Robust and MaxMin Optimization under Matroid and Knapsack Uncertainty Sets*

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Abstract

Consider the following problem: given a set system (U,Ω) and an edge-weighted graph G=(U,E) on the same universe U, find the set $A\in\Omega$ such that the Steiner tree cost with terminals A is as large as possible—"which set in Ω is the most difficult to connect up?" This is an example of a max-min problem: find the set $A\in\Omega$ such that the value of some minimization (covering) problem is as large as possible.

In this paper, we show that for certain covering problems which admit good deterministic online algorithms, we can give good algorithms for max-min optimization when the set system Ω is given by a p-system or knapsack constraints or both. This result is similar to results for constrained maximization of submodular functions. Although many natural covering problems are not even approximately submodular, we show that one can use properties of the online algorithm as a surrogate for submodularity.

Moreover, we give stronger connections between max-min optimization and two-stage robust optimization, and hence give improved algorithms for robust versions of various covering problems, for cases where the uncertainty sets are given by p-systems and q knapsacks.

1 Introduction

Recent years have seen a considerable body of work on the problem of constrained submodular maximization: you are given a universe U of elements, a collection $\Omega \subseteq 2^U$ of "independent" sets and a submodular function $f: 2^U \to \mathbb{R}_{\geq 0}$, and the goal is to solve the optimization problem of maximizing f over the "independent" sets:

$$\max_{S \in \Omega} f(S). \tag{Max-}f)$$

It is a classical result that when f is a linear function and (U,Ω) is a matroid, the greedy algorithm solves this exactly. Furthermore, results from the mid-1970s tell us that even when f is monotone submodular and (U,Ω) is a partition matroid, the problem becomes NP-hard, but the greedy algorithm is a $\frac{e}{e-1}$ -approximation—in fact, greedy is a 2-approximation for monotone submodular maximization subject to any matroid constraint. Recent results have shed more light on this problem: it is now known that when f is a monotone submodular function and (U,Ω) is a matroid, there exists a $\frac{e}{e-1}$ -approximation algorithm. We can remove the constraint of monotonicity, and also generalize the constraint Ω substantially: the most general results say that if f is a non-negative submodular function, and if Ω is a p-system, f then one can approximate Max-f to within a factor of O(p); moreover, if Ω is the intersection of O(1) knapsack constraints then one can approximate Max-f to within a constant factor.

Given this situation, it is natural to ask: For which broad classes of functions can we approximately solve the Maxf problem efficiently? (Say, subject to constraints Ω that form a p-system, or given by a small number of knapsack

^{*}An extended abstract containing the results of this paper and of [15] appeared jointly in *Proceedings of the 37th International Colloquium on Automata, Languages and Programming (ICALP), 2010.*

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¹A p-system is similar to, but more general than, the intersection of p matroids; it is formally defined in Section 5.1

constraints, or both.) Clearly this class of functions includes submodular functions. Does this class contain other interesting subclasses of functions which are far from being submodular?

In this paper we consider the case of "max-min optimization": here f is a monotone subadditive function defined by a minimization covering problem, a natural subset of all subadditive functions. We show conditions under which we can do constrained maximization over such functions f. For example, given a set system (U, \mathcal{F}) , define the "set cover" function $f_{SC}: 2^U \to \mathbb{Z}_{\geq 0}$, where f(S) is the minimum number of sets from \mathcal{F} that cover the elements in S. This function f_{SC} is not submodular, and in fact, we can show that there is no submodular function g such that $g(S) \leq f_{SC}(S) \leq \alpha g(S)$ for sub-polynomial α . (See Section 6.) Moreover, note that in general we cannot even evaluate $f_{SC}(S)$ to better than an $O(\log n)$ -factor in polynomial time. However, our results imply $\max_{S \in \Omega} f_{SC}(S)$ can indeed be approximated well. In fact, the result that one could approximately maximize f_{SC} subject to a cardinality constraint was given by Feige et al. [10]; our results should be seen as building on their ideas. (See also the companion paper [15].)

At a high level, our results imply that if a monotone function f is defined by a (minimization) covering problem, if f is subadditive, and if the underlying (minimization) covering problem admits good deterministic online algorithms, then there exist good approximation algorithms for $\mathsf{Max}\text{-}f$ subject to p-systems and q knapsacks. (All these terms will be made formal shortly.) The resulting approximation guarantee for the max-min problem depends on the competitive ratio of the online algorithm, and p and q. Moreover, the approximation ratio improves if there is a better algorithm for the offline minimization problem, or if there is a better online algorithm for a fractional version of the online minimization problem.

Robust Optimization. Our techniques and results imply approximation algorithms for covering problems in the framework of robust optimization as well. In the robust optimization framework, there are two stages of decision making. E.g., in a generic robust optimization problem, one is not only given a set system (U,Ω) , but also an inflation parameter $\lambda \geq 1$. Then one wants to perform some actions in the first stage, and then given a set $A \in \Omega$ in the second stage, perform another set of actions (which can now depend on A) to minimize

(cost of first-stage actions) +
$$\max_{A \in \Omega} \lambda$$
 · (cost of second-stage actions)

subject to the constraint that the two sets of actions "cover" the demand set $A \in \Omega$. As an example, in robust set cover, one is given another set system (U, \mathcal{F}) : the allowed actions in the first and second stage are to pick some sub-collections \mathcal{F}_1 and \mathcal{F}_2 respectively from \mathcal{F} , and the notion of "coverage" is that the union of the sets in $\mathcal{F}_1 \cup \mathcal{F}_2$ must contain A. (If $\lambda > 1$, actions are costlier in the second stage, and hence there is a natural tension between waiting for the identity of A, and over-anticipating in the first stage without any information about A.)

Note that robust and max-min problems are related, at least in one direction: if $\lambda=1$, there is no incentive to perform any actions in the first stage, in which case the robust problem degenerates into a max-min optimization problem. In this paper, we show a reduction in the other direction as well—if one can solve the max-min problem well (and if the covering problem admits a good deterministic online algorithm), then we get an algorithm for the robust optimization version of the covering problem as well. The paper of Feige et al. [10] gave the first reduction from the robust set-cover problem to the max-min set cover problem, for the special case when $\Omega=\binom{U}{k}$; this result was based on a suitable LP-relaxation. Our reduction extends this in two ways: (a) the constraint sets Ω can now be p-systems and q knapsacks, and (b) much more importantly, the reduction now applies not only to set cover, but to many sub-additive monotone covering problems (those with deterministic online algorithms, as mentioned above). Indeed, it is not clear how to extend the ellipsoid-based reduction of [10] even for the Steiner tree problem; this was first noted by Khandekar et al. [22].

Our Results and Techniques. Our algorithm for the max-min problem is based on the observation that the cost of a deterministic online algorithm for the underlying minimization covering problem defining f can be used as a surrogate for submodularity in certain cases; specifically, we show that the greedy algorithm that repeatedly picks an element maintaining membership in Ω and maximizing the cost of the online algorithm gives us a good approximation to the max-min objective function, as long as Ω is a p-system.

We also show how to reduce the problem of maximizing such a function over the intersection of q knapsacks to $n^{O(1/\epsilon^2)}$ runs of approximately maximizing the function over a single partition matroid at a loss of a factor of $q(1+\epsilon)$, or instead to $n^{O(q/\epsilon^2)}$ runs of approximately maximizing over a different partition matroid at a loss of a factor of $(1+\epsilon)$ —this reduction is fairly general and is likely to be of interest in other contexts as well. These results appear in Section 3.

We then turn to robust optimization. In Section 4, we show that given a deterministic online algorithm for the covering function f, and an approximate max-min optimization algorithm for f over a family Ω , we get an algorithm for two-stage robust version of the underlying covering problem with uncertainty set Ω —the approximation guarantee depends on both the competitive ratio of the online algorithm, as well as the approximation guarantee of the max-min problem.

Note that we can combine this latter reduction (using max-min algorithms to get robust algorithms) with our first reduction above (using online algorithms to get max-min algorithms); in Section 5, we give a more careful analysis that gives a better approximation than that obtained by just naively combining the two theorems together.

Finally, in Section 6, we show that some common covering problems (vertex cover and set cover) give rise to functions f that cannot be well-approximated (in a mutliplicative sense) by any submodular function, but still admit good maximization algorithms by our results in Section 3.

1.1 Related Work

Constrained submodular maximization problems have been very widely studied [24, 11, 29, 6, 30, 23]. However, as we mention above, the set cover and vertex cover functions are far from submodular. Interestingly, in a recent paper on testing submodularity [27], Seshadhri and Vondrak conjecture that the success of greedy maximization algorithms may depend on a more general property than submodularity; this work provides further corroboration for this, since we show that in our context online algorithms can serve as surrogates for submodularity.

Feige et al. [10] first considered the k-max-min set cover subject to $\Omega = \binom{U}{k}$ (the "cardinality-constrained" case)—they gave an $O(\log m \log n)$ -approximation algorithm for the problem with m sets and n elements. They also showed an $\Omega(\frac{\log m}{\log \log m})$ hardness of approximation for k-max-min (and k-robust) set cover. The results in this paper build upon ideas in [10], by handling more general covering problems and sets Ω . To the best of our knowledge, none of the k-max-min problems other than min-cut have been studied earlier; note that the min-cut function is submodular, and hence the associated max-min problem can be solved using submodular maximization.

The study of approximation algorithms for robust optimization was initiated by Dhamdhere et al. [8, 14]: they study the case when the scenarios were explicitly listed, and gave constant-factor approximations for several combinatorial optimization problems. Again, the model with implicitly specified (and exponentially many) scenarios Ω was considered in Feige et al. [10], where they gave an $O(\log m \log n)$ -approximation for robust set cover in the cardinality-constrained case $\Omega = \binom{U}{k}$. Khandekar et al. [22] noted that the techniques of [10] did not seem to imply good results for Steiner tree, and developed new constant-factor approximations for k-robust versions of Steiner tree, Steiner forest on trees and facility location, again for the cardinality-constrained case. We investigate many of these problems in the cardinality-constrained case of both the max-min and robust models in the companion paper [15], and obtain approximation ratios better than the online competitive factors. On the other hand, the goal in this paper is to give a framework for robust and max-min optimization under general uncertainty sets.

2 Preliminaries

2.1 Deterministic covering problems

A covering problem Π has a ground-set E of elements with costs $c: E \to \mathbb{R}_+$, and n covering requirements (often called demands or clients), where the solutions to the i-th requirement is specified—possibly implicitly—by a family $\mathcal{R}_i \subseteq 2^E$ which is upwards closed (since this is a covering problem). Requirement i is satisfied by solution $\mathcal{F} \subseteq E$ iff $\mathcal{F} \in \mathcal{R}_i$. The covering problem $\Pi = \langle E, c, \{\mathcal{R}_i\}_{i=1}^n \rangle$ involves computing a solution $\mathcal{F} \subseteq E$ satisfying

all n requirements and having minimum cost $\sum_{e \in \mathcal{F}} c_e$. E.g., in set cover, "requirements" are items to be covered, and "elements" are sets to cover them with. In Steiner tree, requirements are terminals to connect to the root and elements are the edges; in multicut, requirements are terminal pairs to be separated, and elements are edges to be cut.

The min-cost covering function associated with Π is:

$$f_{\Pi}(S) := \min \left\{ \sum_{e \in \mathcal{F}} c_e : \ \mathcal{F} \in \mathcal{R}_i \text{ for all } i \in S \right\}.$$

2.2 Max-min problems

Given a covering problem Π and a collection $\Omega \subseteq 2^{[n]}$ of "independent sets", the *max-min* problem MaxMin(Π) involves finding a set $\omega \in \Omega$ for which the cost of the min-cost solution to ω is maximized,

$$\max_{\omega \in \Omega} f_{\Pi}(\omega).$$

2.3 Robust covering problems

This problem, denoted Robust(Π), is a *two-stage optimization* problem, where elements are possibly bought in the first stage (at the given cost) or the second stage (at cost λ times higher). In the second stage, some subset $\omega \subseteq [n]$ of requirements (also called a *scenario*) materializes, and the elements bought in both stages must collectively satisfy each requirement in ω . Formally, the input to problem Robust(Π) consists of (a) the covering problem $\Pi = \langle E, c, \{\mathcal{R}_i\}_{i=1}^n \rangle$ as above, (b) an uncertainty set $\Omega \subseteq 2^{[n]}$ of scenarios (possibly implicitly given), and (c) an inflation parameter $\lambda \geq 1$. A feasible solution to Robust(Π) is a set of *first stage elements* $E_0 \subseteq E$ (bought without knowledge of the scenario), along with an *augmentation algorithm* that given any $\omega \in \Omega$ outputs $E_\omega \subseteq E$ such that $E_0 \cup E_\omega$ satisfies all requirements in ω . The objective function is to minimize:

$$c(E_0) + \lambda \cdot \max_{\omega \in \Omega} c(E_\omega).$$

Given such a solution, $c(E_0)$ is called the first-stage cost and $\max_{\omega \in \Omega} c(E_\omega)$ is the second-stage cost.

Note that by setting $\lambda = 1$ in any robust covering problem, the optimal value of the robust problem equals that of its corresponding max-min problem.

As in [15], our algorithms for robust covering problems are based on the following type of guarantee. In [15] these were stated for k-robust uncertainty sets, but they immediately extend to arbitrary uncertainty sets.

Definition 2.1 An algorithm is $(\alpha_1, \alpha_2, \beta)$ -discriminating iff given as input any instance of Robust (Π) and a threshold T, the algorithm outputs (i) a set $\Phi_T \subseteq E$, and (ii) an algorithm Augment $_T : \Omega \to 2^E$, such that:

- A. For every scenario $D \in \Omega$,
 - (i) the elements in $\Phi_T \cup \mathsf{Augment}_T(D)$ satisfy all requirements in D, and
 - (ii) the resulting augmentation cost c (Augment_T(D)) $\leq \beta \cdot T$.
- B. Let Φ^* and T^* (respectively) denote the first-stage and second-stage cost of an optimal solution to the Robust(Π) instance. If the threshold $T \geq T^*$ then the first stage cost $c(\Phi_T) \leq \alpha_1 \cdot \Phi^* + \alpha_2 \cdot T^*$.

Lemma 2.2 ([15]) *If there is an* $(\alpha_1, \alpha_2, \beta)$ -discriminating algorithm for a robust covering problem Robust(Π), then for every $\epsilon > 0$ there is a $\left((1 + \epsilon) \cdot \max\left\{\alpha_1, \beta + \frac{\alpha_2}{\lambda}\right\}\right)$ -approximation algorithm for Robust(Π).

2.4 Desirable Properties of the Covering Problem

We now formalize certain properties of the covering problem $\Pi = \langle E, c, \{\mathcal{R}_i\}_{i=1}^n \rangle$ that are useful in obtaining our results. Given a partial solution $S \subseteq E$ and a set $X \subseteq [n]$ of requirements, any set $E_X \subseteq E$ such that $S \cup E_X \in \mathcal{R}_i \ \forall i \in X$ is called an *augmentation* of S for requirements X. Given X, S, define the min-cost augmentation of S for requirements X as:

$$\mathsf{OptAug}(X \mid S) := \min\{c(E_X) \mid E_X \subseteq E \text{ and } S \cup E_X \in \mathcal{R}_i, \ \forall i \in X\}.$$

Also define $\operatorname{Opt}(X) := \min\{c(E_X) \mid E_X \subseteq E \text{ and } E_X \in \mathcal{R}_i \ \forall i \in X\} = \operatorname{OptAug}(X \mid \emptyset), \text{ for any } X \subseteq [n].$ An easy consequence of the fact that costs are non-negative is the following:

Property 2.3 (Monotonicity) For any requirements $X \subseteq Y \subseteq [n]$ and any solution $S \subseteq E$, $\operatorname{OptAug}(X|S) \leq \operatorname{OptAug}(Y|S)$. Similarly, for any $X \subseteq [n]$ and solutions $T \subseteq S \subseteq E$, $\operatorname{OptAug}(X \mid S) \leq \operatorname{OptAug}(X \mid T)$.

From the definition of coverage of requirements, we obtain:

Property 2.4 (Subadditivity) For any two subsets of requirements $X, Y \subseteq [n]$ and any partial solution $S \subseteq E$, we have $\mathsf{OptAug}(X \mid S) + \mathsf{OptAug}(Y \mid S) \ge \mathsf{OptAug}(X \cup Y \mid S)$.

To see this property: if $\mathcal{F}_X \subseteq E$ and $\mathcal{F}_Y \subseteq E$ are solutions corresponding to $\operatorname{OptAug}(X \mid S)$ and $\operatorname{OptAug}(Y \mid S)$ respectively, then $\mathcal{F}_X \cup \mathcal{F}_Y \cup S$ covers requirements $X \cup Y$; so $\operatorname{OptAug}(X \cup Y \mid S) \leq c(\mathcal{F}_X \cup \mathcal{F}_Y) \leq c(\mathcal{F}_X) + c(\mathcal{F}_Y) = \operatorname{OptAug}(X \mid S) + \operatorname{OptAug}(Y \mid S)$.

We assume two additional properties of the covering problem:

Property 2.5 (Offline Algorithm) There is an α_{off} -approximation (offline) algorithm for the covering problem $\text{OptAug}(X \mid S)$, for any $S \subseteq E$ and $X \subseteq [n]$.

Property 2.6 (Online Algorithm) There is a polynomial-time deterministic α_{on} -competitive algorithm for the online version of $\Pi = \langle E, c, \{\mathcal{R}_i\}_{i=1}^n \rangle$.

2.5 Models of Downward-Closed Families

All covering functions we deal with are monotone non-decreasing. So we may assume WLOG that the collection Ω in both MaxMin(Π) and Robust(Π) is *downwards-closed*, i.e. $A \subseteq B$ and $B \in \Omega \implies A \in \Omega$. In this paper we consider the following well-studied classes:

Definition 2.7 (p-system) A downward-closed family $\Omega \subseteq 2^{[n]}$ is called a p-system iff:

$$\frac{\max_{I \in \overline{\Omega}, I \subseteq A} \ |I|}{\min_{J \in \overline{\Omega}, J \subseteq A} \ |J|} \le p, \quad \text{ for each } A \subseteq [n],$$

where $\overline{\Omega} \subseteq \Omega$ denotes the collection of maximal subsets in Ω . Sets in Ω are called independent sets. We assume access to a membership-oracle, that given any subset $I \subseteq [n]$ returns whether or not $I \in \Omega$.

Definition 2.8 (q-knapsack) Given q non-negative vectors $w^1, \ldots, w^q : [n] \to \mathbb{R}_+$ and capacities $b_1, \ldots, b_q \in \mathbb{R}_+$, the q-knapsack constrained family is:

$$\Omega = \left\{ A \subseteq [n] : \sum_{e \in A} w^j(e) \le b_j, \text{ for all } j \in [q] \right\}.$$

These constraints model a rich class of downward-closed families. Some interesting special cases of p-systems are p-matroid intersection [26] and p-set packing [18, 4]; see the appendix in [6] for more discussion on p-systems. Jenkyns [21] showed that the natural greedy algorithm is a p-approximation for maximizing linear functions over p-systems, which is the best known result. Maximizing a linear function over q-knapsack constraints is the well-studied class of packing integer programs (PIPs), eg. [28]. Again, the greedy algorithm is known to achieve an O(q)-approximation ratio. When the number of constraints q is constant, there is a PTAS [7].

3 Algorithms for Max-Min Optimization

In this section we give approximation algorithms for constrained max-min optimization, i.e. Problem (Max-f) where f is given by some underlying covering problem and Ω is given by some p-system and q-knapsack. We first consider the case when Ω is a p-system. Then we show that any knapsack constraint can be reduced to a 1-system (specifically a partition matroid) in a black-box fashion; this enables us to obtain an algorithm for Ω being the intersection of a p-system and q-knapsack. The results of this section assume Properties 2.4 and 2.6.

3.1 Algorithm for *p*-System Constraints

The algorithm given below is a greedy algorithm, however it is relative to the objective of the online algorithm A_{on} from Property 2.6 rather than the (approximate) function value itself.

Algorithm 1 Algorithm for MaxMin(Π) under p-system

- 1: **input:** the covering instance Π that defines f and p-system Ω .
- 2: **let** current scenario $A_0 \leftarrow \emptyset$, counter $i \leftarrow 0$, input sequence $\sigma \leftarrow \langle \rangle$.
- 3: while $(\exists e \in [n] \setminus A_i \text{ such that } A_i \cup \{e\} \in \Omega)$ do
- 4: $a_{i+1} \leftarrow \arg\max \{c(\mathcal{A}_{on}(\sigma \circ e)) c(\mathcal{A}_{on}(\sigma)) : e \in [n] \setminus A_i \text{ and } A_i \cup \{e\} \in \Omega\}.$
- 5: **let** $\sigma \leftarrow \sigma \circ a_{i+1}, \ A_{i+1} \leftarrow A_i \cup \{a_{i+1}\}, \ i \leftarrow i+1.$
- 6: end while
- 7: **let** $D \leftarrow A_i$ be the independent set constructed by the above loop.
- 8: **output** solution D.

Theorem 3.1 Assuming Properties 2.4 and 2.6 there is a $((p+1)\alpha_{on})$ -approximation algorithm for MaxMin(Π) under p-systems.

Proof: The proof of this lemma closely follows that in [6] for submodular maximization over a p-system. We use slightly more notation that necessary since this proof will be used in the next section as well.

Suppose that the algorithm performed k iterations; let $D=\{a_1,\cdots,a_k\}$ be the ordered set of elements added by the algorithm. Define $\sigma=\langle\rangle$, $G_0:=\emptyset$, and $G_i:=\mathcal{A}_{on}(\sigma\circ a_1\cdots a_i)$ for each $i\in[k]$. Note that $G_0\subseteq G_1\subseteq\cdots\subseteq G_k$. It suffices to show that:

$$\mathsf{OptAug}(B \mid G_0) \le (p+1) \cdot c(G_k \setminus G_0) \qquad \text{for every } B \in \Omega. \tag{3.1}$$

This would imply $\operatorname{Opt}(B) \leq (p+1) \cdot c(G_k) \leq (p+1) \alpha_{\operatorname{on}} \cdot \operatorname{Opt}(D)$ for every $B \in \Omega$, and hence that D is the desired approximation.

We use the following claim proved in [6], Appendix B (this claim relies on the properties of a p-system).

Claim 3.2 ([6]) For any $B \in \Omega$, there is a partition $\{B_i\}_{i=1}^k$ of B such that for all $i \in [k]$,

- 1. $|B_i| < p$, and
- 2. For every $e \in B_i$, we have $\{a_1, \dots, a_{i-1}\} \bigcup \{e\} \in \Omega$.

For any sequence π of requirements and any $e \in [n]$ define $\operatorname{Aug}(e;\pi) := c(\mathcal{A}_{on}(\pi \circ e)) - c(\mathcal{A}_{on}(\pi))$. Note that this function depends on the particular online algorithm. From the second condition in Claim 3.2, it follows that each element of B_i was a feasible augmentation to $\{a_1, \ldots, a_{i-1}\}$ in the i^{th} iteration of the **while** loop. By the greedy choice,

$$c(G_{i}) - c(G_{i-1}) = \operatorname{Aug}(a_{i}; \sigma \circ a_{1} \cdots a_{i-1}) \geq \max_{e \in B_{i}} \operatorname{Aug}(e; \sigma \circ a_{1} \cdots a_{i-1})$$

$$\geq \frac{1}{|B_{i}|} \sum_{e \in B_{i}} \operatorname{Aug}(e; \sigma \circ a_{1} \cdots a_{i-1})$$

$$\geq \frac{1}{|B_{i}|} \sum_{e \in B_{i}} \operatorname{OptAug}(\{e\} \mid G_{i-1}) \qquad (3.2)$$

$$\geq \frac{1}{|B_{i}|} \cdot \operatorname{OptAug}(B_{i} \mid G_{i-1}) \qquad (3.3)$$

$$\geq \frac{1}{n} \cdot \operatorname{OptAug}(B_{i} \mid G_{i-1}). \qquad (3.4)$$

Above equation (3.2) is by the definition of $G_{i-1} = \mathcal{A}_{on}(\sigma \circ a_1 \cdots a_{i-1})$, equation (3.3) uses the subadditivity Property 2.4, and (3.4) is by the first condition in Claim 3.2.

Summing over all iterations $i \in [k]$, we obtain:

$$c(G_k) - c(G_0) = \sum_{i=1}^k \mathsf{Aug}(a_i; \sigma \circ a_1 \cdots a_{i-1}) \geq \frac{1}{p} \sum_{i=1}^k \mathsf{OptAug}(B_i \mid G_{i-1}) \geq \frac{1}{p} \sum_{i=1}^k \mathsf{OptAug}(B_i \mid G_k)$$

where the last inequality follows from monotonicity since $G_{i-1} \subseteq G_k$ for all $i \in [k]$.

Using subadditivity Property 2.4, we get $c(G_k) - c(G_0) \ge \frac{1}{n} \cdot \mathsf{OptAug}(\bigcup_{i=1}^k B_i \mid G_k) = \frac{1}{n} \cdot \mathsf{OptAug}(B \mid G_k)$.

Let $J := \arg \min\{c(J') \mid J' \subseteq E$, and $G_k \cup J' \subseteq \mathcal{R}_e$, $\forall e \in B\}$. i.e. OptAug $(B \mid G_k) = c(J)$. Observe that $J \cup (G_k \setminus G_0)$ is a feasible augmentation to G_0 that covers requirements B. Thus,

$$\mathsf{OptAug}(B \mid G_0) \leq c(J) + c(G_k \setminus G_0) = \mathsf{OptAug}(B \mid G_k) + c(G_k \setminus G_0) \leq (p+1) \cdot c(G_k \setminus G_0).$$

This completes the proof.

3.2 Reducing knapsack constraints to partition matroids

In this subsection we show that every knapsack constraint can be reduced to a suitable collection of partition matroids. This property is then used to complete the algorithm for $\mathsf{MaxMin}(\Pi)$ when Ω is given by a p-system and a q-knapsack. Observe that even a single knapsack constraint need not correspond exactly to a small p-system: eg. the knapsack with weights $w_1 = 1$ and $w_2 = w_3 = \cdots = w_n = \frac{1}{n}$, and capacity one is only an (n-1)-system (since both $\{1\}$ and $\{2,3,\cdots,n\}$ are maximal independent sets). However we show that any knapsack constraint can be approximately reduced to a partition matroid (which is a 1-system). The main idea in this reduction is an enumeration method from Chekuri and Khanna [7].

Lemma 3.3 Given any knapsack constraint $\sum_{i=1}^{n} w_i \cdot x_i \leq B$ and fixed $0 < \epsilon \leq 1$, there is a polynomial-time computable collection $\mathcal{P}_1, \cdots, \mathcal{P}_T$ of $T = n^{O(1/\epsilon^2)}$ partition matroids such that:

1. For every
$$X \in \bigcup_{t=1}^T \mathcal{P}_t$$
, we have $\sum_{i \in X} w_i \leq (1 + \epsilon) \cdot B$.

2.
$$\{X \subseteq [n] \mid \sum_{i \in X} w_i \leq B\} \subseteq \bigcup_{t=1}^T \mathcal{P}_t$$
.

Proof: Let $\delta = \epsilon/6$ and $\beta = \frac{\delta B}{n}$. WLOG we assume that $\max_{i=1}^n w_i \leq B$. Partition the groundset [n] into $G := \lceil \frac{\log(n/\delta)}{\log(1+\delta)} \rceil$ groups as follows.

$$S_k := \begin{cases} \{i \in [n] : w_i \le \beta\} & \text{if } k = 0\\ \{i \in [n] : \beta \cdot (1+\delta)^{k-1} < w_i \le \beta \cdot (1+\delta)^k\} & \text{if } 1 \le k \le G \end{cases}$$

Let T denote the number of non-negative integer partitions of $\lceil G/\delta \rceil$ into G parts. Note that,

$$T := \binom{\lceil G/\delta \rceil + G - 1}{G - 1} \le \exp(\lceil G/\delta \rceil + G - 1) \le n^{O(1/\delta^2)}.$$

We will define a collection of T partition matroids on [n], each over the partition $\{S_0, S_1, \ldots, S_G\}$. For any integer partition $\tau = \{U_k\}_{k=1}^G$ of $\lceil G/\delta \rceil$ (i.e. $U_k \geq 0$ are integers and $\sum_k U_k = \lceil G/\delta \rceil$), define a partition matroid \mathcal{P}_{τ} that has bounds $N_k(\tau)$ on each part S_k , where

$$N_k(\tau) := \begin{cases} \infty & \text{if } k = 0\\ \lfloor \frac{n \cdot (U_k + 1)}{G \cdot (1 + \delta)^{k - 1}} \rfloor & \text{if } 1 \le k \le G \end{cases}$$

Clearly this collection can be constructed in polynomial time for fixed ϵ . We now show that this collection of partition matroids satisfies the two properties in the lemma.

(1) Consider any $X \subseteq [n]$ that is feasible for some partition matroid, say \mathcal{P}_{τ} . The total weight of elements $X \cap S_0$ is at most $n \cdot \beta \leq \delta \cdot B$. For any group $1 \leq k \leq G$, the weight of elements $X \cap S_k$ is at most:

$$|X \cap S_k| \cdot \beta (1+\delta)^k \le N_k(\tau) \cdot \beta (1+\delta)^k \le \delta (1+\delta)(U_k+1) \cdot \frac{B}{G}$$

Hence the total weight of all elements in X is at most:

$$\delta B + \delta (1+\delta) \frac{B}{G} \cdot \left(\sum_{k=1}^{G} U_k + G \right) \leq \delta B + \delta (1+\delta) \frac{B}{G} \cdot \left(\frac{G}{\delta} + 1 + G \right)$$

$$\leq \delta B + \delta (1+\delta) \frac{B}{G} \cdot \left(\frac{G}{\delta} + 2G \right)$$

$$\leq \delta B + (1+\delta) \cdot (B+2\delta B)$$

$$\leq B + 6\delta B.$$

Above we use $\delta \leq 1$. Finally since $\delta = \epsilon/6$, we obtain the first condition.

(2) Consider any $Y\subseteq [n]$ that satisfies the knapsack constraint, i.e. $\sum_{i\in Y}w_i\leq B$. We will show that Y is feasible in \mathcal{P}_{τ} , for some integer partition τ of $\lceil G/\delta \rceil$ as above. For each $1\leq k\leq G$ let Q_k denote the weight of elements in $Y\cap S_k$, and U_k be the unique integer that satisfies $U_k\cdot\frac{\delta B}{G}\leq Q_k<(U_k+1)\cdot\frac{\delta B}{G}$. Define τ to be the integer partition $\{U_k\}_{k=1}^G$. We have $\sum_k U_k\leq G/\delta$, which follows from the fact $B\geq \sum_k Q_k\geq \frac{\delta B}{G}\cdot\sum_k U_k$. By increasing U_k s arbitrarily so that they total to $\lceil G/\delta \rceil$, we obtain a feasible integer partition τ . We now claim that Y is feasible for \mathcal{P}_{τ} . Since each element of S_k has weight at least $\beta\cdot(1+\delta)^{k-1}$, we have

$$|Y \cap S_k| \le \frac{Q_k}{\beta (1+\delta)^{k-1}} \le \frac{(U_k+1) \cdot \delta B/G}{(1+\delta)^{k-1} \cdot \delta B/n} = \frac{n \cdot (U_k+1)}{G \cdot (1+\delta)^{k-1}}.$$

Since $|Y \cap S_k|$ is integral, we obtain $|Y \cap S_k| \leq \lfloor \frac{n \cdot (U_k + 1)}{G \cdot (1 + \delta)^{k - 1}} \rfloor \leq N_k(\tau)$. Thus we obtain the second condition.

Algorithm for *p***-System and** *q***-Knapsack Constraints**

Here we consider MaxMin(Π) when Ω is the intersection of p-system \mathcal{M} and a q-knapsack (as in Definition 2.8). The idea is to reduce the q-knapsack to a single knapsack (losing factor $\approx q$), then use Lemma 3.3 to reduce the knapsack to a 1-system, and finally apply Theorem 3.1 on the resulting p+1 system. Details appear below.

By scaling weights in the knapsack constraints, we may assume WLOG that each knapsack has capacity exactly one; let w^1, \dots, w^q denote the weights in the q knapsack constraints. We also assume WLOG that each singleton element satisfies the q-knapsack; otherwise such elements can be dropped from the groundset.

Algorithm 2 Algorithm for MaxMin(Π) under p-system and q-knapsack

- 1: Approximate the q-knapsack by a single knapsack with weights $\sum_{j=1}^{q} w^{j}$ and capacity q; applying Lemma 3.3 with $\epsilon = \frac{1}{2}$ on this knapsack, let $\{\mathcal{P}_j\}_{j=1}^L$ denote the resulting partition matroids (note $L = n^{O(1)}$).
- 2: For each $j \in [L]$, define $\Sigma_j := \mathcal{M} \cap \mathcal{P}_j$; note that each Σ_j is a (p+1)-system.
- 3: Run the algorithm from Theorem 3.1 under each p+1 system $\{\Sigma_j\}_{j=1}^L$ to obtain solutions $\{E_j \in \Sigma_j\}_{j=1}^L$.
- 4: Let $j^* \leftarrow \arg \max_{j=1}^L c(\mathcal{A}_{on}(E_j))$.
- 5: Partition E_{j^*} into $\{\omega_i\}_{i=1}^{3q+1}$ such that each $\omega_i \in \Omega$, as per Claim 3.5. 6: Output ω_{i^*} where $i^* \leftarrow \arg\max_{i=1}^{3q+1} c\left(\mathcal{A}_{off}(\omega_i)\right)$. Here we use the offline algorithm from Property 2.5.

We now establish the approximation ratio of this algorithm.

Claim 3.4
$$\Omega \subseteq \bigcup_{i=1}^{L} \Sigma_{j}$$
.

Proof: For any $\omega \in \Omega$, we have $\sum_{e \in \omega} w^i(e) \le 1$ for all $i \in [q]$. Hence $\sum_{e \in \omega} \sum_{i=1}^q w^i(e) \le q$, i.e. it satisfies the combined knapsack constraint. Now by Lemma 3.3 (2), we obtain $\omega \in \bigcup_{j=1}^{L} \mathcal{P}_j$. Finally, since $\omega \in \Omega \subseteq \mathcal{M}$, we have $\omega \in \bigcup_{i=1}^L \Sigma_i$.

Claim 3.5 For each $\tau \in \bigcup_{j=1}^L \Sigma_j$ there exists a collection $\{\omega_i\}_{i=1}^{3q+1}$ such that $\tau = \bigcup_{\ell=1}^{3q+1} \omega_\ell$, and $\omega_\ell \in \Omega$ for all $\ell \in [3q+1]$. Furthermore, this is computable in polynomial time.

Proof: Consider any $\tau \in \Sigma := \bigcup_{j=1}^{L} \Sigma_{j}$. Note that $\tau \in \mathcal{M}$, so any subset of τ is also in \mathcal{M} (which is downwardsclosed). We will show that there is a partition of τ into $\{\omega_\ell\}_{\ell=1}^{3q}$ such that each ω_ℓ satisfies the q-knapsack. This suffices to prove the claim. Since $\tau \in \bigcup_{i=1}^L \mathcal{P}_i$, by Lemma 3.3 (1) it follows that $\sum_{e \in \tau} \sum_{i=1}^q w^i(e) \leq \frac{3}{2}q$. Starting with the trivial partition of τ into singleton elements, greedily merge parts as long as each part satisfies the q-knapsack, until no further merge is possible. (Note that the trivial partition is indeed feasible since each element satisfies the q-knapsack.) Let $\{\omega_\ell\}_{\ell=1}^r$ denote the parts in the final partition; we will show $r \leq 3q+1$ which would prove the claim. Consider forming $\lfloor r/2 \rfloor$ pairs from $\{\omega_\ell\}_{\ell=1}^r$ arbitrarily. Observe that for any pair $\{\omega,\omega'\}$, it must be that $\omega \cup \omega'$ violates *some* knapsack; so $\sum_{e \in \omega \cup \omega'} \sum_{i=1}^q w^i(e) > 1$. Thus $\sum_{e \in \tau} \sum_{i=1}^q w^i(e) > \lfloor r/2 \rfloor$. On the other hand, $\sum_{e \in \tau} \sum_{i=1}^q w^i(e) \leq \frac{3}{2}q$, which implies r < 3q + 2.

Theorem 3.6 Assuming Properties 2.4, 2.5 and 2.6, there is an $O((p+1)(q+1)\alpha_{\text{off}}\alpha_{\text{on}})$ -approximation algorithm for $MaxMin(\Pi)$ under a p-system and q-knapsack constraint.

Proof: Let Opt_j denote the optimal value of $\operatorname{MaxMin}(\Pi)$ under p+1 system Σ_j , for each $j \in [L]$. By Claim 3.4 we have $\max_{j=1}^{L} \mathsf{Opt}_{j} \geq \mathsf{Opt}$, the optimal value of $\mathsf{MaxMin}(\Pi)$ under Ω . Observe that Theorem 3.1 actually implies $c(\mathcal{A}_{on}(E_j)) \geq \frac{1}{p+2} \cdot \mathsf{Opt}_j$ for each $j \in [q]$. Thus $c(\mathcal{A}_{on}(E_{j^*})) \geq \frac{1}{p+2} \cdot \mathsf{Opt}$; hence $\mathsf{Opt}(E_{j^*}) \geq \frac{1}{\alpha_{\mathsf{on}}(p+2)} \cdot \mathsf{Opt}$. Now consider the partition $\{\omega_i\}_{i=1}^{3q+1}$ of E_{j^*} from Claim 3.5. By the subadditivity property, $\sum_{i=1}^{3q+1} \mathsf{Opt}(\omega_i) \geq \mathsf{Opt}(E_{j^*})$; i.e. there is some $i' \in [3q+1]$ with $\mathsf{Opt}(\omega_{i'}) \geq \frac{1}{\alpha_{\mathsf{on}}(p+2)(3q+1)} \cdot \mathsf{Opt}$. Thus the i^* found using the offline algorithm (Property 2.5) satisfies $\mathsf{Opt}(\omega_{i^*}) \geq \frac{1}{\alpha_{\mathsf{on}} \, \alpha_{\mathsf{off}} \, (p+2)(3q+1)} \cdot \mathsf{Opt}$

Remark: We can obtain a better approximation guarantee of $O((p+1)(q+1)\alpha_{\text{on}})$ in Theorem 3.6 using randomization. This algorithm is same as Algorithm 2, except for the last step, where we output ω_{ℓ} for $\ell \in [3q+1]$ chosen *uniformly at random*. From the above proof of Theorem 3.6, it follows that:

$$E[\mathsf{Opt}(\omega_\ell)] = \frac{1}{3q+1} \sum_{i=1}^{3q+1} \mathsf{Opt}(\omega_i) \geq \frac{\mathsf{Opt}(E_{j^*})}{3q+1} \geq \frac{1}{\alpha_{\mathsf{on}} \left(p+2\right) (3q+1)} \cdot \mathsf{Opt}.$$

4 General Framework for Robust Covering Problems

In this section we present an abstract framework for robust covering problems under any uncertainty set Ω , as long as we are given access to offline, online and max-min algorithms for the base covering problem. Formally, this requires Properties 2.5, 2.6 and the following additional property (recall the notation from Section 2).

Property 4.1 (Max-Min Algorithm) There is an α_{mm} -approximation algorithm for the max-min problem: given input $S \subseteq E$, $\mathsf{MaxMin}(S) := \max_{X \in \Omega} \min\{c(A) \mid S \cup A \in R_i, \ \forall i \in X\}$.

Theorem 4.2 Under Properties 2.4, 2.5, 2.6 and 4.1, there is an $O(\alpha_{\sf off} \cdot \alpha_{\sf on} \cdot \alpha_{\sf mm})$ -approximation algorithm for the robust covering problem $\mathsf{Robust}(\Pi) = \langle E, c, \{R_i\}_{i=1}^n, \Omega, \lambda \rangle$.

Proof: The algorithm proceeds as follows.

Algorithm 3 Algorithm Robust-with-General-Uncertainty-Sets

- 1: **input:** the Robust(Π) instance and threshold T.
- 2: **let** counter $t \leftarrow 0$, initial online algorithm's input $\sigma = \langle \rangle$, initial online solution $F_0 \leftarrow \emptyset$.
- 3: repeat
- 4: **set** $t \leftarrow t + 1$.
- 5: let $E_t \subseteq [n]$ be the scenario returned by the algorithm of Property 4.1 on MaxMin (F_{t-1}) .
- 6: let $\sigma \leftarrow \sigma \circ E_t$, and $F_t \leftarrow \mathcal{A}_{on}(\sigma)$ be the current online solution.
- 7: **until** $c(F_t) c(F_{t-1}) \leq 2\alpha_{\mathsf{on}} \cdot T$
- 8: **set** $\tau \leftarrow t 1$.
- 9: **output** first-stage solution $\Phi_T := F_{\tau}$.
- 10: **output** second-stage solution Augment_T where for any $\omega \subseteq [n]$, Augment_T(ω) is the solution of the offline algorithm (Property 2.5) for the problem OptAug($\omega \mid \Phi_T$).

As always, let $\Phi^* \subseteq E$ denote the optimal first stage solution (and its cost), and T^* the optimal second-stage cost; so the optimal value is $\Phi^* + \lambda \cdot T^*$. We prove the performance guarantee using the following claims.

Claim 4.3 (General 2nd stage) For any $T \geq 0$ and $X \in \Omega$, elements $\Phi_T \bigcup \mathsf{Augment}_T(X)$ satisfy all the requirements in X, and $c(\mathsf{Augment}_T(X)) \leq 2\alpha_{\mathsf{off}} \cdot \alpha_{\mathsf{mm}} \cdot \alpha_{\mathsf{on}} \cdot T$.

Proof: It is clear that $\Phi_T \bigcup \mathsf{Augment}_T(X)$ satisfy all requirements in X. By the choice of set $E_{\tau+1}$ in line 5 of the last iteration, for any $X \in \Omega$ we have:

$$\mathsf{OptAug}(X \mid F_\tau) \leq \alpha_{\mathsf{mm}} \cdot \mathsf{OptAug}(E_{\tau+1} \mid F_\tau) \leq \alpha_{\mathsf{mm}} \cdot (c(F_{\tau+1}) - c(F_\tau)) \leq 2\alpha_{\mathsf{mm}} \cdot \alpha_{\mathsf{on}} \cdot T$$

The first inequality is by Property 4.1, the second inequality uses the fact that $F_{\tau+1} \supseteq F_{\tau}$ (since we use an online algorithm to augment in line 6),² and the last inequality follows from the termination condition in line 7. Finally, since Augment_T(X) is an α_{off} -approximation to OptAug(X | F_{τ}), we obtain the claim.

²This is the technical reason we need an online algorithm. If instead we had used an offline algorithm to compute F_t in step 6 then $F_t \not\supseteq F_{t-1}$ and we could not upper bound the augmentation cost $\mathsf{OptAug}(E_t \mid F_{t-1})$ by $c(F_t) - c(F_{t-1})$.

Claim 4.4 $\operatorname{Opt}(\cup_{t \le \tau} E_t) \le \tau \cdot T^* + \Phi^*$.

Proof: Since each $E_t \in \Omega$ (these are solutions to MaxMin), the bound on the second-stage optimal cost gives $\operatorname{OptAug}(E_t \mid \Phi^*) \leq T^*$ for all $t \leq \tau$. By subadditivity (Property 2.4) we have $\operatorname{OptAug}(\cup_{t \leq \tau} E_t \mid \Phi^*) \leq \tau \cdot T^*$, which immediately implies the claim.

Claim 4.5
$$\operatorname{Opt}(\bigcup_{t \leq \tau} E_t) \geq \frac{1}{\alpha_{con}} \cdot c(F_{\tau}).$$

Proof: Directly from the competitiveness of the online algorithm in Property 2.6.

Claim 4.6 (General 1st stage) If $T \geq T^*$ then $c(\Phi_T) = c(F_\tau) \leq 2 \alpha_{on} \cdot \Phi^*$.

Proof: We have $c(F_{\tau}) = \sum_{t=1}^{\tau} \left[c(F_t) - c(F_{t-1}) \right] > 2\alpha_{\mathsf{on}}\tau \cdot T \geq 2\alpha_{\mathsf{on}}\tau \cdot T^*$ by the choice in Step (7). Combined with Claim 4.5, we have $\mathsf{Opt}(\cup_{t \leq \tau} E_t) \geq 2\tau \cdot T^*$. Now using Claim 4.4, we have $\tau \cdot T^* \leq \Phi^*$, and hence $\mathsf{Opt}(\cup_{t \leq \tau} E_t) \leq 2 \cdot \Phi^*$. Finally using Claim 4.5, we obtain $c(F_{\tau}) \leq 2\alpha_{\mathsf{on}} \cdot \Phi^*$.

Claim 4.3 and Claim 4.6 imply that the above algorithm is a $(2\alpha_{\sf on},\ 0,\ 2\alpha_{\sf mm}\alpha_{\sf on}\alpha_{\sf off})$ -discriminating algorithm for the robust problem Robust $(\Pi) = \langle E, c, \{R_i\}_{i=1}^n, \Omega, \lambda \rangle$. Now using Lemma 2.2 we obtain the theorem.

Explicit uncertainty sets An easy consequence of Theorem 4.2 is for the *explicit scenario* model of robust covering problems [8, 14], where Ω is specified as a list of possible scenarios. In this case, the MaxMin problem can be solved using the α_{off} -approximation algorithm from Property 2.5 which implies an $O(\alpha_{\text{off}}^2 \alpha_{\text{on}})$ -approximation for the robust version. In fact, we can do slightly better—observing that in this case, the algorithm for second-stage augmentation is the same as the Max-Min algorithm, we obtain an $O(\alpha_{\text{off}} \cdot \alpha_{\text{on}})$ -approximation algorithm for robust covering with explicit scenarios. As an application of this result, we obtain an $O(\log n)$ approximation for robust Steiner forest with explicit scenarios, which is the best known result for this problem.

5 Robust Covering under p-System and q-Knapsack Uncertainty Sets

Recall that any uncertainty set Ω for a robust covering problem can be assumed WLOG to be *downward-closed*, i.e. $X \in \Omega$ and $Y \subseteq X$ implies $Y \in \Omega$. Eg., in the k-robust model $\Omega = \{S \subseteq [n] : |S| \le k\}$. Hence it is of interest to obtain good approximation algorithms for robust covering when Ω is specified by means of general models for downward-closed families. In this section, we consider the two well-studied models of p-systems and q-knapsacks (Definitions 2.7 and 2.8).

The result of this section says the following: if we can solve both the offline and online versions of a covering problem well, we get good algorithms for Robust(Π) under uncertainty sets given by the intersection of p-systems and q-knapsack constraints. Naturally, the performance depends on p and q; we note that this is unavoidable due to complexity considerations. Based on Theorem 4.2 it suffices to give an approximation algorithm for the maxmin problem under p-systems and q-knapsack constraints; so Theorem 3.6 combined with Theorem 4.2 implies an $O\left((p+1)(q+1)\alpha_{\text{on}}^2\alpha_{\text{off}}^2\right)$ -approximation ratio. However, we can obtain a better guarantee by considering the algorithm for Robust(Π) directly. Formally we show that:

Theorem 5.1 Under Properties 2.4, 2.5 and 2.6, the robust covering problem $\mathsf{Robust}(\Pi)\langle E, c, \{\mathcal{R}_i\}_{i=1}^m, \Omega, \lambda \rangle$ admits an $O((p+1) \cdot (q+1) \cdot \alpha_{\mathsf{off}} \cdot \alpha_{\mathsf{on}})$ -approximation guarantee when Ω is given by the intersection of a psystem and q-knapsack constraints.

The outline of the proof is same as for Theorem 3.6. We first consider the case when the uncertainty set is a p-system (subsection 5.1); then using the reduction in Lemma 3.3 we solve a suitable instance of Robust(Π) under a (p+1)-system uncertainty set.

5.1 p-System Uncertainty Sets

In this subsection, we consider Robust(Π) when the uncertainty set Ω is some p-system. The algorithm is a combination of the ones in Theorem 4.2 and Theorem 3.1. We start with an empty solution, and use the online algorithm to greedily try and build a scenario of large cost. If we do find a "violated" scenario which is unhappy with the current solution, we augment our current solution to handle this scenario (again using the online algorithm), and continue. The algorithm is given as Algorithm 4 below.

Algorithm 4 Algorithm Robust-with-p-system-Uncertainty-Sets

```
1: input: the Robust(\Pi) instance and bound T.
 2: let counter t \leftarrow 0, initial online algorithm's input \sigma = \langle \rangle, initial online solution F_0 \leftarrow \emptyset.
 3: repeat
 4:
         set t \leftarrow t + 1.
         let current scenario A_0^t \leftarrow \emptyset, counter i \leftarrow 0.
 5:
         while (\exists e \in [n] \setminus A_i^t \text{ such that } A_i^t \cup \{e\} \in \Omega) do
 6:
            a_{i+1} \leftarrow \arg\max\{c(\mathcal{A}_{on}(\sigma \circ e)) - c(\mathcal{A}_{on}(\sigma)) \mid e \in [n] \setminus A_i \text{ and } A_i \cup \{e\} \in \Omega\}.
 7:
            let \sigma \leftarrow \sigma \circ a_{i+1}, A_{i+1}^t \leftarrow A_i^t \cup \{a_{i+1}\}, i \leftarrow i+1.
 8:
 9:
         end while
         let E_t \leftarrow A_i^t be the scenario constructed by the above loop.
10:
         let F_t \leftarrow \mathcal{A}_{on}(\sigma) be the current online solution.
12: until c(F_t) - c(F_{t-1}) \le 2\alpha_{\sf on} \cdot T
13: set \tau \leftarrow t - 1.
14: output first-stage solution \Phi_T := F_{\tau}.
15: output second-stage solution Augment<sub>T</sub> where for any \omega \subseteq [n], Augment<sub>T</sub>(\omega) is the solution of the offline
     algorithm (Property 2.5) for the problem \mathsf{OptAug}(\omega \mid \Phi_T).
```

We first prove a useful lemma about the behavior of the **while** loop.

Lemma 5.2 (Max-Min Lemma) For any iteration t of the repeat loop, the scenario $E_t \in \Omega$ has the property that for any other scenario $B \in \Omega$, $OptAug(B \mid F_{t-1}) \leq (p+1) \cdot c(F_t \setminus F_{t-1})$.

Proof: The proof is almost identical to that of Theorem 3.1.

Consider any iteration t of the **repeat** loop in Algorithm 4 that starts with a sequence σ of elements (that have been fed to the online algorithm \mathcal{A}_{on}). Let $A = \{a_1, \cdots, a_k\}$ be the ordered set of elements added by the algorithm in this iteration. Define $G_0 := \mathcal{A}_{on}(\sigma)$, and $G_i := \mathcal{A}_{on}(\sigma \circ a_1 \cdots a_i)$ for each $i \in [k]$. Note that $F_{t-1} = G_0$ and $F_t = G_k$, and $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k$. It suffices to show that $\operatorname{OptAug}(B \mid G_0) \leq (p+1) \cdot c(G_k \setminus G_0)$ for every $B \in \Omega$. But this is precisely Equation (3.1) from the proof of Theorem 3.1.

Corollary 5.3 (Second Stage) For any $T \geq 0$ and $B \in \Omega$, elements $\Phi_T \bigcup \mathsf{Augment}_T(B)$ satisfy all the requirements in B, and $c(\mathsf{Augment}_T(B)) \leq 2\alpha_{\mathsf{off}} \cdot \alpha_{\mathsf{on}} \cdot (p+1) \cdot T$.

Proof: Observe that $\Phi_T = F_\tau = \mathcal{A}_{on}(\sigma)$, so the first part of the corollary follows from the definition of Augment_T. By Lemma 5.2 and the termination condition on line 12, we have $\operatorname{OptAug}(B \mid F_\tau) \leq (p+2) \cdot (c(F_{\tau+1}) - c(F_\tau)) \leq 2(p+2)\alpha_{\text{on}} T$. Now Property 2.5 guarantees that the solution $\operatorname{Augment}_T(B)$ found by this approximation algorithm has cost at most $2\alpha_{\text{off}} \cdot \alpha_{\text{on}} \cdot (p+2) T$.

It just remains to bound the cost of the first-stage solution F_{τ} . Below Φ^* denotes the optimal first-stage solution (and its cost); and T^* is the optimal second-stage cost.

```
Lemma 5.4 (First Stage) If T \geq T^* then c(\Phi_T) = c(F_\tau) \leq 2\alpha_{\mathsf{on}} \cdot \Phi^*.
```

Proof: For any set $X \subseteq [n]$ of requirements let $\mathsf{Opt}(X)$ denote the minimum cost to satisfy X. Firstly, observe that $\mathsf{Opt}(\cup_{t \le \tau} E_t) \le \tau \cdot T^* + \Phi^*$. This follows from the fact that each of the τ scenarios E_t are in Ω , so the bound

on the second-stage optimal cost gives $\mathsf{OptAug}(E_t \mid \Phi^*) \leq T^*$ for all $t \leq \tau$. By subadditivity (Assumption 2.4) we have $\mathsf{OptAug}(\cup_{t \leq \tau} E_t \mid \Phi^*) \leq \tau \cdot T^*$, which immediately implies the inequality. Now, we claim that

$$\mathsf{Opt}(\cup_{t \le \tau} E_t) \ge \frac{1}{\alpha_{\mathsf{on}}} \cdot c(F_\tau) \ge \frac{1}{\alpha_{\mathsf{on}}} \cdot 2\alpha_{\mathsf{on}} \tau \cdot T^* = 2\tau \cdot T^*. \tag{5.5}$$

The first inequality follows directly from the competitiveness of the online algorithm in Assumption 2.6. For the second inequality, we have $c(F_{\tau}) = \sum_{t=1}^{\tau} \left[c(F_t) - c(F_{t-1}) \right] > 2\alpha_{\mathsf{on}}\tau \cdot T \geq 2\alpha_{\mathsf{on}}\tau \cdot T^*$ by the terminal condition in Step 12. Putting the upper and lower bounds on $\mathsf{Opt}(\cup_{t \leq \tau} E_t)$ together, we have $\tau \cdot T^* \leq \Phi^*$, and hence $\mathsf{Opt}(\cup_{t \leq \tau} E_t) \leq 2 \cdot \Phi^*$. Using the competitiveness of the online algorithm again, we obtain $c(F_{\tau}) \leq 2\alpha_{\mathsf{on}} \cdot \Phi^*$.

From Corollary 5.3 and Lemma 5.4, it follows that our algorithm is $(2\alpha_{\sf on}, 0, 2\alpha_{\sf off} \alpha_{\sf on} \cdot (p+1))$ -discriminating (cf. Definition 2.1) to Robust(Π). Thus we obtain Theorem 5.1 for the case q=0.

5.2 Algorithm for p-Systems and q-Knapsacks

Here we consider Robust(Π) when the uncertainty set Ω is the intersection of p-system \mathcal{M} and a q-knapsack. The algorithm is similar to that in Subsection 3.3. Again, by scaling weights in the knapsack constraints, we may assume WLOG that each knapsack has capacity exactly one; let w^1, \dots, w^q denote the weights in the q-knapsack constraints. We also assume WLOG that each singleton element satisfies the q-knapsack. The algorithm for Robust(Π) under Ω works as follows.

Algorithm 5 Algorithm Robust with p-system and q-knapsack Uncertainty Set

- 1: Consider a modified uncertainty set Ω' that is given by the intersection of \mathcal{M} and the *single knapsack* with weight-vector $\sum_{j=1}^{q} w^{j}$ and capacity q.
- 2: Applying the algorithm in Lemma 3.3 to this single knapsack with $\epsilon=1$, let $\{\mathcal{P}_j\}_{j=1}^L$ denote the resulting partition matroids (note $L=n^{O(1)}$).
- 3: For each $j \in [L]$, define uncertainty-set $\Sigma_j := \mathcal{M} \cap \mathcal{P}_j$; note that each Σ_j is a (p+1)-system.
- 4: Let $\Sigma \leftarrow \bigcup_{j=1}^{L} \Sigma_{j}$. Solve Robust(Π) under Σ using the algorithm of Theorem 5.6.

Recall Claims 3.4 and 3.5 which hold here as well.

Lemma 5.5 Any α -approximate solution to Robust(Π) under Σ is a $(3q+1)\alpha$ -approximate solution to Robust(Π) under uncertainty-set Ω .

Proof: Consider the optimal first-stage solution Φ^* to Robust(Π) under Ω , let T^* denote the optimal second-stage cost and Opt the optimal value. Let $\tau \in \Sigma$ be any scenario, with partition $\{\omega_i\}_{i=1}^{3q+1}$ given by Claim 3.5. Using the subadditivity Property 2.4, we have $\operatorname{OptAug}(\tau|\Phi^*) \leq \sum_{\ell=1}^{3q+1} \operatorname{OptAug}(\omega_\ell|\Phi^*) \leq (3q+1) \cdot T^*$. Thus the objective value of Φ^* for Robust(Π) under Σ is at most $c(\Phi^*) + \lambda \cdot (3q+1) T^* \leq (3q+1) \cdot \operatorname{Opt}$.

Claim 3.4 implies that for any solution, the objective value of Robust(Π) under Ω is at most that of Robust(Π) under Σ . Thus the lemma follows.

For solving Robust(Π) under Σ , note that although Σ itself is not any p'-system, it is the *union of polynomially-many* (p+1)-systems. We show below that a simple extension of the algorithm in Subsection 5.1 also works for unions of p-systems; this would solve Robust(Π) under Σ .

Theorem 5.6 There is an $O((p+1)\alpha_{\text{off}}\alpha_{\text{on}})$ -approximation for Robust(Π) when the uncertainty set is given by the union of polynomially-many p-systems.

Proof: Let $\Sigma = \bigcup_{j=1}^{L} \Sigma_j$ denote the uncertainty set where each Σ_j is a p-system. The algorithm for Robust(Π) under Σ is just Algorithm 4 where we replace the body of the repeat-loop (ie. lines 4-11) by:

- 1: **set** $t \leftarrow t + 1$.
- 2: **for** $(j \in [L])$ **do**

```
3: let current scenario A_j \leftarrow \emptyset,

4: while (\exists e \in [n] \setminus A_j \text{ such that } A_j \cup \{e\} \in \Sigma_j) do

5: e^* \leftarrow \arg\max\{c(\mathcal{A}_{on}(\sigma \circ A_j \circ e)) - c(\mathcal{A}_{on}(\sigma \circ A_j)) \mid e \in [n] \setminus A_j \text{ and } A_j \cup \{e\} \in \Omega\}.

6: A_j \leftarrow A_j \cup \{e^*\}.

7: end while

8: Let \Delta_j \leftarrow c(\mathcal{A}_{on}(\sigma \circ A_j)) - c(\mathcal{A}_{on}(\sigma)).

9: end for

10: let j^* \leftarrow \arg\max\{\Delta_j \mid j \in [L]\}, and E_t \leftarrow A_{j^*}.

11: let \sigma \leftarrow \sigma \circ E_t and F_t \leftarrow \mathcal{A}_{on}(\sigma) be the current online solution.
```

Consider any iteration t of the repeat loop. By Lemma 5.2 applied to each p-system Σ_i ,

Claim 5.7 For each
$$j \in [L]$$
, we have $\operatorname{OptAug}(B|F_{t-1}) \leq (p+1) \cdot \Delta_j$ for every $B \in \Sigma_j$.

By the choice of scenario E_t and since $\Sigma = \bigcup_{j=1}^L \Sigma_j$, we obtain:

Claim 5.8 For any iteration
$$t$$
 of the repeat loop and any $B \in \Sigma$, $\operatorname{OptAug}(B|F_{t-1}) \leq (p+1) \cdot c(F_t \setminus F_{t-1})$.

Based on these claims and proofs identical to Corollary 5.3 and Lemma 5.4, we obtain the same bounds on the first and second stage costs of the final solution F_{τ} . Thus our algorithm is $(2\alpha_{\sf on},0,2\alpha_{\sf off}\;\alpha_{\sf on}\cdot(p+1))$ -discriminating, which by Lemma 2.2 implies the theorem.

Finally, combining Lemma 5.5 and Theorem 5.6 we obtain Theorem 5.1.

Remark: In Theorem 5.1, the dependence on the number of constraints describing the uncertainty set Ω is inevitable (under some complexity assumptions). Consider a very special case of the robust covering problem on ground-set E, requirements E (where $i \in E$ is satisfied iff the solution contains i), a unit cost function on E, inflation parameter $\lambda = 1$. The uncertainty set Ω is given by the intersection of p different cardinality constraints coming from some *set packing* instance on E. In this case, the optimal value of the robust covering problem is exactly the optimal value of the set packing instance. The hardness result from Håstad [17] now implies that this robust covering problem is $\Omega(p^{\frac{1}{2}-\epsilon})$ hard to approximate. We note that this hardness applies only to algorithms having running time that is sub-exponential in both |E| and p; this is indeed the case for our algorithm.

Results for p**-System and** q**-Knapsack Uncertainty Sets.** We now list some specific results for robust covering under uncertainty sets described by p-systems and knapsack constraints; these follow directly from Theorem 5.1 using known offline and (deterministic) online algorithms for the relevant problems.

Problem	Offline ratio	Online ratio	p-system, q-knapsack Robust
Set Cover	$O(\log m)$	$O(\log m \cdot \log n)$ [2]	$pq \cdot \log^2 m \cdot \log n$
Steiner Tree/Forest	2 [1, 13]	$O(\log n)$ [19, 5]	$pq \cdot \log n$
Minimum Cut	1	$O(\log^3 n \cdot \log \log n) [3, 16]$	$pq \cdot \log^3 n \cdot \log \log n$
Multicut	$\log n$ [12]	$O(\log^3 n \cdot \log \log n) [3, 16]$	$pq \cdot \log^4 n \cdot \log \log n$

6 Non-Submodularity of Some Covering Functions

In this section we show that some natural covering functions are not even approximately submodular. Let $f: 2^U \to \mathbb{R}_{\geq 0}$ be any monotone subadditive function. We say that f is α -approximately submodular iff there exists a submodular function $g: 2^U \to \mathbb{R}_{\geq 0}$ with $g(S) \leq f(S) \leq \alpha \cdot g(S)$ for all $S \subseteq U$.

Consider the min-set-cover function, $f_{SC}(S) = \text{minimum number of sets required to cover elements } S$.

Proposition 6.1 The min-set-covering function is not o(n)-approximately submodular.

Proof: The proof follows from the lower bound on *budget-balance* for *cross-monotone cost allocations*. Immorlica et al. [20] showed that there is no o(n)-approximately budget-balanced cross-monotone cost allocation for the set-cover game. On the other hand it is known (see Chapter 15.4.1 in [25]) that any submodular-cost game admits

a budget-balanced cross-monotone cost allocation. This also implies that any α -approximately submodular cost function (non-negative) admits an α -approximate budget-balanced cross-monotone cost allocation. Thus the min-set-covering function can not be o(n)-approximately submodular.

Similarly, for minimum multicut ($f_{MMC}(S)$ = minimum cost cut separating the pairs in S),

Proposition 6.2 The min-multicut function is not $o(n^{1/3})$ -approximately submodular.

Proof: This uses the result that the vertex-cover game does not admit $o(n^{1/3})$ -approximately budget-balanced cross-monotone cost allocations [20]. Since multicut (even on a star graph) contains the vertex-cover problem, the proposition follows.

On the other hand, some other covering functions are indeed approximately submodular.

- The minimum-cut function $(f_{MC}(S))$ = minimum cost cut separating vertices S from the root) is in fact submodular due to submodularity of cuts in graphs.
- The min-Steiner-tree $(f_{ST}(S))$ = minimum length tree that connects vertices S to the root) and min-Steiner-forest $(f_{SF}(S))$ = minimum length forest connecting the pairs in S) functions are $O(\log n)$ -approximately submodular. When the underlying metric is a tree, these functions are submodular—in this case they reduce to weighted coverage functions. Using probabilistic approximation of general metrics by trees, we can write $g(S) = E_{T \in \mathcal{T}}[f^T(S)]$ where \mathcal{T} is the distribution on dominating tree-metrics (from [9]) and f^T is the Steiner-tree/Steiner-forest function on tree T. Clearly g is submodular. Since there exists \mathcal{T} that approximates distances in the original metric within factor $O(\log n)$ [9], it follows that g also $O(\log n)$ -approximates f_{ST} (resp. f_{SF}).

While approximate submodularity of the covering problem Π (eg. minimum-cut or Steiner-tree) yields direct approximation algorithms for $\mathsf{MaxMin}(\Pi)$, it is unclear whether they help in solving $\mathsf{Robust}(\Pi)$ (even under cardinality-constrained uncertainty sets [15]). On the other hand, the online-algorithms based approach in this paper solves both $\mathsf{MaxMin}(\Pi)$ and $\mathsf{Robust}(\Pi)$, for uncertainty sets from p-systems and q-knapsacks.

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