Minimum Latency Submodular Cover*

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Abstract

We study the Minimum Latency Submodular Cover problem (MLSC), which consists of a metric (V,d) with source $r \in V$ and m monotone submodular functions $f_1, f_2, ..., f_m : 2^V \to [0,1]$. The goal is to find a path originating at r that minimizes the total cover time of all functions. This generalizes well-studied problems, such as Submodular Ranking [1] and Group Steiner Tree [16]. We give a polynomial time $O(\log \frac{1}{\epsilon} \cdot \log^{2+\delta} |V|)$ -approximation algorithm for MLSC, where $\epsilon > 0$ is the smallest non-zero marginal increase of any $\{f_i\}_{i=1}^m$ and $\delta > 0$ is any constant.

We also consider the Latency Covering Steiner Tree problem (LCST), which is the special case of MLSC where the f_i s are multi-coverage functions. This is a common generalization of the Latency Group Steiner Tree [20, 8] and Generalized Min-sum Set Cover [2, 3] problems. We obtain an $O(\log^2 |V|)$ -approximation algorithm for LCST.

Finally we study a natural stochastic extension of the Submodular Ranking problem, and obtain an adaptive algorithm with an $O(\log 1/\epsilon)$ approximation ratio, which is best possible. This result also generalizes some previously studied stochastic optimization problems, such as Stochastic Set Cover [17] and Shared Filter Evaluation [27, 26].

1 Introduction

Ordering a set of elements so as to be simultaneously good for several valuations is an important issue in web-search ranking and broadcast scheduling. A formal model for this was introduced by Azar, Gamzu and Yin [2] where they studied the Multiple Intents Re-ranking problem (a.k.a. Generalized Min Sum Set Cover [3]). In this problem, a set of elements is to be displayed to m different users, each of whom wants to see some threshold number of elements from its subset of interest. The objective is to compute an ordering that minimizes the average (or total) overhead of the users, where the overhead corresponds to the position in the ordering when the user is satisfied.

Subsequently, Azar and Gamzu [1] studied a generalization, the Submodular Ranking problem, where the interests of users are represented by arbitrary (monotone) submodular functions. Again, the objective is to order the elements so as to minimize the total overhead, where now the overhead of a user is the position when its utility function is "covered". An interesting feature of this problem is that it generalizes both the minimum set cover [23] and min-sum set cover [4, 15] problems.

In this paper, we extend both of these models to the setting of metric switching costs between elements. This allows us to handle additional issues such as:

- Data locality: it takes d(i, j) time to read or transmit data j after data i.
- Context switching: it takes d(i,j) time for a user to parse data j when scheduled after data i.

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From a theoretical point of view, these problems generalize a number of previously studied problems and our results unify/extend techniques used in different settings.

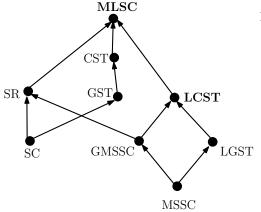
We introduce and study the Minimum Latency Submodular Cover problem (MLSC), which is the metric version of Submodular Ranking [1], and its interesting special case, the Latency Covering Steiner Tree problem (LCST), which extends Generalized Min-Sum Set Cover [2, 3]. The formal definitions follow shortly, in the next subsection. We obtain poly-logarithmic approximation guarantees for both problems. We remark that due to a relation to the well-known Group Steiner Tree [16] problem, any significant improvement on our results would lead to a similar improvement for Group Steiner Tree. The MLSC problem is a common generalization of several previously studied problems [16, 25, 15, 20, 8, 2, 1]; see also Figure 1.

In a somewhat different direction, we also study the Weighted Stochastic Submodular Ranking problem, where elements are stochastic and the goal is to adaptively schedule elements so as to minimize the expected total cover time. We obtain an $O(\log \frac{1}{\epsilon})$ -approximation algorithm for this problem, which is known to be best possible even in the deterministic setting [1]. This result also generalizes many previously studied stochastic optimization problems [17, 27, 26].

1.1 Problem Definitions

We now give formal definitions of the problems considered in this paper. The problems followed by * are those for which we obtain the first non-trivial results. Several other problems are also discussed since those are important special cases of our main problems. The relationships between these problems are shown pictorially in Figure 1.

A function $f: 2^V \to \mathbb{R}_+$ is submodular if, for any $A, B \subseteq V$, $f(A) + f(B) \ge f(A \cup B) + f(A \cap B)$; and it is monotone if for any $A \subseteq B$, $f(A) \le f(B)$. We assume some familiarity with submodular functions [29].



MLSC: minimum latency submodular cover

LCST: latency covering Steiner tree

CST : covering Steiner tree GST : group Steiner tree

SR: submodular ranking

LGST : latency group Steiner tree

GMSSC: generalized min-sum set cover (multiple intents re-ranking)

SC: set cover

MSSC: min sum set cover

Figure 1: An arrow from X to Y means X is a special case of Y.

Minimum Latency Submodular Cover* (MLSC): There is a ground set V of elements/vertices and $d:\binom{V}{2}\to\mathbb{R}_+$ is a distance function. We assume that d is symmetric and satisfies the triangle inequality. In addition there is a specified root vertex $r\in V$. There are m monotone submodular functions $f_1,\ldots,f_m:2^V\to\mathbb{R}_+$ representing the valuations of different users. We assume, without loss of generality by truncation, that $f_i(V)=1$ for all $i\in[m]$. Function f_i is said to be covered (or satisfied) by set $S\subseteq V$ if $f_i(S)=1=f_i(V)$. The cover time of function f_i in a path π is the length of the shortest prefix of π that has f_i value

¹Throughout the paper, for any integer $\ell \geq 1$, we denote $[\ell] := \{1, 2, \dots, \ell\}$.

one, i.e.

min
$$\left\{t: f_i\left(\left\{v \in V: v \text{ appears within distance } t \text{ on } \pi\right\}\right) = 1\right\}$$
.

The objective in the Minimum Latency Submodular Cover problem is to compute a path originating at r that minimizes the sum of cover times of all functions. A technical parameter that we use to measure performance (which also appears in [1, 31]) is ϵ which is defined to be the smallest non-zero marginal increase of any function $\{f_i\}_{i=1}^m$.

Generalized Min-Sum Set Cover (GMSSC): Given a ground set V and m subsets $\{g_i \subseteq V\}_{i=1}^m$ with respective requirements $\{k_i\}_{i=1}^m$, the goal is to find a linear ordering of V that minimizes the sum of cover times. A subset g_i is said to be covered when at least k_i elements from g_i have appeared. Min-Sum Set Cover (MSSC) is the special case when $\max_i k_i = 1$.

Submodular Ranking (SR): Given a ground set V and m monotone submodular functions $f_1, \ldots, f_m : 2^V \to \mathbb{R}_+$, the goal is compute a linear ordering of V that minimizes the sum of cover times of all functions. The cover time of a function is the minimum number of elements in a prefix that has function value at least one. This is a special case of MLSC when metric d is uniform. The set cover problem is a special case of SR when there is a single submodular function (which is also a coverage function). GMSSC is another special case of SR, where each subset g_i corresponds to the submodular function $f_i(S) = \min\{|g_i \cap S|/k_i, 1\}$.

Group Steiner Tree (GST): Given a metric (V, d) with root $r \in V$ and N groups of vertices $\{g_i \subseteq V\}_{i=1}^N$, the goal is to find a minimum length tree containing r and at least one vertex from each of the N groups. Observe that an r-rooted tree can be converted into a path starting from r with at most a factor two loss in the total length, and vice versa. Thus GST is a special case of MLSC when there is only a single submodular function

$$f_1(S) = \frac{1}{N} \sum_{i=1}^{N} \min\{|g_i \cap S|, 1\}.$$

Note that $f_1(S') = 1$ if and only if $S' \cap g_i$ is nonempty for all $i \in [N]$.

Covering Steiner Tree (CST): This is a generalization of GST with the same input as above, where each group g_i is also associated with a requirement k_i . The goal here is to find a minimum length tree that contains r and at least k_i vertices from group g_i , for all $i \in [N]$. We recover CST as a special case of MLSC by setting

$$f_1(S) = \frac{1}{N} \sum_{i=1}^{N} \min \left\{ \frac{|g_i \cap S|}{k_i}, 1 \right\}.$$

Note that now $f_1(S') = 1$ if and only if $|S' \cap g_i| \ge k_i$ for all $i \in [N]$.

Latency Group Steiner Tree (LGST): This is a variant of the group Steiner tree problem. Given a metric (V, d) with root r and N groups of vertices $\{g_i \subseteq V\}_{i=1}^N$, the goal is to find a path π originating from r that minimizes the sum of cover times of the groups. (A group g_i is covered at the shortest prefix of π that contains at least one vertex from g_i .) Note that MSSC is the special case when the metric is uniform.

Latency Covering Steiner Tree* (LCST): The input to this problem is the same as for LGST with additional requirements $\{k_i\}_{i=1}^N$ corresponding to each group. The objective is again a path π originating from r that minimizes the sum of cover times, where group g_i is covered at the shortest prefix of π that contains at least k_i vertices from g_i . Clearly, LGST is the special case of LCST where all requirements $k_i = 1$. GMSSC is also a special case when the metric is uniform. We obtain LCST as a special case of MLSC with m = N functions and $f_i(S) = \min\{|g_i \cap S|/k_i, 1\}$ for all $i \in [N]$.

Weighted Stochastic Submodular Ranking* (WSSR): This is a stochastic generalization of the submodular ranking problem. We are given a set V of stochastic elements (random variables), each having an independent distribution over a certain domain Δ . The submodular functions are also defined on the ground set Δ , i.e. $f_1, ..., f_m : 2^{\Delta} \to [0, 1]$. In addition, each element $i \in V$ has a deterministic time ℓ_i to be

scheduled. The realization (from Δ) of any element is known immediately after scheduling it. The goal is to find an adaptive ordering of V that minimizes the total expected cover time. Since elements are stochastic, it is possible that a function is never covered: in such cases we just fix the cover time to be $\sum_{i \in V} \ell_i$ (which is the total duration of any schedule).

We will be concerned with *adaptive* algorithms. Such an algorithm is allowed to decide the next element to schedule based on the instantiations of the previously scheduled elements. This models the setting where the algorithm can benefit from user feedback.

WSSR generalizes the Stochastic Set Cover studied in [17]. Interestingly, it also captures some variants of Stochastic Set Cover that have applications in processing multiple queries with probabilistic information [27, 26]. Various applications of WSSR are discussed in more detail in Section 5.

1.2 Our Results and Techniques

Our first result is on the Minimum Latency Submodular Cover problem (MLSC) problem.

Theorem 1.1. For any constant $\delta > 0$, there is an $O(\log \frac{1}{\epsilon} \cdot \log^{2+\delta} |V|)$ -approximation algorithm for the Minimum Latency Submodular Cover problem.

Note that in the special case of Group Steiner Tree, this result is larger only by a factor of $O(\log^{\delta}|V|)$ than its best known approximation ratio of $O(\log N \log^2 |V|)$, due to Garg, Konjevod and Ravi [16]. Our algorithm uses the framework of [1] and the Submodular Orienteering problem (SOP) [12] as a sub-routine. The input to SOP consists of metric (V, d), root r, monotone submodular function $f: 2^V \to \mathbb{R}_+$ and length bound B. The goal is to find a path originating at r having length at most B that maximizes f(S), where $S \subseteq V$ is the set of vertices visited in the path. Specifically, we show that a (ρ, σ) -bicriteria approximation algorithm² for SOP can be used to obtain an $O(\rho \sigma \cdot \log \frac{1}{\epsilon})$ -approximation algorithm for MLSC. To obtain Theorem 1.1 we use an $(O(1), O(\log^{2+\delta}|V|))$ -bicriteria approximation for SOP that follows from [6, 11].

Our algorithm for MLSC is an extension of the elegant "adaptive residual updates scheme" of Azar and Gamzu [1] for Submodular Ranking (i.e. uniform metric MLSC). As shown in [1], an interesting aspect of this problem is that the natural greedy algorithm, based on absolute contribution of elements, performs very poorly. Instead they used a modified greedy algorithm that selects one element at a time according to residual coverage. In the MLSC setting of general metrics, our algorithm uses a similar residual coverage function to repeatedly augment the solution. However our augmentations are paths of geometrically increasing lengths, instead of just one element. A crucial point in our algorithm is that the residual coverage functions are always submodular, and hence we can use Submodular Orienteering (SOP) in the augmentation step.

We remark that the approach of covering the maximum number of functions within geometrically increasing lengths fails because the residual coverage function here is non-submodular; in fact as noted in [3] this subproblem contains the difficult dense-k-subgraph problem (even for the special case of Generalized Min-Sum Set Cover with requirement two). We also note that the choice of our (submodular) residual coverage function ultimately draws on the submodular ranking algorithm [1].

The analysis in [1] was based on viewing the optimal and approximate solutions as histograms. This approach was first used in this line of work by Feige, Lov'asz and Tetali [15] for the Min-Sum Set Cover problem (see also [4]). This was also the main framework of analysis in [2] for Generalized Min-Sum Set Cover and then for Submodular Ranking [1]. However, these proofs have been increasingly difficult as the problem in consideration adds more generality. Instead we follow a different and more direct approach that is similar to the analysis of Minimum Latency Travelling Salesman problem, see eg. [10, 13]. In fact, the proof of Theorem 1.1 is enabled by a new simpler analysis of the Submodular Ranking algorithm [1].

Our second result is a better approximation ratio for the Latency Covering Steiner tree (LCST) problem. Note that Theorem 1.1 implies directly an $O(\log k_{max} \cdot \log^{2+\delta} |V|)$ -approximation algorithm for LCST, where $k_{max} = \max_{i=1}^{N} k_i$.

Theorem 1.2. There is an $O(\log^2 |V|)$ -approximation algorithm for Latency Covering Steiner Tree.

²Given any instance of SOP, such an algorithm returns a path of length at most $\sigma \cdot B$ and function value at least OPT/ ρ .

The main idea in this result is a new LP relaxation for Covering Steiner Tree (using Knapsack Cover type inequalities [7]) having a poly-logarithmic integrality gap. This new LP might also be of some independent interest. The previous algorithms [25, 21] for covering Steiner tree were based on iteratively solving an LP with large integrality gap. However, this approach does not seem suitable to the latency version we consider. Our new LP relaxation for Covering Steiner Tree (CST) is crucial for obtaining the approximation stated in Theorem 1.2. As shown in [28], any improvement over Theorem 1.2 even in the $k_{max} = 1$ special case (i.e. Latency Group Steiner Tree) would yield an improved approximation ratio for Group Steiner Tree, which is a long-standing open question.

Our final result is for the Weighted Stochastic Submodular Ranking problem. As shown in [17, 18], even special cases of this problem have polynomially large adaptivity gap (ratio between the optimal non-adaptive and adaptive solutions)³. This motivates adaptive algorithms, and we obtain the following result in Section 5.

Theorem 1.3. There is an adaptive $O(\log \frac{1}{\epsilon})$ -approximation algorithm for the Weighted Stochastic Submodular Ranking problem.

In particular, we show that the natural stochastic extension of the algorithm from [1] achieves this approximation factor. We remark that the analysis in [1] of deterministic submodular ranking required unit costs, whereas Theorem 1.3 holds for the stochastic setting even with non-uniform costs $\{\ell_i\}$.

As mentioned before, our results generalize the results in [17, 27, 26] which study (some variants of) Stochastic Set Cover. Our analysis is arguably simpler and more transparent than [26], which gave the first tight analysis of these problems. We note that [26] used an intricate charging scheme with "dual prices" and it does not seem directly applicable to general submodular functions.

We note that our techniques do not extend directly to the stochastic MLSC problem (on general metrics), and obtaining a poly-logarithmic approximation here seems to require additional ideas.

1.3 Previous Work

The first poly-logarithmic approximation for Group Steiner Tree was $O(\log N \log^2 |V|)$, obtained by Garg et al. [16]. This is still the best known bound. Chekuri, Even and Kortsarz [11] gave a combinatorial algorithm that achieved a slightly weaker approximation ratio (the algorithm in [16] was LP-based). This combinatorial approach was extended in Calinescu and Zelikovsky [6] to the problem of covering any submodular function in a metric space. We use this algorithm in the submodular orienteering (SOP) subroutine for our MLSC result. For SOP an $O(\log |V|)$ -approximation is known due to Chekuri and Pal [12], but with a quasi-polynomial running time. We note that an $\Omega(\log^{2-\delta} |V|)$ hardness of approximation is known for Group Steiner Tree (even on tree metrics) due to Halperin and Krauthgamer [22].

The Covering Steiner Tree problem was introduced by Konjevod, Ravi and Srinivasan [25], which can be viewed as the multicover version of Group Steiner Tree. They gave an $O(\log(Nk_{max})\log^2|V|)$ -approximation using an LP-relaxation. However the LP used in [25] has a large $\Omega(k_{max})$ integrality gap; they got around this issue by iteratively solving a suitable sequence of LPs. They also extended the randomized rounding analysis from [16] to this context. Later, Gupta and Srinivasan [21] improved the approximation bound to $O(\log N \log^2 |V|)$, removing the dependence on the covering requirements. This algorithm was also based on solving a similar sequence of LPs; the improvement was due to a combination of threshold rounding and randomized rounding. In this paper, we give a stronger LP relaxation for Covering Steiner Tree based on so-called Knapsack-Covering-inequalities (abbreviated to KC-inequalities), that has an $O(\log N \log^2 |V|)$ integrality gap.

The Stochastic Set Cover problem (which is a special case of Weighted Stochastic Submodular Ranking) was introduced by Goemans and Vondrák [17]. Here each set covers a random subset of items, and the goal is to minimize the expected cost of a set cover. [17] showed a large adaptivity gap for Stochastic Set Cover, and gave a logarithmic approximation for a relaxed version where each stochastic set can be added multiple

³A non-adaptive solution is just a fixed linear ordering of the elements, whereas an adaptive solution can select the next element based on previous instantiations.

times. A related problem in context of fast query evaluation was studied in [27], where the authors gave a triple logarithmic approximation. This bound was improved to the best-possible logarithmic ratio by Liu, Parthasarathy, Ranganathan and Yang [26]; this result was also applicable to stochastic set cover (where each set can be added at most once). Another related paper is by Golovin and Krause [18], where they defined a general property "adaptive submodularity" and showed nearly optimal approximation guarantees for several objectives (max coverage, min-cost cover and min sum cover). The most relevant result in [18] to WSSR is the 4-approximation for Stochastic Min Sum Set Cover. This approach required a fixed submodular function f such that the objective is to minimize $\mathbb{E}\left[\sum_{t\geq 0} f(\overline{V}) - f(\overline{\pi}_t)\right]$ where $\overline{\pi}_t$ is the realization of elements scheduled within time t and \overline{V} denotes the realization of all elements. However, this is not the case even for the special case of Generalized Min-Sum Set Cover with requirements two. Recently, Guillory and Bilmes [19] studied the Submodular Ranking problem in an online regret setting, which is different from the adaptive model we consider.

1.4 Organization

In Section 2 we revisit the Submodular Ranking problem and give an easier and perhaps more intuitive analysis of the algorithm from [1]. This simpler analysis is then used in the algorithms for Minimum Latency Submodular Cover (Theorem 1.1) and Weighted Stochastic Submodular Ranking (Theorem 1.3), that appear in Sections 3 and 5 respectively. Section 4 contains the improved approximation algorithm for Latency Covering Steiner Tree (Theorem 1.2) which makes use of a new linear programming relaxation for Covering Steiner Tree. The section on LCST can be read independently of the other three sections.

2 Simpler Analysis of the Submodular Ranking Algorithm

In this section, we revisit the Submodular Ranking problem [1]. Recall that the input consists of a ground set V := [n] of elements and monotone submodular functions $f_1, f_2, ... f_m : 2^{[n]} \to [0, 1]$ with $f_i(V) = 1, \forall i \in [m]$. The goal is to find a complete linear ordering of the elements that minimizes the total cover time of all functions. The cover time $cov(f_i)$ of f_i is defined as the smallest index t such that the function f_i has value 1 for the first t elements in the ordering. We also say that an element e is scheduled at time t if it is the t-th element in the ordering. It is assumed that each function f_i satisfies the following property: for any $S \supseteq S'$, if $f_i(S) - f_i(S') > 0$ then it must be the case that $f_i(S) - f_i(S') \ge \epsilon$, where $\epsilon > 0$ is a constant that is uniform for all functions f_i . This is a useful parameter in describing the performance guarantee.

Azar and Gamzu [1] gave a modified greedy-style algorithm with an approximation factor of $O(\log \frac{1}{\epsilon})$ for Submodular Ranking. Their analysis was histogram-based and fairly involved. In this section, we give an alternate shorter proof of their result. Our analysis also extends to the more general MLSC problem which we study in the next section. The algorithm ALG-AG from [1] is given below. In the output, $\pi(t)$ denotes the element that appears in the t-th time slot.

Algorithm 1 ALG-AG

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INPUT: Ground set [n]; monotone submodular functions f_i: 2^{[n]} \to [0, 1], i \in [m]

1: S \leftarrow \emptyset

2: for t = 1 to n do

3: Let f^S(e) := \sum_{i \in [m], f_i(S) < 1} \frac{f_i(S \cup \{e\}) - f_i(S)}{1 - f_i(S)}

4: e = \arg\max_{e \in [n] \setminus S} f^S(e)

5: S \leftarrow S \bigcup \{e\}

6: \pi(t) \leftarrow e

7: end for

OUTPUT: A linear ordering \langle \pi(1), \pi(2), \dots, \pi(n) \rangle of [n].
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Theorem 2.1 ([1]). ALG-AG is an $O(\ln(\frac{1}{\epsilon}))$ -approximation algorithm for Submodular Ranking.

Let $\alpha:=1+\ln(\frac{1}{\epsilon})$. To simplify notation, without loss of generality, we assume that α is an integer. Let R(t) denote the set of functions that are not satisfied by ALG-AG earlier than time t; R(t) includes the functions that are satisfied exactly at time t. For notational convenience, we use $i \in R(t)$ interchangeably with $f_i \in R(t)$. Analogously, $R^*(t)$ is the set of functions that are not satisfied in the optimal solution before time t. Note that algorithm's objective $\text{ALG} = \sum_{t \geq 1} |R(t)|$ and the optimal value $\text{OPT} = \sum_{t \geq 1} |R^*(t)|$. We will be interested in the number of unsatisfied functions at times $\{8\alpha 2^j: j \in \mathbb{Z}_+\}$ by ALG-AG and the number of unsatisfied functions at times $\{2^j: j \in \mathbb{Z}_+\}$ by the optimal solution. Let $R_j:=R(8\alpha 2^j)$ and $R_j^*=R^*(2^j)$ for all integer $j \geq 0$. It is important to note that R_j and R_j^* are concerned with different times. For notational simplicity, we let $R_{-1}:=\emptyset$.

We show the following key lemma. Roughly speaking, it says that the number of unsatisfied functions by ALG-AG diminishes quickly unless it is comparable to the number of unsatisfied functions in OPT.

Lemma 2.2. For any $j \ge 0$, we have $|R_j| \le \frac{1}{4}|R_{j-1}| + |R_j^*|$.

Proof. When j=0 the lemma holds trivially. Now consider any integer $j\geq 1$ and time step $t\in [8\alpha 2^{j-1},8\alpha 2^j)$. Let S_{t-1} denote the set of elements that ALG-AG schedules before time t and let e_t denote the element that ALG-AG schedules exactly at time t. Let E_j denote the set of elements that ALG-AG schedules until time $8\alpha 2^j$. Let E_j^* denote the set of elements that OPT schedules until time 2^j . Recall that ALG-AG picks e_t as an element e that maximizes

$$f^{S_{t-1}}(e) := \sum_{i \in [m]: f_i(S_{t-1}) < 1} \frac{f_i(S_{t-1} \cup \{e\}) - f_i(S_{t-1})}{1 - f_i(S_{t-1})}$$

This leads us to the following proposition.

Proposition 2.3. For any $j \ge 1$, time step $t \in [8\alpha 2^{j-1}, 8\alpha 2^j)$ and $e \in E_j^*$, we have $f^{S_{t-1}}(e_t) \ge f^{S_{t-1}}(e)$.

Proof. Since ALG-AG has chosen to schedule element e_t over all elements $e \in E_j^* \setminus S_{t-1}$, we know that the claimed inequality holds for any $e \in E_j^* \setminus S_{t-1}$. Further, the inequality holds for any element e in S_{t-1} , since $f^{S_{t-1}}(e) = 0$ for such an element e.

By taking an average over all elements in E_i^* , we derive

$$f^{S_{t-1}}(e_t) \ge \frac{1}{|E_j^*|} \sum_{e \in E_j^*} f^{S_{t-1}}(e)$$

$$\ge \frac{1}{|E_j^*|} \sum_{e \in E_j^*} \sum_{i \in R_j \setminus R_j^*} \frac{f_i(S_{t-1} \cup \{e\}) - f_i(S_{t-1})}{1 - f_i(S_{t-1})}$$
(1)

Observe that in (1), the inner summation only involves functions f_i for which $f_i(S_{t-1}) < 1$. This is because for any $i \in R_j$, function f_i is not covered before time $8\alpha 2^j$ and $t < 8\alpha 2^j$. Due to submodularity of each function f_i , we have that

$$(1) \geq \frac{1}{|E_{j}^{*}|} \sum_{i \in R_{j} \setminus R_{i}^{*}} \frac{f_{i}(S_{t-1} \cup E_{j}^{*}) - f_{i}(S_{t-1})}{1 - f_{i}(S_{t-1})} = \frac{1}{|E_{j}^{*}|} \sum_{i \in R_{j} \setminus R_{i}^{*}} 1 \geq \frac{|R_{j}| - |R_{j}^{*}|}{|E_{j}^{*}|}$$

The equality is due to the fact that for any $i \notin R_i^*$, $f_i(E_i^*) = 1$ and each function f_i is monotone. Hence:

$$\sum_{8\alpha \cdot 2^{j-1} < t < 8\alpha \cdot 2^j} f^{S_{t-1}}(e_t) \ge \frac{8\alpha(2^j - 2^{j-1})}{|E_j^*|} (|R_j| - |R_j^*|) = 4\alpha(|R_j| - |R_j^*|), \tag{2}$$

where we used $|E_j^*| = 2^j$. We now upper bound the left-hand-side of (2). To this end, we need the following claim from [1].

Claim 2.4 (Claim 2.3 in [1]). Given a monotone function $f: 2^{[n]} \to [0,1]$ with f([n]) = 1 and sets $\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_\ell \subseteq [n]$, we have (using the convention 0/0 = 0)

$$\sum_{k=1}^{\ell} \frac{f(S_k) - f(S_{k-1})}{1 - f(S_{k-1})} \le 1 + \ln \frac{1}{\delta}.$$

Here $\delta > 0$ is such that for any $A \subseteq B$, if f(B) - f(A) > 0 then $f(B) - f(A) \ge \delta$.

Proof. We give a proof for completeness. We can assume, without loss of generality, that $S_{\ell} = [n]$. Order the values in the set $\{f(S_k) \mid 0 \le k \le \ell\} \setminus \{1\}$ in increasing order to obtain $\beta_0 < \beta_1 < \ldots < \beta_H$. By the assumption, we have $\beta_0 \ge 0$ and $\beta_H \le 1 - \delta$ (moreover, $\beta_h - \beta_{h-1} \ge \delta$, $\forall h \in [H]$). We will show that

$$\sum_{h=1}^{H} \frac{\beta_h - \beta_{h-1}}{1 - \beta_{h-1}} \le \ln \frac{1}{\delta}$$

Since $f(S_{\ell}) = 1$, the summation we want to bound has an additional term of $\frac{1-\beta_H}{1-\beta_H} = 1$. Knowing that the function $u(x) = \frac{1}{1-x}$ is increasing for $x \in [0,1)$, we derive

$$\sum_{h=1}^{H} \frac{\beta_h - \beta_{h-1}}{1 - \beta_{h-1}} = \sum_{h=1}^{H} \int_{x=\beta_{h-1}}^{\beta_h} \frac{1}{1 - \beta_{h-1}} \, \mathrm{d}x \le \sum_{h=1}^{H} \int_{x=\beta_{h-1}}^{\beta_h} \frac{1}{1 - x} \, \mathrm{d}x = \int_{x=0}^{\beta_H} \frac{1}{1 - x} \, \mathrm{d}x$$

$$= \ln \left(\frac{1 - \beta_0}{1 - \beta_H} \right) \le \ln \frac{1}{\delta}$$

This proves the claim.

Note that any function f_i not in R_{j-1} does not contribute to the left-hand-side of (2) since any such function f_i was already covered before time $8\alpha 2^{j-1} \le t$. Further, knowing by Claim 2.4 that each function $f_i \in R_{j-1}$ can add at most $\alpha := 1 + \ln \frac{1}{\epsilon}$, we can upper bound the left-hand-side of (2) by $\alpha |R_{j-1}|$. Formally,

$$\sum_{8\alpha \cdot 2^{j-1} < t \le 8\alpha \cdot 2^{j}} f^{S_{t-1}}(e_{t}) = \sum_{8\alpha \cdot 2^{j-1} < t \le 8\alpha \cdot 2^{j}} \sum_{i \in R_{j-1} : f_{i}(S_{t-1}) < 1} \frac{f_{i}(S_{t-1} \cup \{e_{t}\}) - f_{i}(S_{t-1})}{1 - f_{i}(S_{t-1})}$$

$$\leq \sum_{i \in R_{j-1}} \sum_{t \ge 1 : f_{i}(S_{t-1}) < 1} \frac{f_{i}(S_{t-1} \cup \{e_{t}\}) - f_{i}(S_{t-1})}{1 - f_{i}(S_{t-1})}$$

$$\leq \alpha |R_{j-1}|$$

$$(3)$$

From (2) and (3) we obtain $4\alpha(|R_j| - |R_j^*|) \le \alpha |R_{j-1}|$ which completes the proof of Lemma 2.2.

Now we can prove Theorem 2.1 using Lemma 2.2.

Proof of Theorem 2.1.

3 Minimum Latency Submodular Cover

Recall that in the Minimum Latency Submodular Cover problem (MLSC), we are given a metric (V,d) with root $r \in V$ and m monotone submodular functions $f_1, f_2, ..., f_m : 2^V \to [0,1]$. Without loss of generality, by scaling, we assume that all distances $d(\cdot, \cdot)$ are integers. The objective in MLSC is to find a path starting at r that minimizes the total cover time of all functions.

As mentioned earlier, our algorithm for MLSC uses as a subroutine an algorithm for the Submodular Orienteering problem (SOP). In this problem, given metric (V,d), root r, monotone submodular function $g: 2^V \to \mathbb{R}_+$ and bound B, the goal is to compute a path P originating at r that has length at most B and maximizes g(V(P)) where V(P) is the set of vertices covered by P. We assume a (ρ, σ) -bicriteria approximation algorithm ALG-SOP for SOP. That is, on any SOP instance, ALG-SOP returns a path P of length at most $\sigma \cdot B$ and $g(V(P)) \geq \mathsf{OPT}/\rho$, where OPT is the optimal value obtained by any length B path. We recall the following known results on SOP .

Theorem 3.1 ([6]). For any constant $\delta > 0$ there is a polynomial time $(O(1), O(\log^{2+\delta} |V|))$ bicriteria approximation algorithm for the Submodular Orienteering problem.

Theorem 3.2 ([12]). There is a quasi-polynomial time $O(\log |V|)$ approximation algorithm for the Submodular Orienteering problem.

We now describe our algorithm ALG-MLSC for MLSC that uses the (ρ, σ) bicriteria approximation algorithm ALG-SOP. Here $\alpha = 1 + \ln \frac{1}{\epsilon}$. Note the difference from the submodular ranking algorithm [1]: here each augmentation is a path possibly covering several vertices. Despite the similarity of ALG-MLSC to the min-latency TSP type algorithms [10, 13] an important difference is that we do not try to directly maximize the number of covered functions in each augmentation: as noted before this subproblem is at least as hard as dense-k-subgraph, for which the best approximation ratio known is only polynomial [5]. Instead we maximize in each step some proxy residual coverage function f^S that suffices to eventually cover all functions quickly. This function is a natural extension of the single-element coverage values used in ALG-AG [1]. It is important to note that in Line (4), $f^S(\cdot)$ is defined adaptively based on the current set S of visited vertices in each iteration. Moreover, since each function f_i is monotone and submodular, so is f^S for any $S \subseteq V$. In Step 6, $\pi \cdot P$ denotes the concatenation of paths π and P.

Algorithm 2 ALG-MLSC

```
INPUT: (V, d), r \in V; \{f_i : 2^V \to [0, 1]\}_{i=1}^m.

1: S \leftarrow \emptyset, \pi \leftarrow \emptyset.

2: for k = 0, 1, 2, ... do

3: for u = 1, 2, ..., 4\alpha\rho do

4: Define submodular function
```

$$f^{S}(T) := \sum_{i \in [m], f_{i}(S) < 1} \frac{f_{i}(S \cup T) - f_{i}(S)}{1 - f_{i}(S)}, \text{ for all } T \subseteq V.$$

Use ALG-SOP to find a path P of length at most $\sigma \cdot 2^k$ starting from r that ρ -approximately maximizes $f^S(V(P))$ where V(P) is the set of nodes visited by P.

```
6: S \leftarrow S \cup V(P) \text{ and } \pi \leftarrow \pi \cdot P.
```

7: end for

8: end for

OUTPUT: Output solution π .

We prove the following theorem, which implies Theorem 1.1.

Theorem 3.3. ALG-MLSC is an $O(\alpha \rho \sigma)$ -approximation algorithm for Minimum Latency Submodular Cover.

We now analyze ALG-MLSC. We say that the algorithm is in the j-th phase, when the variable k of the for loop in Step 2 has value j. Note that there are $4\alpha\rho$ iterations of Steps 4-5 in each phase.

Proposition 3.4. Any vertex v added to S in the j-th phase is visited by π within $16\alpha\rho\sigma \cdot 2^{j}$.

Proof. The final solution is a concatenation of the paths that were found in Step 6. Since all these paths are stitched at the root r, the length of π at the end of phase j is at most $\sum_{k=1}^{j} 2 \cdot 4\alpha \rho \cdot \sigma 2^k \leq 16\alpha \rho \sigma \cdot 2^j$. \square

Let R(t) denote the set of (indices of) the functions that are not covered by ALG-MLSC earlier than time t; R(t) includes the functions that are covered exactly at time t as well. We interchangeably use $i \in R(t)$ and $f_i \in R(t)$. Let $R_j := R_j (16\alpha\rho\sigma 2^j)$. Similarly, we let $R^*(t)$ denote the set of functions that are not covered by OPT earlier than time t and let $R_j^* = R^*(2^j)$. Let $R_{-1} := \emptyset$.

We show the following key lemma. It shows that the number of uncovered functions by ALG-MLSC must decrease fast as j grows, unless the corresponding number in the optimal solution is comparable.

Lemma 3.5. For any $j \ge 0$, we have $|R_j| \le \frac{1}{4}|R_{j-1}| + |R_j^*|$.

Proof. The lemma trivially holds when j=0. Now consider any fixed phase $j \geq 1$. Let S_0 denote the set of vertices that were added to S up to the end of phase j-1. Let $H=4\alpha\rho$ and $T_1,T_2,...,T_H$ be the sets of vertices that were added in Line (6) in the j-th phase. Let $S_h=S_0\cup T_1\cup T_2\cup...\cup T_h$, $\forall 1\leq h\leq H$. We prove Lemma 3.5 by lower and upper bounding the quantity

$$\Delta_j := \sum_{h=1}^H f^{S_{h-1}}(T_h) = \sum_{h=1}^H \sum_{i \in [m]: f_i(S_{h-1}) < 1} \frac{f_i(S_h) - f_i(S_{h-1})}{1 - f_i(S_{h-1})},$$

which is intuitively the total amount of "residual requirement" that is covered by the algorithm in phase j. We first lower bound Δ_j . Let T^* denote the set of vertices that OPT visited within time 2^j . Observe that in Line (5), ALG-MLSC could have visited all nodes in T^* by choosing P as OPT's prefix of length 2^j . Via the approximation guarantee of ALG-SOP, we obtain

Proposition 3.6. For any $h \in [H]$, we have $f^{S_{h-1}}(T_h) \geq \frac{1}{\rho} \cdot f^{S_{h-1}}(T^*)$.

We restrict our concern to the functions in $R_j \setminus R_j^*$. Observe that for any $i \in R_j$ and $h \in [H]$, $f_i(S_{h-1}) < 1$ and that for any $i \notin R_j^*$, $f_i(T^*) = 1$. Hence by summing the inequality in the above proposition over all functions f_i in $R_j \setminus R_j^*$, we have

$$\Delta_{j} \geq \frac{1}{\rho} \sum_{h=1}^{H} f^{S_{h-1}}(T^{*}) \geq \frac{1}{\rho} \sum_{h=1}^{H} \sum_{i \in R_{j} \setminus R_{j}^{*}} \frac{f_{i}(T^{*} \cup S_{h-1}) - f_{i}(S_{h-1})}{1 - f_{i}(S_{h-1})} \geq \frac{1}{\rho} \sum_{h=1}^{H} \sum_{i \in R_{j} \setminus R_{j}^{*}} 1$$

$$\geq \frac{H}{\rho} (|R_{j}| - |R_{j}^{*}|) = 4\alpha (|R_{j}| - |R_{j}^{*}|)$$

$$(4)$$

We now upper bound Δ_j . Note that for any $i \notin R_{j-1}$, $f_i(S_0) = 1$ and therefore f_i does not contribute to Δ_j . For any $i \in R_{j-1}$, the total contribution of f_i to Δ_j is at most α by Claim 2.4. Hence,

$$\Delta_i \le \alpha |R_{i-1}| \tag{5}$$

Combining (4) and (5) completes the proof of Lemma 3.5.

Finally, we can use Lemma 3.5 to prove Theorem 3.3 exactly as we proved Theorem 2.1 in the previous section using Lemma 2.2. We omit repeating the calculations here.

4 Latency Covering Steiner Tree

In this section, we give consider the Latency Covering Steiner Tree problem (LCST), which is an interesting special case of MLSC. Recall that the input to LCST consists of a symmetric metric (V, d), root $r \in V$ and a collection $\mathcal G$ of groups, where each group $g \in \mathcal G$ is a subset of vertices with an associated requirement k_g . The goal is find a path staring from r that minimizes the total cover time of all groups. We say that group g is covered at the earliest time t when the path within distance t visits at least k_g vertices in g. We give an $O(\log g_{max} \cdot \log |V|)$ -approximation algorithm for this problem where $g_{max} := \max_{g \in \mathcal G} |g|$ is the maximum group size. This would prove Theorem 1.2.

Simplifying assumptions. Following [25, 21], without loss of generality, we assume that:

- 1. The metric is induced by a tree T = (V, E) with root r and weight w_e on each edge $e \in E$.
- 2. Every vertex in a group is a leaf, i.e. has degree one in T.
- 3. The groups in \mathcal{G} are disjoint.
- 4. Every vertex of degree one lies in some group.

The only non-trivial assumption is the first one, which uses tree embedding [14] to reduce general metrics to trees, at the loss of an $O(\log |V|)$ approximation factor. In the rest of this section, we work with such instances of LCST and obtain an $O(\log g_{max})$ -approximation algorithm.

We first discuss a new LP relaxation for the covering Steiner tree problem in Subsection 4.1, which can be shown to have a poly-logarithmic integrality gap. Next, in Subsection 4.2 extend this idea to obtain an LP relaxation for latency covering Steiner tree. In Subsection 4.3 we present our rounding algorithm for LCST, and finally Subsection 4.4 contains the analysis of the algorithm.

4.1 New LP Relaxation for CST

Recall that the input to Covering Steiner Tree consists of a metric (V, d) with root r and a collection of groups $\mathcal{G} \subseteq 2^V$ where each group $g \in \mathcal{G}$ is associated with a requirement k_g . The goal is to find a minimum cost r-rooted tree that includes r and at least k_g vertices from each group g. Although an $O(\log m \cdot \log g_{max} \cdot \log n)$ -approximation is known for CST [21], there was no (single) linear program known to have a poly-logarithmic integrality gap. Previous results on CST relied on an LP with large $\Omega(k_{max})$ integrality gap [25].

We introduce stronger constraints, that yield an LP for CST with integrality gap $O(\log m \cdot \log g_{max} \cdot \log n)$. This new LP is an important ingredient in our algorithm for LCST, and might also be useful in other contexts.

Let L denote the set of leaves in V. Because of the above simplifying assumptions, we can label each vertex v in a group with a unique leaf-edge incident on it, and vice versa. We abuse notation by allowing $j \in L$ to denote both the leaf-vertex and its unique incident edge. For any subset of leaves $L' \subseteq L$, let $\operatorname{cut}(r, L')$ denote the family of all edge-subsets whose removal separates the root r from all vertices in L'.

We formulate the following linear programming relaxation for CST on tree instances.

$$\min \quad \sum_{e \in E} w_e x_e \tag{LP}_{\mathsf{CST}})$$

s.t.
$$x_{pe(e)} \ge x_e$$
 $\forall e \in E$ (6)

$$(k_g - |A|) \sum_{j \in B \setminus L} x_j + \sum_{j \in B \cap (L \setminus A)} x_j \ge k_g - |A| \qquad \forall g \in \mathcal{G}, \forall A \subseteq g, \forall B \in \mathsf{cut}(r, g \setminus A)$$
 (7)

$$x_e \in [0,1]$$
 $\forall e \in E$

Validity of LP_{CST}. We first argue that this is a valid relaxation. Consider any instance of CST on trees and a fixed feasible solution (tree) τ^* , which gives a natural integral solution: $x_e = 1$ if and only if $e \in \tau^*$. We focus on constraints (7), since the other constraints are obviously satisfied. Consider any $g \in \mathcal{G}$, $A \subseteq g$ and $B \in \operatorname{cut}(r, g \setminus A)$. Let $\tau^*(E \setminus A)$ denote the subtree induced by the edges in both τ^* and $(E \setminus A)$, i.e. $\tau^* \cap (E \setminus A)$. Note that $\tau^*(E \setminus A)$ is connected, since A consists only of leaf edges. Since τ^* has at least k_g edges from g (it is a feasible CST solution), we have $|\tau^*(E \setminus A) \cap (g \setminus A)| \ge k_g - |A|$.

- Suppose that there exists $\bar{j} \in \tau^*(E \setminus A) \cap B$ such that $\bar{j} \notin L$. Then since $\bar{j} \in B \setminus L$, it follows that $(k_g |A|) \sum_{j \in B \setminus L} x_j \geq k_g |A|$, hence the constraint is satisfied.
- The remaining case has $\tau^*(E \setminus A) \cap B \subseteq L$. In words, $B \text{ cuts } g \setminus A$ from r using only leaf edges; so $B \supseteq \tau^*(E \setminus A) \cap (g \setminus A)$. Thus $\sum_{i \in B \cap (L \setminus A)} x_i \ge |\tau^*(E \setminus A) \cap (g \setminus A)| \ge k_g |A|$.

In both the above cases, constraint (7) is satisfied.

Solving LP_{CST}. Since LP_{CST} has exponentially many constraints, in order to solve it in polynomial time, we need a separation oracle. Again we focus on constraints (7), since other constraints are only polynomially many. We observe that this separation oracle reduces to the following problem.

Problem MinCutWithExceptions: Given as input a tree T rooted at r with leaves L and cost $\ell(e)$ on each edge e and an integer $D \ge 0$, the goal is to find a minimum cost cut that separates r from any D leaves.

To see how this suffices to separate constraints (7), consider any fixed $g \in \mathcal{G}$ and all $A \subseteq g$ with $|A| = \eta$ (finally we iterate over all $g \in \mathcal{G}$ and $0 \le \eta \le n$). Then $k_g - |A|$ (the right-hand-side of the constraints) is also fixed. Given x_j values, we would like to find $A \subseteq g$ with $|A| = \eta$ and $B \in \operatorname{cut}(r, g \setminus A)$ that minimizes the left-hand-side, and test if this is smaller than $k_g - \eta$. Formally, we can recast this into MinCutWithExceptions as follows: Remove all edges from E that are not on any path from the root r to a vertex in g, and let T' be the resulting tree and this is the input tree to the problem. Note that leaves of T' are precisely g. For all leaf-edges $j \in g$, let $\ell(j) := x_j$; and for all non-leaf $e \in T' \setminus g$, $\ell(e) := (k_g - \eta) \cdot x_e$. Also set bound $D := |g| - \eta$.

We next show that MinCutWithExceptions can be solved via a dynamic programming.

Lemma 4.1. The problem MinCutWithExceptions can be solved in polynomial time.

Proof. To formally describe our dynamic program, we make some simplifying assumptions. By introducing dummy edges of infinite cost, we assume without loss of generality, that the tree T is binary and the root r is incident to exactly one edge e_r . Hence every non-leaf edge e has exactly two child-edges e_1 and e_2 . For any edge $e \in T$, let T_e denote the subtree of T rooted at e, i.e. T_e contains edge e and all its descendants.

We define a recurrence for C[e, k] which denotes the minimum cost cut that separates the root of T_e from exactly k leaves in T_e . Note that $C[e_r, D]$ gives the optimal value.

For any leaf-edge f set:

$$C[f,k] = \begin{cases} 0 & \text{if } k = 0, \\ \ell(f) & \text{if } k = 1, \text{ and } \\ \infty & \text{otherwise.} \end{cases}$$

For any non-leaf edge e with children e_1 and e_2 , set:

$$C[e,k] = \begin{cases} 0 & \text{if } k = 0;\\ \min\limits_{k_1 + k_2 = k} \{C[e_1, k_1] + C[e_2, k_2]\} & \text{if } 1 \leq k < |L \cap T_e|;\\ \min\limits_{k_1 + k_2 = k} \{\ell(e), & \text{if } k = |L \cap T_e|;\\ \infty & \text{otherwise.} \end{cases}$$

It can be checked directly that this recurrence computes the desired values in polynomial time.

4.2 LP Relaxation for LCST

We formulate the following linear relaxation for tree instances of latency covering Steiner tree.

$$\min \quad \frac{1}{2} \cdot \sum_{\ell > 0} 2^{\ell} \sum_{g \in \mathcal{G}} (1 - y_g^{\ell}) \tag{LP}_{\mathsf{LCST}})$$

s.t.
$$x_{pe(e)}^{\ell} \ge x_e^{\ell}$$
 $\forall \ell \ge 0, e \in E$ (8)

$$\sum_{i \in E} w_e x_e^{\ell} \le 2^{\ell} \tag{9}$$

$$(k_g - |A|) \sum_{j \in B \setminus L} x_j^{\ell} + \sum_{j \in B \cap L \setminus A} x_j^{\ell} \ge (k_g - |A|) \cdot y_g^{\ell} \quad \forall \ell \ge 0, g \in \mathcal{G}, A \subseteq g, B \in \mathsf{cut}(r, g \setminus A) \quad (10)$$

$$y_g^{\ell+1} \ge y_g^{\ell} \qquad \qquad \forall \ell \ge 0, g \in \mathcal{G} \qquad (11)$$

$$x_e^\ell \in [0,1] \hspace{1cm} \forall \ell \geq 0, e \in E$$

$$y_g^\ell \in [0,1] \qquad \qquad \forall \ell \geq 0, g \in \mathcal{G}$$

To see that this is a valid relaxation, let OPT denote the optimal path. For any $\ell \geq 0$ let $\mathsf{OPT}(2^\ell)$ denote the prefix of length 2^ℓ in OPT . We construct a feasible integral solution to $\mathsf{LP}_{\mathsf{LCST}}$ as follows. The variable x_e^ℓ indicates if edge e lies in $\mathsf{OPT}(2^\ell)$. The indicator variable y_g^ℓ has value one if and only if group g is covered by $\mathsf{OPT}(2^\ell)$, i.e. at least k_g vertices of g are contained in $\mathsf{OPT}(2^\ell)$. Constraints (8) follow from the fact that $\mathsf{OPT}(2^\ell)$ is a path starting at r. Constraints (9) say that the edges in $\mathsf{OPT}(2^\ell)$ have a total weight of at most 2^ℓ , which is clearly true. Note that for each $\ell \geq 0$, there is a set of constraints (10) that is similar to the constraints (7) in $\mathsf{LP}_{\mathsf{CST}}$; the validity of these constraints (10) can be shown exactly as for (7). Constraints (11) enforce the fact that if group g is covered by $\mathsf{OPT}(2^\ell)$ then it must be covered by $\mathsf{OPT}(2^{\ell+1})$ as well, which is trivially true. Now consider the objective value: the total contribution of a group g that is covered by OPT at some time $t \in (2^k, 2^{k+1}]$ is $\frac{1}{2} \cdot \sum_{\ell=0}^k 2^\ell \leq 2^k$. Thus the objective value of this integral solution is at most OPT .

We can ensure by standard scaling arguments, at the loss of a 1 + o(1) factor in the objective, that all distances are polynomially bounded. This implies that the length of any optimal path is also polynomial, and so it suffices to consider $O(\log n)$ many values of ℓ . Thus the number of variables in $\mathsf{LP}_{\mathsf{LCST}}$ is polynomial. Note that constraints (10) are exponentially many. However, for each fixed ℓ and g, we can use the same separation oracle that we used for the constraints (7) of $\mathsf{LP}_{\mathsf{CST}}$.

4.3 Rounding Algorithm for LCST

Before presenting our algorithm for LCST, we discuss the basic rounding scheme from [25] (which is an extension of [16]) and some of its useful properties.

Algorithm 3 ALG-KRS [25]

INPUT: Undirected tree T = (V, E) rooted at $r; z_e \in [0, 1]$, such that for all $e \in E, z_{pe(e)} \ge z_e$.

- 1: $S \leftarrow \emptyset$.
- 2: For each $e \in E$ incident to the root r, add e to S with probability z_e .
- 3: For each $e \in E$ such that $pe(e) \in S$, add e to S with probability $\frac{z_e}{z_{pe(e)}}$.

OUTPUT: The connected component (tree) S.

Proposition 4.2 ([25]). Each edge e is included in the final solution of ALG-KRS with probability z_e .

Proof. We prove this by induction on the depth of edge e from r. The base case involves edges incident to the root r, where this property is clearly true. For the inductive step, assume that the parent edge pe(e) of

e is included with probability $z_{pe(e)}$; then by the algorithm description, edge e is included with probability $z_{pe(e)} \cdot \frac{z_e}{z_{pe(e)}} = z_e.$

Definition 4.3 (KRS properties). Consider any $z \in [0,1]^E$, $g \in \mathcal{G}$, $R(g) \subseteq g$ and $0 \le r_g \le |R(g)|$. We say that $(z, R(g), r_g)$ satisfies the KRS properties if it satisfies the following:

$$z_{pe(e)} \ge z_e \qquad \forall e \in E$$
 (12)

$$z_{pe(e)} \geq z_e \qquad \forall e \in E$$

$$\sum_{j \in T(e) \cap R(g)} z_j \leq r_g \cdot z_e \qquad \forall e \in E$$
(12)

where T(e) is the subtree below (and including) edge e.

The first property (12) is the same as the constraints (8). The second property (13) is a Lipschitz-type condition which implies that conditional on any edge e being chosen, its subtree T(e) can contribute at most r_q to the requirement of R(g).

Lemma 4.4 ([25]). Suppose that $(z, R(g), r_g)$ satisfies the KRS properties. Let L_{krs} denote the set of leaves that are covered by ALG-KRS with input $\{z_e : e \in E\}$. Consider any constant $\delta \in [0,1]$. Then for any $g \in \mathcal{G}$,

$$\Pr\left[|L_{krs} \cap R(g)| \le (1 - \delta)\mu_g\right] \le \exp\left(-\frac{\delta^2 \cdot \mu_g}{2 + r_g(1 + \ln|R(g)|)}\right)$$

where $\mu_g := \mathbb{E}[|L_{krs} \cap R(g)|] = \sum_{j \in R(g)} z_j$.

Proof. We only give a sketch of the proof, since this is implicit in [25]. For any $j, j' \in R(g)$, we say that $j \sim j'$ if and only if (1) $j \neq j'$ and (2) the least common ancestor lca(j,j') of j and j' is not r. Define

$$\Delta_g := \sum_{j,j' \in R(g): j \sim j', z_{\mathsf{lca}(j,j')} > 0} \frac{z_j \cdot z_{j'}}{z_{\mathsf{lca}(j,j')}}$$

In Theorem 3.2 in [25], Konjevod et al. showed using the KRS properties that

$$\Delta_g \le \mu_g(r_g - 1 + r_g \ln |R(g)|)$$

We note that the proof of Theorem 3.2 implies this, although it is stated only for $\mu_g = r_g$. Further, they used this bound in Jansen's inequality to obtain for any $\delta \in [0, 1]$,

$$\Pr\left[|L_{krs} \cap R(g)| \le (1 - \delta)\mu_g\right] \le \exp\left(-\frac{\delta^2 \mu_g}{2 + \Delta_g/\mu_g}\right)$$

Above $\mu_g = \mathbb{E}[|L_{krs} \cap R(g)|] = \sum_{j \in R(g)} \Pr[j \in L_{krs}] = \sum_{j \in R(g)} z_j$, by Proposition 4.2. Combining the above two inequalities yields the lemma.

We are now ready to present our algorithm to round $\mathsf{LP}_{\mathsf{LCST}}$, described formally as $\mathsf{ALG}\text{-}\mathsf{LCST}$ below. Let $(\overline{x}, \overline{y})$ denote a fixed optimal solution to LP_{LCST}. The algorithm proceeds in phases $\ell = 0, 1, 2, \cdots$ where the ℓ^{th} phase rounding uses variables with superscript ℓ in LP_{LCST}. Let $\overline{E}^{\ell} := \{e \in E \mid \overline{x}_e^{\ell} \geq 1/4\}$. Observe that \overline{E}^{ℓ} forms a tree rooted at r due to the constraints (8). The edges in \overline{E}^{ℓ} are added to our solution with probability one by the ℓ -th phase of our algorithm. Tree \tilde{T}^{ℓ} is obtained from T by contracting edges \overline{E}^{ℓ} . Let $R^{\ell}(g) := g \setminus \overline{E}^{\ell}$ and $r_g^{\ell} = k_g - |g \cap \overline{E}^{\ell}|$ denote the residual vertices of group g and its residual requirement, in phase ℓ . In the subsequent analysis, we will show that the algorithm satisfies a group with constant probability in every phase when it is substantially covered (say to extent $\frac{1}{2}$) by the fractional solution $(\overline{x}, \overline{y})$.

In each phase $\ell \geq 0$ we preprocess (in Line 6) \bar{x}^{ℓ} to obtain \tilde{x}^{ℓ} as described in the next lemma.

Algorithm 4 ALG-LCST

```
INPUT: Tree T with edge lengths, root r, groups \mathcal{G} and requirements \{k_g\}_{g\in\mathcal{G}}.
  1: \pi \leftarrow \emptyset.
  2: Let (\overline{x}, \overline{y}) be an optimal solution to LP<sub>LCST</sub>.
  3: for \ell = 0, 1, 2, ... do
            \overline{E}^{\ell} \leftarrow \{e \in E \mid \overline{x}_e^{\ell} \geq 1/4\}, \, R^{\ell}(g) \leftarrow g \setminus \overline{E}^{\ell} \text{ and } r_g^{\ell} \leftarrow k_g - |g \cap \overline{E}^{\ell}|.
            Shrink all edges in \overline{E}^{\ell} in T and let \tilde{T}^{\ell} be the resulting tree with the edge set \tilde{E}^{\ell} := E \setminus \overline{E}^{\ell}.
             Obtain solution \tilde{x}^{\ell} from \bar{x}^{\ell} using Lemma 4.5.
  6:
             For each e \in \tilde{E}^{\ell}, z_e^{\ell} \leftarrow 4\tilde{x}_e^{\ell}; note that z_e^{\ell} \in [0,1].
  7:
             \mathcal{S}^{\ell} \leftarrow \emptyset.
  8:
             repeat the following 6(3 + \log g_{max}) times:
 9:
                    	au^\ell \leftarrow 	ext{ the tree produced by ALG-KRS with } z = z^\ell 	ext{ on tree } \tilde{T}^\ell
10:
                    Add \tau^{\ell} to \mathcal{S}^{\ell}
11:
             Combine all trees in \mathcal{S}^{\ell} with \overline{E}^{\ell} and take an Euler tour P^{\ell} of the resulting tree.
12:
             if P^{\ell} has weight at most 192(3 + \log g_{max}) \cdot 2^{\ell} then
13:
                    \pi \leftarrow \pi \cdot P^{\ell}.
14:
15: end for
OUTPUT: Path \pi originating from r.
```

Lemma 4.5. For any $\ell \geq 0$, we can find in polynomial time $\tilde{x}_e^{\ell} \in [0, \overline{x}_e^{\ell}]$, $\forall e \in E \setminus \overline{E}^{\ell}$ such that $\forall g \in \mathcal{G}$:

1. $(\tilde{x}^{\ell}, R^{\ell}(g), r_g^{\ell})$ satisfies the KRS-properties in tree \tilde{T}^{ℓ} .

2.
$$\sum_{j \in R^{\ell}(g)} \tilde{x}_{j}^{\ell} \geq r_{g}^{\ell} \cdot \overline{y}_{g}^{\ell}$$
 (coverage property).

Proof. Fix any $\ell \geq 0$. To reduce notation, we drop the superscript ℓ from \tilde{T} , \overline{E} , \overline{x} , \overline{y} , \tilde{x} , R(g) and r_g throughout this proof. Consider constraints (10) of $\mathsf{LP}_{\mathsf{LCST}}$. Fix a group $g \in \mathcal{G}$ and let $A := g \cap \overline{E}$. Consider tree \tilde{T} as a flow network with each leaf edge f having capacity \overline{x}_f and each non-leaf edge f having capacity f and each non-leaf edge f having capacity the min cut separating f from f from f has value at least f and each non-leaf edge f having capacity f have f is obtained from f by edge-contraction. Hence there must exist a max-flow of volume at least f is obtained from f by edge-contraction. Hence there must exist a max-flow of volume at least f is obtained from f by edge-contraction. Let f denote the volume of this flow into each leaf edge $f \in f$ exist f have have that f in the above network. Let f denote the volume of this flow into each leaf edge $f \in f$ exist f have have that f in the exist f denote the volume of this flow into each leaf edge f is clearly we have that f in the exist f denote the volume of this flow into each leaf edge f is clearly we have that f in the exist f in the exist f in the exist f in the exist f is exist.

$$\sum_{j \in R(g)} \tilde{x}_j \ge r_g \cdot \overline{y}_g. \tag{14}$$

Moreover, by the capacities on non-leaves,

$$\sum_{j \in T(e) \cap R(g)} \tilde{x}_j \le r_g \cdot \overline{x}_e, \quad \forall e \in E \setminus \overline{E}$$
 (15)

We can use the above procedure on each group $g \in \mathcal{G}$ separately, to compute \tilde{x}_f for all leaf edges $f \in E \setminus \overline{E}$; this is well-defined since groups are disjoint. For each non-leaf edge $e \in E \setminus \overline{E}$ set $\tilde{x}_e := \overline{x}_e$. Thus we have $0 \le \tilde{x}_e \le \overline{x}_e$ for all $e \in E \setminus \overline{E}$. Observe that this computation can easily be done in polynomial time.

Now, (15) implies the second KRS property (13). Property (12) follows, since for each $e \in E \setminus \overline{E}$, we have $\tilde{x}_{pe(e)} = \overline{x}_{pe(e)} \ge \overline{x}_e \ge \tilde{x}_e$; the first inequality is due to constraint (8) of LP_{LCST}. Finally, (14) implies the coverage property claimed in the lemma.

⁴In particular every cut B' separating r from $g \setminus A$ in \tilde{T} is also a cut separating r from $g \setminus A$ in T.

4.4 Analysis

For any group g, define $\ell(g)$ to be the smallest $\ell \geq 0$ such that $\overline{y}_g^\ell \geq 1/2$. Then it follows that for any $\ell \geq \ell(g)$, $\overline{y}_g^\ell \geq 1/2$ due to constraints (11) of LP_{LCST}. In words, the optimal fractional solution covers group g to an extent of at least half within time $2^{\ell(g)}$. Consider any group $g \in \mathcal{G}, \ \ell \geq \ell(g)$ and a tree τ^ℓ in Line (10) of ALG-LCST. Since all edges in \overline{E}^ℓ are included in P^ℓ with probability 1, group g is covered by P^ℓ if and only if at least r_g^ℓ vertices in its residual set $R^\ell(g)$ are covered by τ^ℓ . This motivates us to derive the following.

Lemma 4.6. For any $g \in \mathcal{G}$ and $\ell \geq \ell(g)$,

$$\Pr[|\tau^{\ell} \cap R(g)| < r_g] \le \exp\left(-\frac{1}{2(3 + \ln q_{max})}\right).$$

Proof. From Lemma 4.5 it follows that $(\tilde{x}^{\ell}, R^{\ell}(g), r_g^{\ell})$ satisfies the KRS properties on tree \tilde{T}^{ℓ} . Since $z^{\ell} = 4 \cdot \tilde{x}^{\ell}$, $(z^{\ell}, R^{\ell}(g), r_g^{\ell})$ also satisfies the KRS properties. Furthermore, using $\overline{y}_g^{\ell} \geq \frac{1}{2}$ and the coverage property in Lemma 4.5,

$$\mu_g^\ell \ := \ \mathbb{E}[|\tau^\ell \cap R^\ell(g)|] \ = \ \sum_{j \in R^\ell(g)} z_j^\ell = \ 4 \cdot \sum_{j \in R^\ell(g)} \tilde{x}_j^\ell \ \ge \ 4 \cdot r_g^\ell \cdot \overline{y}_g^\ell \ \ge \ 2r_g^\ell.$$

Here we also used Proposition 4.2 that $\Pr[j \in \tau^{\ell}] = z_j^{\ell}$. By applying Lemma 4.4 with $\delta = 1/2$, we have

$$\Pr\left[|\tau^{\ell} \cap R(g)| < r_g\right] \leq \exp\left(-\frac{r_g^{\ell}}{2(2 + r_g^{\ell}(1 + \ln|R^{\ell}(g)|))}\right) \leq \exp\left(-\frac{1}{2(3 + \ln g_{max})}\right).$$

This proves Lemma 4.6.

Lemma 4.7. Consider any group $g \in \mathcal{G}$ and $\ell \geq \ell(g)$. The probability that P^{ℓ} has a total weight of at most $192(3 + \log g_{max}) \cdot 2^{\ell}$ and covers g is at least 3/4.

Proof. By Proposition 4.2, we know that each edge $e \in \tilde{E}^{\ell}$ is included in τ^{ℓ} with probability $z_e^{\ell} = 4\tilde{x}_e^{\ell} \le 4\bar{x}_e^{\ell}$. Since for all $e \in \overline{E}^{\ell}$, $\overline{x}_e \ge 1/4$, the expected total weight of the edges in \overline{E}^{ℓ} and τ^{ℓ} is upper bounded by

$$\sum_{e \in \overline{E}^{\ell}} w_e + \sum_{e \in \tilde{E}^{\ell}} w_e \cdot 4\tilde{x}^{\ell} \quad \leq \quad 4 \sum_{e \in E} w_e \cdot \overline{x}^{\ell}_e \quad \leq \quad 4 \cdot 2^{\ell}$$

The last inequality is due to the constraints (9). Hence the expected cost of P^{ℓ} is at most $24(3 + \log g_{max}) \cdot 2^{\ell}$. Markov's inequality immediately gives that the total weight of P^{ℓ} is greater than $192(3 + \log g_{max}) \cdot 2^{\ell}$ with probability at most 1/8. Since τ_{ℓ} is sampled $6(3 + \log g_{max})$ times independently, from Lemma 4.6, we know that group g is not covered by P^{ℓ} with probability at most $1/e^3 \leq 1/8$. Hence the lemma follows.

Fix any group $g \in \mathcal{G}$, and $\ell \geq \ell(g)$. Among $P^{\ell(g)}$, $P^{\ell(g)+1}$, ..., P^{ℓ} , consider the paths that are added to π . Clearly the total weight of such paths is at most $O(\log g_{max} \cdot 2^{\ell})$. By Lemma 4.7, the probability that none of these paths covers g is at most $\frac{1}{4^{\ell-\ell(g)+1}}$. Hence the expected cover time of g is at most

$$\sum_{\ell > \ell(g)} O(\log g_{max}) \cdot 2^{\ell} \cdot \frac{1}{4^{\ell - \ell(g) + 1}} = O(\log g_{max} \cdot 2^{\ell(g)}).$$

Thus the expected total cover time is at most $O(\log g_{max}) \cdot \sum_{g \in \mathcal{G}} 2^{\ell(g)}$.

By definition of $2^{\ell(g)}$ being the "half completion time" in the LP, we know

$$\mathsf{OPT} \geq \ \frac{1}{2} \cdot \sum_{\ell > 0} 2^{\ell} \ \sum_{g \in \mathcal{G}} (1 - \overline{y}_g^{\ell}) \geq \frac{1}{2} \sum_{g \in \mathcal{G}} 2^{\ell(g) - 1} \left(1 - \overline{y}_g^{\ell(g) - 1} \right) \geq \frac{1}{8} \cdot \sum_{g \in \mathcal{G}} 2^{\ell(g)}.$$

Thus we obtain that ALG-LCST is an $O(\log g_{max})$ -approximation for LCST on tree instances. Using probabilistic tree embedding [14], we conclude that ALG-LCST yields an $O(\log g_{max} \cdot \log |V|)$ -approximation for general metrics, thereby proving Theorem 1.2.

5 Weighted Stochastic Submodular Ranking

In this section, we study the Weighted Stochastic Submodular Ranking problem (WSSR). The input consists of a set $\mathcal{A} = \{X_1, ..., X_n\}$ of n independent random variables (stochastic elements), each over domain Δ , with integer lengths $\{\ell_j\}_{j=1}^n$ (deterministic), and m monotone submodular functions $f_1, ..., f_m : 2^{\Delta} \to [0, 1]$ on groundset Δ . We are also given the distribution (over Δ) of each stochastic element $\{X_j\}_{j=1}^n$. (We assume explicit probability distributions, i.e. for each X_j and $b \in \Delta$ we are given $\Pr[X_j = b]$.) The realization $x_j \in \Delta$ of the random variable X_j is known immediately after scheduling it. Here, X_j requires ℓ_j units of time to be scheduled; if X_j is started at time t then it completes at time $t + \ell_j$ at which point its realization $x_j \in \Delta$ is also known. A feasible solution/policy is an adaptive ordering of A, represented naturally by a decision tree with branches corresponding to the realization of the stochastic elements. We use $\langle \pi(1), \ldots, \pi(n) \rangle$ to denote this ordering, where each $\pi(l)$ is a random variable denoting the index of the l-th scheduled element.

The cover time $\operatorname{cov}(f_i)$ of any function f_i is defined as the earliest time t such that f_i has value one for the realization of the elements completely scheduled within time t. More formally, $\operatorname{cov}(f_i)$ is the earliest time t such that $f_i(\{x_{\pi(1)},...,x_{\pi(k_t)}\})$ is equal to 1 where k_t is the maximum index such that $\ell_{\pi(1)} + \ell_{\pi(2)} + ... + \ell_{\pi(k_t)} \leq t$. If the function value never reaches one (due to the stochastic nature of elements) then $\operatorname{cov}(f_i) = \ell_1 + \ell_2 + ... + \ell_n$ which is the maximum time of any schedule. Note that the cover time is a random value. The goal is to find a policy that (approximately) minimizes the expected total cover time $\mathbb{E}\left[\sum_{i \in [m]} \operatorname{cov}(f_i)\right]$.

5.1 Applications

Our stochastic extension of submodular ranking captures many interesting applications.

Stochastic Set Cover. We are given as input a ground set Δ , and a collection $\mathcal{S} \subseteq 2^{\Delta}$ of (non-stochastic) subsets. There are stochastic elements $\{X_j: j \in [n]\}$, each defined over Δ and having respective costs $\{\ell_j: j \in [n]\}$. The goal is to give an adaptive policy that hits all sets in \mathcal{S} at the minimum expected cost. This problem was studied in [17, 27, 26]. The problem can be shown as an instance of WSSR with a single monotone submodular function $f_1(A) := \frac{1}{|\mathcal{S}|} \sum_{S \in \mathcal{S}} \min\{1, |A \cap S|\}$ and parameter $\epsilon = 1/|\mathcal{S}|$.

Shared Filter Evaluation. This problem was introduced by [27], and the result was improved to an essentially optimal solution in [26]. In this problem, there is a collection of independent "filters" $X_1, X_2,, X_n$, each of which gets evaluated either to True or False. For each filter $j \in [n]$, we are given the "selectivity" $p_j = \Pr[X_i \text{ is true}]$ and the cost ℓ_j of running the filter. We are also given a collection $\mathcal Q$ of queries, where each query Q_i is a conjunction of a subset of queries. We would like to determine each query in $\mathcal Q$ to be True or False by (adaptively) testing filters of the minimum expected cost. In order to cast this problem as WSSR, we use $\Delta = \bigcup_{j=1}^n \{Y_j, N_j\}$; for each $j \in [n], X_j = Y_j$ with probability p_j , and $X_j = N_j$ with the remaining probability $1 - p_j$. We create one monotone submodular function:

$$f_1(A) := \frac{\sum_{Q_i \in \mathcal{Q}} \min \left\{ 1, \ |A \cap \{N_j : j \in Q_i\}| + \frac{1}{|Q_i|} \cdot |A \cap \{Y_j : j \in Q_i\}| \right\}}{|\mathcal{Q}|}$$

(Note that a query Q_i gets evaluated to: False if any one of its filters is False, and True if all its filters are True.) Here the parameter $\epsilon = 1/\left(|\mathcal{Q}| \max_i |Q_i|\right)$.

We note that the Shared Filter Evaluation problem can be studied for a latency type of objective also. In this case, for each query $Q_i \in \mathcal{Q}$, we create a separate submodular function:

$$f_i(A) := \min \left\{ 1, \ |A \cap \{N_j : j \in Q_i\}| + \frac{1}{|Q_i|} \cdot |A \cap \{Y_j : j \in Q_i\}| \right\}$$

In this case, the WSSR problem corresponds precisely to filter evaluation that minimizes the average time to answer queries in Q. The parameter $\epsilon = 1/(\max_i |Q_i|)$.

Stochastic Generalized Min Sum Set Cover. We are given as input a ground set Δ , and a collection $S \subseteq 2^{\Delta}$ of (non-stochastic) subsets with requirement k(S) for each $S \in S$. There are stochastic elements $\{X_j : j \in [n]\}$, each defined over Δ . Set $S \in S$ is said to be completed when at least k(S) elements from S have been scheduled. The goal is to find an adaptive ordering of [n] so as to minimize the expected total completion time. This can be reduced to WSSR by defining function $f^S(A) := \min\{1, |A \cap S|/k(S)\}$ for each $S \in S$; here $\epsilon = 1/k_{max}$ where k_{max} denotes the maximum requirement.

For this problem, our result implies an $O(\log k_{max})$ -approximation to adaptive policies. However, for non-adaptive policies (where the ordering of elements is fixed *a priori*), one can obtain a better O(1)-approximation algorithm by combining the Sample Average Approximation (SAA) method [24, 9] with O(1)-approximations known for the non-stochastic version [3, 30].

We also note that the analysis in [1] for the deterministic submodular ranking was only for elements having unit sizes. Our analysis also holds under non-uniform sizes.

5.2 Algorithm and Analysis

We consider adaptive policies: this chooses at each time $\ell_{\pi(1)} + \ell_{\pi(2)} + ... + \ell_{\pi(k-1)}$, the element

$$X_{\pi(k)} \in \mathcal{A} \setminus \{X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}, ..., X_{\pi(k-1)}\}$$

after observing the realizations $x_{\pi(1)}, ..., x_{\pi(k-1)}$. So it can be described as a decision tree. Our main result is an $O(\log \frac{1}{\epsilon})$ -approximate adaptive policy, which proves Theorem 1.3. This result is again inspired by our simpler analysis of the algorithm from [1].

To formally describe our algorithm, we quickly define the probability spaces we are concerned with. We use $\Omega = \Delta^n$ to denote the outcome space of \mathcal{A} . We use the same notation Ω to denote the probability space induced by this outcome space. For any $S \subseteq \mathcal{A}$ and its realization s, let $\Omega(s)$ denote the outcome subspace that conforms to s. We can naturally define the probability space defined by $\Omega(s)$ as follows: The probability that $w \in \Omega(s)$ occurs is $\Pr_{\Omega}[w]/\Pr_{\Omega}[\Omega(s)]$; we also use $\Omega(s)$ to denote this probability space.

The main algorithm is given below and is a natural extension of the deterministic algorithm [1]. Let $\alpha := 1 + \ln(\frac{1}{\epsilon})$. In the output, $\pi(l)$ denotes the *l*th element in \mathcal{A} that is scheduled.

Algorithm 5 ALG-AG-STO

- 1: **INPUT**: $\mathcal{A} = \{X_1, ..., X_n\}$ with $\{\ell_1, ..., \ell_n\}$; $f_i : 2^{\Delta} \to [0, 1], i \in [m]$.
- 2: $S \leftarrow \emptyset$. (S are the elements completely scheduled so far, and s their instantiation.)
- 3: while there exists function f_i with $f_i(s) < 1$ do
- 4: Choose element X_e as follows,

$$X_e = \arg\max_{X_e \in \mathcal{A} \backslash S} \ \frac{\mathbb{E}_{\ \Omega(s)} \left[\sum_{i \in [m], f_i(s) < 1} \ \frac{f_i(s \cup \{X_e\}) - f_i(s)}{1 - f_i(s)} \right]}{\ell_e}$$

- 5: $S \leftarrow S \bigcup \{X_e\}.$
- 6: $\pi(|S|) \leftarrow X_e$. Schedule X_e and observe its realization.
- 7: end while
- 8: **OUTPUT**: An adaptive ordering π of \mathcal{A} .

Observe that taking expectation over $\Omega(s)$ in Step 4 is the same as expectation over the distribution of X_e since $X_e \notin S$ and the elements are independent. This value can be computed exactly since we have an explicit probability distribution of X_e . Also note that this algorithm implicitly defines a decision tree. We will show that ALG-AG-STO is an $O(\ln(\frac{1}{\epsilon}))$ -approximation algorithm for WSSR.

To simplify notation, without loss of generality, we assume that α is an integer. Let R(t) denote the (random) set of functions that are not satisfied by ALG-AG-STO before time t. Note that the set R(t) includes the functions that are satisfied exactly at time t. Analogously, the set $R^*(t)$ is defined for the optimal policy.

For notational convenience, we use $i \in R(t)$ interchangeably with $f_i \in R(t)$. Let $C(t) := \{f_1, ..., f_m\} \setminus R(t)$ and $C^*(t) := \{f_1, ..., f_m\} \setminus R^*(t)$. Note that all the sets $C(\cdot)$, $C^*(\cdot)$, $R(\cdot)$, $R^*(\cdot)$ are stochastic. We have that $\mathsf{ALG} = \sum_{t \in [n]} |R(t)|$ and $\mathsf{OPT} = \sum_{t \in [n]} |R^*(t)|$ and hence ALG and OPT are stochastic quantities. We show that $\mathbb{E}[\mathsf{ALG}] = O(\alpha) \cdot \mathbb{E}[\mathsf{OPT}]$ which suffices to prove the desired approximation ratio.

We are interested in the number of unsatisfied functions at times $\{8\alpha 2^j: j \in \mathbb{Z}_+\}$ by ALG-AG-STO and the number of unsatisfied functions at times $\{2^j: j \in \mathbb{Z}_+\}$ by the optimal policy. Let $R_j := R(8\alpha 2^j)$ and $R_j^* = R^*(2^j)$. It is important to note that R_j and R_j^* are concerned with different times, and they are stochastic. For notational simplicity, we let $R_{-1} := \emptyset$.

We show the following key lemma. Once we get this lemma, we can complete the proof similar to the proof of Theorem 2.1 via Lemma 2.2.

Lemma 5.1. For any $j \ge 0$, we have $\mathbb{E}[|R_j|] \le \frac{1}{4}\mathbb{E}[|R_{j-1}|] + \mathbb{E}[|R_j^*|]$.

Proof. The lemma trivially holds for j=0, so we consider any $j\geq 1$. For any $t\geq 1$, we use s_{t-1} to denote the set of elements completely scheduled by ALG-AG-STO by time t-1 along with their instantiations; clearly this is a random variable. Also, for $t\geq 1$ let $\sigma(t)\in [n]$ denote the (random) index of the element being scheduled during time slot (t-1,t]. Since elements have different sizes, note that $\sigma(t)$ is different from $\pi(t)$ which is the t-th element scheduled by ALG-AG-STO. Observe that s_{t-1} determines $\sigma(t)$ precisely, but not the instantiation of $X_{\sigma(t)}$.

Let $E_j^* \subseteq \mathcal{A}$ be the (stochastic) set of elements that is completely scheduled by the optimal policy within time 2^j . For a certain stochastic set (or elements) S, we denote its realization under an outcome w as S(w). For example, $X_i(w) \in \Delta$ is the realization of element X_i for outcome w; and $E_j^*(w)$ is the set of first 2^j elements completely scheduled by OPT (under w) along with their realizations.

For any time t and corresponding outcome s_{t-1} , define a set function:

$$f^{s_{t-1}}(D) := \sum_{i \in [m], f_i(s_{t-1}) < 1} \frac{f_i(s_{t-1} \cup D) - f_i(s_{t-1})}{1 - f_i(s_{t-1})}, \quad \forall D \subseteq \Delta.$$

We also use $f_i^{s_{t-1}}(D)$ to denote the term inside the above summation.

The function $f^{s_{t-1}}: 2^{\Delta} \to \mathbb{R}_+$ is monotone and submodular since it is a summation of monotone and submodular functions. We also define

$$F^{s_{t-1}}(X_e) := \mathbb{E}_{w \leftarrow \Omega(s_{t-1})} [f^{s_{t-1}}(X_e(w))], \quad \forall X_e \in \mathcal{A}.$$
 (16)

Observe that this is zero for elements $X_e \in s_{t-1}$.

Proposition 5.2. Consider any time t and outcome s_{t-1} . Note that s_{t-1} determines $\sigma(t)$. Then:

$$\frac{1}{\ell_{\sigma(t)}} \cdot F^{s_{t-1}}(X_{\sigma(t)}) \geq \frac{1}{\ell_i} \cdot F^{s_{t-1}}(X_i), \qquad \forall X_i \in \mathcal{A}$$

Proof. At some time $t' \leq t$ (right after s_{t-1} is observed) ALG-AG-STO chose to schedule element $X_{\sigma(t)}$ over all elements $X_i \in \mathcal{A} \setminus s_{t-1}$. By the greedy rule we know that the claimed inequality holds for any $X_i \in \mathcal{A} \setminus s_{t-1}$. Furthermore, the inequality holds for any element $X_i \in s_{t-1}$, since here $F^{s_{t-1}}(X_i) = 0$.

We now define the *expected gain* by ALG-AG-STO in step t as:

$$G_t := \mathbb{E}_{s_{t-1}} \left[\frac{1}{\ell_{\sigma(t)}} F^{s_{t-1}}(X_{\sigma(t)}) \right].$$
 (17)

And the expected total gain:

$$\Delta_j := \sum_{t=8\alpha 2^{j-1}}^{8\alpha 2^j} G_t \tag{18}$$

We complete the proof of Lemma 5.1 by upper and lower bounding Δ_i .

Upper bound for Δ_j . Fix any outcome $w \in \Omega$. Below, all variables are *conditioned on* w and hence they are all deterministic. (For ease of notation we do not write w in front of the variables).

$$\Delta_{j} := \sum_{t=8\alpha 2^{j-1}}^{8\alpha 2^{j}} \frac{1}{\ell_{\sigma(t)}} f^{s_{t-1}}(x_{\sigma(t)}) = \sum_{t=8\alpha 2^{j-1}}^{8\alpha 2^{j}} \frac{1}{\ell_{\sigma(t)}} \sum_{i \in [m]: f_{i}(s_{t-1}) < 1} f_{i}^{s_{t-1}}(x_{\sigma(t)})$$

$$\leq \sum_{t=8\alpha 2^{j-1}}^{8\alpha 2^{j}} \frac{1}{\ell_{\sigma(t)}} \sum_{i \in R_{j-1}} f_{i}^{s_{t-1}}(x_{\sigma(t)}) \leq \sum_{t \geq 1} \frac{1}{\ell_{\sigma(t)}} \sum_{i \in R_{j-1}} f_{i}^{s_{t-1}}(x_{\sigma(t)})$$

$$= \sum_{i \in R_{i-1}} \sum_{k=1}^{n} \frac{f_{i}(T_{k}) - f_{i}(T_{k-1})}{1 - f_{i}(T_{k-1})}$$

The first inequality uses the fact that any $i \notin R_{j-1}$ has f_i already covered before time $8\alpha 2^{j-1}$, and so it never contributes to Δ_j . In the last expression, $T_k := \{x_{\pi(1)}, ..., x_{\pi(k)}\} \subseteq \Delta$, the first k instantiations seen under w. The equality uses the fact that for each $\sum_{j=1}^{k-1} \ell_{\pi(j)} < t \le \sum_{j=1}^{k} \ell_{\pi(j)}$ we have $s_{t-1} = T_{k-1}$ and $\sigma(t) = k$. Finally, by Claim 2.4, the contribution of each function $f_i \in R_{j-1}$ is at most $\alpha := 1 + \ln \frac{1}{\epsilon}$. Thus we obtain $\Delta_j(w) \le \alpha |R_{j-1}(w)|$, and taking expectations,

$$\Delta_j \leq \alpha \mathbb{E}[|R_{j-1}|] \tag{19}$$

Lower bound for Δ_j . Consider any $8\alpha 2^{j-1} \leq t \leq 8\alpha 2^j$. We lower bound G_t . Condition on s_{t-1} ; this determines $\sigma(t)$ (but not $x_{\sigma(t)}$). Note that $\sum_{i=1}^n \ell_i \cdot \Pr[X_i \in E_j^* | s_{t-1}] \leq 2^j$ by definition of E_j^* being the elements that are completely scheduled by time 2^j in OPT. Hence we have

$$\sum_{X_i \in \mathcal{A}} \frac{\ell_i}{2^j} \cdot \Pr[X_i \in E_j^* | s_{t-1}] \le 1.$$

By applying Proposition 5.2 with the convex multipliers (over i) given above.

$$\frac{1}{\ell_{\sigma(t)}} F^{s_{t-1}}(X_{\sigma(t)}) \ge \sum_{X_i \in \mathcal{A}} \frac{\ell_i}{2^j} \Pr[X_i \in E_j^* | s_{t-1}] \cdot \frac{1}{\ell_i} F^{s_{t-1}}(X_i)$$

$$= \frac{1}{2^j} \sum_{X_i \in \mathcal{A}} \Pr[X_i \in E_j^* | s_{t-1}] \sum_{x_i \in \Delta} \Pr[X_i = x_i | s_{t-1}] \cdot f^{s_{t-1}}(x_i)$$

$$= \frac{1}{2^j} \sum_{X_i \in \mathcal{A}} \sum_{x_i \in \Delta} \Pr[X_i \in E_j^* \wedge X_i = x_i | s_{t-1}] \cdot f^{s_{t-1}}(x_i)$$

$$= \frac{1}{2^j} \sum_{w \in \Omega(s_{t-1})} \Pr[w | s_{t-1}] \sum_{X_i \in E_j^*(w)} f^{s_{t-1}}(X_i(w))$$
(20)

The first equality is by definition of $F^{s_{t-1}}(\cdot)$ from (16). The second equality holds since the optimal policy must decide whether to schedule X_i (by time 2^j) without knowing the realization of X_i . Now for each $w \in \Omega(s_{t-1})$, due to submodularity of the function $f^{s_{t-1}}(\cdot)$, we get

$$\sum_{X_i \in E_j^*(w)} f^{s_{t-1}}(X_i(w)) \ge f^{s_{t-1}}(E_j^*(w)) = \sum_{i \in [m], f_i(s_{t-1}) < 1} \frac{f_i(E_j^*(w)) - f_i(s_{t-1})}{1 - f_i(s_{t-1})} \ge |C_j^*(w)| - |C(t, w)|. \tag{21}$$

Recall that $E_j^*(w)$ denotes the set of elements scheduled by time 2^j in OPT (conditional on w), as well as the realizations of these elements. The equality comes from the definition of $f^{s_{t-1}}$. The last inequality

holds because $C(t, w) = \{i \in [m] : f_i(s_{t-1}) = 1\}$ and set $E_j^*(w)$ covers functions $C_j^*(w)$. Combining (20) and (21) gives:

$$\frac{1}{\ell_{\sigma(t)}} F^{s_{t-1}}(X_{\sigma(t)}) \geq \frac{\left(\mathbb{E}\left[|C_{j}^{*}| \mid s_{t-1} \right] - \mathbb{E}\left[|C(t)| \mid s_{t-1} \right]\right)}{2^{j}}.$$

By deconditioning the above inequality (taking expectation over s_{t-1}) and using (17), we derive:

$$G_t \geq \frac{1}{2^j} \cdot \left(\mathbb{E}[|C_j^*|] - \mathbb{E}[|C(t)|] \right) \geq \frac{1}{2^j} \cdot \left(\mathbb{E}[|C_j^*|] - \mathbb{E}[|C_j|] \right),$$

where the last inequality uses $\mathbb{E}[C(t)]$ is non-decreasing and $t \leq 8\alpha 2^{j}$.

Now summing over all $t \in [8\alpha 2^{j-1}, 8\alpha 2^j)$ yields:

$$\Delta_j = \sum_{t=8\alpha 2^{j-1}}^{8\alpha 2^j} G_t \geq 4\alpha \left(\mathbb{E}[|C_j^*|] - \mathbb{E}[|C_j|] \right) = 4\alpha \left(\mathbb{E}[|R_j|] - \mathbb{E}[|R_j^*|] \right). \tag{22}$$

Combining (22) and (19), we obtain:

$$4\alpha(\mathbb{E}[|R_i|] - \mathbb{E}[|R_i^*|]) \leq \alpha \mathbb{E}[|R_{i-1}|]$$

which simplifies to the desired inequality in Lemma 5.1.

Using exactly the same calculations as in the proof of Theorem 2.1 from Lemma 2.2, Lemma 5.1 implies an $O(\alpha)$ -approximation ratio for ALG-AG-STO. This completes the proof of Theorem 1.3.

6 Conclusion

In this paper we considered the minimum latency submodular cover problem in general metrics, which is a common generalization of many well-studied problems. We also studied the stochastic submodular ranking problem, which generalizes a number of stochastic optimization problems. Both results were based on a new analysis of the algorithm for submodular ranking [1]. Our result for stochastic submodular ranking is tight, and any significant improvement (more than a $\log^{\delta} |V|$ factor) of the result for minimum latency submodular cover would also improve the approximation ratio for Group Steiner Tree, which is a long-standing open problem. An interesting open question is to obtain a poly-logarithmic approximation for stochastic minimum latency submodular cover (on general metrics).

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