Robust Satisfiability of Systems of Equations *

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Abstract

We study the problem of robust satisfiability of systems of nonlinear equations, namely, whether for a given continuous function $f: K \to \mathbb{R}^n$ on a finite simplicial complex K and $\alpha > 0$, it holds that each function $g: K \to \mathbb{R}^n$ such that $||g - f||_{\infty} \leq \alpha$, has a root in K. Via a reduction to the extension problem of maps into a sphere, we particularly show that this problem is decidable in polynomial time for every fixed n, assuming dim $K \leq 2n-3$. This is a substantial extension of previous computational applications of topological degree and related concepts in numerical and interval analysis.

Via a reverse reduction we prove that the problem is undecidable when dim $K \ge 2n - 2$, where the threshold comes from the *stable range* in homotopy theory.

For the lucidity of our exposition, we focus on the setting when f is piecewise linear. Such functions can approximate general continuous functions, and thus we get approximation schemes and undecidability of the robust satisfiability in other possible settings.

1 Introduction

In many engineering and scientific solutions, a highly desired property is the resistance against noise or perturbations. We can only name a fraction of the instances: stability in data analysis [9], robust optimization [3], image processing [27], or stability of numerical methods [29]. Some of the most crucial tools for robust design come from topology, which can capture the stable properties of spaces and maps. Famous concepts using topological methods in computer

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science are fixed point theory [13], fair division theory [14], persistent homology [16], or discrete Morse theory [19].

In this paper, we take the robustness perspective on solving systems of nonlinear equations, a fundamental problem in mathematics and computer science. The tools of algebraic topology will enable us to quantify and compute the robustness of a solution—its resistance against perturbations of a given system of equations. The robustness of a root is a favourable property as the system may come from imprecise measurements or from a model with inherent uncertainty.¹ In case that the system is given by arithmetic expressions, we can use less precise but fast floating point operations and still identify the root, if it is a robust one.

Much research has been done in this direction, and as far as we know, the topological degree has always been the essential core of the approaches. The topological degree is indeed a powerful and practically usable tool for proving the existence of robust solutions of systems of the form f = 0 for (nonlinear) continuous functions $f: \mathbb{R}^n \to \mathbb{R}^n$. Developing techniques for such proofs is a major theme in the interval computation community [37, 31, 24, 22, 12], although the emphasis is usually not put on the completeness of the tests,² but on their usability within numerical solvers. Particularly, efficient formulas for topological degree has been devised in the case where the map f is polynomial [18, 46].

Our contribution. The main ingredient of our results is the replacement of the topological degree by the *extendability* of maps into spheres. The extension problem is given by a continuous map g defined on a subspace A of a topological space X (in our case the map will always take values in a sphere S^d) and the question is whether g can be continuously extended to whole of X.³ Extendability provides a solution that combines the following three properties:

- 1. We quantify the robustness of the roots. Formally, for a given $\alpha > 0$ we say that $f: K \to \mathbb{R}^n$ on a compact domain $K \subseteq \mathbb{R}^m$ has an α -robust root whenever every α -perturbation g of f—that is, a function $g: K \to \mathbb{R}^n$ with $||f - g|| := \max_{x \in K} |f(x) - g(x)| \le \alpha$ —has a root.
- 2. We show that the extendability is a **complete criterion** that verifies or disproves that a given function $f: K \to \mathbb{R}^n$ has an α -robust root for some given $\alpha > 0$ (see Lemma 3.3).
- 3. Our solution includes **underdetermined equations**,⁴ that is, the case with dim K > n. On the positive side, we show an algorithm that works

¹For example, settings of a problem may evolve over time and we would like to know if the problem has a solution also tomorrow.

 $^{^{2}}$ See [22, 21] for some incompleteness results.

³The topological degree can be understood as a solution for a special case of the extension problem: namely, the degree of a map $\partial B^{d+1} \rightarrow S^d$ from the boundary of a (d+1)-disk to a *d*-sphere is zero if and only if the map can be extended to whole of the disk.

 $^{^4}$ Formally, our solution also includes overdetermined systems, but such systems never have a robust solution.



Figure 1: Illustration of one-dimensional functions with and without an α -robust root.

for dim $K \leq 2n - 3$ (here the running time is polynomial when dim K is fixed) or n even (see Theorem 1.2). On the negative side, we prove undecidability when dim $K \geq 2n - 2$ and n is odd (see Theorem 1.3).

The core of the algorithm and the undecidability depends on recent computational complexity studies of the extension problem [7, 8, 47]. We will state the relevant results in Section 2.

The computational setting. In this paper, we assume the simple yet useful setting where the function $f: K \to \mathbb{R}^n$ is piecewise linear (PL) on a finite simplicial complex K.⁵ That means, f is linear on every simplex of K. Every such function f is uniquely determined by its values on the vertices of K—as the linear interpolation on every simplex of K. The usefulness of the PL setting can be seen in the facts that

- such functions *f* can be naturally obtained for instance from measurements in every vertex of a grid or every vertex of a triangulation of a domain of interest, and,
- arbitrary continuous function can be approximated arbitrarily precisely by a PL-function and thus our results in the PL setting yield analogous results in other possible settings, see Section 5.

For easy computational representation of the input, we assume that our algorithms are given only PL functions with rational values on the vertices.⁶ We emphasize that the above mentioned set of α -perturbations of a given function

⁵A (geometric) simplicial complex K is a collection of simplices in some \mathbb{R}^d such that for every two simplices $\Delta, \Delta' \in K$ their intersection $\Delta \cap \Delta'$ is also a simplex of K or an empty set. The underlying topological space $|K| := \bigcup K$ will be also denoted by K when no confusion can arise.

 $^{^{6}\}mathrm{We}$ could use arbitrary subset of real numbers where the computations of absolute values and comparisons are possible in polynomial time.

 $f\colon K\to \mathbb{R}^n$ contains continuous functions that are neither PL nor have rational values on the vertices.

Every simplicial complex can be encoded as a hereditary set system K on a set of vertices V(K)—such K is called *abstract simplicial complex* [25, p. 359]. Every abstract simplicial complex can be realized as a geometric simplicial complex uniquely up to a homoemorphism.⁷ We will use the notation K for both the abstract simplicial complex and its geometric realization where no confusion is possible.

Throughout this paper, $|\cdot|$ will denote a fixed norm on \mathbb{R}^n . Its choice can be arbitrary for the undecidability results. For the algorithmic results we will need that

for every rational point $y \in \mathbb{Q}^n$, the value |y| can be compared to any given rational number α , and, (1)

for every PL map $f: \Delta \to \mathbb{R}^n$ with rational values on the vertices of a simplex Δ , the point $\arg\min_{x \in \Delta} |f(x)|$ can be computed. (2)

When the dimension of Δ is fixed, the computation can be done in polynomial time for ℓ_1 , ℓ_{∞} (the simplex method for linear programming) and ℓ_2 norms (Lagrange multipliers). As already introduced above, for a function $f: K \to \mathbb{R}^n$, the notation ||f|| stands for the L_{∞} norm of f, that is, $||f|| := \max_{x \in K} |f(x)|$.

The algorithmic results. The next definition introduces the main problem we study.

Definition 1.1. Assume that a norm on \mathbb{R}^n is fixed. The Robust Satisfiability Problem (ROB-SAT) is the problem of deciding, for a given continuous function $f : |K| \to \mathbb{R}^n$ on a simplicial complex K and a number $\alpha > 0$, whether each α -perturbation of f has a root.

Let \mathcal{E} be a set of functions with some agreed upon encoding. We say that the ROB-SAT problem for \mathcal{E} is decidable, if there exists an algorithm that correctly decides the above problem for each $f \in \mathcal{E}$ and $\alpha > 0$.

Theorem 1.2. Let \mathcal{E} be the set of all PL functions $f: K \to \mathbb{R}^n$ where K is a finite simplicial complex, $n \in \mathbb{N}$ and either dim $K \leq 2n-3$ or $n \leq 2$. Assume that a norm $|\cdot|$ on \mathbb{R}^n is fixed and that we have an oracle for (1) and (2).⁸ Then the ROB-SAT problem for \mathcal{E} is decidable. Moreover, for each fixed n > 0, the running time is polynomial.

Additionally, ROB-SAT is decidable for all PL-functions $f: K \to \mathbb{R}^n$ for n even without any restriction on the dimension of the simplicial complex K.

The decision procedure involves recent algorithmic results from computational homotopy theory, namely the algorithms for the *extension problem* [8, 47].

⁷Every such a realization is given by a convenient embedding of the set of vertices in some \mathbb{R}^d . Then the collection of abstract simplices of K determines the collection of the geometric simplices of the realization. The required condition is that the intersection of arbitrary two geometric simplices of the realization also belongs to the realization or is empty.

⁸As we mentioned above, such an oracle can be implemented in polynomial time in the case of ℓ_1 , ℓ_2 and ℓ_{∞} norms.

The limit for the dimension of K comes from so-called *stable range* in homotopy theory.

Locating a robust root. In fact, our algorithm yields approximations of one or more connected components within K such that every α -perturbation of f has a root in each of the components. Conversely, for every point x of each of the components, there is an α -perturbation with a root in x. Thus the algorithm approximately localizes the roots within the precision that the parameter α allows.

Computing the robustness of a root. Let $f: K \to \mathbb{R}^n$ be a PL function. It is easy to show that the set of all $\alpha \geq 0$, such that every α -perturbation of f has a root, is either empty, or a closed and bounded set. Let us denote, if it is nonempty, the maximum of this set by $\operatorname{rob}(f)$ —the robustness of the root of f. We will prove that for any PL function $f: K \to \mathbb{R}^n$ that has a root, its robustness $\operatorname{rob}(f)$ is equal to $\min_{x \in \Delta} |f(x)|$ for some simplex $\Delta \in$ K. Consequently, by a simple binary search, we can compute the exact value $\operatorname{rob}(f).$ ⁹

It is worthwhile to identify other natural classes \mathcal{E} of functions on a compact domain for which the robustness of the roots can be computed exactly. However, we omit this direction in this paper. Arbitrary continuous functions on a compact domain can be approximated by piecewise linear ones up to arbitrary level of precision, and thus the robustness of their roots can be approximated arbitrarily precisely as well, see Section 5.

Inequalities. We can generalize our results to systems of equations and *inequalities*. Such system can be formally described by $f = 0 \land g \leq 0$ for some $(f,g): K \to \mathbb{R}^n \times \mathbb{R}^k$. An α -perturbation of this system is a system $\tilde{f} = 0 \land \tilde{g} \leq 0$ such that $\|f - \tilde{f}\| \leq \alpha$ and $\|g - \tilde{g}\| \leq \alpha$. Expectably, the robustness of a satisfiable system $f = 0 \land g \leq 0$ is defined to be the maximal $\alpha > 0$ such that every α -perturbation of the system has a solution.

If we assume that we use the max-norm in \mathbb{R}^n , then we will derive as a corollary of Theorem 1.2, that the following problem is decidable: given PL functions $(f,g): K \to \mathbb{R}^n \times \mathbb{R}^k$ such that dim $X \leq 2n-3$ or $n \leq 2$, decide whether each α -perturbation of $f = 0 \land g \leq 0$ is satisfiable or not. This will be discussed at the end of Section 3.

Undecidability. The main question now is what happens in the remaining case when dim $K \ge 2n - 2$ and n is odd. We claim that no approximation of the robustness of roots is algorithmically possible here. The formal statement follows:

Theorem 1.3. Let n > 2 be odd and $|\cdot|$ be an arbitrary norm in \mathbb{R}^n . Then there is no algorithm that, given a finite simplicial complex K of dimension 2n - 2 and a PL function $f : K \to \mathbb{R}^n$, correctly chooses at least one of the following answers:

⁹Here we use the fact that if $|\cdot|$ is the max-norm, then |f| takes the minimum on each simplex in a computable rational point $x^* \in \mathbb{Q}^d$. The same is true for the Euclidean norm $|\cdot|_2$, but the minimum $\min_{\Delta} |f|_2$ is a square root of a rational number.

- $\operatorname{rob}(f) > 0$
- $\operatorname{rob}(f) < 1$

In particular, ROB-SAT for PL-functions is undecidable. Furthermore, it is easy to see that we have similar undecidability result for any class of functions \mathcal{E} such that, for any PL function f and $\epsilon > 0$, we might algorithmically find $g \in \mathcal{E}$ such that $||g - f|| < \epsilon$. For instance, the robustness of the roots cannot be approximated up to any constant factor for systems of polynomial equations, which is quite surprising in light of the fact that the first order theory (\mathbb{R}, \leq) is decidable. While we can always decide the root existence problem for polynomials, we cannot, in general, decide, whether the robustness of the root is greater than ϵ or less than 1.

The proof of Theorem 1.3 is based on a recent undecidability result for the topological extension problem [8].

If we consider systems of equations and *inequalities*, an analogous undecidability result holds even for homotopically trivial domains. Let n > 2 be odd. Then there is no algorithm that, given $m, k \in \mathbb{N}$, a triangulation T of $[-1, 1]^m$ and a pair of PL functions $(f, g) : T \to \mathbb{R}^n \times \mathbb{R}^k$, correctly chooses at least one of the following:

- the robustness of $f = 0 \land g \leq 0$ is greater than 0, or,
- the robustness of $f = 0 \land g \leq 0$ is less than 1.

An immediate consequence is that the ROB-SAT problem is undecidable for systems of polynomial equations and inequalities, if all variables are from the interval [-1,1] and no other constraint on the domain is given. The details are at the end of Section 4.

Related work. By the famous result of Tarski, the first order theory of real number is decidable. On the countrary, in the unbounded case, the root existence problem is undecidable even for functions $f : \mathbb{R} \to \mathbb{R}$ that are expressed in closed forms containing polynomials and the sin function. A carefull examination of the proof in [48] shows that also the ROB-SAT problem for such functions is undecidable.

As we mentioned, topological methods have been extensively used to identify roots of systems of equations, particularly in numerical and interval analysis [39, 15, 23, 2, 24, 44].

Our work was motivated by preceding papers [12, 21, 20], where the topological degree was proved to be a complete criterion for detecting *infinitesimally robust* roots (i.e., the roots corresponding to the case rob(f) > 0) of nonlinear systems of n equations over n variables. The approach extends to formulas obtained from systems of equations by adding the inequalities and the universal quantifiers.

The quantification of the robustness of the roots in our sense already appeared in the theory of *well groups* [17]. Well groups measure in some sense the robustness of intersection of the image of a function $f: X \to Y$ with a closed

subdomain $A \subseteq Y$. As a special but important case with $X = \mathbb{R}^m, Y = \mathbb{R}^n$ and $A = \{0\}$, the zero-dimensional well group capture the robustness of a root of a function f. Particularly, Theorem 1.3 implies incomputability of the well groups in the above special case with $m \ge 2n - 2$ and n odd. However, the algorithms for computing well groups described in [4, 10] do not reach beyond the case m = n, that is, the case with the same number of equations and variables. See [43, 11] for some of the follow-up work.

We also mention the concept of Nielsen root number which is a computable number defined for maps between manifolds of the same dimension. It approximates a lower bound of the number of different solutions of f(x) = c [5, 6, 26]. Like the topological degree, it is a homotopy invariant and does not change under small perturbations of f. For pairs of maps $f_1, f_2: M^m \to N^n$ between compact manifolds of different dimensions m, n, Nielsen theory has been applied to compute a lower bound for the number of connected components of the coincidence set $f'_1 = f'_2$, where $f'_i \simeq f_i$ are homotopic for i = 1, 2 [32, 33]. Surprisingly, the author of [32] shows that under the condition $m \leq 2n-3$ —the same as our stable dimension range—a computable number $N(f_1, f_2)$ coincides with the minimal number of connected components of $\{f'_1 = f'_2\}$ for f'_1 resp. f'_2 homotopic to f_1 resp. f_2 .

Open problems.

- To what extent can be the present algorithms implemented and used on concrete instances? So far only the computation of the degree has been implemented and practically used for detecting robust roots¹⁰ [20]. Can higher non-extendability tests via Steenrod squares and Adem's operation¹¹ be practically used?
- The undecidability result in Theorem 1.3 applies to a general class of functions whose domains may have very complicated topology. If we restrict the function space to functions defined on boxes (products of compact intervals), then the decidability of the ROB-SAT problem for such functions is open.
- A natural generalization of the ROB-SAT problem is to consider firstorder formulas, obtained from PL (in)equalities by adding conjunctions, negations and quantifiers. For a rigorous statement see [21]. Our current solution covers the existentially quantified (conjunctions of) equations.
- In the cases where we are able to verify the existence of a robust solution of f = 0, a natural problem is to describe the zero set. It is worth investigating the computable topological invariants of the solution set that

 $^{^{10} {\}rm The}$ topological degree gives a complete criterion in the case $\dim K = n$ but it can detect robust roots in other cases as well.

¹¹The general algorithm for deciding extendability can be seen as a hierarchy of subsequent tests such that dim K - n of them is needed to get a complete answer. This algorithm has not been implemented yet, but in the case of a sphere, the second and the third tests can be obtained via Steenrod squares and Adem's operation [36]. Here the formulas are known and it might be relatively easy to get an efficient implementation.

are robust with respect to perturbations of f. In particular, can well groups be computed for $m \leq 2n - 3$?

2 Topological preliminaries

In this section we introduce some definitions from algebraic topology that we need throughout the proofs. The details can be found in standard textbooks, such as [28, 34, 45].

We remind that for a simplicial complex K, |K| will refer to the underlying topological space, that is, the union of the simplices in K. If no confusion can arise, we will denote by K both the space and the simplicial complex itself. All simplicial complexes are assumed to be finite, without explicitly saying it.

Star, link and subdivision. Let $A \subseteq K$ be simplicial complexes. We define the star(A, K) to be the set of all faces of all simplices in K that have nontrivial intersection with A, and $link(A, K) := \{\sigma \in star(A, K) | \sigma \cap |A| = \emptyset\}$. Both star(A, K) and link(A, K) are simplicial complexes. The difference $star^{\circ}(A, K) := |star(A, K)| \setminus |link(A, X)|$ is called *open star*. A simplicial complex K' is called a *subdivision* of K whenever |K'| = |K| and each $\Delta' \in K'$ is contained in some $\Delta \in K$. If $a \in |K|$, than we may construct a subdivision of K by replacing the unique Δ containing a in its interior by the set of simplices $\{a, v_1, \ldots, v_k\}$ for all $\{v_1, \ldots, v_k\}$ that span a face of Δ , and correspondingly subdividing each simplex containing Δ . This process is called *starring* Δ at a. If we fix a point a_{Δ} in the interior of each $\Delta \in K$, we may construct a *derived subdivision* K' by starring each Δ at a_{Δ} , in an order of decreasing dimensions.

Homotopy extension property. We say that $f, g: X \to Y$ are homotopic, if there exists a map $H: X \times [0, 1] \to Y$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. The map H is called a homotopy. Let $A \subseteq X$ and Y be topological spaces. A map $F: X \to Y$ is called an extension of $f: A \to Y$, if the restriction $F|_A = f$. The problem of deciding, whether there exists an extension F of $f: A \to Y$, is called the topological extension problem. Let $H: A \times [0, 1] \to Y$ be a homotopy. We say that the pair (X, A) has the homotopy extension property (HEP) with respect to Y, if for any homotopy $h: A \times [0, 1] \to Y$ and an extension $F: X \to Y$ of $h(\cdot, 0)$, there exists a homotopy $H: X \times [0, 1] \to Y$ that is an extension of h. The majority of common pairs (X, A) possess this property with respect to any Y [1, p. 76] and if Y is finitely triangulable (e.g. the sphere), then any closed subspace A of a metric space X has the HEP with respect to Y [30, p. 14]. If follows that the existence of an extension F of $f: A \to Y$ depends only on the homotopy class of f.

Extendability of maps into a sphere. A map between simplicial complexes X and Y is called *simplicial*, if the image of each simplex is a simplex and its restriction to each simplex is a linear map.¹² Every simplicial map $f: X \to Y$

¹²This means, $f(\sum_j a_j v_j) = \sum_j a_j f(v_j)$ within a simplex with vertices v_j , $a_j \ge 0$ and $\sum_j a_j = 1$.

is uniquely defined by a mapping from the vertices of X to the vertices of Y (and thus can easily be described combinatorially). Each continuous map $f: |K| \to |L|$ between simplicial complexes K and L can be approximated by a simplicial map $f^{\Delta}: K' \to L$, where K' is a subdivision of K. In particular, if $f(|star^{\circ}(v)|) \subseteq |star^{\circ}(f^{\Delta}(v))|$, then f and f^{Δ} are homotopic [40, p. 137].

A space Y is called (d-1)-connected iff every map $S^i \to Y$ for $i \leq d-1$ is homotopic to a constant map. Every d-sphere S^d is (d-1)-connected. The core of the proofs of Theorems 1.2 and 1.3 depends on the following recent results from computational homotopy theory.

Theorem 2.1 ([7, Theorem 1.4]). Let $d \ge 2$ be fixed. Then there is a polynomialtime algorithm that, given finite simplicial complexes X, Y, a subcomplex $A \subseteq X$, and a simplicial map $f \colon A \to Y$, where dim $(X) \le 2d - 1$ and Y is (d - 1)connected, decides whether f admits an extension to a (not necessarily simplicial) map $X \to Y$.

The bound $\dim(X) \leq 2d - 1$ might seem too restrictive but it cannot be removed for even dimensional spheres, for example.

Theorem 2.2 ([8, Theorem 1.1 (a)]). Let $d \ge 2$ be even and S^d be a simplicial complex representing the d-sphere. Then the following problem is undecidable: given a simplicial pair (X, A) such that dim X = 2d and a simplicial map $f : A \to S^d$, decide whether there exists a continuous extension $F : X \to S^d$ of f or not.

We remark that in the case of (d-1)-connected spaces with d odd, the undecidability holds for the wedge of two spheres $S^d \vee S^d$ [8, Theorem 1.1 (b)]. The interesting question on the complexity of the case of odd dimensional spheres, where homotopy groups $\pi_n(S^d)$, n > d are all finite, was solved recently by Vokřínek [47].

Theorem 2.3 ([47, Theorem 1]). There exists an algorithm that, given a pair of finite simplicial sets (X, A), a finite d-connected simplicial set Y, $d \ge 1$, with homotopy groups $\pi_n(Y)$ finite for all $2d \le n \le \dim X$ and a simplicial map $f: A \to Y$, decides the existence of a continuous extension $g: X \to Y$ of f.

We remark that there is no running time bound in Theorem 2.3 even for fixed dimensions of X. We also believe that there is much more hope for practical implementations in the stable situation (Theorem 2.1) than in the unstable situation (Theorem 2.3).

We will need a slight extension of Theorem 2.1 for maps to low-dimensional spheres S^1 and S^0 . The results of [7] do not formally cover these cases mainly because they are too particular and not difficult at the same time. We will need some basic terminology and facts from algebraic topology in the proof, such as homotopy and cohomology theory. We point the interested reader to the textbooks [28, 38].

Lemma 2.4. There is a polynomial-time algorithm that, given a simplicial complex X (of any dimension), $k \leq 1$ and a simplicial map $f : A \to S^k$ from a subcomplex A of X. decides whether there exists a continuous extension (not necessarily simplicial) $F: X \to S^k$ of f.

Proof. First consider the case k = 0. We identify the connected components X_1, X_2, \ldots, X_j of the complex X and the subcomplexes $A_1, \ldots, A_j \subseteq A$ such that $A_i = X_i \cap A$ for $i = 1, \ldots, j$. It is straightforward that a simplicial map $f: A \to S^0$ can be extended to X iff f is constant on every subcomplex A_i for $i=1,\ldots,j.$

Now let k = 1. It is well known fact that the circle S^1 is the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$ and ¹³ that there is a natural bijection $[A, S^1] \to H^1(A; \mathbb{Z})$.¹⁴ The bijection sends the homotopy class of a continuous map $f: A \to S^1$ to the cohomology class $f^*(\xi)$ where $f^*: H^1(S^1; \mathbb{Z}) \to H^1(A; \mathbb{Z})$ is the homomorphism induced by f and $\xi \in H^1(S^1; \mathbb{Z})$ is a certain distinguished cohomology class. In our case where $S^1 = |\Sigma^1|$, the cohomology class ξ can be represented by a cocycle that assigns 1 to the directed edge (ordered simplex) $\overrightarrow{e_1e_2}$ of a simplicial complex Σ^1 and assigns 0 to all the remaining directed edges. Thus $f^*(\xi)$ can be represented by a cocycle that assigns 1 to every directed edge \vec{uv} such that $f(u) = e_1$ and $f(v) = e_2$ and 0 to all the remaining ones. In the end, the question of extendability of a map $f: A \to S^1$ reduces to extendability of the cocycle on A representing $f^*(\xi)$ to a cocycle on X. This problem reduces to a system of linear Diophantine equations that can be solved in polynomial time [42, Chapter 5].

3 Decidability

Realization of the sphere. Let us represent the sphere S^{n-1} as the boundary of the cross polytope, i.e., the convex hull of 2n coordinate \pm unit vectors $\pm e_1, \ldots, \pm e_n$. This coincides with the set of unit vectors in \mathbb{R}^n with respect to the ℓ_1 -norm $|x|_1 := \sum_i |x_i|$. Let

$$r: \mathbb{R}^n \setminus \{0\} \to S^{n-1}, \quad x \mapsto \frac{x}{|x|_1}.$$

This map is a homotopy equivalence and the embedding $i: S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is its homotopy inverse. This implies that, for any simplicial complex pair (X, A), there exists an extension $F: X \to \mathbb{R}^n \setminus \{0\}$ of $f: A \to \mathbb{R}^n \setminus \{0\}$ iff there exists an extension $\tilde{F}: X \to S^{n-1}$ of $r \circ f: A \to S^{n-1}$. We choose the triangulation Σ^{n-1} of S^{n-1} consisting of all simplices spanned

by subsets of the 2n vertices $\{\pm e_i\}$ that do not contain a pair of antipodal points $\{e_i, -e_i\}$. For each index i and sign s, $star(se_i)$ is a triangulation of the hemisphere $\{x \in S^{n-1} \mid s x_i \ge 0\}.$

¹³The Eilenberg–MacLane space $K(\mathbb{Z},1)$ is defined by $\pi_1(K(\mathbb{Z},1)) = \mathbb{Z}$ and $\pi_i(K(\mathbb{Z},1)) = 0$ for every $i \neq 1$. ¹⁴For spaces X and Y, we let [X, Y] denote the set of homotopy classes of maps $X \to Y$.

Robustness of the root. Let K be a compact space, $f : K \to \mathbb{R}^n$ be a continuous function and g be an α -perturbation of f with no root. Then $\min_K |q| > 0$, so there exists an $\epsilon > 0$ such that $\epsilon f + (1 - \epsilon)q$ has no root. However, $\|\epsilon f + (1-\epsilon)g - f\| = (1-\epsilon)\|f - g\| \le (1-\epsilon)\alpha$, so the set of all α' such that there exists an α' -perturbation of f with no root, is an open set. It follows that the set of α such that each α -perturbation of f has a root, is a closed set. It is clearly bounded, which justifies the following definition.

Definition 3.1. For a compact space K and a continuous $f: K \to \mathbb{R}^n$ that has a root in K, we define

 $rob(f) := max\{\alpha \mid each \; \alpha \text{-perturbation of } f \text{ has a root}\}.$

For the purposes of this paper, we will need the following definition.

Definition 3.2. Let K be a simplicial complex and $f : K \to \mathbb{R}^n$ be a PL function. We will call $a \in \mathbb{R}$ a critical value of f, if $a = \min_{x \in \Delta} |f(x)|$ for some $\Delta \in K$.

We will show later that rob(f) is always a critical value of f.

Reduction of the ROB-SAT problem to the extension problem. Assume that $f: K \to \mathbb{R}^n$ is an arbitrary continuous function on a compact space K. There is the following general criterion for the existence of a robust root.

Lemma 3.3 (Extendability criterion). A continuous function $f: K \to \mathbb{R}^n$ has an α -robust root if and only if its restriction to $|f|^{-1}\{\alpha\}$ cannot be extended to a map $|f|^{-1}[0,\alpha] \to \mathbb{R}^n \setminus \{0\}.$

Proof. We will prove the equivalent statement that there exists an α -perturbation of f with no root iff there exists an extension $F: |f|^{-1}[0, \alpha] \to \mathbb{R}^n \setminus \{0\}$ of $f|_{|f|^{-1}\{\alpha\}}$

Assume that $\tilde{f}: |f|^{-1}[0,\alpha] \to \mathbb{R}^n \setminus \{0\}$ is an α -perturbation of f with no root and let $H(x,t) := t\tilde{f}(x) + (1-t)f(x)$ be a straight-line homotopy. We will show that $H(x,t) \neq 0$ for $(x,t) \in |f|^{-1}\{\alpha\} \times [0,1]$. Using $|\tilde{f}(x) - f(x)| \leq \alpha$,

$$H(x,t)| = |t\tilde{f}(x) + (1-t)f(x)|$$

= $|f(x) + t(\tilde{f}(x) - f(x))|$
 $\geq |f(x)| - t|\tilde{f}(x) - f(x)|$
 $\geq \alpha - t\alpha = (1-t)\alpha.$

This is positive for t < 1. For t = 1, $|H(x,1)| = |\tilde{f}(x)| > 0$, because \tilde{f} has no root. So, $f|_{|f|^{-1}\{\alpha\}}$ and $\tilde{f}|_{|f|^{-1}\{\alpha\}}$ are homotopic maps from $|f|^{-1}\{\alpha\}$ to $\mathbb{R}^n \setminus \{0\}$. The map $\tilde{f}: |f|^{-1}[0,\alpha] \to \mathbb{R}^n \setminus \{0\}$ is an extension of $\tilde{f}|_{|f|^{-1}\{\alpha\}}$ and due to the homotopy extension property of the pair $(|f|^{-1}[0,\alpha],|f|^{-1}\{\alpha\})$ wrt. the sphere, there exists an extension $F: |f|^{-1}[0,\alpha] \to \mathbb{R}^n \setminus \{0\}$ of $f|_{|f|^{-1}\{\alpha\}}$. Conversely, assume that there exists an extension $F: |f|^{-1}[0,\alpha] \to \mathbb{R}^n \setminus \{0\}$

of $f|_{|f|^{-1}\{\alpha\}}$. Let U be an open neighborhood of $|f|^{-1}\{\alpha\}$ in $|f|^{-1}[0,\alpha]$ such

that for $x \in U$, $|F(x) - f(x)| < \alpha/2$. Due to the compactness of $|f|^{-1}[0, \alpha]$, there exists $\epsilon \in (0, \alpha/2)$ such that $|f|^{-1}[\alpha - \epsilon, \alpha] \subseteq U$ (otherwise, there would exist a sequence $x_n \notin U$, $|f(x_n)| \to \alpha$ and a convergent subsequence $x_{j_n} \to x_0$, whereas $x_0 \in |f|^{-1}\{\alpha\} \subseteq U$, contradicting $x_{j_n} \notin U$). The map $F|_{|f|^{-1}\{\alpha-\epsilon\}}$ is an $\alpha/2$ -perturbation of $f|_{|f|^{-1}\{\alpha-\epsilon\}}$, and an elementary calculation shows that they are homotopic as maps from $|f|^{-1}\{\alpha - \epsilon\}$ to $\mathbb{R}^n \setminus \{0\}$ by the straight-line homotopy. The homotopy extension property implies that $f_{|f|^{-1}\{\alpha-\epsilon\}}$ can be extended to a map $F_1 : |f|^{-1}[0, \alpha] \to \mathbb{R}^n \setminus \{0\}$. Without loss of generality, we may assume that $F_1(x) = f(x)$ for $x \in |f|^{-1}[\alpha - \epsilon, \alpha]$.

Let $g: |f|^{-1}[0,\alpha] \to (0,1]$ be a scalar function such that g(x) = 1 on $|f|^{-1}\{\alpha\}$ and $g(x) < \frac{\epsilon}{2|F_1(x)|}$ on $|f|^{-1}[0,\alpha-\epsilon)$. Define the function $F_2(x) := g(x)F_1(x)$ from $|f|^{-1}[0,\alpha]$ to $\mathbb{R}^n \setminus \{0\}$. If $x \in |f|^{-1}[\alpha-\epsilon,\alpha]$, then

$$|F_2(x) - f(x)| = |f(x)g(x) - f(x)| = |f(x)| |g(x) - 1|$$

$$\leq |f(x)| < \alpha$$

and if $x \in |f|^{-1}[0, \alpha - \epsilon)$, then

$$|F_2(x) - f(x)| \le |F_2(x)| + |f(x)| \le |F_1(x)| \frac{\epsilon}{2|F_1(x)|} + \alpha - \epsilon = \alpha - \frac{\epsilon}{2} < \alpha$$

which shows that F_2 is an α -perturbation of f. By definition, $F_2(x) = f(x)$ for $x \in |f|^{-1}\{\alpha\}$, so the function $F_3: K \to \mathbb{R}^n$ defined by

$$F_3(x) := \begin{cases} F_2(x) & \text{for } x \in |f|^{-1}[0,\alpha] \\ f(x) & \text{for } x \in |f|^{-1}(\alpha,\infty) \end{cases}$$

is a continuous α -perturbation of f with no root.

The straightforward aim would be to apply the algorithm of Theorem 1.2 to the extension problem from the previous lemma. We would need to triangulate the spaces $|f|^{-1}(\alpha)$ and $|f|^{-1}[0,\alpha]$ —a feasible goal in the case $|\cdot| = |\cdot|_{\infty}$. However, we prefer a solution that covers larger class of norms, and in particular, the Euclidean norm. There it is less obvious how to work with the relevant spaces and maps. But we can use the fact that extendability is invariant under homotopy equivalence. It will be enough to apply the algorithm of Theorem 1.2 on "combinatorial approximations" of the spaces and maps from the previous lemma.

Let $f: K \to \mathbb{R}^n$ be a PL function and assume, in addition, that for each $\Delta \in K$, the function |f| takes both the minimum and maximum on Δ in a vertex. This can be achieved for an arbitrary PL function by taking a derived subdivision of K. (The procedure will be detailed in the proof of Theorem 1.2.) We define an auxiliary PL function $\chi: K \to \mathbb{R}$ by

$$\chi(v) := \begin{cases} 0 & \text{when } |f(v)| < \alpha; \\ 1/2 & \text{when } |f(v)| = \alpha; \\ 1 & \text{when } |f(v)| > \alpha. \end{cases}$$

Let us put $X := \chi^{-1}[0, 1/2]$ and $A := \chi^{-1}(1/2)$.

Lemma 3.4 (Combinatorial approximation). Let X and A be defined as above. Then there is a homeomorphism $h: X \to |f|^{-1}[0, \alpha]$ such that $h|_A$ is a homeomorphism $A \to |f|^{-1}\{\alpha\}$. Moreover, there is a homotopy $H: h \sim \text{id}$ such that $f \circ H$ has no root on A.

Proof. The homeomorphism h and the homotopy H are defined simplexwise. Let Δ be a simplex of K. Let the face of Δ spanned by vertices v with $\chi(v) = 0$, resp. $\chi(v) = 1$, $\chi(v) = 1/2$ be denoted by σ , resp. τ, ρ . First, if τ is empty, we define H constant on every point of $\Delta = \Delta \cap X$. Second, if σ is empty, then $\Delta \cap X = \rho = \Delta \cap |f|^{-1}\{\alpha\}$, as α is the minimum of |f| on Δ and is attained in every point of ρ . In this case we define H constant as well. Note that if both ρ and σ were empty, the minimum of |f| of Δ would be greater than α and thus $X \cap \Delta = \emptyset$.

Assume that σ, τ are nonempty and $x \in \Delta \cap X$. If $x \in \sigma$ or $x \in \rho$, then we define the homotopy H(x,t) = x consistently with the above description of H on the faces τ , ρ . Assume that x is neither in σ nor in ρ . If ρ is empty, then there exist unique numbers a, b > 0, a+b = 1 such that x = ay+bz for uniquely defined points $y \in \sigma$ and $z \in \tau$. Similarly, if σ, τ, ρ are all nonempty, then there exist unique coefficients a, b, c > 0, a+b+c = 1 such that x = ay+bz+cw for uniquely defined $y \in \sigma, z \in \tau, w \in \rho$.

We define a line segment s by s(t) = (1-c)(ty + (1-t)z) + cw for $t \in [0, 1]$. It is the segment in Δ parallel to \overline{yz} passing through x. Let y' := s(0) denote the starting point and z' := s(1) denote the endpoint of s. We observe that

- s intersects A in a unique point $x_A := s(1/2);$
- s intersects $|f|^{-1}\{\alpha\}$ in a unique point x_{∂} . This holds because |f| is convex on Δ , $|f(y')| < \alpha$ and $|f(z')| > \alpha$.

A simple situation with all σ, τ and ρ being singletons can be seen in Figure 2. We define

$$H(x,t) := x + t(x_{\partial} - x_A) \frac{|y' - x|}{|y' - x_A|}$$

This homotopy continuously stretches the segment s in such a way that y' is fixed and x_A is sent to x_∂ in t = 1. We have that $H(\cdot, 0)$ is the identity and the final map $h = H(\cdot, 1)$ is a bijection—it bijectively stretches each segment $\overline{y'x_A}$ onto $\overline{y'x_\partial}$ (see Figure 2). For each $x \in \Delta \cap X$, $x \notin \sigma \cup \rho$, there exist unique y', z', x_A and x_∂ and they depend continuously on x. Moreover, it is routine to check that the definition of H on each Δ is compatible with its definition on every face $\Delta' < \Delta$. This altogether proves that both H is continuous. So, the map $H(t, \cdot)$ is a continuous bijection of X and its image and the compactness of X implies that $H(t, \cdot)^{-1}$ is continuous as well. In particular, $H(1, \cdot) = h$ is a continuous bijection of X and $|f|^{-1}[0, \alpha]$ and hence a homeomorphism. Properties $h(|A|) \subseteq$ $|f|^{-1}\{\alpha\}$ and $h^{-1}(|f|^{-1}\{\alpha\}) \subseteq |A|$ follow from the construction.

Finally, once σ is empty, then trivially $f(H(x,t)) \neq 0$. Otherwise for $x \in A$ the value H(x,t) always belongs to the line segment $\overline{x_A x_\partial}$ that doesn't contain a root of f.



Figure 2: Illustration of the Lemma 3.4 in the case $|\cdot| = |\cdot|_2$. The arrows represent the map $h = H(\cdot, 1)$.

Corollary 3.5. There exists an extension $F : |f|^{-1}[0, \alpha] \to \mathbb{R}^n \setminus \{0\}$ of $f|_{|f|^{-1}\{\alpha\}}$ iff there exists an extension $G : X \to \mathbb{R}^n \setminus \{0\}$ of $f|_A$.

Proof. We proved that $h = H(1, \cdot)$ is a homeomorphism of the pairs (X, A) and $(|f|^{-1}[0, \alpha], |f|^{-1}\{\alpha\})$. It follows that there exists an extension $F : |f|^{-1}[0, \alpha] \to \mathbb{R}^n \setminus \{0\}$ of $f|_{|f|^{-1}\{\alpha\}}$ iff there exists an extension $G_1 : X \to \mathbb{R}^n \setminus \{0\}$ of $(f \circ h)|_A$.



But $f \circ H|_{A \times [0,1]}$ is a homotopy between $f \circ h|_A : A \to \mathbb{R}^n \setminus \{0\}$ and $f|_A : A \to \mathbb{R}^n \setminus \{0\}$, so there exists an extension $G_1 : X \to \mathbb{R}^n \setminus \{0\}$ of $(f \circ h)|_A$ iff there exists an extension $G : X \to \mathbb{R}^n \setminus \{0\}$ of $f|_A$.

Corollary 3.6. If $f : K \to \mathbb{R}^n$ is satisfiable, then the robustness rob(f) is a critical value of f.

Proof. Assume that α is not a critical value of f and each α -perturbation of f has a root. There are only finitely many critical values, so there exists a $\beta > \alpha$ such that $[\alpha, \beta]$ doesn't contain any critical value of f. Assume that |f| contains both the maximum and minimum on each simplex in a vertex, as in the construction of Lemma 3.4. Then for each vertex v it holds that $|f(v)| < \alpha$

iff $|f(v)| < \beta$, $|f(v)| > \alpha$ iff $|f(v)| > \beta$ and $\alpha \neq |f(v)| \neq \beta$, so, the simplicial sets $A \subseteq X$ are identical for α and β . It follows from Lemma 3.3 and 3.4 $f: A \to \mathbb{R}^n \setminus \{0\}$ cannot be extended to a map $X \to \mathbb{R}^n \setminus \{0\}$, and each β -perturbation of f has a root as well. Thus $\alpha \neq \operatorname{rob}(f)$.

The algorithm. It remains to show that the reduction from the α -robust satisfiability of f = 0 to the extension problem as described in Theorem 2.1 can be done in an algorithmic way. In order to do this, we will need to convert the spaces $A \subseteq X$ constructed in Lemma 3.4 into simplicial complexes and the PL function $f|_A : A \to \mathbb{R}^n \setminus \{0\}$ into an equivalent simplicial map to the (n-1)-sphere.

In Lemma 3.4 we assumed that |f| contains the minimum on each simplex in some vertex. This can be achieved by a starring each simplex Δ of K at the point $\arg \min_{\Delta} |f|$ whenever it belongs to the interior of Δ . The point $\arg \min_{\Delta} |f|$ can be computed by the assumption (2).

Proof of Theorem 1.2. First we compute $\arg \min_{\Delta} |f|$ for each $\Delta \in K$ and construct a derived subdivision of K by starring each simplex Δ in $\arg \min_{\Delta} |f|$. Then |f| contains the minimum on each simplex in a vertex. Due to the linearity of f on each simplex Δ , $|f(ax + by)| = |af(x) + bf(y)| \le a|f(x)| + b|f(y)|$ for $x, y \in \Delta$, $a, b \ge 0$, and a + b = 1, so |f| is a convex function and hence takes the maximum on each simplex in a vertex too.

In Lemma 3.3 we reduced the ROB-SAT problem for f and α to the extendability of $f|_{|f|^{-1}\{\alpha\}} : |f|^{-1}\{\alpha\} \to \mathbb{R}^n \setminus \{0\}$ to a function $|f|^{-1}[0,\alpha] \to \mathbb{R}^n \setminus \{0\}$. By Corollary 3.5, this is equivalent to the extendability of $f|_A : A \to \mathbb{R}^n \setminus \{0\}$ to a function $X \to \mathbb{R}^n \setminus \{0\}$, where X and A are constructed as in Lemma 3.4 with the use of the assumption (1). We will construct a subdivision K' of K containing a triangulation of X and A. To this end, K' is constructed by starring each simplex $\Delta \in K$ that intersects $\chi^{-1}(1/2)$ in a point $x_\Delta \in \Delta \cap \chi^{-1}(1/2)$. Now χ is a PL function on K' that assigns values in $\{0, 1/2, 1\}$ to the vertices and, additionally, no simplex has a pair of vertices evaluated to 0 and to 1. Thus Xis spanned by vertices evaluated to 0 or 1/2, and A is a simplicial subcomplex of X spanned by vertices evaluated to 1/2.

The extendability of $f : A \to \mathbb{R}^n \setminus \{0\}$ to a function $X \to \mathbb{R}^n \setminus \{0\}$ is further equivalent to the extendability of $r \circ f|_A : A \to S^{n-1}$ to a function $X \to S^{n-1}$, where $r(x) := x/|x|_1$ is the homotopy equivalence $\mathbb{R}^n \setminus \{0\} \to S^{n-1}$.

Next, we can algorithmically construct a subdivision $A' \subseteq X'$ of A and X such that for each $v \in A'$, there exists a pair $(i_v, s_v) \in \{1, \ldots, n\} \times \{+, -\}$ such that f_{i_v} has constant sign s_v on star(v, A). Let Σ^{n-1} be a simplicial representation of the sphere, as defined at the beginning of Section 3. Let $f^{\Delta} : A' \to \Sigma^{n-1}$ be a simplicial map that maps each vertex $v \in A'$ to $s_v e_{i_v}$. This map is well defined, because if $a, b \in \Delta \in A'$, then $f^{\Delta}(b) \neq -f^{\Delta}(a)$, so f^{Δ} maps simplices to simplices. It follows that $(r \circ f)(star(v, A)) \subseteq star^{\circ}(f^{\Delta}(v), \Sigma^{n-1})$ for each vertex $v \in A'$, so $f^{\Delta} \simeq r \circ f : A' \to S^{n-1}$ are homotopic by the simplicial approximation theorem. The pair (X', A') has the homotopy extension property, so there exists an extension $X \to S^{n-1}$ of $r \circ f|_A$ iff there exists an extension

 $X' \to S^{n-1}$ of the simplicial map f^{Δ} . In the case dim K < 2n - 2 = 2(n - 1), it is decidable by Theorem 2.1, in the case $n \leq 2$, it is decidable by Lemma 2.4 and for n odd, it is decidable by Theorem 2.3.

It remains to prove polynomiality for the cases dim K < 2n-2 or $n \leq 2$ for fixed n. The input contains some encoding of the PL function f, which includes the information about K. In the algorithm, we first subdivided K by starring it in $\arg \min_{\Delta} |f|$ for each $\Delta \in K$ and then again by triangulating the sets Aand X defined in 3.4.

Further, we construct a subdivision X', A' of the pair X, A such that f can be represented by a simplicial map from A'. This can be achieved as follows. Let $i \in \{1, \ldots, n\}$. For each edge ab of A that intersects $f_i^{-1}\{0\}$ in its interior and is not contained in $f_i^{-1}\{0\}$, we choose a point $v_{ab} \in f_i^{-1}(0) \cap A$, and then star X in each of this point (in some order). Consequently, the open star of each point in A' is mapped by f into a star of some vertex in S^{n-1} . So for a fixed dimension n, we need only a fixed number of subdivisions of (X, A) in order to construct the simplicial approximation f^{Δ} as in the proof of Theorem 1.2.

We reduced the ROB-SAT problem to the extension problem for $f^{\Delta} : A' \to S^{n-1}$ and $X' \supseteq A'$ and the number of simplices in X' depends polynomially on the number of simplices in the original simplicial complex K. Finally, we use the fact that the decision procedure for the extension problem for maps to spheres is polynomial in the number of simplices in X', assuming that the dimension of X' is fixed (Theorem 2.1).

Note that if f has a root, then the robustness rob(f) can be computed exactly by deciding, whether each α -perturbation of f has a root for α ranging over the computable finite set of critical values. Moreover, if n is fixed, then rob(f) can be computed in polynomial time.

Inequalities. Here we show, how to reduce the robust satisfiability of systems of equations and inequalities to the ROB-SAT problem for a system of equations. In this section, we assume that the norm $\|\cdot\|$ on \mathbb{R}^n -valued functions (implicitly used in the definition of α -perturbation) is derived from the max-norm in \mathbb{R}^n , i.e. $\|f\| := \sup_{x \in K} \max_i |f_i(x)|$. Contrary to other parts of this paper, the proof of the following lemma uses the choice of the norm.

Lemma 3.7. Let $(f,g): K \to \mathbb{R}^n \times \mathbb{R}^k$ and $\alpha > 0$. Let $U := \{x \in K | g(x) \le -\alpha\}^{15}$. Then each α -perturbation of $f = 0 \land g \le 0$ is satisfiable, iff each α -perturbation $h: U \to \mathbb{R}^n$ of $f|_U$ has a root.

Proof. The function $\tilde{g}(x) := g(x) + \alpha$ is an α -perturbation of g. So, if each α -perturbation of $f = 0 \land g \leq 0$ is satisfiable, then each α -perturbation \tilde{f} of f has a root in $U = \{x \in K \mid \tilde{g} \leq 0\}$.

For the other implication, suppose that each α -perturbation of f has a root in U and let $\tilde{f} = 0 \land \tilde{g} \leq 0$ be an α -perturbation of $f = 0 \land g \leq 0$. Then $U \subseteq \{x \in K \mid \tilde{g}(x) \leq 0\}$, so $\tilde{f} = 0$ has a solution on $\{x \in K \mid \tilde{g}(x) \leq 0\}$.

¹⁵The notation $g(x) \leq -\alpha$ means that $g_i(x) \leq -\alpha$ for each component g_i of g.

If f, g are PL function and dim $K \leq 2n - 3$ or $n \leq 2$, then we can construct a triangulation of the set $U = \{x \in K | g(x) \leq -\alpha\}$ and use Theorem 1.2 to decide whether each α -perturbation of f has a root in U.

4 Undecidability

Here we show that the robustness of roots of a PL function cannot be approximated in general. We cannot algorithmically distinguish functions f with rob(f) = 0 from functions g with $rob(g) \ge 1$.

We will continue to represent the sphere S^{n-1} as a triangulation of $\{x \in \mathbb{R}^n \mid |x|_1 = 1\}$ as defined at the beginning of Section 3. All norms in \mathbb{R}^n are equivalent, so for the norm $|\cdot|$ there exist numbers $\kappa_n > 0$ and $\lambda_n > 0$ such that $|x|_1 \leq \kappa_n |x|$ and $|x| \leq \lambda_n |x|_1$ for all $x \in \mathbb{R}^n$. For the case of the max-norm, κ_n can be chosen to be n and λ_n can be chosen to be 1.

Proof of Theorem 1.3. The proof proceeds by reduction from the extension problem as stated in Theorem 2.2. Let X be a simplicial complex of dimension 2(n-1) and let $f: A \to \Sigma^{n-1}$ be a simplicial map from some $A \subseteq X$ to the (n-1)-sphere. We assume that $n-1 \ge 2$ is even, so by Theorem 2.2, we cannot algorithmically decide, given X, A and f, whether there exists an extension $X \to |\Sigma^{n-1}|$ of f or not.

We will assume that A is *full* in X, that is, for each simplex Δ it holds that $\Delta \in A$ iff all vertices of Δ are in A. (If A is not full in X, then X may be algorithmically subdivided into a complex X' containing A, such that A is full in X' [41, Lemma 3.3].)

Define a PL function $f': X \to \mathbb{R}^n$ given by its values on the vertices of X as follows:

$$f'(v) := \begin{cases} \kappa_n f(v); & \text{when } v \in A \\ 0 & \text{when } v \notin A. \end{cases}$$

We will show the following statements:

- A. if there exists a 1-perturbation of f' without a root, then f can be extended to a function $X \to S^{n-1}$,
- B. if f has an extension $F: X \to S^{n-1}$, then for every $\epsilon > 0$ there exists an ϵ -perturbation of f' without a root.

So, an algorithm defined in Theorem 1.3 could decide the topological extension problem for $f: A \to \Sigma^{n-1}$ and $X \supseteq A$, which is impossible by Theorem 2.2.

It remains to show A. and B. We remind that for our realization of the sphere $S^{n-1} \subseteq \mathbb{R}^n$, we have that $|f'(x)| = \kappa_n |f(x)| \ge |f(x)|_1 = 1$ for each $x \in |A|$.

A. Let g' be a 1-perturbation of f' without a root. It holds that $g'|_A$ is homotopic to $f'|_A = \kappa_n f$ as functions from A to $\mathbb{R}^n \setminus \{0\}$ via the straight-



Figure 3: Visualization of the map r used in the proof of Theorem 1.3.

line homotopy, because

$$\begin{aligned} |tg'(x) - (1-t)f'(x)| &= |f'(x) + t(g'(x) - f'(x))| \\ &\geq |f'(x)| - t|g'(x) - f'(x)| \geq 1 - t \end{aligned}$$

which is positive for t < 1 and for t = 1, $g'(x) \neq 0$ by assumption. Since g' is an extension of $g'|_A$, the map f' can also be extended as a map into $\mathbb{R}^n \setminus \{0\}$ and f can be extended to a map from |X| into $|\Sigma^{n-1}| = S^{n-1}$.

B. We need the auxiliary PL function $\chi_A \colon X \to \mathbb{R}$ that is 1 on the vertices of A and 0 on the remaining vertices of X. We will perturb f' on a subdomain where it is "tiny", namely, on $\chi_A^{-1}[0, \delta]$ for $\delta := \epsilon/(2 \kappa_n \lambda_n)$ and leave f' unchanged on $\chi_A^{-1}[\delta, 1]$ where no root occurs. Indeed, for every point x with $\chi_A(x) > 0$ either $x \in A$ (then $f'(x) \neq 0$) or x belongs to a simplex $\Delta \notin A$ and then x lies in the interior of a segment \overline{yz} where

$$y \in \chi_A^{-1}(0)$$
 and $z \in |A|$

(here we use that A is full in X). That means, x = cy + dz for d > 0 and thus $f'(x) \neq 0$.

To define the perturbation we need an auxiliary continuous "blow-up" map $r: \chi_A^{-1}[0, \delta] \to X$ defined on each x = cy + dz with c + d = 1, $d \leq \delta$ and y and z as above (but here possibly d = 0) by

$$r(x) := c'y + (d/\delta)z$$

where $c' + d/\delta = 1$. The points y and z depend continuously on x, so r is continuous as well.

Let $F \colon |X| \to |\Sigma^{n-1}|$ be the extension of f. We define an ϵ -perturbation g' of f' by

$$g'(x) := \begin{cases} \delta \kappa_n F(r(x)) & \text{when } x \in \chi_A^{-1}[0, \delta]; \\ f'(x) & \text{otherwise.} \end{cases}$$

To see that g' is continuous, it is enough to verify that $\delta \kappa_n F(r(x)) = f'(x)$ for $x \in \chi_A^{-1}(\delta)$, that is, when $x = (1 - \delta)y + \delta z$ for some y and z as above. Indeed,

$$\delta \kappa_n F(r(x)) = \delta \kappa_n F(0y+1z) = \delta \kappa_n f(z), \text{ and} f'(x) = (1-\delta) f'(y) + \delta f'(z) = 0 + \delta \kappa_n f(z).$$

Finally, we check that g' is an ϵ -perturbation of f' by the following estimation for every $x = cy + dz \in \chi_A^{-1}[0, \delta]$:

$$\begin{aligned} |\delta \kappa_n F(r(x)) - f'(x)| &\leq \delta \kappa_n ||F|| + |f'(cy + dz)| \\ &\leq \delta \kappa_n \lambda_n + d|f'(x)|\delta \kappa_n \lambda_n + d \kappa_n |f(z)| \\ &\leq \delta \kappa_n \lambda_n + d \kappa_n \lambda_n \leq 2\delta \kappa_n \lambda_n \leq \epsilon, \end{aligned}$$

because necessarily $d \leq \delta$.

Inequalities. An immediate consequence is that for systems of equations and inequalities, we get an undecidability result even for homotopically trivial domains such as products of compact intervals. Formally, there is no algorithm that, given a triangulation T of $[-1, 1]^d$, PL functions $f, g: T \to \mathbb{R}^n$, correctly chooses at least one of the following options:

- the robustness of $f = 0 \land g \leq 0$ is greater than 0, or,
- the robustness of $f = 0 \land g \leq 0$ is less than 1.

This is proved by a reduction from Theorem 1.3 as follows. Let K be a simplicial complex and $f: K \to \mathbb{R}^n$ a PL function. We can algorithmically construct a PL embedding $K \to [-1, 1]^d$ for some d and a triangulation T of $[-1, 1]^d$ containing a subdivision K' of K [41, p. 16]. Furthermore, we assume that K' is full in T (otherwise we would subdivide T once more). Define a scalar valued PL function $g: T \to \mathbb{R}$ to be -1 on the vertices of K' and 1 on the vertices of $T \setminus K'$. We immediately see that $\{x \mid g(x) \leq -1\} = |K'|$. Extend f to a PL function $f_T: T \to \mathbb{R}^n$, by setting f(v) = 0 for each vertex $T \setminus K'$.

If some 1-perturbation of the system $f_T = 0 \land g \leq 0$ is not satisfiable, then it follows that some 1-perturbation of f has no root. Conversely, assume that, for some $\epsilon \in (0, 1)$, some ϵ -perturbation \tilde{f} of f has no root. Then we can extend \tilde{f} to a function $\tilde{f}_T : T \to \mathbb{R}^n$ as follows:

- for $x \in T \setminus star(K', T)$, $\tilde{f}_T(x) := 0$, and
- for $x \in \Delta \in star(K',T) \setminus K'$, we have $x = tx_K + (1-t)x_T$ for uniquely determined $x_K \in \Delta \cap K'$, $x_T \in \Delta \cap (star(K',T) \setminus K')$ and t > 0; then we define $\tilde{f}_T(x) := t\tilde{f}(x_K)$.

The resulting function \tilde{f}_T is an ϵ -perturbation of \tilde{f} , nonzero on the open star $star^{\circ}(K',T)$. Clearly, $\{x \mid g(x) \leq 0\} \subseteq star^{\circ}(K',T)$ and thus $\tilde{f}_T = 0 \land g \leq 0$ is an unsatisfiable ϵ -perturbation of $f_T = 0 \land g \leq 0$.

We reduced the problem of deciding, whether $\operatorname{rob}(f)$ is greater that 0 or less than 1, to the problem of deciding, whether the robustness of $f_T = 0 \land g \leq 0$ is greater than 0 or less than 1. The former is undecidable by Theorem 1.3, so it follows that the latter is undecidable as well.

5 Nonlinear functions

The decidability of the ROB-SAT problem for the class of PL functions defined in Theorem 1.2 implies analogous results for larger function spaces that can be uniformly approximated by PL functions. For example, any function f: $K \to \mathbb{R}^n$ containing expressions such as polynomials, division, sin, exp, log and π can be uniformly approximated by a PL function up to any $\epsilon > 0$ using *interval arithmetic* [35]. This implies the existence of an algorithm that takes such function $f: K \to \mathbb{R}^n$ and two constants $\alpha, \epsilon > 0$, and correctly chooses at least one of the following:

- Each α -perturbation of f has a root,
- There exists an $(\alpha + \epsilon)$ -perturbation of f with no root,

assuming that dim $K \leq 2n-3$. The decision procedure for this problem can construct an $\epsilon/2$ -approximation PL approximation f^{PL} of f, defined on a triangulation K_{Δ} of K, which reduces the above problem to the ROB-SAT problem for f^{PL} and α . Further, it can be shown that if n is fixed and f is smooth, then the size of K_{Δ} depends polynomially on J/ϵ , where J is the upper bound on second partial derivatives $\partial f_i/\partial x_j$. This yields an estimate on the algorithm complexity. If |K| is an m-box (product of m compact intervals), than f = 0represents a system of n equations in m variables, with a given upper and lower bound for each variable.

Conversely, the undecidability result of Theorem 1.3 generalizes to any class of nonlinear functions \mathcal{E} such that any PL function can be algorithmically uniformly approximated by some $g \in \mathcal{E}$. Given $\epsilon > 0$, each *PL* function *f* can be algorithmically approximated by a component-wise polynomial function *p* such that $||f - p|| < \epsilon$, which immediately implies that the ROB-SAT problem is undecidable for general systems of polynomial equations defined on simplicial complexes, once we exceed the dimensions of the stable range. The undecidability result can be slightly strengthened to functions defined on smooth manifolds, because each pair of simplicial complexes (X, A) can be algorithmically embedded to a pair of manifolds (X^m, A^m) in some \mathbb{R}^m with the same homotopical properties. The interesting case of functions defined on *m*-boxes $[-1, 1]^m$ was mentioned in the *open problems* paragraph at the end of Section 1.

References

- M. Arkowitz. Introduction to Homotopy Theory. SpringerLink : Bücher. Springer New York, 2011.
- [2] Thomas Beelitz, Andreas Frommer, Bruno Lang, and Paul Willems. Symbolic-numeric techniques for solving nonlinear systems. *PAMM*, 5(1):705–708, 2005.
- [3] A. Ben-Tal, L.E. Ghaoui, and A. Nemirovski. *Robust Optimization*. Princeton Series in Applied Mathematics. Princeton University Press, 2009.
- [4] Paul Bendich, Herbert Edelsbrunner, Dmitriy Morozov, and Amit Patel. The robustness of level sets. volume 6346 of *Lecture Notes in Computer Science*, pages 1–10. Springer Berlin Heidelberg, 2010.
- [5] R. Brooks. Certain subgroups of the fundamental group and the number of roots of f(x)=a. American Journal of Mathematics, 95(4):pp. 720–728, 1973.
- [6] R. Brown and H. Schirmer. Nielsen root theory and Hopf degree theory. *Pacific Journal of Mathematics*, 198:pp. 49–80, 2001.
- [7] M. Čadek, M. Krčál, J. Matoušek, L. Vokřínek, and U. Wagner. Polynomial-time computation of homotopy groups and Postnikov systems in fixed dimension. Preprint, arXiv:1211.3093, 2012.
- [8] M. Čadek, M. Krčál, J. Matoušek, L. Vokřínek, and U. Wagner. Extendability of continuous maps is undecidable. *Discr. Comput. Geom.*, 2013. To appear. Preprint arXiv:1302.2370.
- [9] G Carlsson. Topology and data. Bull AMS, 46:255–308, 2009.
- [10] Frédéric Chazal, Amit Patel, and Primoz Skraba. Computing the robustness of roots.
- [11] Frédéric Chazal, Primoz Skraba, and Amit Patel. Computing well diagrams for vector fields on Rⁿ. Applied Mathematics Letters, 25(11):1725 – 1728, 2012.
- [12] Pieter Collins. Computability and representations of the zero set. Electron. Notes Theor. Comput. Sci., 221:37–43, December 2008.
- [13] J. Cronin. Fixed Points and Topological Degree in Nonlinear Analysis. American Mathematical Society. Mathematical surveys. American Mathematical Society, 1964.
- [14] M. de Longueville. A Course in Topological Combinatorics. Universitext. Springer, 2012.

- [15] Jianwei Dian and R. B. Kearfott. Existence verification for singular and nonsmooth zeros of real nonlinear systems. *Mathematics of Computation*, 72(242):757–766, 2003.
- [16] H. Edelsbrunner and J. Harer. Persistent homology—a survey. In Surveys on discrete and computational geometry, volume 453 of Contemp. Math., pages 257–282. Amer. Math. Soc., Providence, RI, 2008.
- [17] Herbert Edelsbrunner, Dmitriy Morozov, and Amit Patel. Quantifying transversality by measuring the robustness of intersections. Foundations of Computational Mathematics, 11(3):345–361, 2011.
- [18] David Eisenbud, Harold I Levine, and Bernard Teissier. An algebraic formula for the degree of a \mathbb{C}^{∞} map germ. The Annals of Mathematics, 106(1):19–44, 1977.
- [19] Robin Forman. A user's guide to discrete morse theory. Sém. Lothar. Combin, 48:B48c, 2002.
- [20] Peter Franek and Stefan Ratschan. Effective topological degree computation based on interval arithmetic. CoRR, abs/1207.6331, 2012. To appear in AMS Math. of Compu., preprint available on http://arxiv.org/abs/ 1207.6331.
- [21] Peter Franek, Stefan Ratschan, and Piotr Zgliczynski. Quasi-decidability of a fragment of the analytic first-order theory of real numbers, 2012. Preprint available at http://www2.cs.cas.cz/~ratschan/papers/ quasidec_constr.pdf.
- [22] A. Frommer, F. Hoxha, and B. Lang. Proving the existence of zeros using the topological degree and interval arithmetic. *Journal of Computational* and Applied Mathematics, 199:397–402, February 2007.
- [23] A. Frommer, B. Lang, and M. Schnurr. A comparison of the Moore and Miranda existence tests. *Computing*, 72(3-4):349–354, May 2004.
- [24] Andreas Frommer and Bruno Lang. Existence tests for solutions of nonlinear equations using Borsuk's theorem. SIAM Journal on Numerical Analysis, 43(3):1348–1361, 2005.
- [25] W. Fulton. Algebraic Topology: A First Course. Graduate Texts in Mathematics. Springer-Verlag, 1995.
- [26] Daciberg Gonalves and Peter Wong. Wecken property for roots. Proceedings of the American Mathematical Society, 133(9):pp. 2779–2782, 2005.
- [27] F. Goudail and P. Réfrégier. Statistical Image Processing Techniques for Noisy Images: An Application-Oriented Approach. Kluwer Academic / Plenum Publishers, 2004.

- [28] A. Hatcher. Algebraic Topology. Cambridge University Press, Cambridge, 2001.
- [29] N.J. Higham. Accuracy and Stability of Numerical Algorithms: Second Edition. Society for Industrial and Applied Mathematics, 2002.
- [30] S.T. Hu and V. Hu. *Homotopy theory*. Pure and Applied Mathematics. Elsevier Science, 1959.
- [31] R. Baker Kearfott and Jianwei Dian. Existence verification for higher degree singular zeros of nonlinear systems. SIAM Journal on Numerical Analysis, 41(6):pp. 2350–2373, 2004.
- [32] U. Koschorke. Nielsen coincidence theory in arbitrary codimensions. Journal fur die reine und angewandte Mathematik, 596:211–236, 2006.
- [33] Ulrich Koschorke. Selfcoincidences and roots in Nielsen theory. Journal of Fixed Point Theory and Applications, 2(2):241–259, 2007.
- [34] S.V. Matveev. Lectures on Algebraic Topology. EMS Series of Lectures in Mathematics. European Mathematical Society, 2006.
- [35] Ramon E. Moore, R. Baker Kearfott, and Michael J. Cloud. Introduction to Interval Analysis. SIAM, 2009.
- [36] R. E. Mosher and M. C. Tangora. Cohomology operations and applications in homotopy theory. Harper & Row Publishers, New York, 1968.
- [37] Arnold Neumaier. Interval Methods for Systems of Equations. Cambridge Univ. Press, Cambridge, 1990.
- [38] V. V. Prasolov. *Elements of Homology Theory*. Graduate Studies in Mathematics. American Mathematical Society, 2007.
- [39] L. Rall. A comparison of the existence theorems of Kantorovich and Moore. SIAM Journal on Numerical Analysis, 17(1):148–161, 1980.
- [40] J.J. Rotman. An Introduction to Algebraic Topology. Graduate Texts in Mathematics. Springer, 1988.
- [41] C.P. Rourke and B.J. Sanderson. *Introduction to piecewise-linear topology*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1982.
- [42] A. Schrijver. Theory of linear and integer programming. John Wiley & Sons, Inc., New York, NY, USA, 1986.
- [43] Primos Skraba, Wang Bei, Chen Guoning, and Paul Rosen. 2d vector field simplification based on robustness, 2014. to appear in IEEE Pacific Visualization (PacificVis).

- [44] Andrew Paul Smith. Enclosure methods for systems of polynomial equations and inequalities, 2012. Dissertation thesis, available on http://kops.ub.uni-konstanz.de/bitstream/handle/urn:nbn:de: bsz:352-208986/Diss_Smith.pdf?sequence=1.
- [45] E.H. Spanier. Algebraic Topology. McGraw-Hill series in higher mathematics. Springer, 1994.
- [46] Zbigniew Szafraniec. Topological degree and quadratic forms. Journal of Pure and Applied Algebra, 141(3):299–314, 1999.
- [47] L. Vokřínek. Decidability of the extension problem for maps into odddimensional spheres. ArXiv e-prints, January 2014.
- [48] Paul S. Wang. The undecidability of the existence of zeros of real elementary functions. J. ACM, 21(4):586–589, 1974.