo \leq I_i(s).hi$ . Furthermore, for any state t such that  $I_i(s).lo \leq t \leq I_i(s).hi$ ,  $t \not\leq s$  and  $s \not\leq t$  holds. Now that our claim holds, we know that  $I_i(s)$  and  $I_i(s')$  are both non-empty. We show that  $I_i(s) \cap I_i(s') \neq \emptyset$ . If not, without loss of generality assume that  $I_i(s).hi < I_i(s').lo$ . Now there are two possible cases.

**Case 1**:  $index(I_i(s).hi) + 1 = index(I_i(s').lo)$ .

Let  $I_i(s).hi$  be  $r^{th}$  state on  $S_i$ , i.e.,  $I_i(s).hi = [i, r]$ . Then,  $I_i(s').lo = [i, r + 1]$ . Suppose that s = [j, u] and s' = [k, v]. From the definition of  $I_i(s).hi$  we get that [j, u] < [i, r + 1]. From the definition of  $I_i(s').lo$  we get that [i, r] < [k, v]. Hence, from  $\omega_3$ , we get that [j, u] < [k, v] — contradicting that s and s' are concurrent.

**Case 2:** There exists  $v : I_i(s).hi < v < I_i(s').lo$ . This implies that s < v (because  $I_i(s).hi$  precedes v) and v < s' (because v precedes  $I_i(s').lo$ ). Thus s < s', a contradiction with A being an antichain. Therefore,  $I_i(s) \cap I_i(s') \neq \emptyset$ .

Because any interval  $I_i(s)$  is a total order, it follows that:

$$\bigcap_{s \in A} I_i(s) \neq \emptyset$$

We now choose any state in  $\bigcap_{s \in A} I_i(s)$  to extend A.  $\Box$ 

We have established that every poset that provides the three conditions  $\omega_1, \omega_2$ , and  $\omega_3$  is width-extensible. We now show

the converse — every width-extensible poset guarantees these three conditions.

**Theorem 3.** Let (S, <) be a width-extensible poset. Consider any chain-partition  $\tau$  of (S, <). Then,  $\hat{S} = (S, <, \tau)$  satisfies  $\omega_1, \omega_2$  and  $\omega_3$ .

*Proof.* We show the contrapositive. If  $\omega_1$  is violated, then there exists an initial state t such that there exists a state s different from t which is less than t. Then, s is less than all states in the process containing t. Therefore, the antichain  $\{t\}$  cannot be extended to a width-antichain. The proof for  $\omega_2$  is dual.

If  $\omega_3$  is violated, then there exist [i, s], [j, t] and [k, u], where  $i \neq j \land j \neq k$ , such that [i, s] < [j, t] and [j, t-1] < [k, u] but  $[i, s] \not\leq [k, u]$ . We now do a case analysis on the relationship between [i, s] and [k, u].

**Case 1**: [k, u] < [i, s]. (Illustrated in Fig. 6, Appendix B). In this case we claim that there is no width-antichain that contains [i, s]. Since [i, s] < [j, t], for any state w on process  $C_j$  that is concurrent with [i, s], we get  $w \le [j, t - 1]$ . Since [j, t - 1] < [k, u] none of the states on process  $C_k$  greater than [k, u] are eligible to be in the width-antichain with w. Furthermore, all states less than or equal to [k, u] are ineligible because [k, u] < [i, s].

**Case 2**: [k, u] is incomparable with [i, s]. In this case we claim that there is no width-antichain that includes both [k, u] and [i, s]. No state greater than or equal to [j, t] can be included from  $C_j$  because [i, s] < [j, t]. No state less than or equal to [j, t-1] can be included from  $C_j$  because [j, t-1] < [k, u].

Note that  $\omega_3$  only requires  $i \neq j \land j \neq k$ . It is possible that i = k; the proof still holds.

With Theorems 2 and 3, we have established that conditions  $\omega_1, \omega_2$  and  $\omega_3$  are necessary and sufficient for a poset to be width-extensible. We now show that width-extensibility is a sufficient condition for modeling a concurrent computation under the state based model. First, we outline how to generate an event based model of a concurrent computation from (S, <). Let  $\tau$  be any chain partition of (S, <). We construct an event based model  $(E', \rightarrow, \pi')$  of a concurrent computation by applying the *SE* transform (a reverse transform to *ES*) whose steps are shown in Algorithm 1.

In the algorithm, lines 1 - 11 perform a reversal of steps of ES transform. Lines 13 - 18 try to collapse events that are 'shared' between processes by performing a strongly connected component (SCC) decomposition, and using the SCCs for identifying shared events. If an SCC has events from the same process, then that results in a same process cycle an invalid event based computation. If we represent  $\hat{S}$  as a directed graph with m = |S| vertices and d directed edges, then the complexity of SE transform is  $\mathcal{O}(m+d)$ , i.e. linear in size of the graph.

See Appendix B for some illustrations of SE transform's application to examples discussed in this paper.

#### Algorithm 1 SE (State to Event) Transform

**Input:** State Based Model  $\hat{S} = (S, <, \tau)$ **Output:** Event Based Model  $\hat{E} = (E', \rightarrow, \pi)$ 1:  $E'_i \leftarrow \{\}$ 2: for i = 1 to n do 3: for k = 1 to  $n_i$  do 4: Add (i, k) to  $E'_i$ 5: for k = 0 to  $n_i - 1$  do 6: Define  $(i, k) \rightarrow (i, k+1)$  in  $E'_i$ 7:  $\triangleright E'_i$  is now  $(|S_i| - 1)$ -element chain 8:  $E'_{temp} \leftarrow \bigcup_{i=1}^{n} E'_{i}$ 9: for i = 1 to n do 10:for j = 1 to  $n \land j \neq i$  do if [i, r-1] < [j, s] in S then 11: Define  $(i, r) \to (j, s)$  in  $E'_{temp}$ 12:13:  $E' \leftarrow E'_{temp}$ 14: for all  $C_s$  in SCC-Decomposition of  $E'_t emp$  do 15:if each node is  $C_s$  lies on diff. chains then 16:Replace  $C_s$  with one element e in E'17:Assign all labels of nodes in  $C_s$  to e18:else 19:Report S as **not** width-extensible

The next theorem shows that width-extensibility is sufficient for modeling concurrent computations under the state based model.

**Theorem 4.** Let (S, <) be any width-extensible poset. Then, there exists a concurrent computation for which it is the state based model.

*Proof.* We show that there exists a concurrent computation in the event based model such that when we convert that event based computation to state based model, we get the poset (S, <).

We first create a width chain partition  $\tau$  of (S, <) to get  $(S, <, \tau)$ . We then generate an event based model  $\hat{E'} = (E', \rightarrow, \pi')$  from  $(S, <, \tau)$  using SE transform. It can be easily verified that applying the ES transform to  $(E', \rightarrow, \pi')$  leads to  $(S, <, \tau)$ . It suffices to show that  $(E', \rightarrow)$  is a partial order.

Irreflexivity: Assume,  $(i, r) \to (i, r)$  in  $E'(\pi')$ . This would require r < r in  $\hat{S}$  — a contradiction.

Transitivity: Consider  $(i, r) \rightarrow (j, s) \wedge (j, s) \rightarrow (k, t)$ , in E'. First, let us look at the case where  $i \neq j \wedge j \neq k$ .  $(i, r) \rightarrow (j, s)$  in the event based model is possible only if [i, r-1] < [j, s] in  $\hat{S}$ . Similarly, we also get [j, s-1] < [k, t]. Hence:

$$[i, r-1] < [j, s] \land [j, s-1] < [k, t]$$

By using  $\omega_3$  on  $\hat{S}$  we get [i, r-1] < [k, t] in  $\hat{S} \equiv (i, r) \rightarrow (k, t)$  in E'.

When i = j = k, the transitivity of states the same chain is trivial. Now let us consider the case when  $i = j \land j \neq k$ . Then,  $(i,r) \rightarrow (j,s) \land (j,s) \rightarrow (k,t)$  in E' requires r < s, as i = j, and [j, s - 1] < [k,t] in  $\hat{S}$ . Observe that i = jand r < s means that r - 1, s - 1, s form a totally ordered set, such that  $r - 1 \leq s - 1$ . Hence, we get  $[i, r - 1] \leq$  $[j, s - 1] \land [j, s - 1] < [k, t]$ . By transitivity of < in  $\hat{S}$ , this leads to [i, r - 1] < [k, t] which is the desired condition for  $(i, r) \rightarrow (k, t)$  in E'. The proof for the case of  $i \neq$  j, j = k is similar. Finally, consider the case when  $i = k, i \neq j \land r = t$ . In such a case, the original condition in the E' becomes  $(i, r) \rightarrow (j, s) \land (j, s) \rightarrow (i, r)$ . Given that we have  $i \neq j$ , the condition is only possible if (i, r) and (j, s) represent the same shared event — shared between processes/chains i and j. Now that (i, r) and (j, s) represent the same shared event of transitivity on this event is trivially held.

The following lemma combines the results established earlier to show that ES and SE transforms are inverse functions of each other.

**Lemma 4.** Let  $\hat{E} = (E, \rightarrow, \pi)$  be an event based model for some computation and let  $\hat{S}$  be the result of applying ES transform to  $\hat{E}$ . Then, applying SE transform on  $\hat{S}$  results in  $\hat{E}$ .

*Proof.* Follows directly from lemmas 1, 2, and 3 combined with theorems 2, 3, and 4.  $\hfill \Box$ 

Thus, we have established that  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  properties provide a complete characterization of a state based model for a concurrent computation. In the next section, we discuss asynchronous computations, and show that their state based models are a special case of models of concurrent computations formalized in this section.

## 5. CHARACTERISTICS OF STATE BASED MODELS OF ASYNCHRONOUS CONCURRENT COMPUTATIONS

Asynchronous concurrent computations, which are common in distributed systems, are a special type the concurrent computations that cannot have any 'shared' events. Shared events are only possible when the communication between processes is synchronous. Thus, the event based model of asynchronous computations is defined based on a chain partition  $\pi$  in which all chains are disjoint. The event based model of asynchronous concurrent computations(we use the short-form notation ASC from here on) is given by the following definition:

**Definition 6** (Event based model of ASC). An event based model of an ASC on n processes is is a tuple  $(E, \rightarrow, \pi)$  where E is the set of events,  $\rightarrow$  is the happened-before relation on E, and  $\pi$  is a map from E to  $\{1..n\}$  such that for all distinct events  $e, f \in E : \pi(e) = \pi(f) \Rightarrow (e \rightarrow f) \lor (f \rightarrow e).$ 

Thus,  $\pi$  partitions E such that every block of the partition is totally ordered under  $\rightarrow$ .

Such an event based model, with no 'shared' events, leads to a state based model that satisfies stronger properties than those satisfied by the state based model of the previous section. Intuitively, given that the communication between processes is asynchronous, no two processes can make a 'jump' together from their individual states to next states as if there was a 'shared' execution. Hence, the poset (S, <) exhibits a property that we call 'interleaving-consistency'. **Definition 7** (Interleaving-consistent Poset). A poset (X, < ) is interleaving-consistent if for every width-antichain W that is not equal to the biggest width-antichain, there exists a width-antichain W' > W such that  $|W \cap W'| = |W| - 1$ .

Let  $\mathcal{A}(X)$  be the set of all width-antichains of a poset (X, <). The biggest width-antichain of (X, <) is the width-antichain  $A \in \mathcal{A}(X)$  such that  $\nexists A' \in \mathcal{A}(X) : A < A'$ . Informally, interleaving-consistency requires that any possible cut (modeled as a width-antichain) can be advanced on some process to reach another possible cut. Fig. 4a shows an ASC under the event based model, and the corresponding poset of the state based model in Fig. 4b is interleaving-consistent. In contrast, the event based computation in Fig. 4c is not an ASC, and thus the resulting state based model's poset in Fig. 4d is not interleaving-consistent — the processes make a 'jump' together from states [1, 1], [2, 1] to [1, 2], [2, 2].

ASCs are a special kind (subset) of concurrent computations, and thus a partial order modeling states of an ASC must satisfy  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ . In addition, it should also be *interleaving-consistent*. Formally, *interleaving-consistency* of  $\hat{S}$  is captured by the condition  $\psi$  as follows:

$(\psi)$ for $1 \le i, j \le n$ ,	if $i \neq j$ ,	then	$[i,s-1] < [j,t] \Rightarrow$
$\neg ([j, t-1] < [i, s]).$			

Thus, for an ASC, a poset (S, <) that models its states is characterized by  $\omega_1, \omega_2, \omega_3$ , and  $\psi$ . The state based model for ASCs is formally defined as:

**Definition 8** (State based model of ASCs). An asynchronous distributed computation on n processes is modeled by  $\hat{S} = (S, <, \pi)$ , where S is the set of states, < is an irreflexive partial order relation on S such that (S, <) is a width-extensible and interleaving-consistent poset, and  $\pi$  maps every state to a process from  $\{1...n\}$  such that for all  $i \in \{1...n\}, S_i = \{s \in S \mid i \in \pi(s)\}$  is totally ordered under <.

The following set of results establish the properties of state based models of ASCs.

**Lemma 5.** Suppose  $\hat{S} = (S, <, \tau)$  is obtained by applying ES transform on an ASC's event based model  $\hat{E} = (E, \rightarrow, \pi)$ . Then  $\hat{S}$  satisfies  $\omega_1, \omega_2, \omega_3$ , and  $\psi$ .

*Proof.* Since ASCs are a subset of concurrent computations, the conditions  $\omega_1, \omega_2, \omega_3$  continue to be satisfied as shown in Theorem 3. Suppose (S, <) doesn't satisfy  $\psi$  and thus we have  $[i, s-1] < [j, t] \Rightarrow ([j, t-1] < [i, s])$  in (S, <). But this would require  $(i, s) \rightarrow (j, t) \land (j, t) \rightarrow (i, s)$  in E, which is a contradiction.

**Lemma 6.** Let  $\hat{S} = (S, <, \tau)$  be as defined in Lemma 5. Then, (S, <) is interleaving-consistent.

*Proof.* Suppose (S, <) satisfies  $\psi$ , but is not interleavingconsistent. Hence, there is some antichain A of (S, <) that is not the biggest, and still can not be extended along just one process to form another antichain A'. Let  $[i, a_i]$  be the element from chain i that belongs to A. Our assumption means that  $\nexists i : A - \{[i, a_i]\} + \{[i, a_i+1]\}$  is a width-antichain. Hence  $\forall i, \exists j \neq i : [i, a_i] < [j, a_j + 1]$ . Given that S is finite (we can not keep on finding a 'new' j for every 'new' i we consider), we know that to satisfy this requirement there must exist  $k, k \neq j \land k \neq i$  such that  $[j, a_j] < [i, a_i + 1] \land [k, a_k] < [j, a_j + 1] \land [i, a_i] < [k, a_i + 1]$ . See Fig. 7 in Appendix B for an illustration.

From the previous lemma, we know that (S, <) satisfies  $\omega_1, \omega_2, \omega_3$ . Applying  $\omega_3$  we get  $[k, a_k] < [i, a_i + 1]$ . But this leads to  $[i, a_i] < [k, a_i + 1] \land [k, a_k] < [i, a_i + 1] - a$  contradiction with  $\psi$ .

**Theorem 5.** Let (S, <) be any interleaving-consistent and width-extensible poset. Consider any chain partition  $\tau$  of (S, <). Then,  $\hat{S} = (S, <, \tau)$  satisfies  $\omega_1, \omega_2, \omega_3$ , and  $\psi$ .

*Proof.* Width-extensibility guarantees  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ . Suppose  $\psi$  is not satisfied, i.e., there exist [i, s], [j, t], for  $i \neq j$ , such that  $[i, s - 1] < [j, t] \land ([j, t - 1] < [i, s])$ . Let [j, r] be the largest state on  $S_j$  that is incomparable with [i, s - 1]. Note that  $r \leq t - 1$ . It is clear that [j, r] < [j, t] because [i, s - 1] < [j, t]. Since ([j, t - 1] < [i, s]), we also get that [j, r] < [i, s].

Let  $\mathcal{W}$  be the set of all width-antichains that include both [i, s - 1] and [j, r]. Let A be the biggest antichain in  $\mathcal{W}$ . We claim that there does not exist any width-antichain  $A' \geq A$  such that |A' - A| = 1, and thus not satisfying  $\psi$  contradicts with interleaving-consistency. If A' differs from A on a chain different from i and j, then it violates that A is the biggest antichain that contains [i, s - 1] and [j, r]. Hence, to satisfy interleaving-consistency, A' must differ from A on either i or j. Suppose A' - A = [i, s] then, because A' is a width-antichain, we get that [j, r] is incomparable with [i, s], a contradiction. If A' - A = [j, r + 1], then we get that [i, s - 1] is incomparable with [j, r + 1], which contradicts the definition of [j, r].

**Theorem 6.** Let (S, <) be any poset that is width-extensible and interleaving-consistent. Then, there exists an ASC for which it is the state-based model.

*Proof.* Let  $\tau$  be any chain partition of (S, <). Apply SE transform on  $(S, <, \tau)$  to generate an event based model  $(E', \rightarrow)$ . It is trivial to verify that applying ES transform to  $(E', \rightarrow)$  leads to (S, <). It suffices to show that  $(E', \rightarrow)$  is a partial order.

*Irreflexivity:* can be proved using exactly the same argument used in Theorem 4.

Transitivity: Except the case of  $i = k, i \neq j \land r = t$ , apply the same argument as in Theorem 4. For  $i = k, i \neq j \land r = t$ , we use a different argument. In this case, the left hand side of  $(i, r) \rightarrow (j, s) \land (j, s) \rightarrow (k, t)$  is equivalent to  $[i, r - 1] < [j, s] \land [j, s - 1] < [i, r]$  as i = k, r = t. But,  $\psi$  prohibits this case — hence the left hand side is false and the constraint holds trivially.

Similar to Lemma 4, we can now verify the following result.

**Lemma 7.** Let  $\hat{E} = (E, \rightarrow, \pi)$  be an event based model for some ASC and let  $\hat{S}$  be the result of applying ES transform to  $\hat{E}$ . Then, applying SE transform on  $\hat{S}$  results in  $\hat{E}$ .

*Proof.* Follows directly from lemmas 2, 5, and 6 combined with theorems 2, and 6.  $\hfill \Box$ 

#### 6. APPLICATIONS

To conclude, we now discuss two applications of duality between state and event based models of concurrent computations.

#### 6.1 **Predicate Detection**

Our theory applies to detection of global predicates that depend only on the latest events in ASCs. For example, consider a set of processes that execute three kinds of events: internal, message send and *blocking receive*. The blocking receive event blocks the process until it receives a message from some process. It is clear that in absence of in-transit messages, and the last executed event at all processes being a receive event, the system has a communication deadlock. In this example, we require that the last event at each process be a blocking receive. Even if one process is left out, that process could send messages to all other processes to unblock them.

Recall that an ideal Q of a poset  $P = (X, \leq)$  is a widthideal if the set of all maximal elements in Q, denoted by maximal(Q), is a width-antichain of P. Let B be a predicate, and G be a global state of a computation, then B(G)denotes that B is true on G. A width-predicate is defined as follows.

**Definition 9** (Width-Predicate). A global predicate B in a distributed computation on n processes is a width-predicate if  $B(G) \Rightarrow |maximal(G)| = n$ .

Some examples of width-predicates are:

1. *Barrier synchronization*: "Every process has made a call to the method barrier."

2. *Deadlock for Dining Philosophers*: "Every philosopher has picked up a fork".

3. Global Availability: "Every process has an active session and the total number of permits with processes is less than k."

Note that 1 and 2 are also conjunctive predicates and can already be detected efficiently. But even if B is not stable or conjunctive, as in example 3, we can use our theory to detect it. Clearly, to detect a width-predicate, it is sufficient to construct or traverse the lattice of the width-ideals. The following result, based on [8, 9], gives an idea for an algorithm to construct or traverse the lattice.

**Theorem 7.** Given any finite width-extensible poset P, there exists an algorithm to enumerate all its width-antichains in  $O(n^2L)$  time where n is the width of the poset and L is the size of the lattice of width-antichains.

*Proof.* We exploit the bijection between the set of all downsets of  $(E, \rightarrow, \pi)$  and the set of all width antichains of (S, <)

 $(\tau, \tau)$  (Lemma 1). Given the poset P, we apply the SE transform (Algorithm 1) to get another poset P' such that enumerating consistent cuts of P' is equivalent to enumerating all width-antichains of P. We can now use algorithms in [8, 9] on P' to enumerate all down-sets in  $O(n^2L)$  time.

#### 6.2 Better Understanding of Checkpointing

Checkpointing [1] is widely used for fault tolerance in distributed systems. In *uncoordinated* checkpointing [15], processes take checkpoints independently, without any group communication and coordination. In a distributed computation with n processes, let  $L_i$  denote the sequence of local checkpoints of process  $C_i$ . Note that any checkpoint  $lc \in L_i$ is a local state of process  $C_i$ . Hence,  $L_i$  is a state chain that is totally ordered under the "<" relation that we have used for comparing states in this paper. It is common to assume that the initial state and the final state in each process are checkpointed [15, 12]. Let the set of all local checkpoints be L, i.e.,  $L = \bigcup L_i$ . The set of checkpoints L, together with the existed-before relation "<", forms a state based model  $\hat{L} = (L, <, \tau)$ , where  $\tau$  partitions L into chains. A subset  $G \subseteq L$  is a global checkpoint iff  $\forall c, d \in G : c \parallel d$  and |G| = n. Hence, a global checkpoint is equivalent to a consistent global state in a state based model over checkpoints of the computation. A local checkpoint is 'useless' if it cannot be part of any global checkpoint. Netzer et al. [15] established results on useless checkpoints using the notion of zigzag paths. Wang [17] used a construction called *R*-graph (or, rollback-dependency graph) to devise an algorithm for detection of useless checkpoints. Although, both [15, 17] have made important contributions, they do not clearly highlight the fundamental concept that checkpoints are states of a distributed computation, and reasoning about checkpoints is in effect reasoning over the state based model of an ASC. Using the theory established in this paper, one can easily understand the intuition behind constructions of *zig-zag* paths and *R*-graphs. In short, by viewing a checkpointing computation as a state based model, the interpretation of *zig-zag* paths, useless checkpoints, and R-graphs is as follows.

- Absence of *zig-zag* paths between checkpoints (states) in a computation means that the checkpoints can be part of a width-antichain (consistent cut). Their presence between checkpoints indicates that the checkpoints cannot be part of a width-antichain. A useless checkpoint is a state of a computation that cannot belong to any width-antichain of the poset under the state based model.
- The *R*-graph construction on a checkpoint computation essentially generates an event based model from the state based model that is the original computation. Hence, the algorithm to identify useless checkpoints (in [17]) effectively tries to check if the model of the computation is legal under the event based model when a particular checkpoint is included. Thus, it applies the *SE* transform (Alg. 1) on the state based model imposed by the checkpoints. The *R*-graph construction and detection algorithm (by finding cycles) and the *SE* transform have the same computation complexity  $\mathcal{O}(k+m)$ , where k is the number of checkpoints (states) in the computation and m is the number of messages.

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## APPENDIX A. PROOF OF LEMMA 1

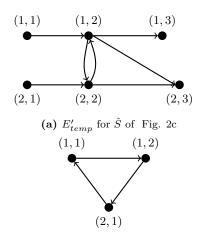
**Lemma 1.** Let  $\hat{E} = (E, \rightarrow, \pi)$  and  $\hat{E} = (S, <, \tau)$  be event and state based models of a computation. Then there is a bijection between consistent cuts of  $\hat{E}$  and  $\hat{S}$ .

Proof. Let G be any consistent cut of  $(E, \rightarrow, \pi)$ . We will show how to construct the corresponding consistent cut T of  $(S, <, \tau)$ . Suppose that G contains at least one event from  $C_i$ . Then, let (i, k) be the largest event from process  $C_i$ . In this case, we add [i, k] to T. If G does not contain any event from  $C_i$ , then we add [i, 0] to T. Clearly, T has exactly n states, one from each process. We show that the cut T is also consistent. If not, suppose [i, s] and [j, t] be two states in T such that [i, s] < [j, t]. This implies that  $(i, s+1) \rightarrow (j, t)$ , under the event based model, contradicting that G is consistent because G contains (j, t) but does not contain (i, s + 1). It is also easy to verify that the mapping from the set of consistent cuts is one-to-one.

Conversely, given a consistent cut T in the state based model, we construct a consistent cut in event based model in 1-1manner as follows. For all states  $[i, k] \in T$  we include all events (i, k') such that  $k' \leq k$ . Note that when k equals 0, no events from  $C_i$  are included. It can again be easily verified that whenever T is a consistent cut in state model, G is a consistent cut in event model.  $\Box$ 

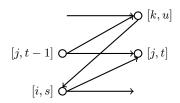
#### **B. ILLUSTRATIONS**

Fig. 5a shows the  $E'_{temp}$  (and not the final E') generated during the execution when SE transform is applied to  $\hat{S}$  given by Fig. 4d. After the SCC decomposition based 'collapsing' on this  $E'_{temp}$ , the generated E' is same as Fig. 4c. Recall that we claimed invalidity of a state based model poset shown in Fig. 3a claiming that such a state model would cause cycles when converted to an event based model. Let us assign state labels to the states shown in that figure: a = [1,0], b = [1,1], c = [1,2], d = [2,0], e = [2,1]. Now apply the SE transform of Alg. 1 to this poset on states. The resulting  $(E, \rightarrow)$  would be the one shown in Fig. 5b. Such a cycle can not exist in a valid event based model.



(b) Event model generated from states of Fig. 3a

Figure 5: SE transform applied to earlier examples



**Figure 6:** Case 1 when  $\omega_3$  is violated (in proof of Theorem 3)

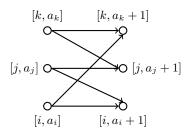


Figure 7: Illustration: Case 1 in proof of Lemma 5