Private Multiplicative Weights Beyond Linear Queries

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Abstract

A wide variety of fundamental data analyses in machine learning, such as linear and logistic regression, require minimizing a convex function defined by the data. Since the data may contain sensitive information about individuals, and these analyses can leak that sensitive information, it is important to be able to solve convex minimization in a privacy-preserving way.

A series of recent results show how to accurately solve a single convex minimization problem in a differentially private manner. However, the same data is often analyzed repeatedly, and little is known about solving multiple convex minimization problems with differential privacy. For simpler data analyses, such as linear queries, there are remarkable differentially private algorithms such as the private multiplicative weights mechanism (Hardt and Rothblum, FOCS 2010) that accurately answer exponentially many distinct queries. In this work, we extend these results to the case of convex minimization and show how to give accurate and differentially private solutions to *exponentially many* convex minimization problems on a sensitive dataset.

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1 Introduction

Consider a dataset $D = (x_1, ..., x_n) \in \mathcal{X}^n$ in which each of the *n* rows corresponds to an individual's record, and each record consists of an element of some data universe \mathcal{X} . The goal of privacy-preserving data analysis is to enable rich statistical analyses on such a dataset while protecting the privacy of the individuals. It is especially desirable to achieve *differential privacy* [DMNS06], which guarantees that no individual's data has a significant influence on the information released about the dataset.

In this work we consider differentially private algorithms that answer *convex minimization (CM)* queries on the sensitive dataset. A CM query is specified by a convex loss function $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$, where Θ is a convex set, and the corresponding query $q_{\ell} : \mathcal{X}^* \to \Theta$ selects the point $\theta \in \Theta$ that minimizes the average loss on the rows of D. That is,

$$q_{\ell}(D) = \operatorname*{argmin}_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} \ell(\theta; x_i).$$

These queries capture fundamental data analyses such as linear and logistic regression and support vector machines. For example, we may have a dataset consisting of *n* labeled examples $(x_1, y_1), \ldots, (x_n, y_n)$ from the data universe $\mathcal{X} = \mathbb{R}^d \times \mathbb{R}$ (corresponding to *d* attributes and a single label per individual), and wish to compute the linear regression

$$\theta^* = \operatorname*{argmin}_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n \left(\langle \theta, x_i \rangle - y_i \right)^2$$

Starting with the results of Dwork and Lei [DL09] and Chaudhuri, Monteleone, and Sarwate [CMS11], there has been a long line of work [KST12, TS13, JT14, BST14] showing how to compute an accurate and differentially private answer to a single CM query. However, in practice the same sensitive dataset will be analyzed by many different analysts, and together these analysts will need answers to a large number of distinct CM queries on the dataset. Any algorithm for solving a single CM query can be applied repeatedly to answer multiple CM queries using the well known composition properties of differential privacy. However, this straightforward approach incurs a significant loss of accuracy, and renders the answers meaningless after a small number of queries (roughly n^2 in most natural settings).

Fortunately, for many interesting types of queries, there are remarkable differentially private algorithms [BLR08, DNR⁺09, DRV10, RR10, HR10, GRU12, HLM12] that are capable of giving accurate answers to *exponentially many* different queries—far greater than what can be achieved using straightforward composition. The most extensively studied case is *linear queries*, which are specified by a property *p* and ask "What fraction of rows in *D* satisfy *p*?" It is also known how to answer exponentially many arbitrary Lipschitz, real-valued queries [DRV10], which generalize linear queries. There are, however, no known non trivial algorithms for privately and accurately answering large sets of CM queries.

In this work we show for the first time that it is possible to give accurate and differentially private answers to exponentially many convex minimization queries. We do so via an extension of the simple and elegant private multiplicative-weights framework of Hardt and Rothblum [HR10], which is known to achieve asymptotically optimal worst-case accuracy [BUV14] and worst-case running time [Ull13] for answering large families of linear queries. Moreover, private multiplicative weights was shown to have a number of practical advantages [HLM12], including good accuracy and running time in practice on low-dimensional datasets, parallelism, and simple implementation, all of which are preserved by our extension. We believe that our technique for adapting the private multiplicative weights framework beyond linear queries may be useful in the future design of differentially private algorithms for other types of non linear queries.

1.1 Our Results

We can now state our results for answering large numbers of CM queries. In order to answer even a single CM query, we need to place some sort of restrictions on the loss function ℓ . In particular, we consider the following types of restrictions on ℓ :

Restrictions	<i>n</i> Needed for a Single Query	<i>n</i> Needed for <i>k</i> Queries
Linear Queries	$O\left(\frac{1}{\alpha}\right)$ [DMNS06]	$\tilde{O}\left(\frac{\sqrt{\log \mathcal{X} } \cdot \log k}{\alpha^2}\right) [\text{HR10}]$
Lipschitz, d-Bounded	$\tilde{O}\left(\frac{\sqrt{d}}{\alpha}\right)$ [BST14]	$\tilde{O}\left(\max\left\{\frac{\sqrt{d \cdot \log \mathcal{X} }}{\alpha^2}, \frac{\log k \cdot \sqrt{\log \mathcal{X} }}{\alpha^2}\right\}\right)$
Lipschitz, <i>d</i> -Bounded, UGLM	$\tilde{O}\left(\frac{1}{\alpha^2}\right)$ [JT14]	$\left \tilde{O}\left(\max\left\{ \frac{\sqrt{\log \mathcal{X} }}{\alpha^3}, \frac{\log k \cdot \sqrt{\log \mathcal{X} }}{\alpha^2} \right\} \right) \right $
Lipschitz, <i>d</i> -Bounded, σ -Strongly Convex	$\tilde{O}\left(\sqrt{\frac{d}{\sigma \alpha}}\right)$ [BST14]	$\tilde{O}\left(\max\left\{\sqrt{\frac{d \cdot \log \mathcal{X} }{\sigma \alpha^3}}, \frac{\log k \cdot \sqrt{\log \mathcal{X} }}{\alpha^2}\right\}\right)$

Table 1: Accuracy guarantees for answering various families of CM queries under differential privacy. New results are shown in green. Error bounds for linear queries, which are a special case of Lipschitz, 1-bounded CM queries are shown for comparison. Error bounds for answering a single CM query under each restriction is also shown for comparison. All results are stated for (ε, δ) -differential privacy for ε constant and δ a negligible function of *n*.

- Lipschitz. ||∇ℓ(θ;x)||₂ ≤ 1 for every θ ∈ Θ, x ∈ X (where the gradient is taken with respect to θ for fixed x).
- *d*-Bounded. $\Theta \subseteq \{\theta \in \mathbb{R}^d \mid ||\theta||_2 \le 1\}.$
- σ -Strongly Convex. $\ell(\theta'; x) \ge \ell(\theta; x) + \langle \nabla \ell(\theta; x), \theta' \theta \rangle + \frac{\sigma}{2} ||\theta' \theta||_2^2$ for every $\theta, \theta' \in \Theta, x \in \mathcal{X}$ (where, again, the gradient is taken with respect to θ for fixed x).
- Unconstrained Generalized Linear Models (UGLM). $\Theta = \mathbb{R}^d$, $\mathcal{X} \subseteq \mathbb{R}^d$ and $\ell(\theta; x) = \ell'(\langle \theta, x \rangle)$ for a convex function $\ell' : \mathbb{R} \to \mathbb{R}$.

The constant 1 in the the Lipschitz and boundedness conditions is arbitrary. One can obtain more general statements in terms of these parameters by rescaling. For simplicity, we will assume throughout that all loss functions ℓ are differentiable, and thus will freely use the gradient operator. However, for all our algorithms and theorems, the assumption that ℓ is differentiable is unnecessary and $\nabla \ell$ can be replaced with an arbitrary subgradient of ℓ .

Table 1 summarizes our results for these different restrictions on the loss functions. In all cases our algorithms are interactive. They take a dataset $D \in \mathcal{X}^n$ as input, interact with a data analyst who chooses a sequence of loss functions ℓ^1, \ldots, ℓ^k , and return answers $\hat{\theta}^1, \ldots, \hat{\theta}^k \in \Theta$ such that for every $j = 1, \ldots, k$

$$\frac{1}{n}\sum_{i=1}^{n}\ell^{j}(\hat{\theta}^{j};x_{i}) \leq \left(\min_{\theta\in\Theta}\frac{1}{n}\sum_{i=1}^{n}\ell^{j}(\theta;x_{i})\right) + \alpha$$

for some error parameter α . We note that the data analyst may be *adaptive*, meaning the choice of ℓ_j can depend on the previous losses and answers $\ell^1, \hat{\theta}^1, \dots, \ell^{j-1}, \hat{\theta}^{j-1}$. Differential privacy becomes easier to achieve as *n* becomes larger. Thus, we ask how big *n* has to be to achieve a given level of accuracy α for answering *k* queries from a family of loss functions \mathcal{L} .

Our results are summarized in the following table. We emphasize that if one were to use an algorithm for answering a single CM query repeatedly via composition, then required database size n would depend polynomially on k, whereas the error depends only polylogarithmically on k in each of our results.

Our algorithms have running time poly($n, |\mathcal{X}|, k$) assuming oracle access to ℓ and its gradient for every ℓ . Thus, our algorithms are not generally efficient, as $|\mathcal{X}|$ will often be exponential in the dimensionality of the data. For example, if $\mathcal{X} = \{0, 1\}^d$, then the dataset consists of nd bits yet our algorithms run in time 2^d , even when k is polynomial and every loss function and its gradient can be efficiently computed. Unfortunately this exponential running time is inherent, under widely believed cryptographic assumptions. Even answering $n^{2+o(1)}$ linear queries, which are a special case of Lipschitz, 1-Bounded CM queries, requires exponential time [Ull13].

Additionally, our algorithms require significantly more error than answering a single CM query. For example, in the case of Lipschitz, *d*-Bounded CM queries, a single query can be answered with a dataset of size $n = \tilde{O}(\sqrt{d}/\alpha)$, whereas answering poly(*n*) queries with our algorithm requires a dataset of size $n = \tilde{O}(\sqrt{\log |\mathcal{X}|} \cdot \log k/\alpha^2)$. By the results of Kasiviswanathan, Rudelson, and Smith [KRS13], a database of size at least $n = \Omega(1/\alpha^2)$ is necessary when answering $\gg 1/\alpha^2$ queries. See Section 4.3 for a more detailed discussion of the lower bounds and computational complexity issues that arise.

Since the error bounds and running time of our algorithm both depend on $|\mathcal{X}|$, our error guarantees may appear vacuous when \mathcal{X} is infinite. For example, in many common applications $\mathcal{X} = \{\theta \in \mathbb{R}^d \mid ||\theta||_2 \le 1\}$ is the *d*-dimensional unit ball. However, in many settings it is essentially without loss of generality (up to, say, a factor of 2 in the error) to round the data points to some finite, data universe. Typically if the data points lie in a *d*-dimensional space, the size of such a data universe will be $(d/\alpha)^{O(d)}$. We leave it for future work to find algorithms that apply to continuous data universes in a more natural way.

1.2 Techniques

In order to describe our algorithms, it will be helpful to start by sketching the private multiplicative weights framework of Hardt and Rothblum [HR10] for answering linear queries. Here, we focus on the "offline" variant from [GHRU11, GRU12, HLM12], in which the *k* loss functions $\mathcal{L} = \{\ell^1, \dots, \ell^k\}$ are specified in advance by the analyst. The offline variant contains the main novel ideas, although we will present our algorithm for the online case.

The algorithm receives as input a dataset $D \in \mathcal{X}^n$ and a set of queries \mathcal{Q} . It will be useful to represent D as a "histogram" over \mathcal{X} , which is a vector indexed by \mathcal{X} where the *x*-th entry is the probability that a random row of D has type x. In this representation, a linear query q can be written as $\langle q, D \rangle$.

The algorithm begins with a hypothesis dataset D^1 , which represents an uneducated guess about D. It then produces a sequence of T differentially private hypotheses D^1, \ldots, D^T that are increasingly good approximations to D. In each round $t = 1, \ldots, T$, the algorithm will privately find the query $q^t \in Q$ such that D^t gives a maximally inaccurate answer. That is, $|\langle q^t, D^t \rangle - \langle q^t, D \rangle|$ is as large as possible. Finding this query can be done privately using a standard application of the exponential mechanism [MT07]. The algorithm then generates D^{t+1} using D^t and q^t via the multiplicative weights update rule.

One can show that after a small number of rounds T, the hypothesis D^T answers every query accurately. The key to the analysis is the following standard fact about the multiplicative-weights update rule: if one can find a vector u^t such that $|\langle u^t, D^t \rangle - \langle u^t, D \rangle|$ is large, then the distance between D^{t+1} and D decreases significantly. Notice that this condition on u^t is precisely that u^t is a linear query for which D^t is inaccurate. Thus, when answering linear queries, we can simply take u^t to be q^t .

In the case of CM queries, we can still use the exponential mechanism to find a loss function $\ell^t \in \mathcal{L}$ such that the minimizer of ℓ^t on D^t is not a good minimizer of the loss on the true dataset D. However, since CM queries are non linear, this information does not immediately give us a suitable vector u^t for the multiplicative-weights update. The key new step in our algorithm is a differentially private way to find a suitable vector u^t . Specifically, we show how to take a query q_ℓ such that $q_\ell(D^t)$ is inaccurate for the true dataset D, and a differentially private approximation to the correct answer $q_\ell(D)$, and use it to find a differentially private vector u^t such that the error $|\langle u^t, D^t \rangle - \langle u^t, D \rangle|$ is large. As with linear queries, having such vectors is sufficient to argue accuracy of the algorithm.

Our approach is inspired by the work of Kasiviswanathan, Rudelson, and Smith [KRS13] who prove lower bounds on the error required for answering certain CM queries. Specifically, they use sufficiently accurate answers to non linear CM queries to extract linear constraints on the dataset, and these linear constraints can then be combined with linear reconstruction attacks to violate privacy. For our results, we use the information that D^t gives an inaccurate answer to a non linear CM query to find a linear query that D^t also answers inaccurately. To do so, we make use of the "dual certificate" style of argument from convex optimization. That is, we derive and analyze the linear query using the first-order optimality conditions on the gradient of ℓ .

1.3 Connection to Generalization Error in Adaptive Data Analysis

Very recently, Dwork et al. [DFH⁺15] and Hardt and Ullman [HU14] showed a connection between differential privacy and *generalization error* in adaptive data analysis, in which the analyst asks an adaptively chosen sequence of queries. By generalization error, we mean the difference between the answers to the queries on the dataset *D* and the answers to the queries on the unknown population from which *D* was drawn. Dwork et al. showed that differentially private algorithms that have low error with respect to the dataset *D* also have low generalization error. Surprisingly, using known differentially private algorithms for answer linear queries yields state-of-the-art bounds on the generalization error required to answer an interactive sequence of linear queries. Bassily et al. [BSSU15] extended the connection between differential privacy and generalization error to the more general family of CM queries. Plugging the results of this paper into their theorem yields state-of-the-art bounds on the generalization error required to answer adaptively chosen CM queries.

2 Preliminaries

2.1 Datasets ,Histograms, and Differential Privacy

We define a *dataset* $\mathcal{D} \in \mathcal{X}^n$ to be a vector of *n* rows $\mathcal{D} = (x_1, \dots, x_n) \in \mathcal{X}^n$ from a *data universe* \mathcal{X} . We say that two datasets $\mathcal{D}, \mathcal{D}' \in \mathcal{X}^n$ are *adjacent* if they differ on only a single row, and we denote this by $\mathcal{D} \sim \mathcal{D}'$.

Definition 2.1 (Differential Privacy [DMNS06]). An algorithm $\mathcal{A} : \mathcal{X}^n \to \mathcal{R}$ is (ε, δ) -differentially private if for every two adjacent datasets $\mathcal{D} \sim \mathcal{D}'$ and every subset $S \subseteq \mathcal{R}$,

$$\Pr\left(\mathcal{A}(\mathcal{D}) \in S\right) \le e^{\varepsilon} \cdot \Pr\left(\mathcal{A}(\mathcal{D}') \in S\right) + \delta.$$

In our algorithm and analysis it will be useful to represent a dataset by its *histogram*. In the histogram representation, the dataset \mathcal{D} is viewed as a probability distribution over \mathcal{X} . We represent this probability distribution as a vector in $D \in \mathbb{R}^{\mathcal{X}}$ where for every $x \in \mathcal{X}$, $D(x) = \Pr_{x' \leftarrow_{\mathbb{R}} \mathcal{D}}(x' = x)$. The condition that $\mathcal{D} \sim \mathcal{D}'$ implies that their histograms satisfy $||D - D'||_1 \leq 1/n$. In the technical sections of this work we will assume all datasets are represented as histograms.

2.2 Convex Minimization (CM) Queries and Accuracy

In this work we are interested in algorithms that answer *convex minimization* (*CM*) *queries* on the dataset. A CM query is defined by a convex *loss function* $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^d$ is a convex set. The associated query $q_\ell : \mathcal{X}^* \to \Theta$ seeks to find $\theta \in \Theta$ that minimizes the expected loss. Formally,

$$q_{\ell}(D) = \operatorname*{argmin}_{\theta \in \Theta} \mathbb{E}_{x \leftarrow_{\mathbb{R}} D} \left(\ell(\theta; x) \right) = \operatorname*{argmin}_{\theta \in \Theta} \sum_{x \in \mathcal{X}} D(x) \cdot \ell(\theta; x)$$

We will use $\mathcal{L} = \{\ell_1, \ell_2, ...\}$ to denote a set of convex loss functions and $\mathcal{Q}_{\mathcal{L}} = \{q_{\ell_1}, q_{\ell_2}, ...\}$ to denote the associated set of convex minimization queries. We will often want to think of ℓ as a function of θ , with *x* fixed. To this end, we will write $\ell_x(\theta) = \ell(\theta; x)$. We will also abuse notation and write $\ell(\theta; D) = \sum_{x \in \mathcal{X}} D(x) \cdot \ell(\theta; x)$ and $\ell_D(\theta) = \ell(\theta; D)$.

In order to define what it means to answer a CM query accurately, we define the following notion of *error*, also known as "excess empirical risk".

Definition 2.2 (Error of an Answer). For a loss function $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$, database $D \in \mathcal{X}^*$, and answer $\hat{\theta} \in \Theta$, we define *the error of* $\hat{\theta}$ *on* ℓ *with respect to* D to be

$$\operatorname{err}_{\ell}(D,\hat{\theta}) = \ell(\hat{\theta};D) - \min_{\theta \in \Theta} \ell(\theta;D).$$

It will also be useful in describing an analyzing out algorithm to define the notion of *error of a database* as follows.

Definition 2.3 (Error of a Database). For a loss function $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$, database $D \in \mathcal{X}^*$, and another database $D' \in \mathcal{X}^*$, we define *the error of* D' *on* ℓ *with respect to* D to be

$$\operatorname{err}_{\ell}(D,D') = \ell_D\left(\operatorname*{argmin}_{\theta'\in\Theta}\ell_{D'}(\theta')\right) - \min_{\theta\in\Theta}\ell_D(\theta).$$

We now define what it means for an algorithm A to be *accurate* for answering a sequence of CM queries from a family \mathcal{L} . We do so by means of a game between A and an adversary \mathcal{B} , defined in Figure 1.

 $\begin{array}{l} \mathcal{B} \text{ chooses } D \in \mathcal{X}^n. \\ \text{For } j = 1, \dots, |\mathcal{L}| \\ \mathcal{B} \text{ outputs a loss function } \ell^j \in \mathcal{L}. \\ \mathcal{A}(D, \ell^j) \text{ outputs } \hat{\theta}^j. \\ \text{ (As } \mathcal{B} \text{ and } \mathcal{A} \text{ are stateful, } \ell^j \text{ and } \hat{\theta}^j \text{ may depend on the history } \ell^1, \hat{\theta}^1, \dots, \ell^{j-1}, \hat{\theta}^{j-1}. \end{array}$

Figure 1: The Sample Accuracy Game $Acc_{n,k,\mathcal{L}}[\mathcal{A},\mathcal{B}]$

Definition 2.4 (Accuracy). Let \mathcal{L} be a set of convex loss functions and $\mathcal{Q}_{\mathcal{L}}$ be the associated set of CM queries. Let $0 < \alpha, \beta \le 1$ and $k, n \in \mathbb{N}$ be parameters. We say that an algorithm \mathcal{A} is (α, β) -accurate for answering k CM queries from $\mathcal{Q}_{\mathcal{L}}$ given a database of size n if for every adversary \mathcal{B} ,

$$\Pr_{\mathsf{Acc}_{n,k,\mathcal{L}}}\left(\max_{j=1,\ldots,k}\operatorname{err}_{\ell^j}(D,\hat{\theta}^j) \leq \alpha\right) \geq 1-\beta.$$

3 Online Private Multiplicative Weights for CM Queries

In this section we present and analyze a differentially private algorithm that answers any family of CM queries provided black-box access to a differentially private algorithm that answers any single CM query from the family.

3.1 The Online Sparse Vector Algorithm

Just like when using private multiplicative weights to answer linear queries, a key ingredient in our algorithm is the *online sparse vector algorithm*. At a high level, the online sparse vector algorithm takes a database $D \in \mathcal{X}^n$ and a sequence of queries q_1, \ldots, q_k , but it provides only a very weak accuracy guarantee. Each query is answered with a single bit in $\{\top, \bot\}$. For a given query q and some threshold α , if $q(D) \ge \alpha$ then the algorithm answering \top and if $q(D) \le \alpha/2$ it answers \bot . If the answer is in $(\alpha/2, \alpha)$ any answer is allowed. The key feature of the online sparse vector algorithm is that the size of the dataset n only needs to be proportional to $\sqrt{T} \cdot \log k$, where T is the number of queries whose answer is above the threshold. In contrast, approximately answering every query requires n to grow like \sqrt{k} .

To maintain brevity, and since the algorithm is standard (see [DR14] for a textbook treatment), we will not specify the algorithm. Instead we will define its properties as a black box. We define the guarantees of the sparse vector algorithm via the following game between the online sparse vector algorithm SV and an adversary B.

The requirement that \mathcal{B} outputs a (3/n)-sensitive query means that q satisfies $|q(D) - q(D')| \leq 3S/n$ for every pair of neighboring databases $D \sim D' \in \mathcal{X}^n$. The choice of (3S/n) can be replaced with any parameter Δ , but we fix it to 3S/n to cut down on notation, since we'll use that choice in the next section.

 $\begin{array}{l} \mathcal{B} \text{ chooses a dataset } D \in \mathcal{X}^n. \\ \text{For } j = 1, \dots, k: \\ \mathcal{B} \text{ outputs a } (3S/n) \text{-sensitive query } q^j \\ (\text{The query } q^j \text{ may depend on the previous queries and answers } q^1, a^1, \dots, q^{j-1}, a^{j-1}.) \\ \mathcal{SV} \text{ returns an answer } a^j \in \{\top, \bot\}. \end{array}$

Figure 2: ThresholdGame_{*n*,*T*,*k*, α [*SV*,*B*]}

Theorem 3.1. There is an algorithm $SV = SV(T, k, \alpha, \varepsilon, \delta)$ such that for every $T, k \in \mathbb{N}$ and $\alpha, \varepsilon, \delta > 0$, the following three conditions hold.

- 1. SV is (ε, δ) -differentially private.
- 2. SV halts if T queries are answered with \top .

3. If

$$n \ge \frac{256 \cdot S \cdot \sqrt{T \cdot \log(2/\delta)} \cdot \log(4k/\beta)}{\varepsilon \alpha}$$

then

$$\Pr_{\text{ThresholdGame}_{n,T,k,\tau}[S\mathcal{V},\mathcal{B}]}\left(\forall j \in [k], \begin{array}{l} q^{j}(D) \geq \alpha \Longrightarrow a^{j} = \top \\ q^{j}(D) \leq \alpha/2 \Longrightarrow a^{j} = \bot \end{array}\right) \geq 1 - \beta.$$

3.2 The Algorithm

We are now ready to describe our algorithm for answering exponentially many convex minimization queries from some family $\mathcal{L} = \{\ell : \Theta \times \mathcal{X} \to \mathbb{R}\}$. Assume every $\ell \in \mathcal{L}$ satisfies the scaling condition

$$\max_{x \in \mathcal{X}, \theta, \theta' \in \Theta} \left| \langle \theta - \theta', \nabla \ell_x(\theta) \rangle \right| \le S.$$

The algorithm is defined in Figure 3. Note that in the algorithm there are two sequences of queries that it will be useful to distinguish. The first is the set of queries actually issued by the analyst, which are index by the letter j and are ℓ^1, \ldots, ℓ^k . There is also the *subsequence* of queries such that $a^j = \top$ and lead to updates. We use the letter t to index these queries, which are ℓ^1, \ldots, ℓ^T (there cannot be more than T such queries, since SV would halt, though there may be fewer). Sometimes it will be useful to consider only the subsequence of queries that are used for updates, which is why we use a separate index for this sequence.

3.3 Accuracy Analysis

In this section, we prove that our algorithm is accurate for any family of CM queries \mathcal{L} , provided that the oracle \mathcal{A}' is accurate for any single CM query from \mathcal{L} . As with previous variants of private multiplicative weights [HR10, GHRU11, GRU12, HLM12], we will derive the accuracy guarantee using the well known "bounded regret" property of the multiplicative weights update rule, combined with the utility guarantees of the online sparse vector algorithm.

To start the analysis we will assume that two conditions are satisfied. First, we assume that SV answered accurately—formally, we assume that

$$\forall j \in [k], \quad \frac{\operatorname{err}_{\ell^j}(D; \hat{D}^t) \ge \alpha \Longrightarrow a^j = \top}{\operatorname{err}_{\ell^j}(D; \hat{D}^t) \le \alpha/2 \Longrightarrow a^j = \bot}$$
(1)

where \hat{D}^t is the current dataset \hat{D}^t that is in use at the time the loss function ℓ^j is considered. By the accuracy of the online sparse vector algorithm SV (Theorem 3.1), the event (1) holds with probability at least $1 - \beta/2$ as long as *n* is sufficiently large.

Input and parameters: A dataset $D \in \mathcal{X}^n$, parameters $\varepsilon, \delta, \alpha, \beta, S, k > 0$, and oracle access to \mathcal{A}' , an $(\varepsilon_0, \delta_0)$ -differentially private algorithm that is (α_0, β_0) -accurate for one convex minimization query in \mathcal{L} on datasets of size n', for parameters $\varepsilon_0, \delta_0, \alpha_0, \beta_0$.

$$T = \frac{64S^2 \log |\mathcal{X}|}{\alpha^2} \qquad \eta = \sqrt{\frac{\log |\mathcal{X}|}{T}}$$
$$\varepsilon_0 = \frac{\varepsilon}{\sqrt{8T \log(4/\delta)}} \qquad \delta_0 = \frac{\delta}{4T} \qquad \alpha_0 = \frac{\alpha}{4} \qquad \beta_0 = \frac{\beta}{2T}$$

Let $SV = SV(T, k, \alpha, \varepsilon/2, \delta/2)$ be the online sparse vector algorithm (Section 3.1). Let t = 1. Let $\hat{D}^t \in \mathbb{R}^{\mathcal{X}}$ be the uniform histogram over \mathcal{X} . For j = 1, ..., k: Receive loss function $\ell = \ell^j \in \mathcal{L}$. Let q^j be the (3/n)-sensitive query $q^j(D) = \operatorname{err}_{\ell}(D, \hat{D}^t)$. Run SV on q^j , to obtain an answer $a^j \in \{\top, \bot\}$. (If SV halts, then halt.) If $a^j = \bot$: Output the answer $\hat{\theta}^j = \operatorname{argmin}_{\theta \in \Theta} \ell(\theta; \hat{D}^t)$. Else if $a^j = \top$: Let $\ell^t = \ell$. Let $\theta^t \leftarrow_{\mathbb{R}} \mathcal{A}'(D, \ell^t)$ be a private estimate of the minimizer of ℓ^t on D. Output the answer $\hat{\theta}^j = \theta^t$. Update \hat{D}^t : Let $\hat{\theta}^t = \operatorname{argmin}_{\theta \in \Theta} \ell(\theta; \hat{D}^t)$ and let $u^t \in [-S, S]^{\mathcal{X}}$ be the vector $u^{t}(x) = \left\langle \theta^{t} - \hat{\theta}^{t}, \nabla \ell_{x}^{t}(\hat{\theta}^{t}) \right\rangle$ Let $\hat{D}^{t+1}(x) \propto \exp(\eta \cdot u^t(x)) \cdot \hat{D}^t(x)$ Let t = t + 1. (Note that $t \le T$, otherwise SV would have halted.)

Figure 3: Online Private Multiplicative Weights for CM Queries

Second, we will assume that every time $a^j = \top$ and $\mathcal{A}'(D, \ell^j)$ is called, it returns an accurate answer—formally,

$$\forall j \text{ such that } a^j = \top, \operatorname{err}_{\ell j}(D, \theta^t) \le \alpha_0.$$
(2)

Since \mathcal{A}' is assumed to be (α_0, β_0) accurate for one query provided that $n \ge n'$, and \mathcal{A}' is called at most T times, we can conclude that the event (2) holds with probability at least $1 - \beta/2$. The following claim is immediate.

Claim 3.2. If

$$n \ge \max\left\{n', \frac{512 \cdot \sqrt{T \cdot \log(4/\delta)} \cdot \log(8k/\beta)}{\varepsilon \alpha}\right\},\,$$

then with probability at least $1 - \beta$, the events (1) and (2) both hold.

Thus, we are justified proving that the online private multiplicative weights algorithm is accurate conditioned on (1) and (2). We start by observing that the algorithm can only fail to be accurate if it halts before the entire sequence of k queries has been asked (because t = T updates have been performed and SV halted).

Claim 3.3. Assume that the algorithm does not terminate before answering k queries, and that (1) and (2) both hold. Then the algorithm answers every query with error at most α . That is,

$$\forall j \in [k], \operatorname{err}_{\ell j}(D, \hat{\theta}^j) \leq \alpha.$$

Proof of Claim 3.3. If the algorithm has not terminated, then each query ℓ^j is answered in one of two ways. If $a^j = \bot$, then we answer with $\hat{\theta}^j = \operatorname{argmin}_{\theta \in \Theta} \ell(\theta; \hat{D}^t)$. In this case, since (1) holds, and $a^j = \bot$, we have $\operatorname{err}_{\ell^j}(D, \hat{\theta}^j) \leq \alpha$. But, by definition, $\operatorname{err}_{\ell^j}(D, \hat{D}^t) = \operatorname{err}_{\ell^j}(D, \hat{\theta}^j)$. So the algorithm answers accurately in the case where $a^j = \bot$. $\operatorname{err}_{\ell^j}(D, \hat{\theta}^j) \leq \alpha$

If $a^j = \top$, then we answer with $\hat{\theta}^j = \theta^j = \mathcal{A}'(D, \ell^j)$. Since (2) holds, we have

$$\operatorname{err}_{\ell j}(D, \hat{\theta}^{j}) = \operatorname{err}_{\ell j}(D, \theta^{t}) \leq \alpha_{0} \leq \alpha_{0}$$

as desired.

To complete the proof, it suffices to show that the algorithm does not terminate early. Here is where we rely on the "bounded regret" property of the multiplicative weights update rule.

Lemma 3.4. [See e.g. [AHK12]] For every sequence $u^1, \ldots, u^T \in [-S, S]^{\mathcal{X}}$,

$$\frac{1}{T} \sum_{t=1}^{T} \left\langle u^{t}, \hat{D}^{t} - D \right\rangle \leq 2S \sqrt{\frac{\log |\mathcal{X}|}{T}}$$

Recall that the algorithm only terminates early if there are *T* queries ℓ^j such that $a^j = \top$, and by (1), $a^j = \top$ only if the error of \hat{D}^t on ℓ^j is at least $\alpha/2$. Thus, in light of the preceding lemma, we would like to show that if \hat{D}^t has error $\alpha/2$ for a query ℓ , then $\langle u^t, \hat{D}^t - D \rangle$ is also large, say $\alpha/4$. If we can show such a statement, then by our choice of *T*, it will be impossible to perform a sequence of *T* updates, and thus the algorithm will not terminate early.

The key lemma, and the main novelty in our analysis, is to relate $\langle u^t, \hat{D}^t - D \rangle$ to the error of \hat{D}^t on a query ℓ^j . We show that $\langle u^t, \hat{D}^t - D \rangle$ is at least the additional loss incurred by $\hat{\theta}^t$ over that of θ^t .

Claim 3.5. *For every* t = 1, ..., T*,*

$$\left\langle \boldsymbol{u}^{t}, \hat{\boldsymbol{D}}^{t} - \boldsymbol{D} \right\rangle \geq \ell_{\boldsymbol{D}}^{t}(\hat{\boldsymbol{\theta}}^{t}) - \ell_{\boldsymbol{D}}^{t}(\boldsymbol{\theta}^{t})$$

Recall that θ^t is an approximation to the optimal solution for ℓ_D^t , whereas $\hat{\theta}^t$ has large error with respect to *D*. Thus we expect the right hand side of the expression to be positive and large.

Proof of Claim 3.5. Recall that we chose

$$\hat{\theta}^t = \operatorname*{argmin}_{\theta \in \Theta} \ell^t_{\hat{D}^t}(\theta).$$

By the first-order optimality condition, and the fact that $\theta^t, \hat{\theta}^t \in \Theta$ for a convex set Θ , the directional derivative of $\ell^t_{\hat{D}t}$ at $\hat{\theta}^t$ in the direction of $\theta^t - \hat{\theta}^t$ will be positive. So we have

$$0 \leq \left\langle \theta^{t} - \hat{\theta}^{t}, \nabla \ell_{\hat{D}^{t}}^{t}(\hat{\theta}^{t}) \right\rangle = \sum_{x \in \mathcal{X}} \hat{D}^{t}(x) \cdot \left\langle \theta^{t} - \hat{\theta}^{t}, \nabla \ell_{x}^{t}(\hat{\theta}^{t}) \right\rangle$$
$$= \left\langle u^{t}, \hat{D}^{t} \right\rangle. \tag{3}$$

The first equality uses linearity of the gradient and the definition $\ell_{\hat{D}^t}^t(\cdot) = \sum_{x \in \mathcal{X}} \hat{D}^t(x) \cdot \ell_x^t(\cdot)$

Similarly, we can look at the directional derivative of ℓ_D^t again taken at $\hat{\theta}^t$ and in the direction of $\theta^t - \hat{\theta}^t$.

$$\left\langle \theta^{t} - \hat{\theta}^{t}, \nabla \ell_{D}^{t}(\hat{\theta}^{t}) \right\rangle = \sum_{x \in \mathcal{X}} D(x) \cdot \left\langle \theta^{t} - \hat{\theta}^{t}, \nabla \ell_{x}^{t}(\hat{\theta}^{t}) \right\rangle$$
$$= \left\langle u^{t}, D \right\rangle.$$
(4)

If $\hat{\theta}^t$ is far from optimal for the input dataset *D*, then moving in the direction of $\theta^t - \hat{\theta}^t$ should significantly decrease the loss. Thus, since ℓ is convex, this directional derivative must be significantly negative. Specifically, since ℓ_D^t is convex, ℓ_D^t lies above all of its tangent lines. Thus,

$$\ell_D^t(\theta^t) \ge \ell_D^t(\hat{\theta}^t) + \left\langle \theta^t - \hat{\theta}^t, \nabla \ell_D^t(\hat{\theta}^t) \right\rangle = \ell_D^t(\hat{\theta}^t) + \left\langle u^t, D \right\rangle.$$

where the equality is from (4) Rearranging terms, we have

$$-\left\langle u^{t}, D\right\rangle \geq \ell_{D}^{t}(\hat{\theta}^{t}) - \ell_{D}^{t}(\theta^{t}).$$

$$\tag{5}$$

Combining (3) and (5), we have

$$\left\langle u^{t}, \hat{D}^{t} - D \right\rangle \geq \ell_{D}^{t}(\hat{\theta}^{t}) - \ell_{D}^{t}(\theta^{t}),$$

which completes the proof.

Using Claim 3.5, and the guarantees (1) and (2), we can now lower bound $\langle u^t, \hat{D}^t - D \rangle$.

Claim 3.6. For every t = 1, ..., T, if the algorithm has not terminated, and (1) and (2) both hold, then

$$\langle u^t, \hat{D}^t - D \rangle > \alpha/4$$

Proof of Claim 3.6. Our goal is to lower bound $\langle u^t, \hat{D}^t - D \rangle$ by the quantity $\operatorname{err}_{\ell^t}(D, \hat{D}^t) = \ell_D^t(\hat{\theta}^t) - \min_{\theta \in \Theta} \ell_D^t(\theta)$. This condition is almost implied by Claim 3.5, except with $\ell_D^t(\theta^t)$ in place of the minimum. In the next claim, we extend the previous claim to handle an approximate minimizer.

However, by (2), $\theta^t = \mathcal{A}'(D, \ell^t)$ is an approximate minimizer. That is,

$$\ell_D^t(\theta^t) \le \min_{\theta \in \Theta} \ell_D^t(\theta) + \alpha_0.$$
(6)

Combining Claim 3.5 with (6) we conclude that if $n \ge n'$, then for every t = 1, ..., T, with probability at least $1 - \beta_0$,

$$\left\langle u^{t}, \hat{D}^{t} - D \right\rangle \ge \ell_{D}^{t}(\hat{\theta}^{t}) - \left(\min_{\theta \in \Theta} \ell_{D}^{t}(\theta) + \alpha_{0} \right) = \operatorname{err}_{\ell^{t}}(D, \hat{D}^{t}) - \alpha_{0}$$

$$\tag{7}$$

Given (7) we would like to show that $\operatorname{err}_{\ell^t}(D, \hat{D}^t)$ is large. But, by (1), we would only do an update if $\operatorname{err}_{\ell^t}(D, \hat{D}^t) > \alpha/2$. Therefore we must have

$$\langle u^t, \hat{D}^t - D \rangle \ge \ell_D^t(\hat{\theta}^t) - \left(\min_{\theta \in \Theta} \ell_D^t(\theta) + \alpha_0\right) > \alpha/2 - \alpha_0 = \alpha/4,$$

as desired.

We are now ready to show that the online private multiplicative weights algorithm does not terminate early.

Claim 3.7. If (1) and (2) both hold, then the algorithm does not terminate before answering k queries.

Proof of Claim 3.7. Assume for the sake of contradiction that the algorithm does terminate early because of the condition t = T. Then, by Claim 3.6, there is a sequence of T queries such that for every query

$$\langle u^t, \hat{D}^t - D \rangle \ge \alpha/4.$$

Then, using the bounded-regret property of multiplicative weights (Lemma 3.4), we must have

$$\alpha/4 < \frac{1}{T} \sum_{t=1}^{T} \left\langle u^{t}, \hat{D}^{t} - D \right\rangle$$

$$\leq 2S \sqrt{\frac{\log |\mathcal{X}|}{T}}$$

$$\leq \alpha/4, \qquad (Lemma 3.4)$$

which is a contradiction.

The analysis of this section immediately implies the following theorem

Theorem 3.8. The online private multiplicative weights algorithm is (α, β) -accurate for answering k CM queries from $Q_{\mathcal{L}}$ given a dataset of size n for

$$n = \max\left\{n', \frac{4096 \cdot S^2 \cdot \sqrt{\log|\mathcal{X}| \cdot \log(4/\delta)} \cdot \log(8k/\beta)}{\varepsilon \alpha^2}\right\}$$

3.4 **Privacy Analysis**

In this section we show that our algorithm (Figure 3) is differentially private. Privacy will follow rather easily from privacy of the online sparse vector algorithm, privacy of A', and well known composition properties of differential privacy.

Theorem 3.9. If \mathcal{A}' is $(\varepsilon_0, \delta_0)$ -differentially private, for ε_0, δ_0 as stated, then the algorithm in Figure 3 is (ε, δ) -differentially private.

3.4.1 Composition of Differential Privacy

Before proceeding to the privacy analysis of our algorithm, we recall the composition properties of differential privacy.

A well-known fact about differential privacy is that the parameters ε , δ degrade gracefully under composition. Specifically, we will make use of the strong composition theorem due to Dwork, Rothblum, and Vadhan [DRV10]. Formally, we say that an algorithm \mathcal{A} is a *T*-fold adaptive composition of (ε_0 , δ_0)differentially private algorithms if \mathcal{A} can be expressed as an instance of the following game for some adversary \mathcal{B} :

Let *D* be a database, let *B* be an adversary, *T* be a parameter For t = 1, ..., T $\mathcal{B}(z_1, ..., z_{t-1})$ outputs an $(\varepsilon_0, \delta_0)$ -DP \mathcal{A}_t Let $z_t = \mathcal{A}_t(D)$ Output $z_1, ..., z_T$

Figure 4: T-Fold Adaptive Composition

Theorem 3.10 ([DRV10]). For every $T \in \mathbb{N}$ and $0 \le \varepsilon_0, \delta_0, \delta' \le 1/2$, if \mathcal{A} is a T-fold adaptive composition of $(\varepsilon_0, \delta_0)$ -differentially private algorithms, then \mathcal{A} is $(\varepsilon, \delta' + T\delta_0)$ -differentially private for

$$\varepsilon = \sqrt{2T\log(1/\delta')} \cdot \varepsilon_0 + 2T \cdot \varepsilon_0^2$$

In particular, if A is a T-fold adaptive composition of $(\varepsilon_0, \delta_0)$ -differentially private algorithms, where

$$\varepsilon_0 = \frac{\varepsilon}{\sqrt{8T\log(2/\delta)}} \qquad \delta_0 = \frac{\delta}{2T}$$

then A is (ε, δ) -differentially private.

3.4.2 Proof of Theorem 3.9

There are only two places where the algorithm uses the private dataset *D*: (1) when using the online sparse vector algorithm to answer the queries $q^j = \text{err}_{\ell i}(D, \hat{D}^t)$, and (2) when using \mathcal{A}' to obtain a private approximation to the minimizer of some loss function ℓ^t . First, we will show that the online sparse vector algorithm is $(\varepsilon/2, \delta/2)$ -differentially private. This claim will follow immediately from Theorem 3.1

provided that the queries q^j are indeed (3S/n)-sensitive. To show this, first, observe that if $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$ satisfies

$$\max_{x \in \mathcal{X}, \theta, \theta' \in \Theta} \left| \langle \theta - \theta', \nabla \ell_x(\theta) \rangle \right| \le S,$$

then for every $x \in \mathcal{X}$, there exists $b_x \in \mathbb{R}$ such that for every $\theta \in \Theta$, $\ell(\theta, x) \in [b_x, S]$. That is, for every x, there is some interval of width S that bounds the loss $\ell(\theta, x)$. With this information we can bound the sensitivity of the error function in the following way: Fix any $\ell \in \mathcal{L}$. Let $\overline{\ell}(\theta, x) = \ell(\theta, x) - b_x$. Let $\overline{\theta} = \operatorname{argmin}_{\theta \in \Theta} \ell_{\hat{D}^t}(\theta)$.

$$\begin{split} \max_{D,D'\in\mathcal{X}^n} \left| \operatorname{err}_{\ell}(D,\hat{D}^t) - \operatorname{err}_{\ell}(D',\hat{D}^t) \right| \\ &= \max_{D,D'\in\mathcal{X}^n} \left| \left(\ell_D(\overline{\theta}) - \min_{\theta\in\Theta} \ell_D(\theta) \right) - \left(\ell_{D'}(\overline{\theta}) - \min_{\theta\in\Theta} \ell_{D'}(\theta) \right) \right| \\ &= \max_{D,D'\in\mathcal{X}^n} \left| \left(\overline{\ell}_D(\overline{\theta}) - \min_{\theta\in\Theta} \overline{\ell}_D(\theta) \right) - \left(\overline{\ell}_{D'}(\overline{\theta}) - \min_{\theta\in\Theta} \overline{\ell}_{D'}(\theta) \right) \right| \\ &= \max_{D,D'\in\mathcal{X}^n} \left| \left(\overline{\ell}_D(\overline{\theta}) - \ell_{D'}(\overline{\theta}) \right) \right| + \left| \left(\min_{\theta\in\Theta} \overline{\ell}_D(\theta) - \min_{\theta\in\Theta} \overline{\ell}_{D'}(\theta) \right) \right| \\ &\leq \frac{S}{n} + \frac{2S}{n} = \frac{3S}{n}. \end{split}$$

Since this bound holds for every $\ell \in \mathcal{L}$, we have

$$\max_{\ell \in \mathcal{L}} \max_{D, D' \in \mathcal{X}^n} \left| \operatorname{err}_{\ell}(D, \hat{D}^t) - \operatorname{err}_{\ell}(D', \hat{D}^t) \right| \leq \frac{35}{n}.$$

Thus, the queries given to SV are indeed (3S/n)-sensitive and we are justified in assuming that SV is an $(\varepsilon/2, \delta/2)$ -differentially private algorithm.

Now, we return to analyzing the privacy loss of \mathcal{A}' . By assumption, for every fixed ℓ^t , the choice of $\theta^t = \mathcal{A}'(D, \ell^t)$ is $(\varepsilon_0, \delta_0)$ -differentially private with respect to the input *D*. Moreover, the choice of ℓ^t depends only on the output of $S\mathcal{V}$, which we have already argued is $(\varepsilon/2, \delta/2)$ -differentially private. Therefore, we can view all of the calls to \mathcal{A}' as a single *T*-fold adaptive composition of $(\varepsilon_0, \delta_0)$ -differentially private algorithms. For ε_0, δ_0 as specified in the online private multiplicative weights algorithm, the result will be $(\varepsilon/2, \delta/2)$ -differentially private. Since these are the only two ways in which the private dataset *D* is used, we have proven that the entire algorithm is (ε, δ) -differentially private.

4 Applications of Theorem 3.8

In this section we give some interpretation of Theorem 3.8 and show how it can be applied to specific interesting cases that have been considered in the literature on differentially private convex minimization in order to obtain the results stated in the introduction.

4.1 Interpreting Theorem 3.8

In Theorem 3.8, we have assumed that there exists an $(\varepsilon_0, \delta_0)$ -differentially private algorithm \mathcal{A}' that is (α_0, β_0) -accurate for any one ℓ from \mathcal{L} given n' samples. By a standard argument, if there exists a $(1, \delta_0)$ -differentially private algorithm \mathcal{A}'' that is (α_0, β_0) -accurate for ℓ given n'' samples, then there exists an $(\varepsilon_0, \delta_0)$ -differentially private algorithm with the same accuracy given $O(n''/\varepsilon_0)$ samples. Applying this observation, simplifying, and dropping the dependence on $\beta, \varepsilon, \delta$, we can write the requirement in Theorem 3.8 as

$$n \gtrsim \max\left\{\frac{n''}{\varepsilon_0}, \frac{S^2 \cdot \log k}{\alpha^2}\right\} \lesssim \frac{S \cdot \sqrt{\log|\mathcal{X}|} \cdot \log k}{\alpha} \cdot \max\left\{n'', \frac{S}{\alpha}\right\}$$

The first term in the max is just the size of dataset required to answer a single convex minimization query in \mathcal{L} with $\varepsilon = 1$. The second term in the max can be either larger or smaller than n''. However, for the most basic setting of a single, Lipschitz loss function over a bounded domain, $n'' \gg S/\alpha$, so the second term will be dominated by the first term.

Thus, in some cases, Theorem 3.8 can be interpreted as saying that the amount of data required to answer *k* queries from \mathcal{L} is only a factor of $\approx (S \cdot \sqrt{\log |\mathcal{X}|} \cdot \log k)/\alpha$ larger than the amount of data required to both answer a single query in \mathcal{L} . Using the simple composition approach where each of the *k* queries is answered independently would require a factor of $\approx \sqrt{k}$ more data than answering a single query. Thus our algorithm is a substantial improvement when $\sqrt{k} \gg (S \cdot \sqrt{\log |\mathcal{X}|} \cdot \log k)/\alpha$.

4.2 Applications

We now show how to instantiate Theorem 3.8 with various differentially private algorithms for answering convex minimization queries to obtain the results in the Introduction.

4.2.1 Lipschitz and Bounded Loss Functions.

In much of the work on differentially private convex minimization, the queries are normalized so that the parameter θ lies in a unit L_2 ball, and the loss function ℓ satisfies a Lipschitz condition. Bassily, Smith, and Thakurta [BST14] recently showed optimal upper and lower bounds for answering a single query from this family. Formally,

Theorem 4.1 ([BST14]). Let $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$ be a convex loss function where $\Theta \subseteq \{\theta \in \mathbb{R}^d \mid ||\theta||_2 \leq 1\}$ and for every $\theta \in \Theta$, $x \in \mathcal{X}$, $||\nabla \ell_x(\theta)||_2 \leq 1$. Let q_ℓ be the associated CM query. There is a $(\varepsilon_0, \delta_0)$ -differentially private algorithm that is (α_0, β_0) -accurate for q_ℓ on datasets of size n for

$$n = O\left(\frac{\sqrt{d}}{\alpha_0\varepsilon_0}\right) \cdot \operatorname{polylog}\left(\frac{1}{\delta_0}, \frac{1}{\beta_0}\right).$$

Note that if Θ is contained in a unit L_2 ball and ℓ is 1-Lipschitz, then the scaling parameter *S* is at most 2. Combining Theorem 3.8 and Theorem 4.1 yields the following result.

Theorem 4.2. Let \mathcal{L} be the set of convex loss functions $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$ for $\Theta \subseteq \{\theta \in \mathbb{R}^d \mid \|\theta\|_2 \leq 1\}$ such that for every $\ell \in \mathcal{L}$, $\theta \in \Theta$, $x \in \mathcal{X}$, $\|\nabla \ell_x(\theta)\|_2 \leq 1$. Let $\mathcal{Q}_{\mathcal{L}}$ be the associated family of CM queries. There is an (ε, δ) -differentially private algorithm that is (α, β) -accurate for k CM queries from $\mathcal{Q}_{\mathcal{L}}$ on datasets of size n for

$$n = \tilde{O}\left(\frac{\sqrt{\log|\mathcal{X}|}}{\alpha^{2}\varepsilon} \cdot \max\left\{\sqrt{d}, \log k\right\}\right) \cdot \operatorname{polylog}\left(\frac{1}{\delta}, \frac{1}{\beta}\right).$$

4.2.2 Generalized Linear Models.

Using the algorithm of Theorem 4.1, *n* must grow polynomially with *d* to solve even a single CM query in dimension *d*, and this was shown to be inherent by Bassily et al. [BST14] (building on [BUV14]). However, the work of Jain and Thakurta [JT14] shows that dependence on *d* can be avoided for the important class of *unconstrained generalized linear models*. For example, logistic regression and linear regression are generalized linear models. A convex loss function $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$ is a generalized linear model if $\Theta \subseteq \mathbb{R}^d$, $\mathcal{X} \subseteq \mathbb{R}^d$, and $\ell(\theta, x)$ depends only on the inner product of θ and *x*. That is, there exists a convex function $\ell' : \mathbb{R} \to \mathbb{R}$ such that $\ell(\theta, x) = \ell'(\langle \theta, x \rangle)$. We say that the generalized linear model is unconstrained if there are no constraints other than boundedness. That is, $\Theta = \{\theta \in \mathbb{R}^d \mid ||\theta||_2 \le 1\}$.

Theorem 4.3 ([JT14]). Let $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$ be an unconstrained generalized linear model with the domain $\Theta = \{\theta \in \mathbb{R}^d \mid ||\theta||_2 \le 1\}$ and for every $\theta \in \Theta$, $x \in \mathcal{X}$, $||\nabla \ell_x(\theta)||_2 \le 1$. Let q_ℓ be the associated CM query. There

is a $(\varepsilon_0, \delta_0)$ -differentially private algorithm that is (α_0, β_0) -accurate for q_ℓ on datasets of size n for

$$n = O\left(\frac{1}{\alpha_0^2 \varepsilon_0}\right) \cdot \operatorname{polylog}\left(\frac{1}{\delta_0}, \frac{1}{\beta_0}\right).$$

Combining Theorem 3.8 and Theorem 4.3 yields the following result.

Theorem 4.4. Let \mathcal{L} be the set of unconstrained generalized linear models $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$ with the domain $\Theta \subseteq \{\theta \in \mathbb{R}^d \mid \|\theta\|_2 \leq 1\}$ such that for every $\ell \in \mathcal{L}, \theta \in \Theta, x \in \mathcal{X}, \|\nabla \ell_x(\theta)\|_2 \leq 1$. Let $\mathcal{Q}_{\mathcal{L}}$ be the associated family of CM queries. There is an (ε, δ) -differentially private algorithm that is (α, β) -accurate for k CM queries from $\mathcal{Q}_{\mathcal{L}}$ given n records for

$$n = \tilde{O}\left(\frac{\sqrt{\log|\mathcal{X}|}}{\alpha^{2}\varepsilon} \cdot \max\left\{\frac{1}{\alpha}, \log k\right\}\right) \cdot \operatorname{polylog}\left(\frac{1}{\delta}, \frac{1}{\beta}\right).$$

4.2.3 Strongly Convex Loss Functions.

Stronger accuracy guarantees for answering a single CM query are also achievable in the common special case where ℓ is strongly convex. Informally, ℓ is strongly convex if it can be lower bounded by a quadratic function. Specifically, for a parameter $\sigma \ge 0$, the function $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$ is 2σ -strongly convex if for every $\theta, \theta' \in \Theta$ and $x \in \mathcal{X}, \ell(\theta'; x) \ge \ell(\theta; x) + \langle \theta' - \theta, \nabla \ell(\theta; x) \rangle + \sigma \|\theta' - \theta\|_2^2$. In the previous statement, the gradient is with respect to θ .

Theorem 4.5 ([BST14]). Let $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$ be a σ -strongly convex loss function where $\Theta \subseteq \{\theta \in \mathbb{R}^d \mid ||\theta||_2 \leq 1\}$ and for every $\theta \in \Theta$, $x \in \mathcal{X}$, $||\nabla \ell_x(\theta)||_2 \leq 1$. Let q_ℓ be the associated CM query. There is a $(\varepsilon_0, \delta_0)$ -differentially private algorithm that is (α_0, β_0) -accurate for q_ℓ on datasets of size n for

$$n = O\left(\frac{\sqrt{d}}{\sqrt{\sigma \alpha_0}\varepsilon_0}\right) \cdot \operatorname{polylog}\left(\frac{1}{\delta_0}, \frac{1}{\beta_0}\right)$$

Combining Theorem 3.8 and Theorem 4.5 yields the following result.

Theorem 4.6. Let \mathcal{L} be the set of σ -strongly convex loss functions $\ell : \Theta \times \mathcal{X} \to \mathbb{R}$ for $\Theta \subseteq \{\theta \in \mathbb{R}^d \mid \|\theta\|_2 \leq 1\}$ such that for every $\ell \in \mathcal{L}, \theta \in \Theta, x \in \mathcal{X}, \|\nabla \ell_x(\theta)\|_2 \leq 1$. Let $\mathcal{Q}_{\mathcal{L}}$ be the associated family of CM queries. There is an (ε, δ) -differentially private algorithm that is (α, β) -accurate for k CM queries from $\mathcal{Q}_{\mathcal{L}}$ on datasets of size n for

$$n = \tilde{O}\left(\frac{\sqrt{\log|\mathcal{X}|}}{\varepsilon} \max\left\{\frac{\sqrt{d}}{\sqrt{\sigma}\alpha^{3/2}}, \frac{\log k}{\alpha^2}\right\}\right) \cdot \operatorname{polylog}\left(\frac{1}{\delta}, \frac{1}{\beta}\right)$$

4.3 Running Time and Discussion of Computational Complexity

In this section we discuss the computational complexity of the algorithm. To do so, we assume $\Theta \subseteq \mathbb{R}^d$, and for simplicity and concreteness we consider the natural choice of data universe $\mathcal{X} = \{0, 1\}^d$, or equivalently $\mathcal{X} = \left\{\frac{\pm 1}{\sqrt{d}}\right\}^d$. Since our algorithm uses the ability to solve a single CM query in \mathcal{L} as a blackbox, we assume that this step can be done in poly(*n*,*d*) time both privately and non-privately. For this informal discussion, we also ignore the dependence in running time on $S, \alpha, \beta, \varepsilon, \delta$, which will not substantially affect the conclusions.

There are three main steps that dominate the running time of each of the *k* iterations:

- 1. Running the online sparse vector algorithm SV on q^j . This step can be done in time poly(*n*, *d*).
- 2. If $a^j = \top$, finding a private approximate minimizer of ℓ^j . By assumption, this step can be done in time poly(*n*, *d*).
- 3. If $a^j = \top$, computing the new histogram \hat{D}^{t+1} . This step requires time $\tilde{O}(2^d)$.

Since each of these steps is carried out for k steps, the overall running time is $poly(n, 2^d, k)$. Even tough it was useful to think of the database as a histogram, which is a vector of length 2^d , the input database D would more naturally be represented as a collection of records $D \in (\{0,1\}^d)^n$. Thus it is natural to look for an algorithm with running time poly(n, d, k). In summary, even when the individual loss functions can be privately minimized in poly(n, d) time, our algorithm requires time $poly(n, 2^d, k)$, which is exponential in the dimension of the data. More generally, there is a polynomial dependence on $|\mathcal{X}|$, where one would hope for a polylogarithmic dependence.

Unfortunately, this exponential running time is inherent. Since CM queries generalize the well studied class of linear queries, we can carry over the hardness results of Ullman [Ull13] to this setting. Specifically, assuming the existence of one-way functions, there is no poly(n,d)-time algorithm that takes as input a set of k arbitrary differentiable convex loss functions, and a database $D \in (\{0,1\}^d)^n$ for $n \le k^{1/2-o(1)}$, and and outputs answers that are even 1/100-accurate for each query in \mathcal{L} .

Although the hardness result rules out an efficient mechanism for answering an arbitrary large set of CM queries, more efficient algorithms may be possible for specific families \mathcal{L} . In the setting of counting queries, such algorithms are known for special cases such as *interval queries* [BNS13] and *marginal queries* [GHRU11, HRS12, TUV12, CTUW14, DNT13]. It would be interesting to see if techniques from those works can be applied to give more efficient algorithms for natural families of CM queries. We remark that Ullman and Vadhan [UV11] show that efficient algorithms that output synthetic data cannot be accurate even for very simple families of counting queries, and thus also for certain very simple families of CM queries. Our algorithm indeed can be modified to output a synthetic dataset (namely, the final histogram \hat{D}^t used in the execution of the algorithm), and thus substantially different techniques would be required to answer interesting classes of CM queries more efficiently. We leave it as an interesting direction for future work to improve the running time of our algorithm for interesting restricted families of CM queries.

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