Rational Parameterization of Real Algebraic Surfaces

Josef Schicho*

Research Institute for Symbolic Computation
Johannes Kepler University
A-4040 Linz, Austria
Josef.Schicho@risc.uni-linz.ac.at
http://www.risc.uni-linz.ac.at/people/jschicho

Abstract

The parameterization problem asks for a real parameterization of an implicitly given real algebraic surface, in terms of rational functions in two variables. We give an algorithm for the parameterization of tubular surfaces. Also, it is shown that many instances of the parameterization problem can be reduced to the tubular case.

1 Introduction

Given an algebraic surface in terms of its equation, the parameterization problem asks for a parameterization with rational functions in two parameters. For instance, the unit sphere with equation $x^2 + y^2 + z^2 - 1 = 0$ has the parameterization

$$(x, y, z) = \left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1}, \frac{s^2+t^2-1}{s^2+t^2+1}\right).$$

Parameterizations allow to produce points on the surface easily. They are very popular in the form of in NURBS-(non-uniform rational B-spline) -represention, which are used for many applications in Computer Aided Design and Manufacture, such as reliable surface plotting and display, motion display (computing transformations), computing cutter offset surfaces, computing curvatures for shading and colouring, and many others (see also [5, 3, 17, 8, 11]).

Not every algebraic surface admits a parameterization. Those that have one are called unirational. A parameterization algorithm should decide whether a parameterization exists, and produce one in the affirmative case.

The complex theory is much better understood than the real theory. We have Castelnuovo's criterion $p_a = P_2 = 0$ (see [6, 27]) which is necessary and sufficient for unirationality. Moreover, any unirational surface allows a parameterization with the feature that we can express the parameters in terms of rational functions on the surface (proper parameterization).

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Algorithmically, the problem was solved by the author (see [19, 20]). We can decide Castelnuovo's criterion. If it is fulfilled, then we can produce a proper parameterization.

In the real case, Castelnuovo's criterion is necessary for unirationality. Sufficiency is still an open question. Also, there are unirational surfaces which do not have a proper parameterization (see [13, 23]).

The parameterization problem for real algebraic surfaces is still open algorithmically. There are partial algorithms that produce parameterizations for particular classes of surfaces, e.g. conicoids and cubicoids [1, 21] or canal surfaces [16]. In this paper, we give a parameterization algorithm for the class of tubular surfaces, which are surfaces with an equation of the form $A(t)x^2 + B(t)y^2 + C(t) = 0$.

Tubular surfaces are important because one can reduce many instances of the parameterization problem to the tubular case. Our result that tubular surfaces are unirational implies that any surface with a pencil of rational curves is unirational. This is a real analogon to a classical theorem [14], which is a cornerstone in the complex theory of algebraic surfaces. It also suggests some strategy to solve the parameterization problem over the reals completely (see remark 1).

Tubular surfaces have been investigated in another context, viewing them as quadratic forms over function fields. Hillgarter gave an algorithm that decides whether a given quadratic form $Ax^2 + By^2 + Cz^2$ over $k(t)$ is isotropic, and produces an isotropic vector in the affirmative case (see [9, 10]). His method works for arbitrary fields k in which we can perform field operations and factor univariate polynomials, e.g. $k = \mathbf{Q}$. In the case $k = \mathbf{R}$, the method presented here (lemma 5) is more efficient because it does not require to find zeroes of a zero-dimensional ideal in several variables, as it would be necessary in the general case.

A related problem is the parameterization problem for real algebraic curves. Here, the vanishing of the genus is a necessary and sufficient condition for unirationality (see [23]). Any unirational curve has a proper parameterization (Lüroth's theorem, see [26]). There are algorithms for genus computation [25], proper parameterization [18, 22], and for turning an arbitrary parameterization into a proper one [2].

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2 The Problem

Throughout, let $\mathbf{R}$ be a computable real closed field, for which we can solve univariate equations (representing complex solutions as pairs of reals), for instance the field of real algebraic numbers.

A set is called an algebraic surface iff it is the zero set of some prime ideal in some affine or projective space and it is two-dimensional (in the topological sense, with the usual topology). In most cases, we consider surfaces in 3-space, and the surface is the zero set of a real irreducible polynomial. (Note that irreducibility implies absolute irreducibility if the zero set is two-dimensional.)

The condition on two-dimensionality is hard to check algorithmically. Fortunately, we have a nice criterion.

Proposition 1 *Let F be a squarefree. The zero set of F is two-dimensional iff F has a smooth point (i.e. a point where $F = 0$ and $\mathrm{grad} F \neq 0$).*

Proof: See [12], theorem XI.3.6.

The parameterization problem is the following.

Input: A surface S, given by its equation.
Output: A rational map defined in some dense subset of $\mathbf{R}^2$, such that the image is a two-dimensional subset of S, if there exists one, `NotExist` otherwise.

A parameterization is called *proper* iff it is birational, i.e. iff we may express the parameters in terms of rational functions defined on a dense subset of the surface. Because the rational functions defining the inverse are defined almost everywhere, proper rational maps have dense image.

Surfaces with a parameterization are called *unirational*, surfaces with a proper parameterization are called *rational*.

3 Known Results for the Complex Case

Needless to say, the real case is the important case for all geometric applications. However, a close examination of the complex case is helpful for understanding the real case.

By a theorem of Castelnuovo [6], a surface is rational iff it is unirational iff the arithmetical genus p_a and the second plurigenus P_2 are both zero (see [23] for a definition of these notions).

Algorithmically, the situation is as follows. Given a complex surface, we can compute the numbers p_a and P_2, and therefore decide (uni)rationality. In the rational case, we can compute a proper parameterization [19, 20].

A great deal of the algorithm [20] uses only field arithmetic and can therefore be used for arbitrary computable fields (especially for $\mathbf{R}$). To explain what can be done for arbitrary fields, we have to introduce some concepts.

If k is a field, then $\overline{k}$ is the algebraic closure of k.

Suppose that we work over some ground field k. By abuse of language, a *generic element of* k is a new transcendental constant, i.e. $\tau \in k(\tau)$. Obviously, any two generic elements have the same properties, and so we commonly speak of *the* generic element.

The class of *Del Pezzo surfaces* is a class of surfaces of degree ≤ 9, which will be discussed later.

A closer examination of the algorithm [20] shows the following.

Theorem 1 *Given a surface S over an arbitrary field k, and a smooth point on S, one can compute the arithmetical genus p_a and second plurigenus P_2 of S. If both numbers are zero, then one can algorithmically do one of the following:*

1. *Construct a proper parameterization of S.*
2. *Construct a rational map $f : S \to k$, such that the generic fiber $f^{-1}(\tau)$, as a curve over $\overline{k(\tau)}$, is a rational curve.*
3. *Construct a birational map $f : S \to P$, where P is a Del Pezzo surface.*

Proof: See [20].

In case (2), we say that S has a pencil of rational curves. Actually, one can show that almost all fibers, considered as curves over $\overline{k}$, are rational curves.

In the complex case, one can show directly - i.e. without using Castelnuovo's criterion, and in a constructive way - that any surface with a pencil of rational curves is rational (see [14]), and that any Del Pezzo surface is rational (see [7]). These constructions are used in the parameterization algorithm [20].

4 Application to the Real Case

In the real case, Castelnuovo's criterion $p_a = P_2 = 0$ is necessary for unirationality, because $p_a = P_2 = 0$ is equivalent to the existence of a parameterization with complex coefficients. The numbers p_a and P_2 do not depend on the choice of the ground field. So, we first evaluate these numbers. If one of them is different from 0, we return `NotExist`. Otherwise, we can perform one of the constructions in theorem 1. We distinguish three cases.

Case (1) (we can construct a proper parameterization) is already finished.

Let us shortly discuss case (3), i.e. we can construct a birational map to a Del Pezzo surface. For a broader discussion, including precise definition, we refer to [13] or [19].

We can subdivide the class of Del Pezzo surfaces into two subclasses; call them type A and type B. (The usual terminology is "Del Pezzo surfaces of degree d, $2 \leq i \leq 9$" for type A and "Del Pezzo surfaces of degree 1" for type B. However, the meaning of the word "degree" is not the usual one.)

Del Pezzo surfaces of type A are unirational. There is a construction of a parameterization using only field operations, when a smooth point is provided (see [13, 19]). One can give examples which are not rational (see [13, 23]).

The question of unirationality of Del Pezzo surfaces of type B is an open problem. In order to facilitate attempts to solve it, we give a definition for Del Pezzo surfaces of type B: these are the surfaces defined by an equation $x^2 + y^3 + A(t)y + B(t) = 0$, where A is a polynomial of degree ≤ 4 and B is a polynomial of degree ≤ 6, such that the bivariate polynomial $y^3 + A(t)y + B(t)$ is irreducible.

Remark 1 The impact for solving this problem is the following (we anticipate our result for the remaining cases): A positive answer would imply that Castelnuovo's criterion is necessary and sufficient for unirationality in the real case. An algorithmic answer would imply the solution of the parameterization problem for the real case.

Let us discuss case (2), i.e. we can construct a rational map $f : S \to \mathbf{R}$, such that the generic fiber, as a curve over $\overline{\mathbf{R}(\tau)}$, is a rational curve.

The following theorem is a cornerstone in the theory of rational curves.

Theorem 2 *Let k be an arbitrary field. Let C be a rational curve over $\overline{k}$ which is defined over k. Then there is a birational map, defined over k, from C to an irreducible conic curve Q.*

Proof: See [14], [22], or [24], for algorithmic proofs.

We use theorem 2 to simplify our surface parameterization problem.

Lemma 1 *Let S be a surface with a pencil of rational curves. Then S is birationally equivalent to some surface T with equation $A(t)x^2 + B(t)y^2 + C(t) = 0$, where $A, B, C \in \mathbf{R}[t]$.*

Proof: Let $F(u, v, w) = 0$ be the equation of S. Let $P \in \mathbf{R}(u, v, w)$ be a defining rational function for the rational map $f : S \to \mathbf{R}$. The generic fiber $C := f^{-1}(\tau)$ is a curve over $\overline{\mathbf{R}(\tau)}$, but it is defined over $\mathbf{R}(\tau)$. By theorem 2, we have a birational map $g : C \to Q$, where Q is an irreducible conic. Without loss of generality, we may assume that Q is defined by an equation $A(\tau)x^2 + B(\tau)y^2 + C(\tau) = 0$, where $A, B, C \in \mathbf{R}[\tau]$ (this can be achieved by a projective transformation). Let T be the surface defined by $A(t)x^2 + B(t)y^2 + C(t) = 0$.

Suppose that g is given by the two rational functions $X, Y \in \mathbf{R}(\tau)(u, v, w)$, and g^{-1} is given by three rational functions $U, V, W \in \mathbf{R}(\tau)(x, y)$. Substituting τ by $P(u, v, w)$ in X, Y, we get a rational map from S to T. It has an inverse, which is obtained by substituting τ by t in U, V, W.

We call surfaces with an equation of the form $A(t)x^2 + B(t)y^2 + C(t) = 0$ *tubular*. They will be discussed in the next section.

5 Parameterization of Tubular Surfaces

In this section, we will parameterize "tubular surfaces", given by an irreducible equation $A(t)x^2+B(t)y^2+C(t) = 0$, where $A, B, C \in \mathbf{R}[t]$.

It is convenient to pass to the projective case, so that we have an equation $A(t)x^2 + B(t)y^2 + C(t)z^2 = 0$. The variable t has degree 0. The surface is the zero set of the above equation in $\mathbf{P}^2 \times \mathbf{R}$.

A tubular surface S with equation $A(t)x^2 + B(t)y^2 + C(t)z^2$ is called *normalized* iff A, B, C are relatively prime and squarefree.

Lemma 2 *Any tubular surface is birationally equivalent to a normalized one.*

Proof: Suppose that $A = A'D^2$. Then we set $x' = Dx$ and get a birationally equivalent surface with equation $A'x'^2 + By^2 + Cz^2 = 0$.

Suppose that $A = A'D$ and $B = B'D$. Since the equation is irreducible, we know that D and C are relatively prime. Then we set $x' = Dx$, $y' = Dy$, and get a birationally equivalent surface with equation $A'x'^2 + B'y'^2 + CDz^2 = 0$.

For any $t_0 \in \mathbf{R}$, we have a section S_{t_0} with equation $A(t_0)x^2 + B(t_0)y^2 + C(t_0)z^2 = 0$. It may be reducible or even empty. The generic section S_τ is a conic curve over $\overline{\mathbf{R}(\tau)}$.

Lemma 3 *If the generic section S_τ has a point defined over $\mathbf{R}(\tau)$, then S is rational.*

Proof: Because the equation of S is absolutely irreducible, the equation of S_τ is also absolutely irreducible. Thus, any given point is necessarily smooth. Using the point, we can parameterize properly: translate the point to $(0 : 1 : 0)$, the transformed equation is linear in y, and y can be eliminated with a rational function. This method requires only field arithmetic, hence the parameterization is defined over $\mathbf{R}(\tau)$.

Replacing τ by t in the parameterization of S_τ, we get a proper parameterization of S.

Remark 2 The converse of lemma 3 does not hold: The sphere $x^2 + y^2 + t^2 - 1 = 0$ is rational, but one can easily show that there is the generic section has no point defined over $\mathbf{R}(\tau)$.

For a tubular surface S, we define the *spine* $sp(S)$ as the set of all $t_0 \in \mathbf{R}$ with non-empty section. It is a union of intervals which are limited by zeroes of A, B, C.

Lemma 4 *Let S be a normalized tubular surface with full spine $sp(S) = \mathbf{R}$. Then the following hold.*

1. *Any section of a zero of A, B, or C, is a union of two lines.*
2. *The signs of the three numbers $lcoeff(A), lcoeff(B), lcoeff(C)$ are not all equal.*

Proof: Obvious.

Lemma 5 *Let S be a normalized tubular surface. The generic section S_τ has a point defined over $\mathbf{R}(\tau)$ iff T has full spine.*

Proof: Only if: Suppose that S_τ has a point $(X(\tau) : Y(\tau) : Z(\tau))$. Since we are in the projective case, we may assume that X, Y, Z are polynomials and they have no common factor. For any $t_0 \in \mathbf{R}$, the point $(X(t_0) : Y(t_0) : Z(t_0))$ is on the section, hence $t_0 \in sp(S)$.

If: Suppose that S has full spine. We define $d := \deg A + \deg B + \deg C$. We distinguish 3 cases.

Case 1: the parities of $\deg A, \deg B, \deg C$ are not all equal (i.e. one is even and two are odd or vice vera), and at most one of them is 0. We define

$$l := \lfloor \frac{\deg B + \deg C - 1}{2} \rfloor,$$

$$m := \lfloor \frac{\deg A + \deg C - 1}{2} \rfloor,$$

$$n := \lfloor \frac{\deg A + \deg B - 1}{2} \rfloor.$$

An easy calculation shows

$$2l + \deg A < d, 2m + \deg B < d, 2n + \deg C < d, \quad (1)$$

$$l + m + n = d - 2. \quad (2)$$

Let X, Y, Z be indeterminate polynomials of degree l, m, n, respectively. Let t_0 be a zero of A. By lemma 4, the quadratic form $B(t_0)y^2 + C(t_0)z^2$ splits into two linear factors; let $b_0y + c_0z$ be one of them. We demand that the polynomials X, Y, Z fulfill the condition

$$b_0Y(t_0) + c_0Z(t_0) = 0. \quad (3)$$

We impose the same demand for any zero of A, B, C.

Now, let z_0 and $\overline{z_0}$ be two conjugated complex zeroes of A. The complex quadratic form $B(z_0)y^2+C(z_0)z^2$ splits into two complex linear factors; let u_0y+v_0z be one of them. We demand that the polynomials X, Y, Z fulfill the conditions

$$u_0Y(z_0)+v_0Z(z_0)=\overline{u_0}Y(\overline{z_0})+\overline{v_0}Y(\overline{z_0})=0. \qquad (4)$$

Note that the two equations are conjugate, so they are equivalent to a system of two equations with real coefficients. - We impose the same demand for any pairs of conjugated complex zeroes of A, B, C.

Collecting all these demands, we have a system of d linear equations for $l+m+n+3$ unknowns, the indeterminate coefficients of X, Y, Z. Since $l+m+n+3=d+1$ by (2), there is a nontrivial solution. We set X, Y, Z to be such a solution.

Let $P:=AX^2+BY^2+CZ^2\in\mathbf{R}[t]$. By (1), P is a polynomial of degree $<d$. It has at least d different complex zeroes, namely all zeroes of A, B, C. Thus, $P=0$, and $(X(\tau):Y(\tau):Z(\tau))$ is a point on the generic section defined over $\mathbf{R}(\tau)$.

Case 2: the parities of $\deg A, \deg B, \deg C$ are all equal (i.e. all are even or all are odd), and not all three are 0. W.l.o.g., we suppose that $\deg A>0$. Also w.l.o.g., we suppose that $\mathit{lcoeff}(A)>0$. By lemma 4, one of the other two leading coefficients must be negative. W.l.o.g., suppose that $\mathit{lcoeff}(B)<0$. We define

$$l:=\frac{\deg B+\deg C}{2},$$

$$m:=\frac{\deg A+\deg C}{2},$$

$$n:=\frac{\deg A+\deg B}{2}-1.$$

Let X, Y, Z be indeterminate polynomials of degree l, m, n, respectively. As in case 1, we obtain d linear equations for the $l+m+n+3=d+2$ coefficients of X, Y, Z. One additional linear equation is obtained in the following way: let x_l be the coefficient of t^l in X, and let y_m be the coefficient of t^m in Y. Then the quadratic form $\mathit{lcoeff}(A)x_l^2+\mathit{lcoeff}(B)y_m^2$ splits into two linear factors, and we take one of them.

The polynomial $P:=AX^2+BY^2+CZ^2$ has degree $\le d$, but the coefficient of t^d vanishes because of the last equation, so it has degree $<d$. As in case 1, it has d different zeroes, and therefore vanishes. Hence $(X(\tau):Y(\tau):Z(\tau))$ is a point on the generic section defined over $\mathbf{R}(\tau)$.

Case 3: Two of A, B, C are constant and the third has odd degree. Suppose that A, B are constant. Since the spine is full, A and B do not have the same sign. W.l.o.g., assume $A=1, B=-1$. Let

$$X:=\frac{1-C}{2}, Y:=\frac{1+C}{2}, Z:=1.$$

Then $(X(\tau):Y(\tau):Z(\tau))$ is a point on the generic section defined over $\mathbf{R}(\tau)$.

Case 4: A, B, C are constant. Trivial.

Lemma 6 *Let S be a tubular surface. Let $[a,b]$ be an interval which is contained in $sp(S)$. Then there exists a normalized tubular surface S' with full spline, and a rational map $f:S'\to S$, such that the image $f(S')$ is dense in the slice set $S_{[a,b]}:=\{(p,t)\in S\mid t\in[a,b]\}$.*

Proof: Let $F(t'):=a+\frac{b-a}{t'^2+1}\in\mathbf{R}(t')$. Take the equation

$$A(F(t'))x^2+B(F(t'))y^2+C(F(t'))z^2=0,$$

and clear the denominators. The result defines a tubular surface S'. For any t'_0, the section $S'_{t'_0}$ is equal to $S_{F(t'_0)}$, which is non-empty because $F(t'_0)\in[a,b]$. Hence S' has full spine.

The rational map f is defined by $(p,t')\mapsto(p,F(t'))$.

Now, S' may not be normalized. But normalization is birational and preserves the spine up to isolated points, hence we may normalize without changing the desired property.

Theorem 3 *Any tubular surface is unirational.*

Proof: Let S be a tubular surface. If the spine would be empty or finite, then S would be empty or a finite union of curves, and not a surface. Therefore, we have an interval $[a,b]\in sp(S)$.

By lemma 6, there exists a normalized tubular surface S' with full spine and a rational map $f:S'\to S$ with two-dimensional image. By lemma 5, the generic section of S' has a point defined over $\mathbf{R}(\tau)$. By lemma 3, there exists a proper parameterization $g:\mathbf{R}^2\to S'$. Because proper parameterizations have dense image, the composed map $f\circ g:\mathbf{R}^2\to S$ has two-dimensional image, hence is a parameterization.

Remark 3 A necessary criterion for a surface to be rational is that its projectivization has exactly one connected component of dimension 2. One can show that the number of connected components of dimension 2 of (the projectivization of) a tubular surface S is equal to the number of intervals in $sp(S)$, where $[-\infty,a]$ and $[b,\infty]$ count as one "interval" $[b,a]$ (including $\pm\infty$) whenever $a<b$. Using this, one can easily construct non-rational tubular surfaces with arbitrary many components.

However, one can show that any of the connected components is the image of a suitable parameterization (apply the construction in the proof of theorem 3 with a carefully chosen interval).

Remark 4 In order to be able to make the interval as large as possible, we also give functions rational F with image containing infinite "intervals". If $a=-\infty$, we can choose $F(t')=-t'^2+b$. If $b=\infty$, we can choose $F(t')=t'^2+a$. If $a>b$ (i.e. the "interval" contains $\pm\infty$ as an inner point), we can choose $F(t')=a+\frac{a-b}{t^2-1}$.

Theorem 4 *Any surface with a pencil of rational curves is unirational.*

Proof: By lemma 1 and theorem 3.

6 The Parameterization Algorithm

The proofs we gave are constructive, i.e. they implicitly contain an algorithm. Here is an explicit version.

Algorithm *Parameterize.*
Input: A surface S.
A rational map $f:S\to\mathbf{R}$, such that the generic fiber is a rational curve.
Output: A parameterization of S.

Parameterize(S, f):
$C :=$ the generic fiber of f;
Compute a birational map $g : Q \to C$, where Q is an irreducible conic;
Transform Q to the form
$A(\tau)x^2 + B(\tau)y^2 + C(\tau)z^2 = 0$;
[The map g has to be updated, too.]
$T :=$ the tubular surface defined by
$(A(t)x^2 + B(t)y^2 + C(t)z^2)$;
$h := ParameterizeTubular(T)$;
Substitute $\tau \leftarrow t$ in g;
[The result is a birational map $g : T \to S$.]
return $g \circ h : \mathbf{R}^2 \to S$.

Algorithm *ParameterizeTubular.*
Input: A tubular surface S.
Output: A parameterization of S.

ParameterizeTubular(S):
Compute the spine $sp(S)$;
$[a, b] :=$ an interval contained in $sp(S)$;
$F(t') := a + \frac{b-a}{t'^2+1}$;
Substitute $t \leftarrow F(t')$ in the equation of S and clear denominators;
$S' :=$ the tubular surface defined by the result;
Normalize S';
$S'_\tau :=$ the generic section of S';
$p_\tau := FindPoint(S'_\tau)$;
Translate p_τ to $(0 : 1 : 0)$;
[Now, the equation of S'_τ is linear in y.]
Eliminate y and set $(x : z) := (s : 1)$;
$g :=$ the resulting parameterization of S'_τ;
Transform g back along the translation map;
Transform g back along the normalization map;
Substitute $\tau \leftarrow t'$ in g;
return $(g, F(t'))$.

Algorithm *FindPoint*:
Input: A conic S_τ defined over $\mathbf{R}(\tau)$.
S_τ has to be the generic section of a normalized tubular surface with full spine.
Output: A point on S_τ defined over $\mathbf{R}(\tau)$.

Findpoint(S_τ):
Let $A, B, C \in \mathbf{R}[t]$ be the coefficients of x^2, y^2, z^2;
if (among A, B, C, we have two constants with different sign) then
[We assume $A > 0$ and $B < 0$ are constant, the other case are treated analogously.]
Apply a linear transformation to achieve $A = 1, B = -1$;
return $(\frac{1-C(\tau)}{2} : \frac{1+C(\tau)}{2} : 1)$ and exit.
Define l, m, n as in the proof of lemma 5;
$X, Y, Z :=$ indeterminate polynomials of degree l, m, n with coefficients $x_l, \ldots, x_0, y_m, \ldots, y_0, z_n, \ldots, z_0$;
, := the empty list;
for each real zero t_0 of A, B, C do
[We assume $A(t_0) = 0$, the other cases are are treated analogously.]
Factor $(B(t_0)y^2 + C(t_0)z^2)$;
Let $b_0 y + c_0 z$ be a linear factor;
Add $b_0 Y + c_0 Z = 0$ to , ;
for each pair $(z_0, \overline{z_0})$ of complex zeroes of A, B, C do
[We assume $A(z_0) = 0$, the other cases are are treated analogously.]
Factor $(B(z_0)y^2 + C(z_0)z^2)$ over $\mathbf{C}$;
Let $u_0 y + v_0 z$ be a linear factor;
$R :=$ the real part of $b_0 Y + c_0 Z = 0$;
$I :=$ the imaginary part of $b_0 Y + c_0 Z = 0$;
Add $R = 0$, $I = 0$ to , ;
if the parities of $\deg A, \deg B, \deg C$ are equal then
Factor $(\mathit{lcoeff}(A)x_l^2 + \mathit{lcoeff}(B)y_m^2)$;
Let L be a linear factor;
Add $L = 0$ to , ;
Compute a nontrivial solution of , ;
[Now, X, Y, Z are determined.]
return $(X\tau) : Y(\tau) : Z(\tau))$.

The correctness of the algorithms follows from the fact that they exactly perform the constructions in the proofs of lemma 1 (algorithm *Parameterize*, theorem 3 (algorithm *ParameterizeTubular*), and lemma 5 (algorithm *FindPoint*).

Example 1 Let S be the cubic surface with equation

$$x^2 + y^2 - z^3 - 2z^2 - 3z = 0.$$

The projection to the z-axis is a rational map with generic fiber a conic, with equation

$$x^2 + y^2 + (-\tau^3 - 2\tau^2 - 3\tau) = 0.$$

Since Q already has the desired diagonal form, the transformation step is void. The equation of T, already homogenized, is

$$x^2 + y^2 + (-t^3 - 2t^2 - 3t)z^2 = 0.$$

This is our input for the algorithm *ParameterizeTubular.*

The spine is $[-1, 0] \cup [1, \infty]$. We choose the infinite interval $[1, \infty]$ and get $F(t') = t'^2 + 1$. The equation of T' is

$$x^2 + y^2 + (-t'^6 - 5t'^4 - 4t'^2)z^2 = 0.$$

We normalize by introducing $z' = t'z$ and get the normalized tubular surface T'' with equation

$$x^2 + y^2 + (-t'^4 - 5t'^2 - 4)z'^2 = 0.$$

This is our input for the algorithm *FindPoint.*

The degrees of the coefficient polynomials A, B, C are even, and $\deg C > 0$. Hence, we choose $l := \frac{\deg B + \deg C}{2} = 2$, $m := \frac{\deg B + \deg C}{2} - 1 = 1$, $n := \frac{\deg B + \deg C}{2} = 0$, which yields

$$X := x_0 + x_1 t' + x_2 t'^2, Y := y_0 + y_1 t', Z := z_0,$$

$$P := X^2 + Y^2 + (-t'^4 - 5t'^2 - 4)Z^2.$$

The zeroes of C are $\pm i$, $\pm 2i$, where $i = \sqrt{-1}$. We evaluate P at i and factorize:

$$P(i) = (x_0 + x_1 i - x_2)^2 + (y_0 + y_1 i)^2 =$$

$$(x_0 + x_1 i - x_2 + y_0 i - y_1)(x_0 + x_1 i - x_2 - y_0 i + y_1).$$

The choice of the first factor yields the two linear equations

$$x_0 - x_2 - y_1 = 0, x_1 + y_0 = 0.$$

Now, we evaluate P at $2i$ and factorize:

$$P(2i) = (x_0 + 2x_1 i - 4x_2)^2 + (y_0 + 2y_1 i)^2 =$$

$$(x_0 + 2x_1 i - 4x_2 + y_0 i - 2y_1)(x_0 + 2x_1 i - 4x_2 - y_0 i + 2y_1).$$

The choice of the first factor yields the two linear equations

$$x_0 - 4x_2 - 2y_1 = 0, 2x_1 + y_0 = 0.$$

Finally, we factor the leading coefficient of P:

$$x_2^2 - z_0^2 = (x_2 + z_0)(x_2 - z_0).$$

The choice of the first factor yields the linear equation

$$x_2 + z_0 = 0.$$

This system of linear equations has the nontrivial solution

$$x_0 = 2, x_1 = 0, x_2 = -1, y_0 = 0, y_1 = 3, z_0 = 1.$$

Thus, we have a point $(2-t'^2 : 3t' : 1)$ on the generic section of T''. Its transform on T' is

$$(x : y : z) = (2t' - t'^3 : 3t'^2 : 1).$$

By translating this point to $(0 : 1 : 0)$, elimination of the second coordinate, and translating back, we obtain the parameterization

$$x = -t'^3 + 2t' + \frac{2t'^3 - 4t' - 6t'^2 s}{1 + s^2},$$

$$y = 3t'^2 + \frac{(2t'^3 - 4t' - 6t'^2 s)s}{1 + s^2},$$

$$z = 1.$$

We dehomogenize and add the component $F(t')$, which yields the parameterization

$$x = -t'^3 + 2t' + \frac{2t'^3 - 4t' - 6t'^2 s}{1 + s^2},$$

$$y = 3t'^2 + \frac{(2t'^3 - 4t' - 6t'^2 s)s}{1 + s^2},$$

$$t = t'^2 + 1$$

of T'. Replacing t by z, we get a parameterization of the given cubic surface.

Note that the projectivization of this surface has two smooth connected components (see [15]). Because the number of connected components is a birational invariant for smooth projective surfaces (see [4]), the surface is not rational, i.e. there exists no proper parameterization.

Let us assume that field elements are represented with constant lenghth, and that we can perform field operations in constant time and solve univariate algebraic equations in polynomial time of the degree. Then it is easy to check that the algorithm *ParameterizeTubular* is polynomial. But when we do exact arithmetic, then this assumption is unrealistic (especially when algebraic numbers are involved). Therefore, it is important to analyze the bit complexity of the algorithm. This will be a topic of future research.

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