arXiv:1505.01954v1 [math.AC] 8 May 2015

GREEN'S FUNCTIONS FOR STIELTJES BOUNDARY PROBLEMS

M. ROSENKRANZ AND N. SERWA

Abstract. Stieltjes boundary problems generalize the customary class of well-posed two-point boundary value problems in three independent directions, regarding the specification of the boundary conditions: (1) They allow more than two evaluation points. (2) They allow derivatives of arbitrary order. (3) Global terms in the form of definite integrals are allowed. Assuming the Stieltjes boundary problem is regular (a unique solution exists for every forcing function), there are symbolic methods for computing the associated Green's operator.

In the classical case of well-posed two-point boundary value problems, it is known how to transform the Green's operator into the so-called Green's function, the representation usually preferred by physicists and engineers. In this paper we extend this transformation to the whole class of Stieltjes boundary problems. It turns out that the extension (1) leads to more case distinction, (2) implies ill-posed problems and hence distributional terms, (3) has apparently no effect on the structure of the Green's function.

1. Introduction

Boundary problems for linear ordinary differential equations (LODEs) or linear partial differential equations (LPDEs) are certainly among the most important model types in the engineering sciences. Interestingly, their systematic treatment in Symbolic Computation started rather recently [17]. For handling the central problems of solving and factoring boundary problems, a *differential algebra setting* for LODEs is employed in [20, 19] and for LPDEs in [21, 18]. An overarching abstract framework based only on Linear Algebra is developed in [15]. For the classical treatment of boundary problems in Analysis, we refer to [7, 10, 22, 24].

In this paper we restrict ourselves to LODEs, where the "industrial standard" for solving boundary problems is their so-called *Green's function.* This is in stark contrast to the operator-based methodology used

Date: 16 April 2015.

in the above references. In fact, given a fundamental system, the algorithm of [17, 20] computes the solution of a boundary problem in the form of its *Green's operator*. In the classical setting of well-posed two-point boundary value problems (see Section 2), this algorithm admits an optional extra step for extracting the corresponding Green's function. Our goal here is to extend this postprocessing step to the considerably larger class of Stieltjes boundary problems (see Section 2).

One way to understand the relationship between Green's operators and functions is to view the latter as a certain canonical form. For making this precise we equip the ring of integro-differential operators with a slightly different set of reduction rules favoring multiply initialized integrals, leading to the ring of *equitable integro-differential operators* (Section 3).

A *simple example* will make this clear—in fact the simplest of all honest boundary problems [17, §3.2]. Given a forcing function $f \in C^\infty[0,1]$, we want to find $u \in C^\infty[0,1]$ such that

$$\boxed{\begin{array}{l} u'' = f, \\ u(0) = u(1) = 0. \end{array}}$$

The Green's operator $G\colon C^\infty[0,1] \to C^\infty[0,1]$ of this problem is defined by $Gf = u$. Using the standard reduction system of [20], we would distinguish one integral like ${\textstyle\int} f := \int_0^x f(\xi)\, d\xi$ and then obtain the Green's operator in the canonical form

$$G = x{\textstyle\int} - {\textstyle\int} x + x\lfloor 1\rfloor {\textstyle\int} x - x\lfloor 1\rfloor {\textstyle\int}, \tag{1}$$

where $\lfloor\alpha\rfloor$ denotes the evaluation functional $f \mapsto f(\alpha)$ for any $\alpha \in \mathbb{R}$, in analogy to the multiplier notation of [17]. For extracting the Green's function, however, it is more useful to use the alternative canonical form

$$G = x{\textstyle\int_0} x - x{\textstyle\int_1} x - {\textstyle\int_0} x + x{\textstyle\int_1} \tag{2}$$

where ${\textstyle\int_\alpha} f := \int_\alpha^x f(\xi)\, d\xi$ now denotes the integral initialized at the point $\alpha \in \{0,1\}$. In fact, this is the form given in [17], and we shall see in Section 3 that the setting of biintegro-differential operators used there is essentially a special case of the equitable operator ring employed in this paper. The point of the canonical form (2) is that it allows us to apply the defining relation $Gf(x) = \int_0^1 g(x,\xi)\, f(\xi)\, d\xi$ of the Green's function directly to obtain the latter as

$$g(x,\xi) = \begin{cases} (x-1)\xi & \text{for } 0 \le \xi \le x \le 1, \\ x(\xi-1) & \text{for } 0 \le x \le \xi \le 1. \end{cases} \tag{3}$$

Heuristically speaking, one moves the $\int_0$ terms to the upper and the $\int_1$ terms to the lower branch, at the same time translating the "x" after the integrals into ξ. Note incidentally that $g(x, \xi) = g(\xi, x)$ in the above Green's function (3). As is well known in Analysis [5, §7] [24, §5], this is a consequence of the self-adjoint nature of this boundary problem—a topic that we would wish to investigate in the future for the more general class of Stieltjes boundary Problems (Section 6).

We will elaborate on the above principles to *generalize* it in three "orthogonal directions": (1) We allow more than two evaluation points, leading to an increased number of case branches. (2) Using derivatives of arbitrary order in the boundary conditions leads to distributional terms. (3) Boundary conditions with integral terms (so-called "non-local problems") are also included; they do not lead to further complications.

2. Stieltjes Boundary Problems

For giving a precise definition of the class of admissible boundary problems, we follow the setting of [20]. Hence let $(\mathcal{F}, \partial, \int)$ be a fixed ordinary integro-differential K-algebra (here ordinary means $\ker \partial = K$). Later we shall specialize this to $\mathcal{F} = C^\infty(\mathbb{R})$, the real- or complex-valued smooth function. This is theoretically convenient but of course needs to be replaced by a suitable constructive subalgebra for actual computations.

The *ring of integro-differential operators* over $\mathcal{F}$, introduced in [20, §3], will be denoted here by $\mathcal{F}_\Phi[\partial, \int]$ to emphasize its dependence of the chosen set of characters Φ, and also to mark the contrast to the equitable operator ring $\mathcal{F}[\partial, \int_\Phi]$ to be introduced in Section 3, where the integral operators are parametrized by Φ. In the case of $\mathcal{F} = C^\infty(\mathbb{R})$, these characters will be evaluations at given points of $\mathbb{R}$ so that we may take $\Phi \subseteq \mathbb{R}$.

We recall the *standard decomposition*

$$\mathcal{F}_\Phi[\partial, \textstyle\int] = \mathcal{F}[\partial] \dotplus \mathcal{F}[\int] \dotplus (\Phi), \tag{4}$$

where $\mathcal{F}[\partial]$ denotes the subalgebra of differential operators (the K-subalgebra of $\mathcal{F}_\Phi[\partial, \int]$ generated by $\mathcal{F}$ and ∂), $\mathcal{F}[\int]$ the nonunital subalgebra of integral operators (the nonunital K-subalgebra of $\mathcal{F}_\Phi[\partial, \int]$ generated by $\mathcal{F}$ and $\int$), and (Φ) the two-sided ideal of $\mathcal{F}_\Phi[\partial, \int]$ generated by the characters in Φ. The corresponding *right* ideal $|\Phi) = \Phi \cdot \mathcal{F}_\Phi[\partial, \int]$ is known as the ideal of *Stieltjes conditions*, and one may check that (Φ) is in fact the left $\mathcal{F}$-module generated by the Stieltjes conditions.

From the viewpoint of applications, Stieltes conditions $\beta \in (\Phi)$ are easier to comprehend in terms of their $\mathcal{F}_\Phi[\partial, \int]$-normal form: They can be described uniquely as sums

$$\beta = \sum_{\varphi \in \Phi} \sum_{i \geq 0} a_{\varphi,i} \varphi \partial^i + \sum_{\varphi \in \Phi} \varphi {\textstyle\int} f_\varphi \tag{5}$$

with only finitely many $a_{\varphi,i} \in K$ and $f_\varphi \in \mathcal{F}$ nonzero. The double sum in (5) is called the *local part* of β, the subsequent sum its *global part.* In the important $C^\infty(\mathbb{R})$ case with distinguished integral ${\textstyle\int} = \int_0^x$, this yields

$$\beta(u) = \sum_{\varphi,i} a_{\varphi,i} u^{(i)}(\varphi) + \sum_{\varphi} \int_0^\varphi f_\varphi(\xi)\, u(\xi)\, d\xi,$$

for certain $a_{\varphi,i} \in \mathbb{R}$ and $f_\varphi \in C^\infty(\mathbb{R})$.

An n-th order *Stieltjes boundary problem* is a pair $(T, \mathcal{B})$ with a monic differential operator $T \in \mathcal{F}[\partial]$ of order n and a boundary space $\mathcal{B} \leq \mathcal{F}^*$ given as linear span $\mathcal{B} = [\beta_1, \dots, \beta_n]$ of n linearly independent Stieltjes conditions. In traditional representation, such a boundary problem is displayed as

$$\boxed{\begin{array}{l} Tu = f, \\ \beta_1 u = \cdots = \beta_n u = 0, \end{array}} \tag{6}$$

with the understanding that $u \in \mathcal{F}$ is desired for any prescribed forcing function $f \in \mathcal{F}$. For the (usual) Green's operator to be well-defined, we need the boundary problem (6) to be *regular* in the sense that $\ker T \dotplus \mathcal{B}^\perp = \mathcal{F}$, where $\mathcal{B}^\perp = \{u \in \mathcal{F} \mid \beta(u) = 0 \text{ for all } \beta \in \mathcal{B}\}$ is the corresponding space of admissible functions. Regularity is equivalent to the requirement that (6) has a unique solution $u \in \mathcal{F}$ for every given $f \in \mathcal{F}$. An algorithmic method for testing regularity starts from a fundamental system $u_1, \dots, u_n \in \mathcal{F}$ for T, meaning a K-basis of $\ker T$. Then (6) is regular iff the evaluation matrix

$$\beta(u) = \begin{pmatrix} \beta_1(u_1) & \cdots & \beta_1(u_n) \\ \vdots & \ddots & \vdots \\ \beta_n(u_1) & \cdots & \beta_n(u_n) \end{pmatrix} \in K^{n \times n} \tag{7}$$

is regular; see (15) of [20]. Given a fundamental system $u_1, \dots, u_n$ for T, the solution algorithm of [20] computes the Green's operator of any regular Stieltjes boundary problem as an integro-differential operator $G \in \mathcal{F}_\Phi[\partial, {\textstyle\int}]$.

Within the class of Stieltjes boundary problems, we make the following distinctions in order to characterize the *classical scenario* as a certain special case.

Definition 1. A Stieltjes boundary problem $(T, \mathcal{B})$ of order n with $\mathcal{B} = [\beta_1, \ldots, \beta_n]$ is called *well-posed* if the β_i can be chosen with all derivatives having order below n; otherwise it is called *ill-posed*. Furthermore, we call $(T, \mathcal{B})$ an *m-point boundary problem* if the maximal number of evaluation points occurring in any K-basis (β_i) of $\mathcal{B}$ is m, and we call $(T, \mathcal{B})$ *local* if the β_i can be chosen without global parts.

Let us digress a bit on the notion of ill-posed boundary problems. Following Hadamard, a problem is generally called *well-posed* [8, p. 86] if it is regular (meaning its solution u exists and is unique for all given data f) as well as stable (meaning u depends continuously on f). Otherwise, one speaks of an *ill-posed* problem. In the case of boundary problems (6), we search for $u \in C^\infty(\mathbb{R})$, and the data is given by the forcing function $f \in C^\infty(\mathbb{R})$. Stability—and hence well-posedness—depends on the topology chosen for the function space $C^\infty(\mathbb{R})$. Using the L^2 norm as in many application problems, the distinction between well- and ill-posed boundary problems coincides with the one given above.

Since local boundary problems involve only evaluations of the unknown function (rather than definite integrals), we also call them "boundary *value* problems". We can now characterize the classical case, described for example in [5, §7], by the following three-fold restriction: They are the <u>well-posed</u> <u>two</u>-point boundary <u>value</u> problems. (Sometimes one meets the further restriction to self-adjoint boundary problems.)

The classical case (in the above sense) is clearly the most frequent case in the applications (but this could also be due to a selection bias: having a well-equipped toolbox for classical problems might tempt engineers to restrict their attention to classical problems). Nevertheless, *multi-point boundary value problems* are also important for some applications [1, Ex. 1.6], [23], [13], [14], [3]. Boundary problems with nonlocal conditions are more seldom, they are usually studied for nonlinear equations [4], [11]; the linear case serves as the initial approximation. Finally, the case of ill-posed boundary problems is—for obvious reasons—mostly avoided when engineering problems are modelled. However, there are cases where their treatment is inevitable, typically in the context of inverse problems [9]. Since the numerical treatment of such problems is very delicate, it is of paramount importance to have exact symbolic algorithms wherever this is possible.

We will lift all three of these restrictions for the algorithm of *extracting Green's functions*, which will be given below (Section 4). As indicated in the Introduction, the crucial tool for this purpose—even in the classical case—is the ring of equitable integro-differential operators with its alternative canonical forms.

3. Equitable Operators

The passage from the standard integro-differential operator ring $\mathcal{F}_\Phi[\partial, \textstyle\int]$ to its equitable variant $\mathcal{F}[\partial, \textstyle\int_\Phi]$ is based on the *fundamental theorem of calculus* $\int_\varphi^x f'(\xi)\, d\xi = f(x) - f(\varphi)$ for any function $f \in C^\infty(\mathbb{R})$ and initialization point $\varphi \in \mathbb{R}$. Likewise, if $(\mathcal{F}, \partial, \textstyle\int)$ is an arbitary integro-differential K-algebra and φ a character (multiplicative linear functional), one can use the definition $\int_\varphi := (\mathrm{id} - \varphi) \textstyle\int$ to obtain the corresponding relation $\int_\varphi \partial = \mathrm{id} - \varphi$. In some contexts (especially in the presence of several integral operators like $\int^x$ and $\int^y$ on bivariate functions), it may be useful to write the integral $\int_\varphi$ as $\int_\varphi^x$. If ψ is another character, one observes the relation $\psi \int_\varphi = \int_\psi - \int_\varphi$, and it is natural to write $\int_\varphi^\psi$ for both expressions.

Note that $(\mathcal{F}, \partial, \int_\varphi)$ is again an ordinary integro-differential K-algebra, and the preference of $\textstyle\int$ over $\int_\varphi$ can appear arbitrary in certain settings. Accordingly, one may build the ring of integro-differential operators by adjoining *all* $\int_\varphi$ while the characters φ themselves are now redundant due to the above fundamental relation. The precise formulation of the resulting ring $\mathcal{F}[\partial, \int_\Phi]$ as a quotient is described in [16, §5.1]. For our present purposes, we shall only list its relations (see Table 1 where $\cdot$ denotes the natural action of the operators), which are an easy consequence of the relations of the standard integro-differential operator ring $\mathcal{F}_\Phi[\partial, \textstyle\int]$.

fg	$\to$	$f \cdot g$	∂f	$\to$	$\partial \cdot f + f\partial$	$\partial \int_\varphi^x$	$\to$	1
$\int_\varphi^x f \int_\psi^x$	$\to$	$(\int_\varphi^x \cdot f) \int_\psi^x - \int_\varphi^x (\int_\varphi^x \cdot f)$						
$\int_\varphi^x f \partial$	$\to$	$f - \int_\varphi^x (\partial \cdot f) - \varphi \cdot f + (\varphi \cdot f) \int_\varphi^x \partial$						

Table 1. Equitable Integro-Differential Relations

Similar to the standard decomposition (4), we have also the *equitable decomposition* $\mathcal{F}[\partial, \int_\Phi] = \mathcal{F}[\partial] \dotplus \mathcal{F}[\int_\Phi] \dotplus \mathcal{F}[\int_\Phi \partial]$ where $\mathcal{F}[\int_\Phi]$ is the nonunital subalgebra of equitable integral operators $\sum_{i=0}^n f_i \int_\varphi^x g_i$ and $\mathcal{F}[\int_\Phi \partial]$ the $\mathcal{F}$-submodule consisting of $\sum_{i=0}^n f_i \int_\varphi^x \partial^i$; this leads to the obvious normal forms in $\mathcal{F}[\partial, \int_\Phi]$.

The so-called *translation isomorphism* $\iota\colon \mathcal{F}_\Phi[\partial, \textstyle\int] \to \mathcal{F}[\partial, \textstyle\int_\Phi]$ leaves $f \in \mathcal{F}$ and ∂ invariant while using the above fundamental relation via $\iota(\varphi) = \mathrm{id} - \textstyle\int_\varphi \partial$ and $\iota^{-1}(\textstyle\int_\varphi) = (\mathrm{id} - \varphi)\textstyle\int$. Note that this holds also for the character $\text{€} := \mathrm{id} - \textstyle\int\partial$ associated to the distinguished integral $\textstyle\int = \textstyle\int_{\text{€}}$ underlying $\mathcal{F}_\Phi[\partial, \textstyle\int]$.

Specializing to $\mathcal{F} = C^\infty(\mathbb{R})$ and $\Phi = \{0, 1\}$, we can deal with the *example* in the Introduction, where we have the normal form $G \in \mathcal{F}_\Phi[\partial, \textstyle\int]$ in (1) along with its equitable variant $\iota(G) \in \mathcal{F}[\partial, \textstyle\int_\Phi]$ in (2). In such two-point cases with characters $\Phi = \{\alpha, \beta\}$, the equitable operator ring $\mathcal{F}[\partial, \textstyle\int_\Phi]$ is essentially the same as the ring of *biintegro-differential operators*. More precisely, we obtain a biintegro-differential algebra $(\mathcal{F}, \partial, \textstyle\int^*, \textstyle\int_*)$ with integral $\textstyle\int^* := \textstyle\int_\alpha$ and cointegral $\textstyle\int_* := -\textstyle\int_\beta$ in the sense of [16, Def. 3.23]. Note that $\textstyle\int^*$ and $\textstyle\int_*$ are adjoint with respect to the inner product defined by $\langle f|g\rangle := (\textstyle\int^* + \textstyle\int_*)(fg) = \textstyle\int_\alpha^\beta fg$. Incidentally, the notion of biintegro-differential algebra coincides with the (badly named) notion of "analytic algebra" introduced in [17, Def. 2] and replicated in [20, Ex. 5]. Clearly, the operator ring resulting from $\mathcal{F}[\partial]$ by adjoing $\textstyle\int^*$ and $\textstyle\int_*$ is the same as $\mathcal{F}[\partial, \textstyle\int_\Phi]$, modulo the sign change in the cointegral.

Note that also $\textstyle\int_\varphi \in \mathcal{F}_\Phi[\partial, \textstyle\int]$ and $\varphi \in \mathcal{F}[\partial, \textstyle\int_\Phi]$ are legitimate expressions via the above translation isomorphism. They are not in canonical form but we may think of them as a kind of *abbreviation* for the corresponding canonical expression.

In fact, the extraction of Green's functions is based on the following slight variation of the equitable integro-differential operator ring $\mathcal{F}[\partial, \textstyle\int_\Phi]$. Writing any element $U \in \mathcal{F}_\Phi[\partial, \textstyle\int]$ in the form $U = T + K + B$ with $T \in \mathcal{F}[\partial]$, $K \in \mathcal{F}[\textstyle\int]$ and $B \in (\Phi)$ according to (4), we let $T \in \mathcal{F}[\partial, \textstyle\int_\Phi]$ and $K \in \mathcal{F}[\textstyle\int_{\text{€}}] \subseteq \mathcal{F}[\textstyle\int_\Phi]$ invariant while translating $B \in (\Phi)$ as follows. Since (Φ) is the left $\mathcal{F}$-module generated by Stieltjes conditions (5), we may split $B = \lambda + \gamma$ into a left $\mathcal{F}$-linear combination λ of local Stieltjes conditions and a left $\mathcal{F}$-linear combination γ of global Stieltjes conditions. It turns out to be expedient to keep λ in this form, without eliminating the characters via $\varphi = \mathrm{id} - \textstyle\int_\varphi \partial$, but to translate γ via $\varphi \textstyle\int_{\text{€}} = \textstyle\int_{\text{€}} - \textstyle\int_\varphi =: \textstyle\int_{\text{€}}^\varphi$. This is what we mean when referring in the sequel to the *equitable form* of an integro-differential operator U.

4. Extracting Green's Functions

We now turn to the central task of this paper, the extraction of the Green's function $g(x, \xi)$ corresponding to the Green's operator $G \in$

$\mathcal{F}[\partial, \int_\Phi]$ computed by the algorithm of [20] and converted to equitable form as described in Section 3. Hence we specialize now to $\mathcal{F} = C^\infty(\mathbb{R})$. Note that we may think of $g(x, \xi)$ as a kind of *coordinate representation* of the induced operator action $G\colon \mathcal{F} \to \mathcal{F}$; in quantum mechanics this would correspond to the "position basis" (as opposed to the "momentum basis" in the Pontryagin dual reached via the Fourier transform). Hence we will use the notation $g(x, \xi) = G_{x\xi}$, thinking of the x, ξ rather like continuous indices similar to the discrete indices i, j in the matrix elements A_{ij} of some $A \in K^{n \times n}$.

In fact, we will use this notation $G_{x\xi}$ for any equitable integro-differential operator $G \in \mathcal{F}[\partial, \int_\Phi]$. Its result will in general contain Dirac distributions [22, §2] and their derivatives but nothing beyond that. Since all boundary problems considered in this paper have only finitely many evaluation points $\alpha \in \Phi \subset \mathbb{R}$, one may choose an interval $J \subset \mathbb{R}$ containing all the α. Hence the $C(J^2)$-module $\mathcal{G} \subset \mathcal{D}'(J^2)$ generated by the *Dirac distributions* δ_α and their derivatives will be sufficient to capture all Green's "functions" $G_{x\xi} \in \mathcal{G}$. Here and in the sequel we shall follow the common engineering (and also applied maths) practice of referring to distributions like δ_α as functions. In the same vein, we shall also write $\delta(\xi - \alpha)$ in place of δ_α, in view of the defining property $\int_J \delta(\xi - \alpha)\, f(\xi)\, d\xi = f(\alpha)$.

The transformation from Green's operators to Green's functions

$$\mathcal{F}[\partial, \textstyle\int_\Phi] \to \mathcal{G},\ G \mapsto G_{x\xi}$$

is clearly an $\mathbb{R}$-linear map, hence it will be sufficient to define it on the *canonical* $\mathbb{R}$*-basis* of $\mathcal{F}_\Phi[\partial, \int] \cong \mathcal{F}[\partial, \int_\Phi]$. Following the strategy of the example in the Introduction, the easiest part is $\mathcal{F}[\int] \subseteq \mathcal{F}_\Phi[\partial, \int]$, which is handled by setting

$$(f \textstyle\int g)_{x\xi} = f(x)\, g(\xi)\, [0 \le \xi \le x] - f(x)\, g(\xi)\, [x \le \xi \le 0],$$

where we use the Iverson bracket notation $[P]$ signifying 1 if the property P is true and zero otherwise. Note that at most one of the two summands above is nonzero for fixed (x, ξ). Since $(\Phi) \subset \mathcal{F}_\Phi[\partial, \int]$ is a left $\mathcal{F}$-module over $|\Phi)$, we settle this part via

$$(f\, \lfloor\alpha\rfloor \partial^i)_{x\xi} = (-1)^i f(x)\, \delta^{(i)}(\xi - \alpha),$$
$$(f\, \lfloor\alpha\rfloor \textstyle\int g)_{x\xi} = \operatorname{sgn}(\alpha)\, f(x)\, g(\xi)\, [0 \le \xi \le \alpha].$$

Finally, on $\mathcal{F}[\partial]$ we define

$$(f \partial^i)_{x\xi} = (-1)^i\, f(x)\, \delta^{(i)}(x - \xi),$$

and the definition is complete in view of (4). Moreover, it is easy to check that the assignment $G \mapsto G_{x\xi}$ is correct in the sense that $Gf =$

$\int_J G_{x\xi}\, f(\xi)\, d\xi$. The isomorphism ι of Section 3 may now be employed to obtain the required transformation $\mathcal{F}[\partial, \int_\Phi] \to \mathcal{G}$. In fact, the above case $f \int g \in \mathcal{F}[\int]$ generalizes immediately to

$$(f \textstyle\int_\alpha g)_{x\xi} = f(x)\, g(\xi)\, [\alpha \le \xi \le x] - f(x)\, g(\xi)\, [x \le \xi \le \alpha],$$

which will turn out to be the essential clause for extracting Green's functions of (well-posed) multi-point boundary problems. For seeing this, we need a more detailed description of the underlying Green's operators.

We turn first to the easy case of a one-point boundary problem, more appropriately known under the name of *initial value problems* $(T, [\unicode{x20AC}, \dots, \unicode{x20AC}\partial^{n-1}])$ for $T \in \mathcal{F}[\partial]$ of order n. The corresponding Green's operator is called the fundamental right inverse $T^\diamond$ and can be computed easily via the well-known "variation of constants" formula [16, Thm. 6.4]: If $u_1, \dots, u_n$ is a fundamental system for T with Wronskian matrix W, the fundamental right inverse is given by

$$T^\diamond = \sum_{j=1}^{n} u_j \textstyle\int \dfrac{d_j}{d}.$$

Here $d = \det(W)$ and $d_i = \det(W_i)$, where W_i denotes the matrix resulting from W when replacing the i-th column by the n-th unit vector of K^n.

What we shall need in the sequel is how $T^\diamond$ reacts to left multiplication by $\mathcal{F}[\partial]$.

Lemma 2. *Let $T \in \mathcal{F}[\partial]$ be any monic differential operator of order n, and choose a fundamental system $u_1, \dots, u_n$ for T with Wronskian matrix W. Then we have*

$$\partial^k T^\diamond = \sum_{j=1}^{n} u_j^{(k)} \textstyle\int \dfrac{d_j}{d} + \displaystyle\sum_{j=1}^{k} \partial^{k-j} \rho_j \tag{8}$$

$$\textit{with} \qquad \rho_k := \frac{1}{d} \sum_{j=1}^{n} u_j^{(k-1)} d_j \in \mathcal{F},$$

where d and d_i are as above.

Note that $\rho_1 = \dots = \rho_{n-1} = 0$ by the definition of the d_j; hence the second sum in $\partial^k T^\diamond$ is only present for $k \ge n$, and we may equivalently write its range as $j = n, \dots, k$. Furthermore, we have $\rho_n = 1$ from the definition of d. For $k > n$, however, the ρ_k are functions of $\mathcal{F}$, so in general they *do not commute* with the ∂^{k-j} in the second summand of (8).

Proof. We use induction on k. In the base case $k = 0$, this is the usual variation-of-constants formula as given in [5, p. 74]; see [20, Prop. 22] and [16, Thm. 6.4] for its operator formulation. Now assume (8) for fixed $k \geq 0$; we show it for $k + 1$. By the induction hypothesis we obtain

$$\partial^{k+1} T^{\diamond} = \sum_{j=1}^{n} u_j^{(k+1)} {\textstyle\int} \frac{d_j}{d} + \frac{1}{d} \sum_{j=1}^{n} u_j^{(k)} d_j + \sum_{j=1}^{k} \partial^{k-j+1} \rho_j,$$

which is just (8) for $k + 1$ since the middle sum is ρ_{k+1} and can be absorbed into the third. □

Lemma 3. *The Green's operator of any regular Stieltjes boundary problem is contained in $\mathcal{F}[\int_\Phi] + \mathcal{L}$, where $\mathcal{L}$ denotes the left $\mathcal{F}$-module generated by the local Stieltjes conditions.*

Proof. Assume $(T, \mathcal{B})$ is any regular Stieltjes boundary problem of order n with Green's operator G, and let P be the projector onto $\ker T$ along $\mathcal{B}^\perp$. By the proof of [20, Thm. 26] we have $G = (1 - P)T^\diamond$, and we know that P is an $\mathcal{F}$-linear combination of Stieltjes conditions by [20, (16)] in that same proof. From (8) it is clear that $T^\diamond \in \mathcal{F}[\int_\Phi]$, so it suffices to show $PT^\diamond \in \mathcal{F}[\int_\Phi] + \mathcal{L}$. Each summand of P is either of the form $f \lfloor \alpha \rfloor \, \partial^k$ or $f \lfloor \alpha \rfloor \int g = f \int_{€} g - f \int_\alpha g \in \mathcal{F}[\int_\Phi]$. In the latter case we obtain an expression in $\mathcal{F}[\int_\Phi]$ since $\mathcal{F}[\int_\Phi]$ is a (nonunital) subalgebra of $\mathcal{F}[\partial, \int_\Phi]$. It remains to prove $f \lfloor \alpha \rfloor \, \partial^k T^\diamond \in \mathcal{F}[\int_\Phi] + \mathcal{L}$. From (8) we see that

$$f \lfloor \alpha \rfloor \, \partial^k T^\diamond = \sum_{j=1}^{n} f u_j^{(k)}(\alpha) \int_{€}^{\alpha} \frac{d_j}{d} + \sum_{j=1}^{k} f \lfloor \alpha \rfloor \, \partial^{k-j} \rho_j.$$

The first sum is clearly contained in $\mathcal{F}[\int_\Phi]$, while the second is in $\mathcal{L}$ because $\partial^{k-j} \rho_j \in \mathcal{F}[\partial]$ may be rewritten in canonical form as a sum of terms $g_i \partial^i$ so that $\lfloor \alpha \rfloor \, \partial^{k-j} \rho_j$ is a sum of local conditions $\alpha(g_i) \lfloor \alpha \rfloor \, \partial^i$ and hence itself local. □

We are now ready to state the main *structure theorem for Green's functions* of regular Stieltjes boundary problems.

Theorem 4. *The Green's function of any regular Stieltjes boundary problem with m evaluations $\alpha_1, \dots, \alpha_m$ has the form $g(x, \xi) = \tilde{g}(x, \xi) + \hat{g}(x, \xi)$, where the functional part $\tilde{g} \in C(J^2)$ is defined by the $2(m-1)$ case branches*

$$\xi \in [\alpha_i, \alpha_{i+1}] \; (0 < i < m), x \leq \xi;$$
$$\xi \in [\alpha_i, \alpha_{i+1}] \; (0 < i < m), \xi \leq x,$$

while the distributional part $\hat{g}(x,\xi)$ *is an* $\mathcal{F}$*-linear combination of the* $\delta(\xi-\alpha_i)$ *and their derivatives.*

Proof. If G is the Green's operator of the given Stieltjes boundary problem, Lemma 3 says that $G = \tilde{G} + \hat{G}$ with $\tilde{G} \in \mathcal{F}[\textstyle\int_\Phi]$ and $\hat{G} \in \mathcal{L}$. We will show that $\tilde{g}(x,\xi) = \tilde{G}_{x\xi}$ and $\hat{g}(x,\xi) = \hat{G}_{x\xi}$ are as described in the theorem. Starting with the former, we may write

$$\tilde{G} = \sum_{i=1}^{r} f_i \textstyle\int_{\alpha_i} g_i,$$

where $\alpha_i = \alpha_j$ is possible for $i \neq j$. Using the transformation $\mathcal{F}[\partial, \textstyle\int_\Phi] \to \mathcal{G}$, we obtain $\tilde{g}(x,\xi)$ as

$$\begin{aligned}
&\sum_{i=1}^{r} \Big(f_i(x)\, g_i(\xi)\, [\alpha_i \le \xi][\xi \le x] - f_i(x)\, g_i(\xi)\, [\xi \le \alpha_i][x \le \xi] \Big) \\
&\qquad = \sum_{i=1}^{r} \Big(\sum_{\alpha_j \le \alpha_i} f_j(x)\, g_j(\xi) \Big) [\alpha_{j-1} \le \xi \le \alpha_j][\xi \le x] \\
&\qquad\quad - \sum_{i=1}^{r} \Big(\sum_{\alpha_j \ge \alpha_i} f_j(x)\, g_j(\xi) \Big) [\alpha_j \le \xi \le \alpha_{j+1}][x \le \xi],
\end{aligned}$$

where the two inner sums are restricted by $j > 0$ and $j < n$. Collecting terms, this is a sum of $2(m-1)$ characteristic functions over disjoint domains in $\mathbb{R}^2$, hence one may also write $\tilde{g}(x,\xi)$ in terms of a corresponding case distinction with $2(m-1)$ branches.

The distributional part $\hat{g}(x,\xi)$ is even easier. Writing $\hat{G}$ as an $\mathcal{F}$-linear combination of local conditions we obain $\hat{g}(x,\xi)$ via

$$\hat{G}_{x\xi} = \Big(\sum_{\alpha,i} f_{i,\alpha} \alpha \partial^i \Big)_{x\xi} = \sum_{\alpha,i} (-1)^i\, f_{i,\alpha}(x)\, \delta^{(i)}(\xi - \alpha),$$

which is clearly of the stated form. □

The above theorem is constructive, and we plan to implement the underlying algorithm on top of the `Maple` package `IntDiffOp` [12].

Remark 5. If the distinguished character $€ = \lfloor 0 \rfloor$ is not used in the boundary conditions, a straightforward translation of the Green's operator G may introduce two spurious extra case branches in the Green's function $G_{x\xi}$ since $\int_0^x$ occurs in the formula for G. For avoiding this, one has to use a different version of $T^\diamond$ that replaces $\lfloor 0 \rfloor$ by any one of the characters $\lfloor \alpha_i \rfloor$ used in the boundary conditions.

5. Examples

Our first example (in addition to the minimal one from the Introduction) is a four-point boundary value problem taken from [2, (2.1,2.2)], where we have specialized the parameters and rescaled the interval to $J = [0, 1]$ for the sake of simplicity. Hence we are dealing with the boundary problem

$$\boxed{\begin{array}{l} -u'' = f, \\ u(0) + u(1/3) = u(1) + u(2/3) = 0, \end{array}}$$

where we may assume $u, f \in C^\infty[-2, 2]$. Note that this is a well-posed boundary problem, so the Green's function will not have a distributional part. Computing the Green's operator with the `IntDiffOp` package yields after some rearrangements the result

$$\begin{aligned} G = {} & x{\textstyle\int} - {\textstyle\int} x + (-5/24 + x/4)\lfloor 1/3 \rfloor {\textstyle\int} \\ & + (5/8 - 3x/4)\lfloor 1/3 \rfloor {\textstyle\int} x + (1/8 - 3x/4)\lfloor 1 \rfloor {\textstyle\int} x \\ & + (1/12 - x/2)\lfloor 2/3 \rfloor {\textstyle\int} + (-1/8 + 3x/4)\lfloor 2/3 \rfloor {\textstyle\int} x \end{aligned}$$

Transforming G to equitable form is simple, via $\lfloor\alpha\rfloor\int \rightsquigarrow \int_0^x - \int_\alpha^x$. We can then determine the corresponding Green's function $g(x,\xi) = \tilde{g}(x,\xi)$ with 6 cases, and its terms may be computed according to Theorem 4. The result for $g(x,\xi)$ is summarized in the table below.

Case	Term
$0 \le \xi \le 1/3, \xi \le x$	$(3/4)x\xi - (5/8)\xi$
$0 \le \xi \le 1/3, x \le \xi$	$(3/4)x\xi + (3/8)\xi - x$
$1/3 \le \xi \le 2/3, \xi \le x$	$(3/2)x\xi - (5/4)\xi - (1/4)x + 5/24$
$1/3 \le \xi \le 2/3, x \le \xi$	$(3/2)x\xi - (1/4)\xi - (5/4)x + 5/24$
$2/3 \le \xi \le 1, \xi \le x$	$(3/4)x\xi - (9/8)\xi + (1/4)x + 1/8$
$2/3 \le \xi \le 1, x \le \xi$	$(3/4)x\xi - (1/8)\xi - (3/4)x + 1/8$

Our second example is, as it were, totally unclassical: It is ill-posed, has nonlocal conditions and contains three evaluation points $-1, 0, 1$. In our standard notation, we write this boundary problem as

$$\boxed{\begin{array}{l} u'' - u = f, \\ u'''(-1) - \int_0^1 u(\xi)\,\xi\,d\xi = 0, \\ u'(-1) - u''(1) + \int_{-1}^1 u(\xi)\,d\xi = 0, \end{array}}$$

where we assume now $u, f \in C^\infty[-1, 1]$. Using the method of [20], it is straightforward to compute the Green's operator G. In fact, the

`IntDiffOp` package yields the result

$$\begin{aligned}
\sigma\, G &= \sigma/2(e^x {\textstyle\int} e^{-x} - e^{-x} {\textstyle\int} e^x) \\
&+ 2(-e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1})(\lfloor -1 \rfloor \partial + \lfloor 1 \rfloor {\textstyle\int} x) \\
&+ (e-1)(-e^{x+2} - 2e^{x+1} + e^{-x+1})(\lfloor -1 \rfloor {\textstyle\int} + \lfloor 1 \rfloor {\textstyle\int}) \\
&+ (3e^{x+2} - e^{x+1} - 3e^{-x+1} + 3e^{-x}) \lfloor 1 \rfloor {\textstyle\int} e^x \\
&+ (2e^{x+2} - 3e^{x+1})(e^{-1} \lfloor -1 \rfloor {\textstyle\int} e^{-x} + e \lfloor -1 \rfloor {\textstyle\int} e^x) \\
&+ (-e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1}) \lfloor 1 \rfloor
\end{aligned}$$

using the abbreviation $\sigma := 2(2e-3)(e-1)$ while collecting and factoring some terms for enhanced readability. After transforming this to equitable form (which is again straightforward), we can apply Theorem 4 to extract the Green's function $g(x,\xi) = \tilde{g}(x,\xi) + \hat{g}(x,\xi)$ with the distributional part

$$\begin{aligned}
\sigma\, \hat{g}(x,\xi) &= (-e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1})\, \delta(\xi - 1) \\
&+ 2\,(-e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1})\, \delta'(\xi - 1)
\end{aligned}$$

coming from the $(\dots)\,\lfloor 1 \rfloor$ and $(\dots)\,\lfloor 1 \rfloor \partial$ terms, and with the functional part defined by the case distinction for $\sigma\, \tilde{g}(x,\xi)$ as given in the table below.

Case	Term
$-1 \le \xi \le 0, \xi \le x$	$3e^{x+2+\xi} + 3e^{x-\xi} - 2e^{x+1-\xi} - 2e^{3+x+\xi}$ $+e^{3+x} + e^{-x+1} + e^{x+2} - e^{-x+2} - 2e^{x+1}$
$-1 \le \xi \le 0, x \le \xi$	$-2e^{x+1} + 2e^{-x+2+\xi} - 5e^{-x+1+\xi} - 2e^{x+2-\xi}$ $-2e^{3+x+\xi} + 3e^{-x+\xi} + e^{-x+1} + e^{x+2}$ $+e^{3+x} + 3e^{x+1-\xi} + 3e^{x+2+\xi} - e^{-x+2}$
$0 \le \xi \le 1, \xi \le x$	$-2e^{3+x}\xi - 2e^{-x+1}\xi + 2e^{x+2}\xi + 2e^{-x+2}\xi$ $-2e^{x+1}\xi + 3e^{x+2+\xi} + 3e^{x-\xi} - 5e^{x+1-\xi}$ $+2e^{-x+1+\xi} - e^{x+1+\xi} - 2e^{-x+2+\xi} + 2e^{x+2-\xi}$ $-e^{3+x} - e^{-x+1} - e^{x+2} + e^{-x+2} + 2e^{x+1}$
$0 \le \xi \le 1, x \le \xi$	$-2e^{3+x}\xi - 2e^{-x+1}\xi + 2e^{x+2}\xi + 2e^{-x+2}\xi$ $-2e^{x+1}\xi + 3e^{-x+\xi} + 3e^{x+2+\xi} - e^{3+x}$ $-e^{-x+1} - e^{x+2} + e^{-x+2} + 2e^{x+1}$ $-3e^{-x+1+\xi} - e^{x+1+\xi}$

Incidentally, this example shows also that the *representation of Green's operators* in terms of Green's functions—despite its long tradition in engineering and physics—is not always the most useful and economical way of representing the Green's operator. For many purposes it is better to take the Green's operator just as an element of the operator ring $\mathcal{F}_\Phi[\partial, \int]$ or $\mathcal{F}[\partial, \int_\Phi]$.

6. Conclusion and Future Work

While we have focused our attention to semi-inhomogeneous boundary problems (those with an inhomogeneous differential equation and homogeneous boundary conditions), one may also consider the opposite case of semi-homogeneous boundary problems—this is especially important in the case of LPDEs. The corresponding semi-homogeneous Green's operator maps the prescribed boundary values to the solution [18]. In the case of LODEs, one often restricts attention to well-posed two-point boundary value problems (in the sense of Definition 1). Writing the two evaluations as $\lfloor a \rfloor$ and $\lfloor b \rfloor$ and their action on u as $u(a)$ and $u(b)$, one may consider the extended evaluation matrix

$$\lfloor a, b \rfloor(u) := \begin{pmatrix} u_1(a) & \cdots & u_n(a) \\ \vdots & \ddots & \vdots \\ u_1^{(n-1)}(a) & \cdots & u_n^{(n-1)}(a) \\ u_1(b) & \cdots & u_n(b) \\ \vdots & \ddots & \vdots \\ u_1^{(n-1)}(b) & \cdots & u_n^{(n-1)}(b) \end{pmatrix} \in K^{2n \times n}$$

which is similar to (7) except that it is rectangular since we consider more boundary functionals than we could possibly impose for one regular boundary problem. If we do prescribe all $2n$ boundary derivatives, they must satisfy n relations given by the kernel of the map $\mathbb{R}^{2n} \to \mathbb{R}^n, X \mapsto X \cdot \lfloor a, b \rfloor(u)$. For the simple example in Section 1, the extended evaluation matrix for the fundamental system $u_1 = 1, u_2 = x$ is

$$\lfloor 0, 1 \rfloor(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

whose kernel has basis $(-1, 1, 1, 0), (0, -1, 0, 1)$. Written in terms of the boundary functionals, they encode the two relations $u(1) - u(0) = u'(0)$ and $u'(0) = u'(1)$. The analogous case for LPDEs gives rise to the interesting notion of universal boundary problem [25].

There is another, more fundamental, way of extending the results in this paper: Currently our method for extracting Green's functions works for arbitrary Stieltjes boundary problems $(T, \mathcal{B})$, but only in the *standard integro-differential algebra* $\mathcal{F} = C^\infty(\mathbb{R})$. Of course, it requires a Green's operator $G \in \mathcal{F}_\Phi[\partial, \textstyle\int] \cong \mathcal{F}[\partial, \textstyle\int_\Phi]$ and hence a fundamental system for T.

It would be interesting to extend the concept of Green's function and the corresponding extraction method to arbitrary ordinary integro-differential algebras $(\mathcal{F}, \partial, \int)$. For the functional part $\tilde{g}(x, \xi)$, it is clear how to achieve this since one sees from the structure of Green's operators that necessarily $\tilde{g} \in \mathcal{F} \otimes \mathcal{F}$. The ring $\mathcal{F} \otimes \mathcal{F}$ has the structure of a partial integro-differential algebra with derivations and integrals

$$\begin{aligned} \partial_x(f \otimes g) &= (\partial f) \otimes g \quad \text{and} \quad \partial_y(f \otimes g) = f \otimes (\partial g), \\ \textstyle\int^x (f \otimes g) &= (\textstyle\int f) \otimes g \quad \text{and} \quad \textstyle\int^y (f \otimes g) = f \otimes (\textstyle\int g). \end{aligned}$$

This structure will be useful for studying various properties of Green's function, in particular their symmetry: For well-posed two-point boundary value problems $(T, \mathcal{B})$ it is known [22, §3.3] that the Green's function $g(x, \xi)$ is symmetric whenever $(T, \mathcal{B})$ is *self-adjoint*. Otherwise one may associate to $(T, \mathcal{B})$ an *adjoint boundary value problem* whose Green's function is then $g(\xi, x)$. It would be useful to know how these results generalizes to arbitrary Stieltjes boundary problems.

Having an abstract integro-differential algebras $(\mathcal{F}, \partial, \int)$, the other problem is Green's function will in general have a distributional part $\hat{g}(x, \xi)$ that does not fit into $\mathcal{F} \otimes \mathcal{F}$. For accommodating distributions into the setting of integro-differential algebras, it is probably necessary to construct a integro-differential module generated over $\mathcal{F} \otimes \mathcal{F}$ by a suitable notion of *abstract Dirac distributions*. (It is well known that distributions do not enjoy a convenient ring structure, hence it seems to be more reasonable to go for a module. This is also the path followed in the algebraic analysis of $\mathcal{D}$-modules; see [6, §6.1] for example.)

Acknowledgments

We would like to thank the referees for their diligent work and their crucial suggestions. In particular, the reference on the universal boundary value problem (see first paragraph of Section 6) is of considerable interest.

References

[1] R. Agarwal. *Boundary value problems for higher order differential equations*. World Scientific Publishing Co., Teaneck, NJ, 1986.

[2] Z. Bai, W. Ge, and Y. Wang. Multiplicity results for some second-order four-point boundary value problem. *Nonlinear Analysis*, 60:491–500, 2005.

[3] P. R. Beesack. On the Green's function of an N-point boundary value problem. *Pacific J. Math.*, 12:801–812, 1962.

[4] M. Benchohra, J. Henderson, R. Luca, and A. Ouahab. Boundary data smoothness for solutions of second order ordinary differential equations with integral boundary conditions. *Dynam. Systems Appl.*, 23(2-3):133–143, 2014.

[5] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[6] S. C. Coutinho. *A primer of algebraic D-modules*, volume 33 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[7] D. G. Duffy. *Green's functions with applications*. Studies in Advanced Mathematics. Chapman & Hall, Boca Raton, FL, 2001.

[8] H. W. Engl. *Integralgleichungen*. Springer Lehrbuch Mathematik. Springer-Verlag, Vienna, 1997.

[9] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems*, volume 375 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1996.

[10] A. S. Fokas. *A unified approach to boundary value problems*, volume 78 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[11] T. Jankowski. Differential equations with integral boundary conditions. *J. Comput. Appl. Math.*, 147(1):1–8, 2002.

[12] A. Korporal, G. Regensburger, and M. Rosenkranz. A Maple package for integro-differential operators and boundary problems, 2010. Also presented as a poster at ISSAC '10.

[13] W. S. Loud. Self-adjoint multi-point boundary value problems. *Pacific J. Math.*, 24:303–317, 1968.

[14] C. Pang, W. Dong, and Z. Wei. Green's function and positive solutions of nth order m-point boundary value problem. *Appl. Math. Comput.*, 182(2):1231–1239, 2006.

[15] G. Regensburger and M. Rosenkranz. An algebraic foundation for factoring linear boundary problems. *Ann. Mat. Pura Appl. (4)*, 188(1):123–151, 2009. DOI:10.1007/s10231-008-0068-3.

[16] G. Regensburger and M. Rosenkranz. Symbolic integral operators and boundary problems. Lecture Notes, available at `http://www.risc.jku.at/education/courses/ws2009/SioBp2/siobp.pdf`, 2009.

[17] M. Rosenkranz. A new symbolic method for solving linear two-point boundary value problems on the level of operators. *J. Symbolic Comput.*, 39(2):171–199, 2005.

[18] M. Rosenkranz and N. Phisanbut. A symbolic approach to boundary problems for linear partial differential equations. In V. Gerdt, W. Koepf, E. Mayr, and E. Vorozhtsov, editors, *Computer Algebra in Scientific Computing*, volume 8136 of *Lecture Notes in Computer Science*, pages 301–314. Springer, 2013.

[19] M. Rosenkranz and G. Regensburger. Integro-differential polynomials and operators. In D. Jeffrey, editor, *ISSAC'08: Proceedings of the 2008 International Symposium on Symbolic and Algebraic Computation*. ACM Press, 2008.

[20] M. Rosenkranz and G. Regensburger. Solving and factoring boundary problems for linear ordinary differential equations in differential algebras. *Journal of Symbolic Computation*, 43(8):515–544, 2008.

[21] M. Rosenkranz, G. Regensburger, L. Tec, and B. Buchberger. A symbolic framework for operations on linear boundary problems. In V. P. Gerdt, E. W.

Mayr, and E. H. Vorozhtsov, editors, *Computer Algebra in Scientific Computing. Proceedings of the 11th International Workshop (CASC 2009)*, volume 5743 of *LNCS*, pages 269–283, Berlin, 2009. Springer.

[22] I. Stakgold and M. Holst. *Green's functions and boundary value problems.* Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2011.

[23] J.-P. Sun and Q.-Y. Ren. Existence of solution for third-order m-point boundary value problem. *Applied Mathematics E-Notes*, 10:268–274, 2010.

[24] G. Teschl. *Ordinary differential equations and dynamical systems*, volume 140 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2012.

[25] I. Volovich and V. Sakbaev. Universal boundary value problem for equations of mathematical physics. *Proceedings of the Steklov Institute of Mathematics*, 285(1):56–80, 2014. Preprint version at `http://arxiv.org/abs/1312.4302`.