Solving boundary-value problems with perturbations

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ABSTRACT

The program *Macsyma*[1] is used to solve boundary- and initial-value problems for the differential equation $\ddot{y} = f(x)y$. The solution is expanded into a power series and then solved by integration and equating coefficients.

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1. Introduction

The method to solve the differential equation $\ddot{y} = f(x)y$ described herein gives analytical expressions for the terms of a power series. The goal is to write a procedure in *Macsyma* to solve equations with the method given. The coefficients of the power series are obtained by symbolic integration.

2. Description of Problem

Let there be given the following differential equation:

$$\ddot{y} = f(x)y \tag{2-1}$$

(y is the *dependent* variable, x is the *independent* variable. The equation is *linear)* with initial values

$$y(0) = 1,\ \dot{y}(0) = 1 \tag{2-2}$$

or the boundary values

$$y(0) = 1,\ y(A) = 1 \tag{2-3}$$

where $A>0$.

The equation 2-1 is solved in the following four steps:

1) Introduction of a (small) parameter c into 2-1:

$$\ddot{y} = cf(x)y \tag{2-4}$$

2) Power series expansion of the solution $y(x)$ of 2-4

$$y(x) = \sum_{n=0}^{\infty} c^n y_n(x) \tag{2-5}$$

3) The functions $y_n(x)$ are determined by insertion of 2-5 in 2-4 and equating coefficients of the powers of c. We get:

$$y_0(x) = a_0 x + b_0$$
$$y_n(x) = \iint f(x) y_{n-1}(x) d^2x + a_n x + b_n, \quad n>0 \tag{2-6}$$

The coefficients a_n, b_n are determined from the conditions 2-2 or 2-3.

4) The solution of the original problem 2-1 is obtained from 2-4 with $c = 1$.

3. Problems to solve

3.1. Basic problems

1) Determine the equations for the coefficients a_n, b_n for both problem 2-2 and 2-3.
2) Write a procedure in MACSYMA with parameters f and y_n to compute y_{n+1}.
3) For the function $f(x) = -e^{-x}$ and $A = 6$ compute the functions $y_0, \ldots, y_3$ and the approximations s_N of the original problem:

$$s_N = \sum_{m=0}^{N} y_m(x) \tag{3-1}$$

for both conditions 2-2 and 2-3.

4) Solve problem 2-2 for $f(x) = -1$ up to N=6. Solve 2-1 by hand and compare the results.

3.2. Optional problems

5) In what sense is 3-1 an approximation of the correct solution of 2-4 even for $c = 1$?

6) Give a function $f(x)$, where this method breaks down.

4. Hints

Functions in *Macsyma* are defined with ":=", e.g.

```
f(x) := -%e^-x;
```

You can also define *arrays* of functions:

```
y[0](x) := x+1;
y[i](x) := ''proc2a(f, y[i-1]);
```

which is a recursive definition suitable for problem 3), where *proc2a* is your procedure written in problem 2) (the construct '' forces evaluation of the expression. Function definitions are normally not evaluated).

Remember: *If all else fails, read the documentation*[3]*!*

References

1. Symbolics Inc., *An Introduction to MACSYMA*, Symbolics Inc., Massachusetts, 1984.

2. Roman E. Maeder, "Eqn- and ms-Macros for the Mathematical Laboratory," The Mathematical Laboratory No. 001-901, Mathematik ETH, Zurich, 1986.

3. Symbolics Inc., *MACSYMA Reference Manual Version 10*, I, Symbolics Inc., Massachusetts, 1984.