A Method of Trading Diameter for Reduced Degree to Construct Low Cost Interconnection Networks*

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Abstract

Classical network topology has identified many "bottom-up" approaches to designing low cost interconnection networks. The foremost figure of merit is considered to be cost, the product of degree and diameter. We argue that average cost, being the product of average degree and diameter, is more applicable than the conventional cost and propose a general method of "top-down" network construction that provides networks of average cost $\Theta(\log_B N)$ where $B = o(N)$ is a bound on maximum degree and N is the number of nodes. From this we show an example topology that has average cost of $\Theta\left(\frac{\log N}{\log\log N}\right)$. By doing so we examine a class of networks with constant average cost, that is, networks whose average cost is fixed to a constant value as the size of the network increases to infinity. We then identify those aspects that are undesirable, namely some nodes have infinite degree, and show methods of trading an increase in diameter for a reduction in degree.

1 Introduction

Interconnection network topology is a fundamental research topic in the area of parallel and distributed computing. Continual operation of efficient and effective communication between processing elements or *nodes* is a desirable characteristic of all interconnection networks. The foremost figure of merit used to identify networks with efficient communication properties is considered to be cost[16, 15, 9, 5], $c = dk$, defined as the product of degree (d) and diameter (k) and usually stated in terms of n the number of nodes in the network, $c = f(n)$, to facilitate comparison. The degree of a network is the maximum of the number of edges connected to any single node. The diameter of a network is the maximum of the minimum number of edges needed to be traversed to get from any node to any other node. It is desirable from a manufacturing point of view to have a small degree since routers need to be built that can handle the degree of the network and the monetary cost may be quadratically related to the router degree. But small degree typically necessitates a large diameter and compromise is inevitable. Furthermore, other desirable characteristics such as *scalability*[4, 11], *routing*[2], *bi-section width*[8] and *fault-tolerance*[3, 1] may impose further compromise[10, 12, 14].

Our previous research provided a class of networks based on the Υ network[7, 6]. We utilised the concept of *average cost* when comparing the Υ networks since it was seen that the conventional cost metric did not properly exhibit the merits of the Υ structure. The average cost is defined as the product of *average degree* and diameter. Here we establish a firmer understanding of cost and its relationship to average cost. We note that the degree of a network is related to the number of edges. This is because each edge that is added to a network increases the degree of two nodes by one. That is, one may talk about total degree, δ, being the sum of the degree, d_i, for each node,

$$\delta = \sum_{i}^{n} d_i = 2e \qquad (1)$$

where e is the number of edges. The average degree then becomes available: $\bar{d} = \delta/n$.

The average degree seems to provide a better

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quantity for comparison based on the following grounds:

1. the average degree takes into account the number of edges and

2. the physical construction cost of routers has a greater potential for decline than the wiring costs.

In 1 we note that the *maximum* degree, d, does not embody the real number of edges, e, used to construct the network. We can provide many examples, and indeed the Υ network is one, where the maximum degree simply fails to provide a valid quantity of measure. In 2 we appeal to the prediction that the cost of routers will generally decrease faster than the cost of wiring (edges). This has the effect of meaning that to some limit, the cost of routers with degree less than that limit, will be identical or at least related by a constant factor. Advances in optical network construction may well produce such a router that no longer needs electronic switching or physical guides for the signal.

So we are now in a position to claim that the *average cost*, $\bar{c} = \bar{d}k$, is a suitable quantity of measure for the merit of a network topology.

The difference between maximum cost and average cost of some popular networks is actually zero. This is so when the average degree is equal to the maximum degree. Such a case only occurs when the degree of every node in the network is the same. We will refer to networks exhibiting this property as *regular* networks. Almost every popular network is regular: ring, torus, hypercube and CCC[13] for instance. The fat-tree is not regular since the degree of nodes on lower levels decreases. Regularity also tends to decrease the physical cost since every router in the network is identical and thus easier to manufacture. Unfortunately, as will be exemplified later, it seems that regular networks will never actually provide low cost in the average sense. We will show with simple examples that low cost networks can be produced with the tradeoff being the acceptance of non-regular topologies.

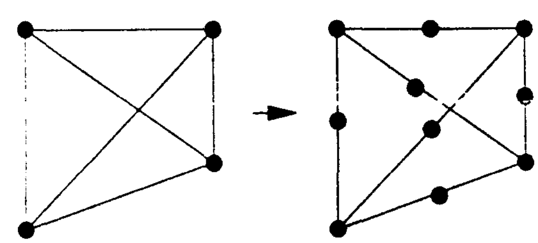
A clear criticism is that some networks may exist with infinite maximum degree but finite average degree. We address this criticism in the first half this paper by showing simple examples of this phenomenon and a way of countering the criticism by stipulating a bound to the maximum degree in terms of the network size.

2 A class of constant average cost networks

We now present an example of networks that provide fixed average cost with an infinite number of nodes. The existence of such networks as purely theoretical entities shows an interesting aspect of network topology and illuminates a method of network comparison that seems under-treated in popular literature.

First consider one of the simplest of networks, the star networks. The average degree of a star network $\bar{d} = \frac{(n-1)+(n-1)}{n}$ multiplied by the diameter, 2, produces an average cost $\bar{c} = \left(4 - \frac{4}{n}\right)$ which approaches 4 as $n \to \infty$. The maximum cost approaches infinity though because the maximum degree is equal to $(n-1)$. The simple example seems to illustrate a short fall when considering average cost as a function of average degree since the star network may immediately be claimed to be near the best in terms of $\bar{c}$ when compared to for instance some more elaborate networks discussed in our previous work. To combat this criticism we ask for the examination of the maximum degree in terms of the number of nodes. The comparison will become clearer with the following example.

Consider the completely connected network, every node is connected to every other node. Before looking at the average cost we transform the network to produce a new network, C, by replacing every edge with a node of degree 2. Figure 1 shows the formation of $C(4)$ network from a 4 node completely connected network.



(a) Completely connected graph (b) C(4) graph

Figure 1: A completely connected network transformed to a C network.

The average degree of C $\bar{d} = \frac{2n(n-1)+2n(n-1)}{2n+n(n-1)}$ multiplied by the diameter, 4, gives $\bar{c} = 4\frac{4(n-1)}{n+1}$ which approaches 16 as $n \to \infty$. Note that the number of nodes is now $N = \frac{n^2+n}{2}$. The maximum degree as a function of the number nodes is no longer $(N-1)$ as it was for the star network but now $d = \epsilon\sqrt{2N} - 1$, where it can be shown through an approximate ($16N^2 \gg 2N$) polynomial solution that $\left(1 - \frac{\sqrt{2N}}{4N}\right) < \epsilon < 1$. This indicates that N for the C network increases proportionally larger than the corresponding N for the star network with increase in maximum degree. In other words a C network with maximum degree d has more nodes than a star network of the same maximum degree. This clearly identifies a statement of comparison in terms of a function that indicates the maximum degree in terms of number of nodes. The maximum degree is not always a function of the number of nodes though. For the hypercube network it is, since the number of nodes dictates the exact size of the network. For the CCC network it is not since the size may vary but the maximum degree remains constant. These things aside, the hypercube and CCC networks do not exhibit finite average cost when the number of nodes approaches infinity.

Let us continue our discussion by modifying the C network. Replace each edge of a completely connected network with not just a single node of degree 2 but a new network, namely the $\Upsilon(2^t)$ network, to form a new C network. The Υ network has average degree 4 with diameter t. It also has the property that the two outer nodes, with degree $2t-1$, are directly connected and that from any node inside the network it takes only $\frac{t}{2}$ steps to get to either of these outer nodes. Figure 2 shows the construction of this network from a completely connected network of size 4. Note that the previous C network is basically an instance of the new C network with $t = 0$.

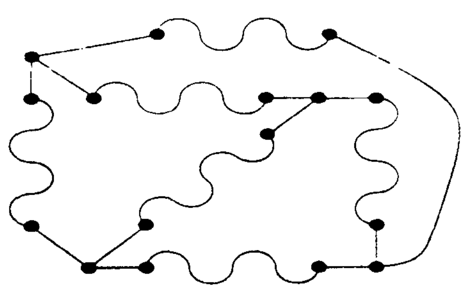


Figure 2: A completely connected network transformed to a C network using $\frac{n(n-1)}{2}$ $\Upsilon(2^t)$ networks.

The average degree of this new C network is found by remembering that the 2^t nodes in each of the $\frac{n(n-1)}{2}$ Υ networks so introduced have average degree 4. The wavy lines represent Υ networks. The remaining n nodes have average degree $(n-1)$. The combined average

$$\bar{d} = \frac{2^t n(n-1)4 + n(n-1)2}{2^t n(n-1) + 2n} \approx 4 \tag{2}$$

is seen to remain at approximately the average of the Υ network. The diameter of C is seen to be no larger than $\frac{t}{2}+1+3+1+\frac{t}{2} = t+5$ and so the average cost is $\bar{c} = 4(t+5)$ which is partially independent of the new number of nodes $N = \frac{n(n-1)}{2}2^t + n$. We now examine the maximum degree as a function of the number of nodes. The maximum degree of this network is seen to be $\max(n-1, 2t)$. If $2t = n-1$ then $d \approx 2\log\left(\frac{N}{n^2}\right) = \bigcirc(\log N)$.

These previous networks provide examples of reducing the maximum degree while attempting to maintain as small as diameter as possible. In general we establish that the average cost is applicable *only if the maximum degree is bounded by some $B = f(N)$ such that B approaches 0 as N approaches infinity.* Also we ensconce that for the case of average cost analysis *only networks with irregular degree topology will provide minimum costs.* If the maximum degree of any network is beyond the bound B then the network is deemed to be ***unrealisable*** or ***high cost.*** An actual order of bound as found in the literature to be acceptable as exhibited by the hypercube and other networks is $B = \Theta(\log N)$. The next logical step is to attempt to reduce the maximum degree to a value less than or equal to B, by sacrificing either size or diameter or both and thus create a usable (***realisable***) network with low cost.

3 General construction of large networks with low average cost

In this section we explain a method of reducing the degree of an arbitrary network by replacing those nodes with degree above some bound with another network. The method is most applicable when applied to those networks that have a constant diameter but suffer from large maximum degree. The idea is to construct a final network from a set of networks with decreasingly smaller degree. If the diameter of these networks is constant then the resulting diameter should be quite small (possibly also constant) but will depend upon the rate at which the maximum degree can be reduced to a bounding value and the corresponding rate at which the diameter increases. This mathematical concept is briefly analysed at the end of the section.

Consider a large network, index by a, of size N_a nodes, $G_a(N_a)$, constructed using some method yielding fixed finite diameter, k_a, and average degree, $\bar{d}_a$. We will use d_a to represent the maximum degree of G_a and $d_{a,i}$ to represent the degree of node i in G_a. We also define a quantity α_a to represent the number of nodes in G_a with degree larger than B, where B is some bound in terms of N_a, above which the degree is deemed high cost. The other important quantities we need to define are the total degree (refer to Equation 1), $\delta_a = \sum_i^{N_a} d_{a,i}$ and a new quantity that we will refer to as *saturated degree* $\varsigma_a = d_a N_a$.

The difference between these values provides us with a *capacity*,

$$\kappa_a = \varsigma_a - \delta_a \tag{3}$$

Let us assume the problem with G_a is that the number of nodes with degree greater than the bounding value, B, is greater than 0 or that $\alpha_a > 0$.

This means that G_a is high in cost and cannot be built. We would like $\alpha_a = 0$. We need to reduce the degree of α_a nodes in G_a to be less than B. Note that some nodes in G_a may have degree less than d_a and greater than B. To reduce the degree of the α_a nodes we will use the technique of replacing each of these nodes with a new network or *super node*. The super node must be constructed of nodes of maximum degree less than the node it is replacing otherwise the replacement defeats itself. In other words, if G_x is the network that replaces a node of degree d_a, then it must be true that

$$d_x < d_a \tag{4}$$

Also, a network cannot be used as a super node if the capacity, κ, of the network is not equal to or larger than the degree of the node being replaced. This is because if the capacity of the network is not larger than the degree of the node it is replacing the maximum degree of the network will need to increase and this increase may then violate the previous condition concerning maximum degree. In other words, if G_x is the network that replaces a node of degree d_a, then it must be true that

$$\kappa_x \geq d_a \tag{5}$$

If these two conditions (4 and 5) are true then the network G_x can become a super node within the network G_a. We now construct a new network, G_b, by replacing all α_a nodes in G_a with an identical copy of a valid super node, namely G_x. To simplify our consideration we make the following assumptions:

- The number of nodes in both G_a and G_x that have maximum degree d_a and d_x is equal to α_a and α_x respectively and
- A traversal through either network between any two nodes that takes no more than k steps passes through *exactly two* of the α nodes (this assumption will be clarified later).

Under these assumptions the G_b network is described by

$$k_b = k_a + 2k_x, \; d_b = d_x, \; \alpha_b = \alpha_a \alpha_x.$$

From the restrictions given in Equation 4 $d_b < d_a$ and thus we have succeeded in reducing the maximum degree of the starting network. The sacrifice of the reduction in degree was an increase in diameter and an increase in the average degree in each of the super node networks. The average degree increases because d_a edges are connected (in an arbitrary manner) to each super node network. If the capacity of the super node network was equal to the degree of the node it replaced then the average degree would be increased to equal the maximum degree. This is not a problem, but may be combated by allowing the capacity of the super node to be some factor larger than the degree of the node it is replacing. Note that both of the networks used in this construction had a near infinite size but fixed finite diameter. The addition of two such fixed finite diameters while creating a total number of nodes equal to $N_a - \alpha_a + N_b\alpha_a$ may seem more of a bonus rather than a detriment. Our interest does not at the moment lie in analysing this cost tradeoff. We are concerned with whether we can reduce the maximum degree to a value lower than or equal to B while maintaining a smaller order of increase in diameter. Note that the network G_b constructed as above does in fact have a lower maximum degree than the starting network, G_a. We thus continue this process of super node substitution until the maximum degree of any node in the network is less than B. To continue we establish B in relation to N_a by providing equation $B = f(N_a)$.

As was stated earlier this function is typically logarithmic. Its exact nature is unimportant at this time though. We now find the number of replacements or the level of expansion needed to achieve bounded degree.

The essence of this technique, and as it seems, the essence of constructing large low cost networks, is the ability to find networks with small maximum degree and further more for the purposes of this technique with finite diameter when an infinite number of nodes is considered. For a set of constant, average cost networks $S = \{S_1, S_2, \ldots, S_n\}$, we arrange them in decreasing order of their degrees such that $\forall_{(x,y)((1<x,y<n)\wedge(x<y))}(d_x > d_y)$, where d_x is the maximum degree of S_x. Of course, other networks would also satisfy the conditions for S_1 such as the completely connected network. We need to apply the other restriction concerning capacity as stated in Equation 5 to this set of networks $\forall_{(x,y)((1<x,y<n)\wedge(x<y))}(\kappa_y \geq d_x)$, where κ is the capacity of the network as stated in Equation 3.

We apply the technique of degree reduction as outlined above with $G_a = S_1$ and $G_r = S_2$ and successively with $G_a = S_2$ and $G_r = S_3$, etc. That is, we use S_2 to reduce S_1, S_3 to reduce S_2 and so forth. We continue to do this until the maximum degree of the S network is less than or equal to B. A network created in this fashion will have a maximum degree no bigger than B, by definition. Given Assumption 3 stated earlier the final diameter however will increase as now $K = g(j)$, where j is the index such that the network S_j has maximum degree no bigger than B. The critical factor in all this is $g(j)$ since it dictates the final diameter and hence the two cost metrics:

$$c = B \cdot g(j), \ \bar{c} = \bar{d} \cdot g(j).$$

of which the second has been shown to be of greater insight. We now show the relationship between the rate at which the maximum degree is reduced to a be no bigger than B and the rate at which the diameter increases. Remember an important point here is that the initial diameter was asymptotically zero (constant).

We introduce a reducing function, $R(d)$, that provides the resulting degree of a network after one phase of degree reduction. If the initial degree was d_0 then after one reduction the new degree d_1 is $R(d_0)$ and after the second reduction the degree d_2 is $R(R(d_0))$ and so after j phases the degree d_j is $R^{(j)}(d_0)$. Essentially we want to find j such that:

$$R^{(j)}(d_0) \leq B \leq f(N_0) \tag{6}$$

where N_0 is the total nodes in the network. We are being a little coarse here since the total nodes actually increases after each phase of reduction but we do not take this into account (a worst case approximation). It seems that for the network structures we will consider (in the next section) the increase in the number of nodes does no more than double the original number. This though is a consideration of classical network topology.

As was stated earlier we would like the reducing function to reduce the maximum degree at a rate faster than the corresponding increase in diameter. This concept is illustrated in the example Figure 3 where the starting degree, d_0, was set to 10000. The figure shows an arbitrary choice of lower bound, B. It is shown that the degree falls below the lower bound at $j = 2$.

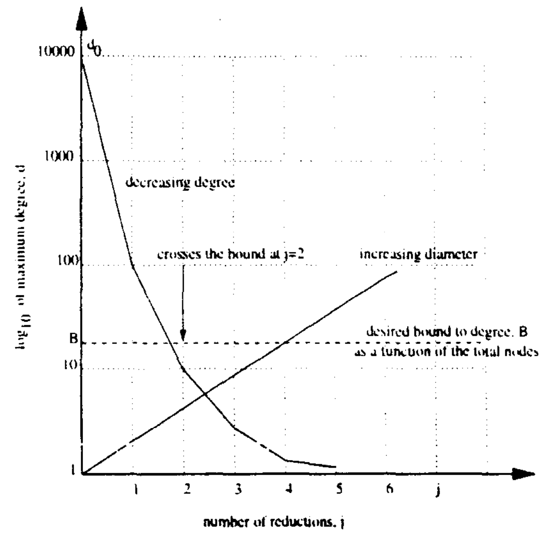


Figure 3: An illustration of a function that reduces the degree while a second function operates to increase the diameter.

To mathematically describe what is happening we state d_0 in terms of N_0, $d_0 = h(N_0)$ and rewrite Equation 6:

$$R^{(j)}(h(N_0)) \leq f(N_0) \tag{7}$$

In other words we may choose to investigate the product $c = g(j)R^{(j)}(d_0)$ and find j_0 such that $\left.\frac{dc}{dj}\right|_{j=j_0} = 0$. Alternatively we may choose to investigate the limit $\omega = \lim_{j\to\infty} \frac{g(j)}{R^{-(j)}(g(0))}$, where $R^{-(j)}$ is the inverse function and the cases when $\omega < 1$, $\omega = 1$ and $\omega > 1$. It is difficult for this mathematics to continue without actual functions. So now we investigate a possible set of networks based on the previous understanding.

4 An example construction

We construct a network based on the general construction method just proposed.

Consider a completely connected network of size n, with each of its edges replaced by a single node of degree 2. This network was examined earlier in the paper and the following properties hold:

$$d = n - 1,\ \bar{d} = \frac{4n - 4}{n + 1},\ k = 4.$$

We assume the problem with this network is that $d \gg B$, where B is some bound to maximum degree, such that the network cannot be feasible built. We state what we would like B to be in terms of the total nodes $B = \log(N)$, where $N \approx \frac{n^2}{2}$ and so $d = 2^{\frac{B+1}{2}}$, which is a value much larger than our desired bounding value. We now construct an appropriate super node to replace each of the n nodes in a manner prescribed earlier. We choose a simple network, constructed using the same method as the starting network but with fewer nodes. In other words we construct a super node from a completely connected network of size n' with its edges replaced by a single node of degree 2 making the total nodes, N', equal to $n' + \frac{n'(n'-1)}{2}$. The capacity of the super node must be equal to or greater than $n - 1$. We can do this by setting $n' = \left\lceil \sqrt{2(n-1)} \right\rceil$.

Now we must establish two things about this super node:

- its maximum degree, d', is less than d and
- its capacity, κ', is greater than or equal to d.

Since both of these are clearly true the super node we have constructed is valid. The good thing is that a whole set of such super nodes must exist that are also valid since the constructed super node is topologically identical to the network that it will become a super node of. Thus, a super super node may be constructed in the same manner. We now calculate the number of times such a replacement must occur. This value is the value for j in the previous section. We note that each time a node of degree d is replaced, it is replaced effectively with a network whose maximum degree, d', may be written $d' \approx \sqrt{d}$.

Assume we need j phases to cause $d \leq B$. The maximum degree, after j phases, of the total network is then $d^{(j)\prime} \approx \sqrt[2^j]{d}$. But $d = 2^{\frac{B+1}{2}}$ so $d^{(j)\prime} = 2^{\frac{B+1}{2^{j+1}}}$, and we would like $2^{\frac{B+1}{2^{j+1}}} \leq B$, which works out easily $j \geq \log\left(\frac{B}{\log B}\right)$.

Since the total nodes has increased (remember we are taking a worse case approximation that does not take into account this increase) it is true that $B \leq \log N$ so

$$j \geq \log\left(\frac{\log N}{\log\log N}\right) \tag{8}$$

Now what does the value of j tell us. It says how many 'levels' of super nodes we must use to get the desired maximum bound on degree. But as we reduce the maximum degree the diameter is increasing. For this example the diameter after j phases of expansion $k_j = \sum_{i=0}^{j} 2^{i+2}$, and asymptotically $k_j = \Theta(2^j)$.

To find the average cost we remember that the average degree must be *constant* for these types of networks from the principles shown earlier. This constant factor, $\bar{d}$, is equal to 4 but its value is unimportant since it is constant. Now we state the average cost $\bar{c} = \Theta(2^j)$ or from Equation 8 $\bar{c} = \Theta\left(\frac{\log N}{\log\log N}\right)$.

We now examine the case when the bound, B, is any lower order function of N: $B = o(N)$. We have, using the same start network and network set, $\mathcal{S}$, $d = n - 1 \approx \sqrt{2N}$, and hfill $d^{(i)\prime} = d^{\frac{1}{2^j}} = (2N)^{\frac{1}{2^{j+1}}}$. From $d^{(i)\prime} \leq B$ and ignoring the factor 2 we have $2^{j+1} \geq \frac{\log N}{\log B} \Rightarrow j = \lceil \log(\log_B N) \rceil$ and $k_j = \Theta(2^j) = \Theta(\log_B N)$, therefore, because the average degree is fixed,

$$\bar{c} = \Theta(\log_B N) \tag{9}$$

For the case when $B = \log^{(i)} N$, $\bar{c} = \Theta\left(\frac{\log N}{\log^{(i+1)} N}\right)$, for any $i \geq 1$. Interestingly the conventional maximum cost is $c = \Theta\left(\frac{\log N}{\log^{(i+1)} N} \log^{(i)} N\right)$. More simply, for the case when $B = N^{\epsilon}$, $0 < \epsilon < 1$, $\bar{c} = \Theta\left(\frac{1}{\epsilon}\right)$, and for $\epsilon = \frac{1}{\log N}$, $\bar{c} = \Theta(\log N)$. Clearly, the rough asymptotic analysis is only a guide towards establishing the actual cost of the final network.

5 Conclusion

Classical network topology has identified many "bottom-up" approaches to designing low cost networks. From this point of view, a network topologist starts with small size low cost network and attempts to add extra nodes while keeping the degree and diameter small. We presented a "top-down" approach to designing networks that starts with an arbitrary large network of arbitrary large degree but constant diameter and proceeds by attempting to reduce the degree to be less than some bound at a rate faster than the corresponding increase in diameter. We provided a general method for doing so that resulted in an average cost of $\Theta(\log_B N)$ for some bound $B = o(N)$ and showed an example based on this method that produces a network with average cost in the order of $\Theta\left(\frac{\log N}{\log\log N}\right)$ where N is the number of nodes in the network (size). We have not yet investigated the opposite alternative of starting with an arbitrary large diameter but constant degree and attempting to reduce the diameter at a rate faster than the corresponding increase in degree nor the apparent zyzigy, or conjunction of these two opposing methods. We leave this for future research.

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