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Output-sensitive algorithms for sumset and sparse polynomial multiplication

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Abstract

We present randomized algorithms to compute the sumset (Minkowski sum) of two integer sets, and to multiply two univariate integer polynomials given by sparse representations. Our algorithm for sumset has cost softly linear in the combined size of the inputs and output. This is used as part of our sparse multiplication algorithm, whose cost is softly linear in the combined size of the inputs, output, and the sumset of the supports of the inputs. As a subroutine, we present a new method for computing the coefficients of a sparse polynomial, given a set containing its support. Our multiplication algorithm extends to multivariate Laurent polynomials over finite fields and rational numbers. Our techniques are based on sparse interpolation algorithms and results from analytic number theory.

1 Introduction

Sparse polynomials are a fundamental object in computer algebra. Computer algebra programs including Maple, Mathematica, Sage, and Singular use a sparse representation by default for multivariate polynomials, and there has been considerable recent work on how to efficiently store and compute with sparse polynomials [Fateman, 2003, Gastineau and Laskar, 2013, Monagan and Pearce, 2009, Roche, 2011, van der Hoeven and Lecerf, 2012].

However, despite the memory advantage of sparse polynomials, the alternative dense representation is still widely used for an obvious reason: speed. It is now classical [Cantor and Kaltofen, 1991] that two degree-*D* dense polynomials can be multiplied in softly linear time: $\mathcal{O}(D \log D \log \log D)$ ring operations, and even better in many cases [Harvey et al., 2014b]. By contrast, two size-*T* sparse polynomials require $\mathcal{O}(T^2)$ operations, and this excludes the potentially significant cost of exponent arithmetic. Much of the recent work on sparse arithmetic has focused on "somewhat dense" or structured cases, where the sparsity of the product is sub-quadratic [Monagan and Pearce, 2009, Roche, 2011, van der Hoeven and Lecerf, 2012]. At the same time, sparse interpolation algorithms, which in the fastest case can learn an unknown *T*-sparse polynomial from O(T) evaluations, have gained renewed interest [Cuyt and Lee, 2008, Javadi and Monagan, 2010, Comer et al., 2012, Arnold et al.].

Most closely related to the current work, [van der Hoeven and Lecerf, 2013] recently presented algorithms to discover the coefficients of a sparse polynomial product, provided a list of the exponents and some preprocessing. In the context of pattern matching problems, [Cole and Hariharan, 2002] gave a Las Vegas algorithm to multiply sparse polynomials with nonnegative integer coefficients whose cost is $\tilde{O}(T \log^2 D)$.

A remaining question is whether output-sensitive sparse multiplication is possible in time comparable to that of dense multiplication. This paper answers that question, with three provisos: First, our complexity is proportional to the "structural sparsity" of the output that accounts for exponent collisions but not coefficient cancellations; second, our algorithms are randomized and may produce incorrect results with controllably low probability; and third, we ignore logarithmic factors in the size of the input.

To explain the first proviso, define for a polynomial *F* its *support* supp(*F*) to be the set of exponents of nonzero terms in *F*. The *sparsity* of *F*, written #F, is exactly # supp(*F*). For two polynomials *F* and *G*, we have # supp(*FG*) $\leq \#F \cdot \#G$. But in many cases the set of *possible exponents*

$$\mathsf{poss}(F,G) \stackrel{\text{der}}{=} \{e_F + e_G : e_F \in \mathsf{supp}(F), e_G \in \mathsf{supp}(G)\}$$

1.0

is much smaller than $\#F \cdot \#G$. This *structural sparsity* T = #poss(F,G), is an upper bound on the actual sparsity S = #supp(FG) of the product. Strict inequality S < T occurs only in the presence of *coefficient cancellations*. Part of our algorithm's cost depends only on the actual sparsity, and part depends on the potentially-larger structural sparsity.

Our algorithms have not yet been carefully implemented, and we do not claim that they would be faster than the excellent software of [Gastineau and Laskar, 2013, Monagan and Pearce, 2013] and others for a wide range of practical problems. However, this complexity improvement indicates that the barriers between sparse and dense arithmetic may be weaker than we once thought, and we hope our work will lead to practical improvements in the near future.

1.1 Our contributions

Our main algorithm is summarized in Theorem 1.1. Here and throughout, we rely on a version of "soft-oh" notation that also accounts for a bound μ on the probability of failure: $\widetilde{\mathcal{O}}_{\mu}(\phi) \stackrel{\text{def}}{=} \mathcal{O}(\phi \cdot \text{polylog}(\phi/\mu))$, for any function ϕ , where polylog means \log^c for some fixed c > 0 [von zur Gathen and Gerhard, 2013, see sec. 25.7].

Theorem 1.1. Given $F, G \in \mathbb{Z}[x]$ with degree bound $D > \deg F + \deg G$ and height bound $C \ge ||F||_{\infty} + ||G||_{\infty}$, and $\mu \in (0,1)$, Algorithm SparseMultZZ correctly computes the product H = FG with probability exceeding $1 - \mu$, using worst-case expected $\widetilde{\mathcal{O}}_{\mu}(S \log C + T \log D)$ bit operations, where $S = \# \operatorname{supp}(FG)$ and T = $\# \operatorname{poss}(F, G)$ are the actual and structural sparsity of the product, respectively.

The linear dependence on $\log D$ and $\log C$ means this result extends easily to multivariate polynomials and finite field coefficients, using the Kronecker substitution.

Our algorithm relies on two subroutines, both of which are based on techniques from sparse interpolation and rely on number-theoretic results on the availability of primes.

The first subroutine Sumset(A, B) computes the *sumset* of two sets of integers A and B, defined as

$$\mathcal{A} \oplus \mathcal{B} \stackrel{\text{def}}{=} \{a+b : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

This algorithm, which may be of independent interest, has softly-linear complexity in the size of the output $\mathcal{A} \oplus \mathcal{B}$.

The second subroutine SparseMulCoeffs(F, G, S) requires a set containing supp(FG) in order to compute FG in time softly-linear in the input and output sizes. It is based on an algorithm in [van der Hoeven and Lecerf, 2013], but is more efficient for large exponents.

The main steps of our multiplication algorithm are:

- 1. Use Sumset to compute poss(F, G).
- 2. Run SparseMulCoeffs with S = poss(F, G) but with smaller coefficients, to discover the true supp(*FG*).
- 3. Run SparseMulCoeffs again, with the smaller exponent set supp(*FG*) but with the full coefficients.

Steps 1 and 2 work with a size-*T* exponent set but with small coefficients, and both contribute $\widetilde{\mathcal{O}}_{\mu}(T \log D)$ to the overall bit complexity. Step 3 uses the size-*S* true support but with the full coefficients, and requires $\widetilde{\mathcal{O}}_{\mu}(S(\log D + \log C))$ bit operations, for a total of $\widetilde{\mathcal{O}}_{\mu}(T \log D + S \log C)$.

1.2 Organization of the paper

Section 2 states our notational conventions and some standard results, and Section 3 contains the technical number theoretic results on which we base our randomizations.

Section 4 revisits and adapts our sparse interpolation algorithm from ISSAC 2014 that will be a subroutine for our sumset algorithm, presented in section 5.

Our new method to find the coefficients, once the support is known, is presented in Section 6. This is then used in concert with our sumset algorithm in Section 7 to describe fully the algorithm of Theorem 1.1, and also to explain how this can be easily extended to output-sensitive sparse multiplication over $R[x_1^{\pm 1}, ..., x_n^{\pm 1}]$, where R is \mathbb{Z}_m , Q, or $GF(p^e)$.

2 Background and Preliminaries

We count the cost of our algorithms in terms of bit complexity on a randomaccess machine. We state their costs using $\widetilde{\mathcal{O}}_{\mu}$ notation, meaning that our algorithms have a factor $\log^c \frac{1}{\mu}$ in the running time. We can make c = 1 by running the entire algorithm with error bound $\frac{2}{3}$ some $\mathcal{O}(\log \frac{1}{\mu})$ times, then returning the most frequent result.

Our main algorithm depends on an unknown number-theoretic constant, as discussed in Section 3. Thus we have proven only the *existence* of a Monte Carlo algorithm. We also discuss how this could be easily handled in practice.

Our randomized procedures return either the correct answer (with controllable probability $1 - \mu$), or an incorrect answer, or the symbol Fail. Whenever a subroutine returns Fail, we assume the calling procedure returns Fail as well.

2.1 Notation and Representations

We let R denote a commutative ring with identity. For $F \in R[x]$ we let $\langle F \rangle \subset R[x]$ denote the ideal generated by *F*. For $n, m \in \mathbb{Z}$, m > 0, we let n rem m and *n* rem *m* denote the integers $s \in [0, m)$ and $t \in [-m/2, m/2)$ respectively, such that $n \equiv s \equiv t \pmod{m}$. We write \mathbb{Z}_m for $\mathbb{Z}/m\mathbb{Z}$, typically represented as $\{n \text{ rem } m | n \in \mathbb{Z}\}$.

Unless otherwise stated we assume $F \in R[x]$ is of the form

$$F(x) = \sum_{1 \le i \le S} c_i x^{e_i},\tag{1}$$

with coefficients $c_i \in \mathbb{R}$ and exponents $e_i \in \mathbb{Z}_{\geq 0}$. Often we assume each $c_i \neq 0$ and the exponents are sorted $e_1 < \cdots < e_S$, but it is sometimes useful to relax these conditions.

We write $S \ge \#F$ and $D > \deg F$ for the sparsity and degree bounds. When $R = \mathbb{Z}$ we write $C > \|F\|_{\infty} \stackrel{\text{def}}{=} \max_i |c_i|$ for the *height* of *F*. We also use the norm $\|F\|_1 = \sum_i |c_i|$.

More generally, we consider multivariate Laurent polynomials

$$F = \sum_{i=1}^{S} c_i x_1^{e_{i1}} \cdots x_n^{e_{in}} \in \mathsf{R}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

In the case $R = \mathbb{Z}$, the *sparse representation* of *F* consists of a tuple (n, C, D, S), followed by a list of *S* tuples $(c_i, (e_{i1}, ..., e_{in}))$, where each c_i is stored using $\Theta(\log C)$ bits and each e_{ij} is stored using $\Theta(\log D)$ bits. When R is instead a finite ring, *C* is omitted and each c_i is stored using $\Theta(\log |\mathsf{R}|)$ bits.

When multiplying $F, G \in \mathbb{Z}[x]$, we assume shared bounds *C* and *D* so that total input/output size is

$$\widetilde{\mathcal{O}}((\#F + \#G + \#(FG))(\log C + \log D)),$$
(2)

given that $||FG||_1 < C^2 \min(\#F, \#G)$.

The *dense representation* of an *n*-variate polynomial *F* is an *n*-dimensional array of D^n coefficients, where exponents are implicitly stored as array indices. In the case that $R = \mathbb{Z}$ with bound *C* as above, this requires $\Theta(D^n \log C)$ bits.

The terms "sparse polynomial" and "dense polynomial" refer only to the choice of representation, and not to the relative number of nonzero coefficients. Typically, we assume F is sparse and reserve \tilde{F} to indicate a dense polynomial. Converting between the sparse and dense representations has softly linear cost in the combined input/output size.

When computing a sumset $A \oplus B$, we assume that every integer in A or B is represented using $\Theta(\log D)$ bits, where $D > \max(||A||_{\infty}, ||B||_{\infty})$. In this setting the combined bit-size of A, B, and $A \oplus B$ is $\widetilde{O}((\#A + \#B + \#(A \oplus B)) \log D)$.

2.2 Integer and Polynomial Arithmetic

We cite the following results from integer and polynomial arithmetic, which we use throughout.

Fact 2.1 ([Harvey et al., 2014a], Thm 9.8; [von zur Gathen and Gerhard, 2013], Cor. 9.9). Let $m, n \in \mathbb{Z}$. Then the following may be computed using $\widetilde{O}(\log m + \log n)$ operations: $m \pm n, mn, m$ rem n, m rem n. Also, arithmetic operations in \mathbb{Z}_n require $\widetilde{O}(\log n)$ bit operations.

Fact 2.2 ([von zur Gathen and Gerhard, 2013], Thm. 10.25). *Given* $m_i \in \mathbb{Z}_{>0}$, and $v_i \text{ rem} m_i$ for $1 \leq i \leq t$ and $M = \prod_{i=1}^t m_i$, one can compute the solution $v \in [0, M)$ to the set of congruences $v \equiv v_i \pmod{m_i}$, $1 \leq i \leq t$ using $\widetilde{\mathcal{O}}(\log M)$ bit operations.

We use Chinese Remaindering to construct exponents from sets of congruences. We also use dense polynomial arithmetic as a subroutine, and cite the following results.

Fact 2.3 ([Harvey et al., 2014b]). Let $F, G \in \mathbb{Z}_m[x]$, deg F, deg G < D. Then the following may be computed using $\tilde{O}(D \log m)$ bit operations: $F \pm G$, FG, F rem G.

In particular, dense arithmetic operations in $\mathbb{Z}_m[x]/\langle x^p - 1 \rangle$ may be computed in $\widetilde{\mathcal{O}}(p \log m)$ bit operations.

Assume $F \in \mathbb{Z}[x]$ as in (1) with bounds D, S, and C as described above. Our algorithm performs arithmetic on modular images of F. For $F \in \mathbb{Z}_m[x]$, we represent $F(x) \mod (x^p - 1) \in \mathbb{Z}_m[x]/\langle x^p - 1 \rangle$ by the remainder from dividing $\mathbb{Z}_m(x)$ by $(x^p - 1)$. Namely $F(x) \mod (x^p - 1)$ is a polynomial with degree less than p. Note we treat $F(x) \operatorname{rem} (x^p - 1)$ and $F(x) \mod (x^p - 1)$ as elements of R[x] and $R[x]/\langle x^p - 1 \rangle$ respectively. To reduce a sparse polynomial $F \mod (x^p - 1)$, we reduce each exponent modulo p, and then add like-degree terms. By Fact 2.1, we have:

Corollary 2.4. Given any $F \in \mathbb{Z}_m[x]$, we can compute $F \mod (x^p - 1)$ using $\widetilde{O}(S(\log D + \log m))$ bit operations.

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Procedure GetPrime(\lambda, \mu)
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Input: \lambda \ge 21; \mu \in (0, 1).

Output: Integer p \in (\lambda, 2\lambda], s.t. \Pr[p \text{ not prime}] < \mu.

1 repeat m = \lceil (5/6) \ln \lambda \ln(1/\mu) \rceil times

2 p \leftarrow \text{random odd integer from } (\lambda, 2\lambda] \cap \mathbb{Z}

3 p \neq p is prime then return p

4 return Fail
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Procedure GetVanishPrime(S, D, γ, μ)

Input: Integers $S, D \in \mathbb{Z}_{>0}$; $\gamma \in (0, 1]$; $\mu \in (0, 1)$. **Output**: Integer p, s.t. for any set S satisfying $\#S \leq S$ and $\|S\|_{\infty} < D$, with probability at least $1 - \mu$, p is a γ -vanish-prime for S. 1 $\lambda \leftarrow \max\left(21, \frac{10}{3\mu}\min\left(S, \frac{1}{1-\gamma}\right)\ln D\right)$ 2 **return** GetPrime($\lambda, \mu/2$)

3 Number-theoretic subroutines

3.1 Choosing primes

We first recall how to choose a random prime number.

Fact 3.1 (Corollary 3, [Rosser and Schoenfeld, 1962]). *If* $\lambda \ge 21$, *then the number of primes in* $(\lambda, 2\lambda]$ *is at least* $3\lambda/(5 \ln \lambda)$.

We test if *p* is prime in $\mathcal{O}(\text{polylog}(p))$ time via the method in [Agrawal et al., 2004]. This test and Fact 3.1 lead to procedure GetPrime.

Lemma 3.2. GetPrime(λ, μ) works as stated and has bit complexity $\widetilde{\mathcal{O}}_{\mu}(\operatorname{polylog}(\lambda))$.

Proof. The stated cost follows from fast primality testing due to [Agrawal et al., 2004]. The probability that any chosen *p* is prime is at least $6/(5 \ln \lambda)$, from Fact 3.1. Therefore, using the fact that $(1 - x) < \exp(-x)$ for any nonzero $x \in \mathbb{R}$, the probability that none of the chosen *p* are prime is at most $(1 - \frac{6}{5 \ln \lambda})^m < \exp(\frac{-6m}{5 \ln \lambda}) \le \mu$, as desired.

It is frequently useful to choose a random prime that divides very few of the integers in some unknown set $S \subset \mathbb{Z}$. If a fraction of γ integers in S do not vanish modulo p, then we call p a γ -vanish-prime for S. We call a 1-vanish-prime for S a *good vanish-prime*. Procedure GetVanishPrime shows how to choose a random γ -vanish-prime.

Lemma 3.3. *Procedure GetVanishPrime works as stated to produce a* γ *-vanish-prime p satisfying*

$$p \in \mathcal{O}\left(\frac{1}{\mu}\min\left(S, \frac{1}{1-\gamma}\right)\log D\right)$$

Procedure GetDiffPrime(S, D, γ, μ)

Input: Integers $S, D \in \mathbb{Z}_{>0}$; $\gamma \in (0, 1]$; $\mu \in (0, 1)$. **Output**: Integer p, s.t. for any set S satisfying $\#S \leq S$ and diam(S) < D, with probability at least $1 - \mu$, p is a γ -difference-prime for S. 1 $\lambda \leftarrow \frac{10}{3\mu}(S-1) \min \left(S, \frac{1}{1-\gamma}\right) \ln D$ 2 **if** $\lambda < 21$ **then return GetPrime**($21, \mu/2$) 3 **else if** $\lambda > D$ **then return GetPrime**($D, \mu/2$) 4 **else return GetPrime**($\lambda, \mu/2$)

and has bit complexity polylog(p).

Proof. Let S be any subset of integers with $\#S \leq S$ and $\|S\|_{\infty} < D$. Write $M = \prod_{a \in S} |a| < D^S$, and write k for the number of "bad primes" for which more than $(1 - \gamma)S$ elements of S vanish modulo p. Since each $p \geq \lambda$, this means that $\lambda^{(1-\gamma)Sk} \leq M$, and because $M < D^S$, $k < \ln D/((1 - \gamma) \ln \lambda)$ is an upper bound on the number of bad primes.

If $1 - \gamma$ is very small, we instead use a similar argument to say that the number of primes for which *any* element of S vanishes is at most $k < S \ln D / \ln \lambda$.

Then Fact 3.1 guarantees the prevalence of bad primes among all primes in $(\lambda, 2\lambda)$ is at most $\mu/2$, so the probability of getting a bad prime, or of erroneously returning a composite p, is bounded by μ .

3.2 Avoiding collisions

A closely related problem is to choose *p* so that most integers in a set S are unique modulo *p*. We say that $a \in S$ collides modulo *p* if there exists $b \in S$ with $a \equiv b \pmod{p}$. We say *p* is a γ -difference-prime for S if the fraction of integers in S which do not collide modulo *p* is at least γ . A 1-difference-prime is called a *good difference-prime for* S. Procedure GetDiffPrime shows how to compute difference-primes, conditioned on the *diameter* of the unknown set S:

$$\mathsf{diam}(\mathbb{S}) \stackrel{\mathsf{def}}{=} \max(\mathbb{S}) - \min(\mathbb{S}).$$

Lemma 3.4. Procedure GetDiffPrime has bit complexity polylog(p) and works as stated to produce a γ -difference-prime p satisfying $p \in O(D)$ and

$$p \in \mathcal{O}\left(\frac{1}{\mu}S\min\left(S,\frac{1}{1-\gamma}\right)\log D\right).$$

Proof. Let S be any set as described. An element $a \in S$ collides modulo p iff the product of differences $\prod_{b \in S, b \neq a} (a - b)$ vanishes modulo p. If p > D this can never happen. Otherwise, as each such product is at most diam $(S)^{S-1} < D^{S-1}$, the result follows from the Lemma 3.3, setting the D of the lemma to D^{S-1} .

Our algorithms often perform arithmetic modulo $(x^p - 1)$. Similar to the notion of collisions above for a set of integers modulo p, we say two distinct terms cx^e and $c'x^{e'}$ of $F \in \mathbb{R}[x]$ collide modulo $(x^p - 1)$ if $e \equiv e' \pmod{p}$.

Example 3.5. Let $F = x + x^6 + 3x^7$. Then $F \mod (x^5 - 1) = 2x + 3x^2$. The term 2x is the image of $x + x^6$. We say x and x^6 collide modulo $(x^5 - 1)$, whereas $3x^7$ uniquely maps to $3x^2$.

Essentially, reduction modulo $(x^p - 1)$ "hashes" exponent $e \in \text{supp}(F)$ to e rem p. If p is a good difference-prime for supp(F) and q is a good vanish-prime for the coefficients of F, then $F \text{ rem } (x^p - 1)$ with coefficients reduced modulo q has the same sparsity as F itself.

3.3 Primes in arithmetic progressions

Sometimes we implicitly reduce exponents modulo p by evaluating at pth roots of unity. In such cases we need to construct primes q such that p|(q-1), and to find pth roots of unity modulo each q.

In principle, this procedure is no different than the previous ones, as there is ample practical and theoretical evidence to suggest that the prevalence of primes in arithmetic progressions without common divisors is roughly the same as their prevalence over the integers in general.

However, the closest to Fact 3.1 that we can get here is as follows, which is a special case of Lemma 7 in [Fouvry, 2013].

Lemma 3.6. There exists an absolute constant λ_0 such that, for all $\lambda \ge \lambda_0$, and for all but at most $\lambda / \ln^2 \lambda$ primes p in the range $(\lambda, 2\lambda]$, there are at least $\lambda^{0.89} / \ln \lambda$ primes q in the range $(\lambda^{1.89}, 2\lambda^{1.89}]$ such that p|(q-1).

Proof. Set K = 0.53, which means $(1.89)^{-1} < K < \frac{17}{32}$. Fixing s = 1, and for any $R \ge 2$, Lemma 7 in [Fouvry, 2013] guarantees the existence of positive constants α_K and x_K such that the following holds: For all $x > \max(x_K, R^{1/K})$, and for all but $R/\ln^2 R$ integers $r \in (R, 2R]$, there are at least $\alpha_K x/(\varphi(r) \ln x)$ primes q in the range (x, 2x] such that r|(q - 1), where $\varphi(r)$ is the Euler totient function.

Setting $\lambda_0 = \max(x_K, 3.78/\alpha_K)$, and letting $R = \lambda$, r = p, and $x = \lambda^{1.89}$, the statement of our lemma holds because

$$\varphi(r)\ln x = (p-1)\ln\lambda^{1.89} < 3.78\lambda\ln\lambda,$$
$$\frac{\alpha_K x}{\varphi(r)\ln x} = \frac{\alpha_K \lambda^{1.89}}{(p-1)\ln\lambda^{1.89}} > \frac{\lambda^{0.89}}{\ln\lambda}.$$

thus

Lemma 3.6 forms the basis for Algorithm GetPrimRoots, where we assume that the constant λ_0 is given. Since this constant has not actually been computed, a reasonable strategy would be to choose some small "guess" for λ_0 and run the algorithm until it does not report failure. If the algorithm fails, it

Procedure GetPrimRoots(D, T, C, µ)

Input: $D \ge \deg F$; $T \ge \#F$; $C \ge \|F\|_{\infty}$; $\mu \in (0, 1)$; where $F \in \mathbb{Z}[x]$ is fixed but unspecified. **Output**: Prime *p*, primes (q_1, \ldots, q_k) , and integers $(\omega_1, \ldots, \omega_k)$; or Fail. 1 $m \leftarrow \lceil \lg \frac{2}{u} \rceil$ 2 $\lambda \leftarrow \max\left(786, \lambda_0, \frac{20}{3u}mT(T-1)\ln D, 1.35\ln^{3.13}(2C)\right)$ $a \leftarrow [1.1 \ln(2C) \ln^2 \lambda]$ 4 repeat *m* times $p \leftarrow \text{GetPrime}(\lambda, \frac{\mu}{4m})$ 5 $\mathcal{A} \leftarrow a$ distinct even integers in [2, 2 $\lambda^{0.89}$] 6 $Q, W \leftarrow \text{empty lists}$ 7 foreach $a \in \mathcal{A}$ do 8 9 $q \leftarrow ap + 1$ $\zeta \leftarrow$ random nonzero element of \mathbb{Z}_q 10 **if** *q is prime and* $\zeta^a \mod q \neq 1$ **then** 11 Add *q* to *Q* and $\omega = \zeta^a$ to *W* 12 if $\prod_{q \in Q} q \ge 2C$ then return p, Q, and W13 14 return Fail

could be due to the random prime *p* being an "exception" in Lemma 3.6, or due to unlucky guesses for the primitive roots ζ , or due to the guessed constant λ_0 being too small. Because our primality tests are deterministic, failure due to λ_0 being too small is detectable by the algorithm returning Fail.

We state the running time and correctness, assuming λ_0 is sufficiently large, as follows.

Lemma 3.7. Procedure GetPrimRoots has worst-case bit complexity

 $\widetilde{\mathcal{O}_{\mu}}\left(\log C \cdot \mathsf{polylog}\left(T + \log D\right)\right)$.

With probability at least $1 - \mu$, it returns a good difference-prime p for F, primes q_1, \ldots, q_k such that $\prod_i q_i \ge 2C$, and pth primitive roots modulo each $q_i, \omega_1, \ldots, \omega_k$.

Proof. The lower bound $\lambda \ge \max(786, 1.35 \ln^{3.13}(2C))$ guarantees that there are sufficiently many even integers in the range $[2, 2\lambda^{0.89}]$ in order for Step 6 to be valid, since for any $\lambda \ge 786$, we have $\lambda^{0.89} > \lambda^{.32} \ln^2 \lambda > 1.1 \ln(2C) \ln^2 \lambda$.

For the running time, the outer loop does not affect the complexity in our notation because $m \in O(\log \frac{1}{u}) \in \widetilde{\mathcal{O}_{\mu}}(1)$. Observe also that

$$\log \lambda \in \operatorname{polylog}\left(\lambda_0 + T + \log D + \log C + \frac{1}{\mu}\right).$$

The running time is dominated by the AKS primality tests in the inner loop, which are performed $O(m \log \text{Cpolylog}(\lambda))$ times, each at cost $O(\text{polylog}(\lambda))$, giving the stated worst-case bit complexity.

All of the checks for primality of p and q_i 's, as well as the test that each ω_i is a pth primitive root of unity modulo q_i , are deterministic. Therefore the only possibility that the algorithm returns an incorrect result other than Fail is the probability that p is not a good difference-prime for supp(F). According to the proof of Lemma 3.4, the condition $\lambda > \frac{20}{3\mu}mT(T-1)\ln D$, and using the union bound over all outer loop iterations, the probability that *any* of the chosen p's is not a good difference-prime is less than $\mu/2$.

Consider next a single iteration of the outer loop. This will produce a valid output unless insufficiently many good q_i 's and ω_i 's are found for that choice of p.

From Fact 3.1 and Lemma 3.6, the probability that *p* is an exception to the lemma is at most $5/(3 \ln \lambda)$, which is less than $\frac{1}{4}$ from the bound $\lambda \ge 786$.

If *p* is not an exception, then Lemma 3.6 tells us that the probability of each *q* being prime is at least $\frac{1}{\ln \lambda}$. When *q* is prime, since prime *p* divides (q - 1), the probability that each ζ^{a} is a *p*-PRU in \mathbb{Z}_{q} is (p - 1)/p, easily making the total probability of successfully adding to *Q* and *W* at each loop iteration at least 0.99/(ln λ).

Let $a \ge 1.1 \ln(2C) \ln^2 \lambda$ be the size of A. By Hoeffding's inequality ([Hoeffding, 1963], Thm. 1), the probability that fewer than $0.03a/(\ln \lambda)$ integers are added to Q after all iterations of the inner loop is at most

$$\exp(-2a(0.96/\ln\lambda)^2) < \exp(-2.02\ln(2C)) < 0.25,$$

where the last inequality holds because $C \ge 1$.

Therefore, with probability at least $\frac{3}{4}$, and using again $\lambda \ge 786$, at least

$$0.03a/\ln\lambda = 0.033\ln(2C)\ln\lambda > \ln(2C)/\ln\lambda$$

integers are added to *Q* each time through the inner loop. Since each $q_i > \lambda$, this means $\prod_i q_i > 2C$, and the algorithm will return on Step 13.

Combining with the probability that *p* is an exception, we conclude that the probability the algorithm does *not* return in each iteration of the outer loop is at most 1/2. As this is repeated $\lceil \lg \frac{2}{\mu} \rceil$ times, the probability is less than $\mu/2$ that the algorithm returns Fail. Using the union bound with the probability that any *p* is not a good difference-prime, we have the overall failure probability less than μ .

4 Multiplying via Interpolation

Let $F, G \in \mathbb{Z}[x]$ be sparse polynomials with $C = ||F||_{\infty} + ||G||_{\infty}$ and $D = \deg F + \deg G$. The subroutine Sumset computes poss(F, G) by first reducing the degrees and heights of the input polynomials and then multiplying them. However, it cannot perform the multiplication using a recursive call to SparseMultZZ because the degrees are never reduced small enough to allow the use of dense arithmetic in a base case.

Procedure SparseInterpBB(F, G, α, r)

Input: $F, G \in \mathbb{Z}_q[x]; \alpha \in \mathbb{Z}_q; r \in \mathbb{Z}_{>0}$. Output: $H(\alpha z) \mod (z^r - 1)$, where H = FG. 1 $(\tilde{F}, \tilde{G}) \leftarrow (F(\alpha z) \operatorname{rem} (z^r - 1), G(\alpha z) \operatorname{rem} (z^r - 1))$ 2 $\tilde{H} \leftarrow \tilde{F} \cdot \tilde{G} \operatorname{rem} (z^r - 1)$ via dense arithmetic 3 return sparse representation of \tilde{H}

Procedure BasecaseMultiply(F, G, S, μ)

Input: $F, G \in \mathbb{Z}[x]; S \ge \#F + \#G + \#(FG); \mu \in (0, 1).$

Output: $H \in \mathbb{Z}[x]$ such that $\Pr[H \neq FG] < \mu$.

1 $q \leftarrow \text{GetPrime}(2S ||F||_{\infty} ||G||_{\infty} + 2 \deg F + 2 \deg G, \frac{\mu}{2})$

² Call procedure MajorityVoteSparseInterpolate from [Arnold et al., 2014], with coefficient field \mathbb{Z}_q , sparsity bound *S*, degree bound deg *F* + deg *G*, error bound $\frac{\mu}{2}$, and black box SparseInterpBB(*F*, *G*, \cdot , \cdot).

Instead, we present here a "base case" algorithm which, given *F*, *G*, and a bound $S \ge \#F + \#G + \#(FG)$, computes *FG*, in time softly linear in *S*, log *C*, and polylog(*D*). Any algorithm with such running time suffices; we will use our own from [Arnold et al., 2014], which is a Monte Carlo sparse interpolation algorithm for univariate polynomials over finite fields.

To adapt [Arnold et al., 2014] for multiplication over $\mathbb{Z}[x]$, we first choose a "large prime" $q > \max(2C, 2D)$ and treat F, G and their product H = FG as polynomials over \mathbb{F}_q . This size of q ensures that no extension fields are necessary. The subroutine SparseInterpBB specifies how the unknown polynomial $H = FG \in \mathbb{F}_q[x]$ will be provided to the algorithm. It exactly matches the sorts of black-box evaluations that [Arnold et al., 2014] requires. The entire procedure is stated as BasecaseMultiply.

Lemma 4.1. The algorithm SparseInterpBB works correctly and uses $O(S \log D \log q + r \log q)$ bit operations.

Proof. Correctness is clear. To compute $F(\alpha z)$, we replace every term cx^e of F and G with $c\alpha^e x^e \in \mathbb{Z}_q$. This costs $\widetilde{O}(S \log D \log q)$ by binary powering. Reducing modulo $(z^r - 1)$ costs $\widetilde{O}(S(\log D + \log q))$ by Corollary 2.4. Dense arithmetic costs $\widetilde{O}(r \log q)$ bit operations by Fact 2.3. Summing these costs yields $\widetilde{O}(S \log D \log q + r \log q)$.

Lemma 4.2. The algorithm BasecaseMultiply correctly returns the product H = FG with probability at least $1 - \mu$, and has bit complexity $\widetilde{O}_{\mu} \left(S \log^2 D \left(\log C + \log D \right) \right)$.

Proof. In order to use SparseInterpBB in the algorithm from [Arnold et al., 2014], we simply replace the straight-line program evaluation on the first line of procedure ComputeImage with our procedure SparseInterpBB. Again, note that

as the prime q was chosen with q > 2D, the MajorityVoteSparseInterpolate algorithm does not need to work over any extension fields.

The correctness is guaranteed by the previous lemma, as well as Theorem 1.1 in [Arnold et al., 2014]. For the bit complexity, in Section 7 of [Arnold et al., 2014], we see that the cost is dominated by $\widetilde{\mathcal{O}}_{\mu}(\log D)$ calls to the black box evaluation function, each of which is supplied $q \in \widetilde{\mathcal{O}}(C+D)$, and $r \in \widetilde{\mathcal{O}}_{\mu}(S \log D)$. Applying the bit complexity of Lemma 4.1 gives the stated result.

5 Sumset Algorithm

Let $\mathcal{A}, \mathcal{B} \in \mathbb{Z} \cap (-D, D)$ be nonempty, $R = #\mathcal{A} + #\mathcal{B}$, and $S = #(\mathcal{A} \oplus \mathcal{B})$ throughout this section. We prove as follows:

Theorem 5.1. Procedure Sumset($\mathcal{A}, \mathcal{B}, \mu$) has bit complexity $\mathcal{O}_{\mu}(S \log D)$ and produces $\mathcal{A} \oplus \mathcal{B}$ with probability at least $1 - \mu$.

We compute the sumset $A \oplus B$ as supp(H), $H = FG \in \mathbb{Z}[x^{-1}, x]$, where $F = \sum_{a \in A} x^a$ and $G = \sum_{b \in B} x^b$. Here *H* has exponents in (-2D, 2D) and $||H||_{\infty} < R$. Thus it suffices to construct the exponents of *H* modulo $\ell \ge 4D$. Moreover, we have supp(H) = poss(F, G), and that

$$R-1 \le \#(\mathcal{A} \oplus \mathcal{B}) = S \le R^2.$$
(3)

5.1 Estimating Sumset Output Size

We first show how to compute a tighter upper bound on the true value of $S = \#H = \#(A \oplus B)$. To this end, let $p \in \mathcal{O}(D)$ be a good difference-prime for supp(*H*), using the naive bound R^2 from (3), and define the $H_1 = F_1G_1$, where $F_1, G_1 \in \mathbb{Z}[x]$ are defined by

$$F_1 = F \operatorname{rem}(x^p - 1), \quad G_1 = G \operatorname{rem}(x^p - 1).$$

Then deg $H_1 < 2p$ and each term cx^e of H corresponds to either one or two terms in H_1 , of degrees $e \overline{\text{rem } p}$ and $(e \overline{\text{rem } p}) + p$. Therefore

$$#H_1/2 \le #H = S \le #H_1.$$

We will compute an approximation $S^* \approx S$ such that $S^*/2 < \#H_1 \leq S^*$, and therefore $S^*/4 < S \leq S^*$. To this end we present a test that, given S^* , always accepts if $\#H_1 \leq S^*$ and probably rejects if $\#H_1 > 2S^*$. We do this for S^* initially 2, doubling whenever the test rejects.

Given the current estimate S^* , we next choose a (1/2)-difference-prime q for the support of any 2*S**-sparse polynomial with degree $2p > \deg H_1$, and compute $H^* = H_1 \mod (x^q - 1)$. We work modulo $m = R^2 > ||H||_1 \ge ||H^*||_{\infty}$, such that none of the coefficients of H^* vanish modulo m. If H_1 is *S**-sparse then H^* is as well. If H_1 has 2*S** terms then, as fewer than *S** terms of H_1 are in

collisions, H^* is *not* S^* -sparse. As no terms of H_1 vanish modulo m, additional terms in H_1 can only increase $#H^*$.

We choose *q* so that the test is correct with probability at least 3/4. By iterating $\lceil 8 \ln(8/\mu) \rceil$ times, by Hoeffding's inequality, the probability is at least $1 - \mu/4$ that the test runs correctly in at least half of the iterations. As $\#H_1 < 2R^2$, it suffices that the test is correct $\lceil \log_2 R + 1 \rceil$ times.

The Sumset procedure performs this test on lines 3–7. By Corollary 2.4 the respective costs of constructing F^* and $F^* \mod (R^2, x^q - 1)$ are $\widetilde{\mathcal{O}}_{\mu}(R \log D)$ and $\widetilde{\mathcal{O}}_{\mu}(R \log p \log R)$, and similarly for G^* . The cost of the dense arithmetic here is $\widetilde{\mathcal{O}}_{\mu}(q \log R)$. Given that $p \in \widetilde{\mathcal{O}}_{\mu}(D), q \in \widetilde{\mathcal{O}}_{\mu}(S \log p)$, the total bit-cost of this part is $\widetilde{\mathcal{O}}_{\mu}(S \log D)$.

5.2 Computing Sumset

Armed with the bound $S^*/4 < S \leq S^*$, we aim to compute H = FG. Our approach is to compute images

$$H_1 = F_1 G_1 \mod (\ell^2, x^p - 1),$$

$$H_2 = F((\ell + 1)x)G((\ell + 1)x) \mod (\ell^2, x^p - 1),$$

for an integer $\ell = 8D \ge \max(\deg H, \|H\|_1)$.

Since the coefficients of H_2 are scaled by powers of $(\ell + 1)$, a single term cx^e in the original polynomial H becomes $cx^{e} \overline{\text{rem }}^p$ in H_1 and $c(\ell + 1)^e x^e \overline{\text{rem }}^p$ in H_2 , and if they are uncollided we can discover $(\ell + 1)^e$ by computing their quotient. Modulo ℓ^2 , this quotient $(\ell + 1)^e$ is simply $e\ell + 1$, from which we can obtain the exponent e. This idea is similar to the "coefficient ratios" technique suggested by [van der Hoeven and Lecerf, 2014], but working modulo ℓ^2 allows us to avoid costly discrete logarithms. Procedure Sumset contains the complete description.

Sumset has four steps that are probabilistic: choosing a good differenceprime p, estimating the sumset size $S = #(A \oplus B)$, and constructing H_1 and H_2 . As each is set to fail with probability less than $\mu/4$, Sumset succeeds with probability at least $1 - \mu$.

We now analyze the total cost of this algorithm. GetDiffPrime produces p of size log $p < \text{polylog}(R + \log D + \frac{1}{\mu})$. Constructing F_1, F_2, G_1, G_2 at the beginning, and the reduction of H_1, H_2 modulo $(\ell^2, x^p - 1)$ at the end, both cost $\widetilde{\mathcal{O}}_{\mu}(S \log D)$ bit operations.

The search for $S^* \cos \widetilde{\mathcal{O}_{\mu}}(S \log D)$ from the previous section. Finally, as $||F_1||_{\infty} < \ell/2$, deg $F_1 < p$, and similarly for F_2 , G_1 , and G_2 , the sparse multiplications due to BasecaseMultiply also costs $\widetilde{\mathcal{O}_{\mu}}(S \log D)$ bit operations. These dominate the complexity as stated in Theorem 5.1.

Procedure Sumset($\mathcal{A}, \mathcal{B}, \mu$)

Input: $\mathcal{A}, \mathcal{B} \subseteq \mathbb{Z}$ with $\#\mathcal{A} + \#\mathcal{B} = R$ and $\max_{k \in A \cup B} |k| < D; \mu \in (0, 1)$. **Output**: Set $\mathbb{S} \subset \mathbb{Z}$ such that $\Pr[\mathbb{S} \neq \mathcal{A} \oplus \mathcal{B}] < \mu$. 1 $p \leftarrow \text{GetDiffPrime}(R^2, 4D, 1, \mu/4)$ $\mathbf{2} \ (F_1, G_1) \leftarrow \left(\sum_{a \in \mathcal{A}} x^{a \text{ rem } p}, \sum_{b \in \mathcal{B}} x^{b \text{ rem } p} \right)$ $s S^* \leftarrow 2$ 4 repeat $\lceil \max(8 \ln(8/\mu), \log_2 R + 1) \rceil$ times $q \leftarrow \text{GetDiffPrime}(2S^*, 2p, \frac{1}{2}, \frac{3}{4})$ $H^* \leftarrow F_1G_1 \mod (R^2, x^q - 1)$, via dense arithmetic if $\#H^* > S^*$ then $S^* \leftarrow 2S^*$ 7 s $\ell \leftarrow 8D$ 9 $(F_2, G_2) \leftarrow \left(\sum_{a \in \mathcal{A}} (a\ell + 1) x^{a \operatorname{\overline{rem}} p}, \sum_{b \in \mathcal{B}} (b\ell + 1) x^{b}\right)$ 10 $H_1 \leftarrow \text{BasecaseMultiply}(F_1, G_1, S^*, \mu/4)$ 11 $H_2 \leftarrow \text{BasecaseMultiply}(F_2, G_2, S^*, \mu/4)$ 12 **for** j = 1, 2 **do** $H_j \leftarrow H_j \mod (\ell^2, x^p - 1)$ 13 $\mathbb{S} \leftarrow$ empty list of integers 14 **for** every nonzero term cx^e of H_1 **do** $c' \leftarrow \text{coefficient of degree-} e \text{ term of } H_2$ 15 if $c \mid c'$ and $\ell \mid (c'/c - 1)$ as integers then 16 Add $(c'/c - 1)/\ell$ to \$ 17 else return Fail //cannot reconstruct an exponent 18 19 return S

6 Multiplication with support

We turn now to the problem of multiplying sparse $F, G \in \mathbb{Z}[x]$, provided some $\mathbb{S} \supseteq \text{supp}(FG)$. This algorithm is used twice in our overall multiplication algorithm: first with large $\mathbb{S} = \text{poss}(F, G)$ but small coefficients, then with the actual support $\mathbb{S} = \text{supp}(FG)$ but full-size coefficients.

The bit-complexity of our algorithm is summarized in the following theorem.

Theorem 6.1. Given $F, G \in \mathbb{Z}[x]$ and a set $\mathbb{S} \subset \mathbb{Z}_{\geq 0}$ such that $\operatorname{supp}(FG) \subseteq \mathbb{S}$, the product FG can be computed in time $\widetilde{\mathcal{O}}_{\mu}((\#F + \#G + \#\mathbb{S})(\log C + \log D))$, where $C = \|F\|_{\infty} + \|G\|_{\infty}$ and $D > \deg F + \deg G$.

This is \mathcal{O}_{μ} -optimal, as it matches the bit-size of the inputs.

Our algorithm requires a small randomly-selected good difference-prime p with $O(\log S + \log \log D)$ bits, and a series of pairs (q, ω) , where each q is a slightly larger prime with $O(\log p)$ bits, and ω is an order-p element in \mathbb{Z}_q .

These numbers are provided by GetPrimRoots (Sec. 3). Our algorithm works by first reducing the exponents modulo p, then repeatedly reducing the coefficients modulo q and performing evaluation and interpolation at powers of ω . This inner loop follows exactly the algorithm of [van der Hoeven and Lecerf, 2013] and [Kaltofen and Yagati, 1989] for applying a transposed Vandermonde matrix and its inverse. Since p is a good difference-prime for the support of the product, there are no collisions and this gives us each coefficient modulo q. The process is then repeated $O(\log C)$ times in order to recover the full coefficients via Chinese remaindering.

6.1 Comparison to prior work

Without affecting the complexity, we may assume that S contains the support of the inputs too, i.e., supp(F) and supp(G). We also assume that $\max S = \deg FG$, such that no $e \in S$ is too large to be an exponent of *FG*. Under these assumptions, and writing S = #S, the stated complexity of our algorithm is simply $\widetilde{\mathcal{O}}_{\mu}(S(\log C + \log D))$.

The problem of computing the coefficients of a sparse product, once the exponents of the product are given, has been recently and extensively investigated by van der Hoeven and Lecerf, where they present an algorithm whose bit complexity (in our notation) is

$$\widetilde{\mathcal{O}}_{\mu}\left(\left(\sum_{e\in\mathbb{S}}\log e\right)\left(\log D + \log C\right)\right)$$

([van der Hoeven and Lecerf, 2013], Corollary 5). As $\sum_{e \in S} \log e \in \mathcal{O}(S \log D)$, the algorithm here saves a factor of at most $\mathcal{O}(\log D)$ in comparison, which could be substantial if the exponents are very large.

Their algorithm is more efficient if the support superset S is *fixed*, in which case they can move the most expensive parts into precomputation and compute the result in the same soft-oh time as our approach, $\widetilde{\mathcal{O}}_{\mu}(S(\log D + \log C))$. Furthermore, the support bit-length $\sum_{e \in S} \log e$ is at most $\mathcal{O}(S \log D)$, but could be as small as $\Omega(S \log S + \log D)$, for example if the support contains only a single large exponent. In such cases our savings is only on the order of $(\log D)/S$.

6.2 Transposed Vandermonde systems

Applying transposed Vandermonde systems, and their inverses, is an important subroutine in sparse interpolation algorithms, and efficient algorithms are discussed in detail by [Kaltofen and Yagati, 1989] and [van der Hoeven and Lecerf, 2013]. We restate the general idea here and refer the reader to those papers for more details.

If \tilde{F} is a dense polynomial, it is well known that applying the Vandermonde matrix $V(\theta_1, \ldots, \theta_D)$ to a vector of coefficients from \tilde{F} corresponds to evaluating \tilde{F} at the points $\theta_1 \ldots, \theta_D$. Applying the inverse Vandermonde matrix corresponds to interpolating \tilde{F} from its evaluations at those points. The product tree method can perform both of these using $\hat{O}(D)$ field operations ([von zur Gathen and Gerhard, 2013], Chapter 10).

If *F* is instead a sparse polynomial $F = \sum_{e \in S} c_1 x^e$, evaluating *F* at consecutive powers of a single high-order element ω corresponds to multiplication with the transposed Vandermonde matrix:

$$V(\omega^{e_1},...,\omega^{e_S})^T(c_1,...,c_S)^T = (F(1),...,F(\omega^{S-1}))^T$$

The transposition principle tells us it is possible to compute the maps V^T and $(V^T)^{-1}$ in essentially the same time as dense evaluation and interpolation. In particular, if the coefficients c_i are in the modular ring \mathbb{Z}_q , then the transposed Vandermonde map and its inverse can be computed using $\widetilde{\mathcal{O}}(S \log q)$ bit operations [van der Hoeven and Lecerf, 2013].

6.3 Statement and analysis of the algorithm

Procedure SparseMulCoeffs(S, F, G, μ) **Input**: Exponents $S = (e_1, e_2, \dots, e_S)$; coefficient lists (f_1, \dots, f_S) and (g_1, \ldots, g_S) , with $F, G \in \mathbb{Z}[x]$ implicitly defined as $F = \sum_{1 \le i \le S} f_i x^{e_i}$ and $G = \sum_{1 \le i \le S} g_i x^{e_i}$; error bound $\mu \in (0, 1)$. **Output**: $(h_1, \ldots, h_S) \in \mathbb{Z}^S$ such that, with probability least $1 - \mu$, $FG = \sum_{1 \le i \le S} h_i x^{e_i}.$ $1 \ C \leftarrow (\max_{1 \le i \le S} |f_i|) (\max_{1 \le i \le S} |g_i|) S$ 2 $p, Q, W \leftarrow \text{GetPrimRoots}(\max \$, \#\$, C, \mu)$ 3 $\mathbb{S}_p \leftarrow (e_1 \operatorname{\overline{rem}} p, e_2 \operatorname{\overline{rem}} p, \dots, e_S \operatorname{\overline{rem}} p)$ 4 $H \leftarrow$ list of S empty lists of integers 5 foreach $(q, \omega) \in Q, W$ do foreach $e_{ip} \in \mathbb{S}_p$ do 6 $v_i \leftarrow \omega^{e_{ip}} \in \mathbb{Z}_q$ by binary powering 7 $\mathbf{a} \leftarrow V(v_1, \ldots, v_S)^T (f_1, \ldots, f_S)^T \in \mathbb{Z}_q^S$ 8 $\mathbf{b} \leftarrow V(v_1, \dots, v_S)^T (g_1, \dots, g_S)^T \in \mathbb{Z}_q^S$ $\mathbf{c} \leftarrow (a_1 b_1, \dots, a_S b_S)^T \in \mathbb{Z}_q^S$ 9 10 if $V(v_1, \ldots, v_S)$ is invertible then 11 $(h_{1p}, \dots, h_{Sp}) \leftarrow (V(v_1, \dots, v_S)^T)^{-1} \mathbf{c} \in \mathbb{Z}_q^S$ for $1 \le i \le S$ do Add h_{ip} to the list H[i]12 13 14 for $1 \le i \le S$ do $h_i \leftarrow$ Chinese remaindering from images H[i] modulo integers in Q 15 16 return $(h_1, ..., h_S)$

Lemma 6.2. Procedure SparseMulCoeffs works as stated when $S \supseteq supp(FG)$. In any case it has bit-complexity

$$\widetilde{\mathcal{O}_{\mu}}\left(\sum_{e\in\mathbb{S}}\log e+S\log C\right).$$

Proof. We first analyze the probability of failure when $S \supseteq \text{supp}(FG)$. The randomization is in the choices of p, q, and ω ; problems can occur if these lack the required properties.

If *p* is a good difference-prime for supp(*FG*), then by definition there will be no collisions in S_p . Furthermore, if ω is a *p*th root of unity modulo *q*, then there are no collisions among the values (v_1, \ldots, v_S) , so $V(v_1, \ldots, v_S)$ is invertible modulo *q*. Algorithm GetPrimRoots ensures this is the case with high probability, and if so the algorithm here faithfully computes each coefficient h_i modulo *q*.

Conversely, if there are no collisions in S_p , and if no zero divisors modulo q are encountered in the application of the Vandermonde matrix and its inverse, then the algorithm correctly computes then values $h_i \mod q$, even if p is not prime or some $\omega \in \mathbb{Z}_q$ is not actually a pth root of unity.

Therefore all failures in choosing tuples p, q, ω are either detected by the algorithm or do not affect its correctness. Since that is the only randomized step, we conclude that the entire algorithm is correct whenever the input exponent set S contains the support of the product.

For the complexity analysis first define $D = \deg(FG)$. Step 2 costs $\mathcal{O}_{\mu}(\log C \cdot \operatorname{polylog}(S + \log D))$ bit operations by Lemma 3.7. Reducing each exponent e_i modulo p, on step 3, can be done for a total of $\widetilde{\mathcal{O}}_{\mu}(\sum_{e \in S} \log e)$ bit operations.

Now we examine the cost of the for loop that begins on step 5. As the exponents are now all less than p, computing each v_i on step 7 requires only $\mathcal{O}(\log p)$ operations modulo q, for a total of $\mathcal{O}(S \log p \log q)$, which is $\widetilde{\mathcal{O}}_{\mu}(S \cdot \text{polylog}(\log C + \log D))$ bit operations. From before we know that applying the transposed Vandermonde matrix and its inverse takes $\widetilde{\mathcal{O}}_{\mu}(S \log q)$, or $\widetilde{\mathcal{O}}_{\mu}(S \cdot \text{polylog}(\log C + \log D))$ bit operations.

Because $\#Q \leq \lceil \log_p(2C) \rceil$, the loop on step 5 repeats $\mathcal{O}(\log C)$ times, for a total cost of $\widetilde{\mathcal{O}}_{\mu}(S \log C \cdot \text{polylog}(\log D))$ bit operations. This also bounds the cost of the Chinese remaindering in the final loop.

7 Multiplication algorithms

The complete multiplication algorithm over $\mathbb{Z}[x]$ that was outlined in the introduction is presented as SparseMultZZ.

Proof of Thm. 1.1. Unless failure occurs, we have $S_1 = poss(F, G)$. Every coefficient in H is a sum of products of coefficients in F and G, so the value C_H computed on step 2 is an upper bound on $||H||_{\infty}$, and p is a good vanish-prime

Procedure SparseMultZZ(*F*, *G*)

Input: Sparse $F, G \in \mathbb{Z}[x]; \mu \in (0, 1)$. Output: Sparse $H \in \mathbb{Z}[x]$, such that $\Pr[H \neq FG] < \mu$. $S_1 \leftarrow \text{Sumset}(\text{supp}(F), \text{supp}(G), \frac{\mu}{4})$ $C_H \leftarrow ||F||_{\infty} ||G||_{\infty} \max(\#F, \#G)$ $p \leftarrow \text{GetVanishPrime}(\#S_1, C, 1, \frac{\mu}{4})$ $H_1 \leftarrow \text{SparseMulCoeffs}(F \text{ rem } p, G \text{ rem } p, S_1, \frac{\mu}{4})$ $S_2 \leftarrow \text{supp}(H_1 \text{ rem } p)$ 6 return SparseMulCoeffs $(F, G, S_2, \frac{\mu}{4}), S_2$

for *H*. Thus $S_2 = \text{supp}(H \text{ rem } p) = \text{supp}(H)$, so the final step correctly computes the product *FG*.

By the union bound, Lemma 3.3 and Theorems 5.1 and 6.1, the probability of failure is less than μ .

Writing T = # poss(F, G) and S = # supp(FG), we see that $\log p \le \text{polylog}(T + \log C + \frac{1}{\mu})$, thus steps 1 and 4 contribute $\widetilde{\mathcal{O}}_{\mu}(T \log D)$ to the overall cost, whereas the last step costs $\widetilde{\mathcal{O}}_{\mu}(S(\log D + \log C))$ bit operations. As $S \le T$, the total bit complexity is $\widetilde{\mathcal{O}}_{\mu}(T \log D + S \log C)$, as required.

We now consider extensions of this algorithm to positive and negative exponents (Laurent polynomials), multiple variables, and other common coefficient rings, using Kronecker substitution. This is stated in the following theorem.

Theorem 7.1. Let $F, G \in R[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ be sparse Laurent polynomials over R, where $R = \mathbb{Z}, \mathbb{Z}_m, GF(q^e)$, or Q. The product FG can be computed using

 $\widetilde{\mathcal{O}}_{u}\left(T\left(n\log D+B\right)\right)$

bit operations, where T = # poss(F, G) is the structural sparsity of the product, $D > max_i | deg_i(FG) |$, and B is the largest bit-length of any coefficient in the input or output.

Proof. Write the output polynomial H = FG as

$$H = \sum_{i=1}^{T} c_i x_1^{e_{i1}} x_2^{e_{i2}} \cdots x_n^{e_{in}},$$

where each $c_i \in \mathsf{R}$ and each e_{ii} satisfies $|e_{ii}| < D$.

We first apply the Kronecker substitution, providing an easily-invertible map between $R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $R[z^{\pm 1}]$: $x_i \mapsto z^{D^{i-1}}$ for $1 \le i \le n$. This increases the degree to D^n , such that the logarithm of this degree $O(n \log D)$, matching the exponent bit-size in the multivariate representation.

The algorithm Sumset already handles negative exponents (i.e., Laurent polynomials) explicitly. The other primary subroutine to procedure SparseMultZZ

is SparseMulCoeffs, which only uses the exponents in the set S_p , which are reduced modulo p and therefore cause no difficulty if they are negative. Thus the multiplication algorithms handle univariate Laurent polynomials without any changes.

To extend the multiplication algorithm and it subroutines beyond $R = \mathbb{Z}$, we use that our algorithm is also softly-linear in the input *heights*. This allows us to adapt to many coefficient domain that provides a natural mapping to the integers, and to preserve softly-linear time if that mapping provides only a softly-linear increase in size.

For a modular ring $R = \mathbb{Z}_m$, we can trivially treat the inputs as actual integers, then reduce modulo *m* after multiplying. For a finite field $R = GF(p^d)$, elements are typically represented as polynomials over $\mathbb{Z}_p[z]$ modulo a degree-*d* irreducible polynomial, so these coefficients can be converted to integers using a low-degree Kronecker substitution. For the rationals R = Q, we might choose a prime *q* larger than the product of the largest numerator and denominator in the output, multiply modulo *q*, then use rational reconstruction to recover the actual coefficients.

In all the above cases, there is growth in the bit-length of coefficients, but only in poly-logarithmic terms of input and output size, therefore not affecting the soft-oh complexity. The only downside is that we are no longer able to split the cost neatly between T = poss(F, G) and S = supp(FG) because the unreduced integer polynomial product might have nonzero coefficients which are really zeros in R.

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