# Sums of Uncertainty: Refinements Go Gradual 

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#### Abstract

A long-standing shortcoming of statically typed functional languages is that type checking does not rule out pattern-matching failures (run-time match exceptions). Refinement types distinguish different values of datatypes; if a program annotated with refinements passes type checking, pattern-matching failures become impossible. Unfortunately, refinement is a monolithic property of a type, exacerbating the difficulty of adding refinement types to nontrivial programs.

Gradual typing has explored how to incrementally move between static typing and dynamic typing. We develop a type system of gradual sums that combines refinement with imprecision. Then, we develop a bidirectional version of the type system, which rules out excessive imprecision, and give a type-directed translation to a target language with explicit casts. We prove that the static sublanguage cannot have match failures, that a well-typed program remains well-typed if its type annotations are made less precise, and that making annotations less precise causes target programs to fail later. Several of these results correspond to criteria for gradual typing given by Siek et al. (2015).


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## 1. Introduction

A central feature of statically typed functional languages is pattern matching over user-defined datatypes that combine several fundamental constructs: sum types (for example, an element of a bool datatype can be either True or False), recursive types (such as lists), and polymorphic types. The aspect of ML datatypes that corresponds to sum types is the focus of this paper.

Static typing is said to catch run-time errors-at least, errors that would manifest in a dynamically typed language as tag check failures, such as subtracting a string from a number. Using the venerable encoding of dynamic typing as injections into a datatype Dynamic (Abadi et al. 1991), these tag check failures become errors raised in the "fall-through" arm of a case expression over Dynamic. The impossibility of such errors is a convincing argument in favour of static typing.

Yet Standard ML programmers frequently write code that is essentially the same as the scorned operations on Dynamic-and that has the same unfortunate risk of run-time errors. The definition of SML (Milner et al. 1997) requires compilers to accept nonexhaustive case expressions, which do not cover all the possible instances of the datatype. A nonexhaustive case expression is isomorphic to an implicit tag check over Dynamic: the non-error case is the only one written out explicitly, while an error case is inserted by the sneaky compiler.

In fairness, the definition encourages compilers to warn about nonexhaustive case expressions. But this only causes programmers to write their own "raise Match" arms, even when the fall-through case is impossible because of an invariant known by the programmer. This leads to verbose code. In response, Freeman and Pfenning (1991) developed datasort refinements that can encode many invariants about datatypes, allowing compilers to accept "nonexhaustive" case expressions when they are known to cover all possible cases. For case analyses of refined types, the nonexhaustiveness warning becomes a nonexhaustiveness error, which the programmer should solve by declaring and using refinements of the datatype.

Unfortunately, this approach is all-or-nothing: either a type is refined and the compiler rejects a nonexhaustive match over it, or the type is not refined and the compiler issues a noncommittal warning. In practice, programmers may want to migrate code written with unrefined types to code that uses refined types; doing this in a single pass over a nontrivial program is extremely difficult. Instead, programmers should be able to add type annotations gradually. This was essentially the motivation for gradual typing (Siek and Taha 2006), except that, where they contemplated migration from dynamically typed code to statically typed code, we are interested in migration from code that is statically typed (modulo nonexhaustiveness) to code that is more statically typed.

Gradual typing is about the possibility of uncertainty: in some cases, one knows exactly what type one has; in other cases, one does not even know whether something is an integer. In this paper, we always know whether something is an integer (or a function, etc.); uncertainty is possible, but only about sum types. This is like the uncertainty of SML datatypes, with one key difference: we allow SML-style uncertainty and refinement-style certainty.

As an example, consider a red-black tree library that passes the SML type checker, but does not use refinement types. Datasort refinements can express the colour invariant, which says that every red node's children must be black. By reasoning about how the library functions should work, a programmer can add annotations that say when the colour invariant should hold, which the refinement type checker will verify. With gradual refinements, this reasoning can be done gradually and in tandem with testing. In fact, the programmer could start by annotating a single function $r$. If all test cases use $r$ in accordance with its refinement type annotation, the programmer gains confidence that the annotation is correct; if any tests violate the annotation, then either the annotation is wrong,
or there is a bug somewhere else. Thus, the more precise invariants guaranteed by refinements can be verified piecemeal.
Contributions. We make the following contributions:

- We define a type assignment system of gradual sums that includes both static refinement sums and dynamic sums. Programs, and even individual types, can be partly static and partly dynamic. However, this system does not readily yield an algorithm, and it allows typing derivations that are gratuitously dynamic (more dynamic than indicated by the programmer's type annotations), which give rise to gratuitous run-time errors.
- We define a bidirectional type system that is easy to implement and suppresses gratuitous dynamism, and prove that it corresponds to the type assignment system. We also prove that a well-typed program remains well-typed if its type annotations are made less precise (more dynamic).
- We define a type-directed translation to a target language with explicit casts. We prove that, given one program with two sets of type annotations (one more precise than the other), the more precisely typed one "fails earlier": either they produce the same result, or they both fail, or the more precisely typed program fails earlier. (For technical reasons, part of this result uses a slightly different version of the translation.)
- We define static and dynamic fragments of the source type system. The static fragment is related to classic datasort refinement type systems; the dynamic fragment is related to Standard ML. We prove that translating a program in the static fragment yields a program that cannot raise Match.


Figure 1. Some key results
Figure 1 depicts some of the results: source programs $e$ are translated to target terms $M$, which step to $M^{\prime}$, preserving typing; source programs $e^{s}$ with only static types are translated to target terms with no match failures.

For space reasons, lemmas, proofs, and a few definitions can be found in the supplementary material.

## 2. Overview

We define a type system that has one of the essential capabilities of datasort refinements: the types can express the knowledge that a value is a particular alternative of a datatype; for example, that a value is not simply a list-either Nil or Cons(...) -but specifically Cons(...). We represent this knowledge through sum types, not through the usual form of datasort refinements, but that is not the important difference.

- Like conventional datatype systems and datasort refinement systems, we can express that a value is either $\operatorname{inj}_{1} e_{1}$ where $e_{1}$ has type $A_{1}$ or $\operatorname{inj}_{2} e_{2}$ where $e_{2}$ has type $A_{2}$. Like datasort refinement systems, we only allow an exhaustive (two-armed) case expression over such a type: if we don't know which
injection it is, the programmer must handle both cases. This is a standard sum type $A_{1}+A_{2}$.
- Like datasort refinement systems, we can express that a value must be a particular injection. We use a subscript sum $A_{1}+{ }_{k} A_{2}$ for the type of the $k$ th injection into $A_{1}+A_{2}$. For example, $\mathrm{inj}_{2}$ True has type Int $+_{2}$ Bool, but $\mathrm{inj}_{1} 5$ has type Int $+_{1}$ Bool. Also like datasort refinement systems, we allow case expressions over such types to have just one arm, because we know which injection we have; there is no need to handle an impossible case.
- Like conventional datatype systems, but unlike datasort refinement systems, we can also express that we don't know which injection we have, but want to allow nonexhaustive matches: the dynamic sum $\mathrm{A}_{1}+{ }^{?} \mathrm{~A}_{2}$ can be deconstructed by a one-armed case expression. If, at run time, the specified arm does not match the scrutinee, it is a run-time error.

The three sum types,$++_{1}$, and +2 are essentially a datasort refinement system. Following datasort refinement systems, $A_{1}+1$ $A_{2}$ and $A_{1}+2 A_{2}$ are subtypes of $A_{1}+A_{2}$.

We can also make $+^{?}$ a subtype of + : the only elimination form permitted for + is a two-armed case, which is always safe. But + ? must not be a subtype of +1 and +2 , because $+^{?}$ contains both left and right injections; through subsumption, we could use a onearmed case on the left injection $\mathrm{inj}_{1}$ to eliminate a value of type $+_{2}$, which would fail at run time.

This yields the following subtype relation:

$$
A_{1}+\left.1 A_{2}^{-}\right|_{A_{1}+{ }^{?} A_{2}} ^{A_{1}+A_{2}} A_{1}+2 A_{2}
$$

For brevity, we can omit $A_{1}$ and $A_{2}$ from the diagram.


Comparison to datasort refinements. Our type $A_{1}+A_{2}$ corresponds to the top datasort of a datatype-the datasort that contains all the values of that datatype. A case expression on + must provide two arms, one for each injection.

Our type $A_{1}+{ }_{1} A_{2}$ corresponds to a datasort that includes exactly the values of the form $c_{1}\left(v_{1}\right)$ where $v_{1}: A_{1}$; similarly, $A_{1}+2 A_{2}$ corresponds to a datasort whose values are $c_{2}\left(v_{2}\right)$ where $v_{2}: A_{2}$.

In contrast, our type $A_{1}+^{?} A_{2}$ corresponds to the unrefined datatype. In datasort refinement systems, unrefined datatypes are part of the unrefined type system; the top datasort for a datatype contains the same values as the unrefined datatype, and is often notated in exactly the same way-but the unrefined datatype is not usable as a datasort. In contrast, both + and $+^{?}$ are types in our system. Moreover, they can be freely combined.

### 2.1 Developing Typing and Subtyping

Verificationists and pragmatists. In the verificationist approach to type theory, followed by Gentzen (1934) and Martin-Löf (1996), introduction forms are taken as the definition of a type; for example, a boolean type is defined by its constructors True and False. The elimination forms are secondary. In the pragmatist approach considered by Dummett (1991) and Zeilberger (2009), elimination forms are taken as the definition, and the introduction forms are
secondary. For example, a boolean type is defined primarily by its elimination form (say, an if-then-else expression).

In our setting, neither strict verificationism nor strict pragmatism seems adequate. Verificationism serves refinements well: the introduction rules directly express the intuition that refinements identify subsets of values. But introduction rules alone cannot distinguish $A_{1}+A_{2}$ and $A_{1}+{ }^{?} A_{2}$, because they have identical sets of inhabiting values (namely, all inj $v_{1}$, and inj ${ }_{2} v_{2}$ such that $v_{1}: A_{1}$ and $v_{2}: A_{2}$ ). The difference must lie in the elimination forms: only a two-armed case can eliminate + , while + ? can be eliminated by a two-armed case or a one-armed case (since the point is to allow nonexhaustive matches). To start from a better-understood foundation, we begin with the introduction rules.

Designing a type system can require trading off simplicity in one set of rules for complexity in another. We choose to minimize the number of typing rules, even though it leads to more complicated subtyping.

Introduction rules. Sum types need introduction forms. Since $+_{1}$ should contain only left injections, and +2 should contain only right injections, we could have a rule

$$
\frac{\Gamma \vdash e: A_{k}}{\Gamma \vdash\left(\mathrm{inj}_{k} e\right):\left(A_{1}+_{k} A_{2}\right)}+{ }_{k} \text { Intro }
$$

(This rule is really two rules, one for $\left(\mathrm{inj}_{1} v\right)$ with a premise $\Gamma \vdash e: A_{1}$ and one for $\left(\mathrm{inj}_{2} \nu\right)$ with a premise $\Gamma \vdash e: A_{2}$.)

Combined with subsumption, this rule gives the desired inhabitants to + , that is, both left and right injections. However, it does not add any inhabitants to $+^{?}$, so we could add another rule:

$$
\frac{\Gamma \vdash e: A_{k}}{\Gamma \vdash\left(\operatorname{inj}_{k} e\right):\left(A_{1}+{ }^{?} A_{2}\right)}+{ }^{?} \text { Intro }
$$

This goes against our goal of minimizing the number of typing rules: now there are two rules that type $\mathrm{inj}_{\mathrm{k}}$ e directly, that is, without using subsumption. The types $+_{k}$ (given by $+_{k}$ Intro) and $+^{?}$ (given by +Intro) are not in a subtyping relation with each otherneither is a subtype of the other. Hence, neither rule encompasses the other, and both are required.

We can avoid this nondeterminism by adding more sum types. By placing the additional sum types at the bottom of the subtyping relation, we can write a single introduction rule that will (through subsumption) populate all of our types with the desired injections.


Now, we need only one introduction rule:

$$
\frac{\Gamma \vdash e: A_{k}}{\Gamma \vdash\left(\operatorname{inj}_{k} e\right):\left(A_{1}+{ }_{k}^{?} A_{2}\right)}+{ }_{k}^{?} \text { Intro }
$$

We can think of $+\frac{?}{i}$ and $+\frac{?}{2}$ as "innate" types: when an injection inj $_{k}$ is created, it has type $+\frac{2}{2}$. Through subtyping, we can interpret $+\frac{k}{k}$ as $+_{k}$, or as the dynamic sum $+^{?}$.

Elimination rules. To design the elimination rules, it is helpful to annotate the subtyping diagram with the elimination forms that each type should allow. We write $L$ for a one-armed case expression
on the left injection ( $\mathrm{inj}_{1}$ ), R for a one-armed case on the right injection ( $\mathrm{inj}_{2}$ ), and B for a two-armed case.


According to this diagram, all types support a two-armed case expression B. The types $+_{1}$ and $+?$ are inhabited only by $\mathrm{inj}_{1}$, so they support the left one-armed case L; similarly, $+_{2}$ and $+\frac{?}{2}$ support the right one-armed case R. However, $+\frac{?}{i}$ and $+\stackrel{?}{2}$ are subtypes of + ?, so by subsumption they also support the "wrong" one-armed cases. The dynamic sum $+^{?}$ supports all three eliminations, with the risk of failing at run time.

Handling the two-armed case expression is straightforward: all the sum types support that elimination form, and all the sum types are subtypes of + , so we can write a single rule that types the scrutinee with + . Given e : $\left(A_{1} \phi A_{2}\right)$ where $\phi$ is any of our sum types, subsumption can be used to derive $e:\left(A_{1}+A_{2}\right)$.

$$
\frac{\Gamma \vdash e:\left(A_{1}+A_{2}\right) \quad \begin{array}{l}
\Gamma, x_{1}: A_{1} \vdash e_{1}: B \\
\Gamma, x_{2}: A_{2} \vdash e_{2}: B
\end{array}}{\Gamma \vdash \operatorname{case}\left(e, \operatorname{inj}_{1} x_{1} \cdot e_{1}, \operatorname{inj}_{2} x_{2} \cdot e_{2}\right): B}+\text { Elim }
$$

One-armed case expressions are more troublesome. Consider a left one-armed case, which matches only values of the form inj $j_{1} v$. Any subtype of $+_{1}$ will work, so we can write a rule that handles $+_{1}$ and $+\frac{?}{i}$ (and symmetrically, +2 and $+\frac{?}{2}$ ). However, $+^{?}$ should support a left one-armed case, but $+^{?}$ is not a subtype of $+_{1}$, leading us to a second rule that handles $+^{?}$.

Since $+{ }^{?}$ supports one-armed cases, it violates a type-theoretic principle: the introduction and elimination rules of a logical connective should be in harmony-that is, they should be locally sound (Dummett 1991) and locally complete (Pfenning and Davies 2001). Local soundness holds when the elimination rules are not more powerful than the introduction rules. Consider some standard rules for pairs:

$$
\frac{\Gamma \vdash e_{1}: A_{1} \quad \Gamma \vdash e_{2}: A_{2}}{\Gamma \vdash\left(e_{1}, e_{2}\right):\left(A_{1} \times A_{2}\right)} \quad \frac{\Gamma \vdash e:\left(A_{1} \times A_{2}\right)}{\Gamma \vdash\left(\operatorname{proj}_{k} e\right): A_{k}}
$$

These rules are locally sound: given something of type $\left(A_{1} \times A_{2}\right)$, projection can only extract things of type $A_{1}$ and $A_{2}$.

Dually, local completeness says that the elimination rules can extract all the information used in the introduction rules. (For a concise explanation of harmony, see Pfenning (2009).)

When the Curry-Howard correspondence holds, a type is inhabited iff the corresponding proposition is provable. Consider the following derivation (eliding empty contexts):

By constructing $\operatorname{inj}_{1} e$, we have shown that $A_{1}$ is inhabited. By subsumption, $\operatorname{inj}_{1} e$ has type $A_{1}+{ }^{?} A_{2}$. An elimination rule for $+{ }^{\text {? }}$ must permit a one-armed case on the second injection, ostensibly having type $A_{2}$. Simply returning $x$ as the result of the case should show that the proposition corresponding to $A_{2}$ is provable. But we never constructed something of type $A_{2}$, so $+^{?}$ does not satisfy local soundness.

As we did for the introduction forms, a single elimination rule can suffice: we just need more sum types. For the introduction forms, we added types at the bottom of the subtyping relation. Since eliminations should behave dually, we will add types at (or, at least, near) the top of the subtyping relation.


The types $+{ }_{1}^{*}$ and $+{ }_{2}^{*}$ support exactly the same eliminations as the subscript sums $+_{1}$ and $+_{2}$, but unlike the subscript sums, they are supertypes of the dynamic sum + ?

Then the single elimination rule for one-armed cases is

$$
\frac{\Gamma \vdash e:\left(A_{1}+{ }_{k}^{*} A_{2}\right) \quad \Gamma, x: A_{k} \vdash e_{k}: B}{\Gamma \vdash \operatorname{case}\left(e, \operatorname{inj}_{k} x \cdot e_{k}\right): B}+{ }_{k}^{*} \operatorname{Elim}
$$

We could simplify the diagram slightly by removing the edge from $+^{?}$ to + , since we now have an alternate routing via the $+_{k}^{*}$ types.

The high-water mark. Have we added enough sum types? We believe so. First, the additional types (beyond $+,+_{1},+_{2}$ and $+^{?}$ ) are motivated by limiting the number of typing rules. Second, there seem to be no other types that could be useful. Consider the following table:

| inhabitants | elimination forms supported |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | B only | $B$ and L | $B$ and R | B, L, and R |
| $\mathrm{inj}_{1}$ | note (a) | +1 | note (b) | + ${ }^{\text {²}}$ |
| $\mathrm{inj}_{2}$ | note (a) | note (b) | +2 | $+{ }_{2}$ |
| $\mathrm{inj}_{1}$ and $\mathrm{inj}_{2}$ | + | $+_{1}^{*}$ | $+_{2}^{*}$ | + ${ }^{\text {+ }}$ |

In the spaces marked "note (a)", such a type would pointlessly restrict the possible elimination forms: the top left space would be a type that could only be eliminated by a two-armed case ("B only"), but was inhabited only by left injections inj ${ }_{1}$.

In the spaces marked "note (b)", such a type would allow onearmed cases that always fail: a left one-armed case L on $\mathrm{inj}_{2}$, or a right one-armed case $R$ on inj $j_{1}$. We provide + ? to give programmers the freedom to use one-armed cases that may fail; it seems pointless to give them one-armed cases that are guaranteed to fail.

If anything, we may have more sum types than we want in practice: having fewer typing rules is good, but showing $+{ }_{1}^{*}$ or $+\frac{?}{2}$ in a compiler error message seems unhelpful.

### 2.2 Developing Precision

Our ultimate goal is a language in which precisely typed code and imprecisely typed code can coexist. In precisely typed code, the impossibility of match failures is a consequence of typing. In imprecisely typed code, bugs may lead to match failures, but imprecisely typed code can be correct: a one-armed case expression may be exhaustive in practice, thanks to some invariant not expressed through the type system.

The approach to typing and subtyping, developed above, already permits some forms of coexistence. For example, if a function $f$ expects a sum type + and we have some $x$ of type $+^{?}$, we can
pass $x$ to $f$. In the derivation below, $\Gamma=f:\left(A_{1}+A_{2}\right) \rightarrow B, x:$ $\left(A_{1}+{ }^{?} A_{2}\right)$.
$\frac{\Gamma \vdash \mathrm{f}:\left(A_{1}+A_{2}\right) \rightarrow B \frac{\Gamma \vdash x: A_{1}+{ }^{?} A_{2} \quad A_{1}+{ }^{?} A_{2} \leq A_{1}+A_{2}}{\Gamma \vdash x: A_{1}+A_{2}}}{\Gamma \vdash \mathrm{fx}: B}$
What about the reverse situation? Suppose a function $g$ from the imprecisely typed part of the program expects $+^{?}$, and we want to pass something of type + . This is possible, but annoying: we have to use a two-armed case to decompose the sum, and immediately rebuild it at type $+{ }^{?}$. Here, $\Gamma=g:\left(A_{1}+{ }^{?} A_{2}\right) \rightarrow B, y:\left(A_{1}+A_{2}\right)$.

$$
\begin{gathered}
\Gamma, x_{1}: A_{1} \vdash \operatorname{inj}_{1} x_{1}:\left(A_{1}+{ }^{?} A_{2}\right) \\
\ldots \quad \Gamma, x_{2}: A_{2} \vdash \operatorname{inj}_{2} x_{2}:\left(A_{1}+{ }^{?} A_{2}\right) \\
\Gamma \vdash g\left(\operatorname{case}\left(y, \operatorname{inj}_{1} x_{1} \cdot \operatorname{inj}_{1} x_{1}, \operatorname{inj}_{2} x_{2} \cdot \operatorname{inj}_{2} x_{2}\right)\right): B
\end{gathered}
$$

To support directly calling imprecise code from precise code, we develop precision relations on sum constructors and types. These relations are inspired by precision relations developed in gradual typing, e.g. Siek and Vachharajani (2008) and Garcia et al. (2016), where? (or $\star$ ) is an unknown, and thus very imprecise, type.

Our static sums $+,+_{1},+_{2}$ are precise in the sense that the "reach" of their information is known. If we have a closed value $v$ of type $A_{1}+A_{2}$, the type system "knows" only that $v$ is either a left or right injection, with no further information. So the type system rejects a one-armed case on $v$.

On the other hand, the dynamic sum $+^{?}$ is imprecise. Some programs that use + ? will have run-time match failures, but some programs that use $+^{?}$ will not have such failures, even some that use one-armed cases. If such one-armed cases always succeed, it is because the program follows invariants that are not expressed in the types-but which may be known by the programmer.

So we would expect + to be more precise than $+^{?}$, notated $+\sqsubseteq+^{?}$ (which can also be read " + is less imprecise than $+^{?}$ "). What about +1 and +2 ? They should be more precise than $+^{?}$; indeed, + ? should be more imprecise than everything else. How do $+_{1}$ and + compare? It is true that +1 has fewer inhabitants than + , but precision is not subtyping. All the static sums have the same degree of certainty: they are equally certain about different propositions (being a left injection, being a right injection, or being either). Thus, we will put $+_{1},+_{2}$ and + together at the bottom of the precision relation $\sqsubseteq$ (they are the least imprecise), with $+^{?}$ at the top:


What properties should precision have? In gradual typing, an important property of precision is that a program should remain welltyped when type annotations are made less precise. In the limit, we should be able to replace all static sums in annotations with $+{ }^{?}$. We call this property varying precision; it is part of the "gradual guarantee" of Siek et al. (2015). (Making annotations more precise does not necessarily preserve typing: for example, changing $a+$ ? annotation on $\mathrm{inj}_{2}()$ to $+{ }_{1}$.)

This property reinforces the intuition that $+^{?}$ should be at the top: this is what lets us substitute $+^{?}$ for more-precise sums. Dually, the static sums should be at the bottom: replacing a sum with a static sum should not, in general, preserve typing.

With this property in mind, how precise are $+_{i}^{?}$ and $+_{i}^{*}$, which we put in to reduce the number of typing rules? It doesn't make sense to "mix subscripts": moving between +2 and $+\frac{?}{i}$ in an annotation, or between $+_{1}$ to $+_{2}^{*}$, never preserves typing. Types with 1 subscripts should stay on the left of the edge from + to $+^{?}$, and 2 subscripts should stay on the right.

Hence, we will place $+{ }_{1}^{3}$ and $+{ }_{1}^{*}$ left of the vertical edge (from + to $+^{?}$ ), and $+_{2}^{?}$ and $+_{2}^{*}$ right of the vertical edge.

Moving to a less precise type should not lose inhabitants, because the lost inhabitants will become ill-typed. Suppose we put $+{ }_{1}^{*}$ below $+{ }_{1}$, making $+_{1}^{*}$ more precise. The sum $+{ }_{1}^{*}$ contains both left and right injections (by the above subtyping relation, $+\stackrel{?}{2} \leq$ $+_{1}^{*}$ ), meaning that $+_{1}^{*}$ has more inhabitants than $+_{1}^{?}$. Therefore, we should not have $+{ }_{1}^{*} \sqsubseteq+{ }_{i}$.

The reverse, where $+{ }_{1}^{?} \sqsubseteq+{ }_{1}^{*}$, is more plausible but would have unfortunate consequences (discussed at the end of this section). So we have no edge between $+{ }_{i}^{3}$ and $+{ }_{1}^{*}$.


Lifting this relation $\sqsubseteq$ on sum constructors to sum types is straightforward: if $\delta^{\prime} \sqsubseteq \delta$ then $\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right) \sqsubseteq\left(A_{1} \delta A_{2}\right)$, provided $A_{1}^{\prime} \sqsubseteq$ $A_{1}$ and $A_{2}^{\prime} \sqsubseteq A_{2}$. For function types, we diverge from subtyping: precision is covariant in the codomain and in the domain. This is consistent with precision in gradual typing, e.g. Siek and Vachharajani (2008) and Garcia et al. (2016), and with the refinement relations of Freeman (1994, p. 31) and Davies (2005).

Can we use this relation to type the above example $g y$, where we want to pass a value of type + to a function expecting something of $+^{?}$ type? Subtyping is internalized through a subsumption rule (the rule on the left); we extend the rule to allow loss of precision: in addition to moving from $A$ to a supertype $B$, we can move from $B$ to a less-precise $B^{\prime}$.

$$
\frac{\Gamma \vdash e: A \quad A \leq B}{\Gamma \vdash e: B} \text { sub. } \frac{\Gamma \vdash e: A \quad A \leq B \quad B \sqsubseteq \mathrm{~B}^{\prime}}{\Gamma \vdash e: \mathrm{B}^{\prime}} \text { sub.+loss }
$$

Imprecision is fundamentally unsound: Using $B \sqsubseteq B^{\prime}$, we move from a precise type (containing, say, + and +2 ) to an imprecise type containing $+^{?}$. Above, we showed that $+^{?}$ does not satisfy local soundness. The purpose of the $\mathrm{B} \sqsubseteq \mathrm{B}^{\prime}$ premise is to allow more-precisely-typed code to interface with less-precisely-typed code. However, a type checker that lost precision wherever possible would behave like a type checker for a system that only had + ?.

In addition to losing precision after subtyping, we allow gaining precision before subtyping:

$$
\frac{\Gamma \vdash e: A^{\prime} \quad A \sqsubseteq A^{\prime} \quad A \leq B \quad B \sqsubseteq B^{\prime}}{\Gamma \vdash e: B^{\prime}} \text { gain+sub.+loss }
$$

Gaining precision is clearly unsound: $A \sqsubseteq A^{\prime}$ allows moving from $+^{?}$ to $+_{1}$ or $+_{2}$. While unsound, this is needed for the property of varying precision: the typing of a single part of a program can become more or less precise, independent of the typing of the rest of the program.

We compose the three premises-gaining precision $A \sqsubseteq A^{\prime}$, subtyping $A \leq B$, and losing precision $B \sqsubseteq B^{\prime}$-into a relation $A^{\prime} \leadsto B^{\prime}$, called directed consistency.

With this relation, allowing $+{ }_{i}^{3} \sqsubseteq+{ }_{1}^{*}$ would nearly erase the distinction between $+_{1}^{*}$ and $+_{2}^{*}$ : first, $+{ }_{1}^{?} \sqsubseteq+{ }_{1}^{*}$; second, $+{ }_{1}^{?} \leq+{ }_{2}^{*}$; third, $+{ }_{2}^{*} \sqsubseteq+{ }_{2}^{*}$. (An earlier version of our system did allow $+\mathfrak{i} \sqsubseteq$ $+{ }_{1}^{*}$-see Appendix C)

Ideally, we should apply imprecision only when the programmer intends it. This goal motivates the bidirectional system in Section4

## 3. Source Type System

|  | i : $:=1 \mid 2$ |
| :---: | :---: |
| Source sums | $\delta::=+\left\|+_{i}\right\|+^{?}\left\|++_{i}^{?}\right\|+_{i}^{*}$ |
| Source expressions | $\begin{aligned} e:: & ()\|x\| \lambda x . e\left\|e_{1} e_{2}\right\|(e:: A) \\ & \mid \operatorname{inj}_{i} e \\ & \mid \operatorname{case}^{2}\left(e, \operatorname{inj}_{1} x_{1} \cdot e_{1}, \operatorname{inj}_{2} x_{2} . e_{2}\right) \\ & \mid \operatorname{case}\left(e, \operatorname{inj}_{i} x . e_{i}\right) \end{aligned}$ |
| Source types | $A, B:=$ Unit $\|A \delta B\| A \rightarrow B$ |
| Source typing conte | $\Gamma::=\cdot \mid \Gamma, x: A$ |

Figure 2. Source syntax
The syntax of the source language is in Figure 2 Here, and throughout the paper, $i$ ranges over 1 and 2 . The symbol $\delta$ ranges over the sum constructors: + is the standard (static) sum, +1 and $+_{2}$ are subscript sums denoting the ith injections, and $+^{?}$ is the gradual or dynamic sum. The final sum constructors, $+_{i}^{?}$ and $+_{i}^{*}$, are motivated by the desire to have the smallest number of introduction and elimination rules, as described in Section[2

Source expressions are the unit (), variables $x$, abstraction $\lambda x . e$ and application $e_{1} e_{2}$, sum injection $\operatorname{inj}_{i} e$, annotation (or ascription) (e :: A), a two-armed case that eliminates + , and a one-armed case that eliminates $+_{i}^{*}$.

Types $A$ and $B$ are Unit, sums $A \delta B$, and functions $A \rightarrow B$. Typing contexts $\Gamma$ are unordered sets of typings $x: A$, where the $x$ are assumed to be distinct.

### 3.1 Subtyping and Precision

Figure 3 gives the rules for a subsum judgment on sum constructors, written $\delta^{\prime} \leq \delta$. These rules follow the diagram in Section 2 The subtyping rule for sum types uses the subsum judgment. As is standard, the subtyping rule for functions is contravariant in the domain $\left(A_{1} \leq A_{1}^{\prime}\right)$ and covariant in the codomain $\left(A_{2}^{\prime} \leq A_{2}\right)$.

Precision on sum constructors (top of Figure 4) corresponds to the diagram from Section 2] On function types, precision is covariant in the domain, as discussed above.

In both subtyping and precision (for types), reflexivity and transitivity are admissible rules. Including transitivity rules would be fine on paper, but hard to implement since the middle type must be guessed. (The relations on sum constructors are a small finite set, so we do include transitivity rules; for an implementation, we would take the transitive closure.)

Subtyping and precision compose to form the directed consistency relation, which has a single rule, DirConsU in Figure 5 The "U" in the name comes from the depiction to the right of the rule. Since precision is reflexive, DirConsU includes all pairs of types that are related by subtyping.

### 3.2 Typing Rules

Typing rules for the source language are shown in Figure 6 The rule for variables, SVar is standard. Rules SAnno and SUnitlntro are standard, as are the rules $S \rightarrow$ Intro and $S \rightarrow$ Elim for functions.

Rule SCSub is a consistent subsumption rule: if $e$ has type $A^{\prime}$ and $A^{\prime}$ is directed consistent (Figure 5) with $A$, then $e$ has type $A$.

The rules for sums SSumIntro SSumElim1 SSumElim2 were developed in Section 2.1

## 4. Bidirectional Source Typing

Motivation. The type assignment system of Section 3includes all the sensible sum types, along with subtyping and precision. By itself, the consistent subsumption rule SCSub makes type inference, and even type-checking, nontrivial: we should apply SCSub only

$$
\begin{aligned}
& \delta^{\prime} \leq \delta \text { Sum } \delta^{\prime} \text { is a subsum of } \delta \\
& \begin{array}{lccc}
\overline{\delta \leq \delta} & \overline{+_{i}^{?} \leq+^{?}} & \overline{+^{?} \leq+_{i}^{*}} & \overline{+_{i}^{?} \leq++_{i}} \\
\overline{+_{i} \leq+_{i}^{*}} & \overline{+_{i}^{*} \leq+} & \frac{\delta^{\prime} \leq \delta_{1}}{\delta_{1} \leq \delta}
\end{array} \\
& A^{\prime} \leq A \text { Type } A^{\prime} \text { is a subtype of } A \\
& \frac{A_{1}^{\prime} \leq A_{1} \quad A_{2}^{\prime} \leq A_{2} \quad \delta^{\prime} \leq \delta}{\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right) \leq\left(A_{1} \delta A_{2}\right)} \\
& \frac{A_{1} \leq A_{1}^{\prime} \quad A_{2}^{\prime} \leq A_{2}}{\left(A_{1}^{\prime} \rightarrow A_{2}^{\prime}\right) \leq\left(A_{1} \rightarrow A_{2}\right)}
\end{aligned}
$$

Figure 3. Source subtyping

$A^{\prime} \sqsubseteq A$ Type $A^{\prime}$ is more precise than $A$

$$
\begin{gathered}
\overline{\text { Unit } \sqsubseteq \text { Unit }} \quad \frac{A_{1}^{\prime} \sqsubseteq A_{1} \quad A_{2}^{\prime} \sqsubseteq A_{2} \quad \delta^{\prime} \sqsubseteq \delta}{\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right) \sqsubseteq\left(A_{1} \delta A_{2}\right)} \\
\\
\\
\\
\\
\left(A_{1}^{\prime} \sqsubseteq A_{1}^{\prime} \rightarrow A_{2}^{\prime}\right) \sqsubseteq\left(A_{1} \sqsubseteq A_{2}\right)
\end{gathered}
$$

Figure 4. Precision

## $A^{\prime} \leadsto B^{\prime}$ Type $A^{\prime}$ is directed consistent

$$
\frac{\mathrm{A} \sqsubseteq \mathrm{~A}^{\prime} \quad \mathrm{A} \leq \mathrm{B}}{\mathrm{~A}^{\prime} \sim \mathrm{B}^{\prime}} \mathrm{B} \sqsubseteq \mathrm{~B}^{\prime}(\text { DirConsU } \quad A \leq \quad \mathrm{A} \leq \mathrm{B}
$$

Figure 5. Directed consistency
where necessary. This problem arises even with ordinary subsumption (subtyping, without changes of precision), which "forgets" that $e$ has a smaller type. Allowing changes of precision makes the problem worse: loss of precision "forgets" that $e$ has a more precise type, while gain of precision may add a downcast that fails at run time.

Such algorithmic difficulties could, perhaps, be resolved through careful design; the real problem with the type assignment system is that it types too many programs. SinceSCSubis always applicable, any expression meant to be typed using only + could be typed using $+^{?}$ instead.

A related problem is that our elimination rules for sums, while elegant, are excessively permissive: since $+\frac{?}{2}$ is a subtype of $+{ }_{1}^{*}$, an expression of type $+\frac{2}{2}$ can be eliminated with a left-arm case-even though such an elimination is guaranteed to cause a match failure at run time. Since this is a consequence of the subtyping part of SCSub it wouldn't help to remove the changes of precision from directed consistency.
$\Gamma \vdash \mathrm{e}: A$ Under typing context $\Gamma$, expression e has type $A$

$$
\begin{aligned}
& \frac{\Gamma(x)=A}{\Gamma \vdash x: A} \text { SVar } \quad \frac{\Gamma \vdash e: A^{\prime} \quad A^{\prime} \leadsto A}{\Gamma \vdash e: A} \text { SCSub } \\
& \frac{\Gamma \vdash e: A}{\Gamma \vdash(e:: A): A} \text { SAnno } \quad \overline{\Gamma \vdash(): \text { Unit }} \text { SUnitIntro } \\
& \Gamma \vdash e_{1}: A \rightarrow B \\
& \frac{\Gamma, x: A \vdash e: B}{\Gamma \vdash(\lambda x . e):(A \rightarrow B)} S \rightarrow \text { Intro } \frac{\Gamma \vdash e_{2}: A}{\Gamma \vdash\left(e_{1} e_{2}\right): B} S \rightarrow \text { Elim } \\
& \frac{\Gamma \vdash e: A_{i}}{\Gamma \vdash\left(\operatorname{inj}_{i} e\right):\left(A_{1}+{ }_{i}^{?} A_{2}\right)} \text { SSumIntro } \\
& \frac{\Gamma \vdash e_{0}: A_{1}+{ }_{i}^{*} A_{2} \quad \Gamma, x: A_{i} \vdash e: A}{\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{i} \text { x.e }\right): A} \text { SSumElim1 } \\
& \frac{\Gamma \vdash e_{0}: A_{1}+A_{2} \quad \begin{array}{l}
\Gamma, x_{1}: A_{1} \vdash e_{1}: A \\
\Gamma, x_{2}: A_{2} \vdash e_{2}: A
\end{array}}{\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{1} x_{1} \cdot e_{1}, \operatorname{inj}_{2} x_{2} \cdot e_{2}\right): A} \text { SSumElim2 }
\end{aligned}
$$

Figure 6. Source typing

We solve all of these problems via a bidirectional version of the system. In many settings, bidirectional typing has been chosen to overcome fundamental limitations of type inference, such as undecidability of inference for object-oriented subtyping (Pierce and Turner 1998), dependent types (Xi and Pfenning 1999: Pientka and Dunfield 2010) and first-class polymorphism (Dunfield and Krishnaswami 2013). It can also be motivated by better localization of type error messages. Our motivation is different: we want to stop the type-checker from doing certain things unless the programmer has signalled that they really want to do those things. Programmers signal their intent through type annotations, which are propagated through the bidirectional typing rules.

In Section 4.3, we show that the bidirectional system is sound and complete (under annotation) with respect to the type assignment system of Section 3

Checking and synthesis. Bidirectional typing splits typing into two judgments. The checking judgment $\Gamma \vdash e \Leftarrow A$ is read " $e$ checks against type $A$ "; the synthesis judgment $\Gamma \vdash e \Rightarrow A$ is read "e synthesizes type $A$ ". Both judgments can be interpreted as saying that $e$ has type $A$; the difference is that in checking, the type $A$ is already known, while synthesis infers $A$ from the available information ( $\Gamma$ and $e$ ). The type in the checking judgment "flows" from some type annotation, either directly or (usually) indirectly.

An important advantage of the bidirectional system is a kind of subformula property (Gentzen 1934; Prawitz 1965). In our case, this property says that in a derivation of $\Gamma \vdash e \Rightarrow A$, every type synthesized or checked against is derived from types found in $\Gamma$ and $e$. For $\Gamma \vdash e \Leftarrow A$, every such type is derived from $\Gamma$, $e$, and $A$. Consequently, dynamic sums cannot appear out of nowhere: they result only from type annotations. We exploit this property in, for example, the proof of Theorem[5]
From type assignment rules to bidirectional rules. As is often the case with bidirectional type systems, our bidirectional rules will strongly resemble our type assignment rules. In general, we construct a bidirectional rule by replacing "." with " $\Leftarrow$ " or " $\Rightarrow$ ". The main question is when to use checking, and when to use synthesis. Checking is more powerful than synthesis; for a premise, we generally prefer to make it a checking judgment, but a checking conclusion may increase the number of required type annotations.

```
\begin{tabular}{l|l}
\(\Gamma \vdash e \Leftarrow A\) & Under context \(\Gamma\), expr. e checks against type \(A\) \\
\(\Gamma \vdash e \Rightarrow A\) & Under context \(\Gamma\), expr. e synthesizes type \(A\)
\end{tabular}
\(\frac{\Gamma(x)=A}{\Gamma \vdash x \Rightarrow A}\) SynVar \(\quad \frac{\Gamma \vdash e \Rightarrow A^{\prime} \quad A^{\prime} \leadsto A}{\Gamma \vdash e \Leftarrow A}\) ChkCSub
    \(\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash(e:: A) \Rightarrow A}\) SynAnno \(\quad \overline{\Gamma \vdash() \Leftarrow \text { Unit }}\) ChkUnitIntro
        \(\frac{\Gamma, x: A \vdash e \Leftarrow B}{\Gamma \vdash(\lambda x . e) \Leftarrow(A \rightarrow B)}\) Chk \(\rightarrow\) Intro
        \(\frac{\Gamma \vdash e_{1} \Rightarrow(A \rightarrow B) \quad \Gamma \vdash e_{2} \Leftarrow A}{\Gamma \vdash\left(e_{1} e_{2}\right) \Rightarrow B}\) Syn \(\rightarrow \operatorname{Elim}\)
        \(\frac{\Gamma \vdash e \Leftarrow A_{i} \quad+_{i}^{?} \leq \delta}{\Gamma \vdash\left(\operatorname{inj}_{i} e\right) \Leftarrow\left(A_{1} \delta A_{2}\right)}\) ChkSumIntro
    \(\Gamma \vdash e_{0} \Rightarrow\left(A_{1} \delta A_{2}\right)\)
    \(\frac{\delta \Rightarrow+_{i}^{*} \quad \Gamma, x: A_{i} \vdash e \Leftarrow A}{\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{i} x . e\right) \Leftarrow A}\) ChkSumElim1
\(\Gamma \vdash e_{0} \Rightarrow\left(A_{1} \delta A_{2}\right) \quad \Gamma, x_{1}: A_{1} \vdash e_{1} \Leftarrow A\)
\(\frac{\delta \Rightarrow+\quad \Gamma, x_{2}: A_{2} \vdash \mathrm{e}_{2} \Leftarrow A}{\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{1} x_{1} \cdot e_{1}, \text { inj }_{2} x_{2} . e_{2}\right) \Leftarrow A}\) ChkSumElim2
\(\delta \Rightarrow \delta^{\prime}\) Sum \(\delta\) synthesizes sum \(\delta^{\prime}\)
\(\overline{+_{i}^{?} \Rightarrow+_{i}^{*}} \quad \overline{+_{i} \Rightarrow+_{i}^{*}} \quad \overline{+^{?} \Rightarrow+_{i}^{*}} \quad \overline{+_{i}^{*} \Rightarrow+_{i}^{*}} \quad \overline{\delta \Rightarrow+}\)
```

Figure 7. Bidirectional typing (source)

For the most part, we follow the recipe of Davies and Pfenning (2000); Dunfield and Pfenning (2004): introduction rules check, and elimination rules synthesize. More precisely, the judgment that includes the relevant connective-the principal judgment-should check for an introduction rule, and synthesize for an elimination rule.

Doing this step naturally determines the directions of many other judgments. For example, in rule Syn $\rightarrow$ Elim the principal judgment is the first premise $\Gamma \vdash e_{1} \Rightarrow\left(A_{1} \rightarrow A_{2}\right)$. Since the type in a synthesis judgment is output, deriving this premise tells us what $A_{1}$ is, enabling us to make the second premise a checking judgment. The premise also tells us what $A_{2}$ is-so we can make the conclusion a synthesis judgment. Consequently, applications $e_{1} e_{2}$ will synthesize a type, without any local annotation, whenever the function $e_{1}$ synthesizes. In rule Chk $\rightarrow$ Intro not following the recipe-by making the conclusion synthesize, $\Gamma \vdash \lambda x . e \Rightarrow\left(A_{1} \rightarrow A_{2}\right)$-means that we don't know $A_{1}$, and cannot construct the context $\Gamma, x: A_{1}$ in the premise. (It may be possible to design a more complicated system in which $\lambda x$. e does synthesize, as Dunfield and Krishnaswami (2013) did for a different type system.)

Rule ChkSumlntro says that inj $j_{1}$ e checks against $A_{1} \delta A_{2}$, where $\delta$ is any sum above $+{ }_{j}$-that is, any sum constructor except $+\frac{3}{2}$ and $+_{2}$. This is a checking rule for two reasons. First, it is an introduction form, so according to the recipe its principal judgment (the conclusion) should check. Second, the simplest synthesizing rule would synthesize $A_{1}+{ }_{i}^{?} A_{2}$. But that is a subtype of $A_{1}+{ }^{?} A_{2}$, introducing a possibly undesired dynamic sum.

In the (one-armed) elimination rule SSumElim1 the principal judgment is the premise $\Gamma \vdash e_{0}: A_{1}+{ }_{i}^{*} A_{2}$. Following the recipe, the corresponding premise of ChkSumElim1 synthesizes. It
would be unfortunate to require it to synthesize exactly $A_{1}+{ }_{i}^{*} A_{2}$ : assuming programmers mostly write type annotations using $+_{1},+2$, + and $+^{?}$, virtually no expressions will synthesize $+_{i}^{*}$. On the other hand, checking e e against $A_{1}+{ }_{i}^{*} A_{2}$ would be too permissive: if we have a left one-armed case case ( $e_{0}, \operatorname{inj}_{1}$ x.e), we would accept $e_{0}$ of type $+\frac{?}{2}$, even though $+\frac{?}{2}$ is a right injection, guaranteeing a runtime failure. Instead, we require that $e_{0}$ synthesize $A_{1} \delta A_{2}$ where $\delta \Rightarrow+{ }_{i}^{*}$. The judgment $\delta \Rightarrow+_{1}^{*}$ is derivable when $\delta$ is $+_{i}^{2},+_{1},+^{?}$ or $+{ }_{1}^{*}$.

For consistency with ChkSumElim1 our two-armed elimination rule ChkSumElim2 has a similar structure (with an additional premise for the second arm) and also uses the $\Rightarrow$ judgment; however, $\delta \Rightarrow+$ is always derivable, because a two-armed case is safe for every sum constructor. We include this premise anyway, to highlight the two rules' similarity.

Several rules are not tied to specific type connectives. An assumption $x: A$ in $\Gamma$ could be read " $x$ synthesizes $A$ ", so SynVar synthesizes its type. Rule SynAnno synthesizes the type given in an annotation ( $e:: A$ ), provided $e$ checks against A. Following earlier bidirectional systems (Davies and Pfenning 2000; Dunfield and Pfenning 2004), the subsumption rule has a checking conclusion and a synthesizing premise. The checking conclusion ensures that subsumption, which loses information, is applied only with the programmer's consent: the type being checked against is derived from a type annotation. The synthesizing premise ensures that we "make progress" as we move from the goal $e \Leftarrow A$ to the subgoal $e \Rightarrow A^{\prime}$ : we cannot use ChkCSub as the concluding rule of its own premise. In addition to subtyping and change of precision, ChkCSub with $A=A^{\prime}$ (using reflexivity) allows us to use a derivation of $\Gamma \vdash e \Rightarrow A$ where we need a derivation of $\Gamma \vdash e \Leftarrow A$. For example, applying a function to a variable requires this rule: SynVar synthesizes, but Syn $\rightarrow$ Elim has a checking premise.

Complexity. Typing in the bidirectional system takes polynomial time. With one exception, the bidirectional rules are in one-to-one correspondence with syntactic forms. The exception is ChkCSub which can be used to check any synthesizing form. So bidirectional typing is syntax-directed in a slightly looser sense than the usual one: For each pair of a syntactic form and a direction (checking or synthesis), exactly one rule applies; if that rule is ChkCSub then exactly one rule applies to derive its synthesizing premise. Thus, the size of a derivation (if one exists) is, at most, twice the size of the expression.

Variations on a theme. Several checking rules could be supplemented with a synthesizing rule, or (in the case of ChkUnitlntro replaced. A synthesizing version of ChkSumIntro however, would be problematic: while we might synthesize the sum constructor $+_{i}$, synthesizing $e$ for $A_{i}$ tells us only one component of the sum. Our system enjoys uniqueness of synthesis: given $\Gamma$ and $e, e$ synthesizes (at most) one type. Synthesizing the other component of the sum would synthesize an infinite number of types. Moreover, a direct implementation would need to guess the other component.

A synthesizing version of ChkSumElim1 would be straightforward; for ChkSumElim2 we could synthesize $e_{1} \Rightarrow B_{1}$ and $e_{2} \Rightarrow B_{2}$ and synthesize their join $B_{1} \vee B_{2}$ in the conclusion.

Except for ChkUnitIntro all of these variations-while perhaps convenient in practice-would make the system larger and more complicated. This paper presents a core calculus; we leave exploration of such variations to future work.

### 4.1 Static System

Two restricted versions of the bidirectional system are of interest. The first is a static system: a simply typed $\lambda$-calculus with sums and refinements over sums, without any dynamic sums. The syntax (Figure (8) is the same as the source language, except for $\delta^{\mathrm{S}}$ which

$$
\begin{aligned}
& \text { Static sums } \\
& \text { Static expressions } \\
& \delta^{S}::=+\mid+_{i} \\
& e^{\mathbf{S}}::=()|x| \lambda x . e^{\mathbf{S}}\left|e_{1}^{\mathbf{S}} e_{2}^{\mathbf{S}}\right| \operatorname{inj}_{i} e^{\mathbf{S}} \mid\left(e^{\mathbf{S}}:: A^{\mathbf{S}}\right) \\
& \left|\operatorname{case}\left(e^{\mathrm{S}}, \mathrm{inj}_{1} \mathrm{x}_{1} . e_{1}^{\mathrm{S}}, \mathrm{inj}_{2} \mathrm{x}_{2} . e_{2}^{\mathrm{S}}\right)\right| \operatorname{case}\left(e^{\mathrm{S}}, \mathrm{inj}_{i} \chi . e_{i}^{\mathrm{S}}\right) \\
& \text { Static types } \\
& A^{S}::=\text { Unit }\left|A_{1}^{S} \delta^{S} A_{2}^{S}\right| A_{1}^{S} \rightarrow A_{2}^{S} \\
& \Gamma^{S}::=\cdot \mid \Gamma^{s}, x: A^{s}
\end{aligned}
$$

$$
A_{1}^{S} \leq_{S} A_{2}^{S} \text { Static type } A_{1}^{S} \text { is a subtype of } A_{2}^{S}
$$

$$
A_{11}^{\mathrm{S}} \leq_{\mathrm{s}} A_{12}^{\mathrm{S}}
$$

$$
\overline{\text { Unit } \leq_{\mathrm{s}} \text { Unit }} \frac{A_{21}^{\mathrm{S}} \leq_{\mathrm{S}} A_{22}^{\mathrm{S}} \frac{\delta_{1}^{\mathrm{S}} \leq_{\mathrm{S}} \delta_{2}^{\mathrm{S}}}{\left(A_{11}^{\mathrm{S}} \delta_{1}^{\mathrm{S}} A_{21}^{\mathrm{S}}\right) \leq_{\mathrm{s}}\left(A_{12}^{\mathrm{S}} \delta_{2}^{\mathrm{S}} A_{22}^{\mathrm{S}}\right)} \frac{A_{12}^{\mathrm{S}} \leq_{\mathrm{S}} A_{11}^{\mathrm{S}}}{\left(A_{11}^{\mathrm{S}} \rightarrow A_{21}^{\mathrm{S}}\right) \leq_{\mathrm{S}}\left(A_{12}^{\mathrm{S}} \leq_{\mathrm{S}} A_{22}^{\mathrm{S}}\right.}}{\left(A_{22}^{\mathrm{S}}\right)}
$$

$\Gamma^{\mathrm{S}} \vdash_{\mathrm{S}} e^{\mathrm{S}} \Leftarrow A^{\mathrm{S}}$ Under typing context $\Gamma^{\mathrm{S}}$, expression $e^{\mathrm{S}}$ checks against type $A^{\mathrm{S}}$ $\Gamma^{\mathrm{S}} \vdash_{\mathrm{S}} e^{\mathrm{S}} \Rightarrow A^{\mathrm{S}}$ Under typing context $\Gamma^{\mathrm{S}}$, expression $e^{\mathrm{S}}$ synthesizes type $A^{\mathrm{S}}$

Figure 8. The static system: the bidirectional system restricted to $+,+_{1},+_{2}$
can only be,$+{ }_{1}$, or $+_{2}$. We follow the bidirectional system in deriving rules for sub-sum, subtyping, and typing; the judgments are decorated with S for "static". The interesting difference is in the typing rules for sums: the introduction rule checks that the sum is above $+_{i}$ (instead of $+_{i}^{?}$ ), and the one-arm elimination StSumElim1 checks that the sum is below $+_{i}$ (instead of $+_{i}^{*}$ ), that is, the sum is exactly $\boldsymbol{t}_{i}$.

### 4.2 Dynamic System

The static system omits dynamic sums; the dynamic system's only sum is the dynamic sum $+^{?}$. Since one-armed cases are allowed on type $+^{?}$, this corresponds to datatypes in Standard ML. The metavariables and judgments are decorated with D for "dynamic". For space reasons, the definition of this system is in the supplementary material (Appendix $\mathbb{A}$.

### 4.3 Metatheory

The bidirectional system is decidable. The $\delta^{\prime} \leq \delta$ judgment is immediately decidable (taking the transitive closure of the rules), and the $A^{\prime} \leq A$ judgment is decidable because each rule moves from larger type expressions to smaller ones. The same holds for $\sqsubseteq$, so directed consistency is decidable. The argument for the typing rules is slightly more interesting, as ChkCSub is a stationary rule (the premise and conclusion type the same expression). However, since this rule moves from checking to synthesis, and no stationary rule moves from synthesis to checking (inSynAnno, the expression becomes smaller), decidability holds.
Theorem 1 (Decidability of bidirectional typing).

1. Given $\Gamma$, $e$ and $A$, the judgment $\Gamma \vdash e \Leftarrow A$ is decidable.
2. Given $\Gamma$ and $e$, the judgment $\Gamma \vdash e \Rightarrow A$ is decidable.

The bidirectional system is sound with respect to the type assignment system: if $e$ is well-typed in the bidirectional system, it is well-typed in the type assignment system. (Proofs can be found in the supplementary material.)
Theorem 2 (Bidirectional soundness).
If $\Gamma \vdash \mathrm{e} \Leftarrow \mathrm{A}$ or $\Gamma \vdash \mathrm{e} \Rightarrow A$ then $\Gamma \vdash \mathrm{e}: А$.
The bidirectional system is also complete: given $e: \mathcal{A}$ in the type assignment system, it is always possible to add annotations that make $e$ well-typed in the bidirectional system. We write $e=: e^{\prime}$ when $e^{\prime}$ is the same as $e$ except that $e^{\prime}$ may have extra annotations.
Theorem 3 (Annotatability).
If $\Gamma \vdash e$ : A then there exist $e^{\prime}$ and $e^{\prime \prime}$ such that (1) $\Gamma \vdash e^{\prime} \Leftarrow \mathcal{A}$ where $e=: e^{\prime}$, and (2) $\Gamma \vdash e^{\prime \prime} \Rightarrow A$ where $e=: e^{\prime \prime}$.

We also show that bidirectional typing derivations are robust under imprecision: if $e^{\prime} \Leftarrow A^{\prime}$, replacing annotations in $e^{\prime}$ with more imprecise types preserves typing. This corresponds to part 1 of the gradual guarantee of Siek et al. (2015, Theorem 5 on p. 11). An example illustrating this theorem's significance appears below in Section 4.4

First, $\Gamma^{\prime} \sqsubseteq \Gamma$ is defined pointwise. Second, let $e^{\prime} \sqsubseteq e$ if, for each annotation ( $e_{0}^{\prime}:: A^{\prime}$ ) in $e^{\prime}$, there is a corresponding annotation ( $e_{0}:: A$ ) in $e$ where $A^{\prime} \sqsubseteq A$. (For full inductive definitions, see Figures 15 and 16 in the supplementary material.)
Theorem 4 (Varying precision of bidirectional typing).

1. If $\Gamma^{\prime} \vdash e^{\prime} \Leftarrow A^{\prime}$ and $e^{\prime} \sqsubseteq e$ and $\Gamma^{\prime} \sqsubseteq \Gamma$ and $A^{\prime} \sqsubseteq A$ then $\Gamma \vdash e \Leftarrow A$.
2. If $\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A^{\prime}$ and $e^{\prime} \sqsubseteq e$ and $\Gamma^{\prime} \sqsubseteq \Gamma$
then there exists $A$ such that $\Gamma \vdash e \Rightarrow A$ and $A^{\prime} \sqsubseteq A$.
The nonempty context is needed for the proof cases for rules whose premises add to $\Gamma^{\prime}$, such as ChkSumElim1

An earlier version of the system, which did not allow gain of precision, has a weaker property: in that system, the given expression $e$ is not necessarily typable, but there exists some "even more imprecise" expression $e_{j}$ that is typable. See Theorem 14 in Appendix C

Static system. As the static system is essentially a restriction of the bidirectional system, it is easy to turn a derivation in the static system into a derivation in the bidirectional system; this is the first part of the following theorem.

Completeness is more interesting: Given a bidirectional derivation whose conclusion is static-that is, the context $\Gamma$, expression $e$, and type $A$ are within the restricted static grammar-we can build a derivation in the static system. This holds because of a subformula property: if there are no dynamic sums in $\Gamma, e$ and $A$, then dynamic sums cannot appear anywhere in the bidirectional derivation.
Theorem 5 (Static soundness and completeness).

## 1. Soundness:

(a) If $\Gamma^{\mathrm{S}} \vdash_{\mathrm{s}} \mathrm{e}^{\mathrm{s}} \Leftarrow A^{\mathrm{s}}$ then $\Gamma^{\mathrm{S}} \vdash \mathrm{e}^{\mathrm{s}} \Leftarrow A^{\mathrm{S}}$
(b) If $\Gamma^{\mathrm{S}} \vdash_{\mathrm{S}} e^{\mathrm{S}} \Rightarrow A^{\mathrm{S}}$ then $\Gamma^{\mathrm{S}} \vdash e^{\mathrm{S}} \Rightarrow A^{\mathrm{S}}$.
2. Completeness:
(a) If $\Gamma^{\mathrm{S}} \vdash e^{\mathrm{S}} \Leftarrow A^{\mathrm{S}}$ then $\Gamma^{\mathrm{S}} \vdash_{\mathrm{S}} e^{\mathrm{S}} \Leftarrow A^{\mathrm{S}}$.
(b) If $\Gamma^{\mathrm{s}} \vdash \mathrm{e}^{\mathrm{s}} \Rightarrow A^{\mathrm{s}}$ then $\Gamma^{\mathrm{s}} \vdash_{\mathrm{s}} e^{\mathrm{s}} \Rightarrow A^{\mathrm{s}}$.

This theorem directly corresponds to part 1 of Theorem 1 of Siek et al. (2015, p. 9) for "fully annotated" expressions. In that work, an expression is fully annotated if it has no gradual type annotations. In our system, expressions without annotations are static.

A corresponding theorem holds for the dynamic system and, in turn, corresponds to part 1 of Theorem 2 of Siek et al. (2015, p. 9). This is a rough correspondence: in our bidirectional system, dynamism is restricted to sum types and arises only through annotations. See Theorem 15 in the appendix.

### 4.4 Example

To see why Theorem 4 matters, consider the following example. Suppose we want to transform a program that uses dynamic sums into one that uses static sums. The program has a function $f$ of type (Unit $+^{?}$ Int) $\rightarrow \operatorname{Int}$, which is called with an argument $x$ of type Unit $+^{?}$ Int.

$$
\begin{aligned}
& \text { let } f=(\lambda y . \cdots)::\left(\text { Unit }+{ }^{?} \ln t\right) \rightarrow \operatorname{Int} \text { in } \ldots \\
& \text { let } x=e_{x}::\left(\text { Unit }+{ }^{?} \ln t\right) \text { in } \\
& \quad f x
\end{aligned}
$$

(We assume that $e_{x}$ is a checking form that needs an annotation; if $e_{x}$ synthesizes (Unit $+{ }^{?} \operatorname{Int}$ ), the annotation could be removed.) The programmer realizes that $f$ only works with a right injection (perhaps its body is a one-armed case on $\mathrm{inj}_{2}$ ), and that $x$ should always be a right injection.

```
let \(f=(\lambda y . \cdots)::\left(\right.\) Unit \(\left.+_{2} \operatorname{Int}\right) \rightarrow\) Int in \(\ldots\)
    let \(x=e_{x}::(\) Unit +2 Int \()\) in
        f \(x\)
```

If this program type-checks and contains no remaining dynamic sum annotations, we know that $f$ and $x$ actually satisfy their annotations, and that the application $f x$ will not cause any match or cast failures. Theorem 4 says that the annotations can be changed one at a time: the program with $+^{?}$ in the type of f but $+_{2}$ in the type of $x$ is well-typed, as is the program with +2 in the type of $f$ but $+^{?}$ in the type of $x$ :

$$
\begin{aligned}
& \text { let } \mathrm{f}=\left(\lambda_{\mathrm{y}}, \cdots\right)::\left(\text { Unit }+_{2} \text { Int }\right) \rightarrow \text { Int in } \ldots \\
& \text { let } x=e_{x}::\left(\text { Unit }+^{?} \text { Int }\right) \text { in } \\
& \quad \mathrm{f}^{x}
\end{aligned}
$$

When synthesizing the type of $\mathrm{f} x$, we use ChkCSub to gain precision in $x$ :


A precise annotation that differs from the correct one, such as Unit $+{ }_{1} \operatorname{Int}$ on $x$, may cause an error-either at type-checking time, or at run time. But a precise annotation that is correct will not cause an error, and constitutes a step towards a completely static program.

## 5. Target Language and Translation

### 5.1 Target Syntax and Semantics

|  | i :: $=1 \mid 2$ |
| :---: | :---: |
| Target sums | ¢ : $:=+\mid+{ }_{\text {i }}$ |
| Target terms | $\begin{aligned} M::= & ()\|x\| \lambda x \cdot M\left\|M_{1} M_{2}\right\| \operatorname{inj}_{i} M \\ & \left.\mid{\operatorname{case}\left(M, i n j_{1}\right.} x_{1} \cdot M_{1}, \operatorname{inj}_{2} x_{2} \cdot M_{2}\right) \\ & \mid \operatorname{case}\left(M, \operatorname{inj}_{i} \cdot M_{i}\right) \\ & \left\|\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle M\right\| \text { matchfail } \end{aligned}$ |
| Values | $W::=()\|x\| \lambda x . M \mid \operatorname{inj}_{i} W$ |
| Target types | $\mathrm{T}::=$ Unit $\left\|\mathrm{T}_{1} \phi \mathrm{~T}_{2}\right\| \mathrm{T}_{1} \rightarrow \mathrm{~T}_{2}$ |
| Target typing c | $\Theta::=\cdot \mid \Theta, x: T$ |

Figure 9. Target syntax
Our target language is a statically typed $\lambda$-calculus with static sum types and a cast construct. The syntax is shown in Figure 9 We write $M$ for target terms (expressions), $W$ for values, and $T$ for target types. The target sum constructors are all the static sum types from the source language:,$++_{1}$, and $+_{2}$. In addition, we have a cast construct $\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle M$, which casts from sum $\phi_{1}$ to $\phi_{2}$. A failing cast, such as $\langle+2 \Leftarrow+\rangle\left(\right.$ inj $\left._{1}()\right)$, steps to the error term matchfail.

Much of the target type system (Figure 10) follows the source type assignment system, if that system were restricted to static sum types. Since the target lacks any dynamic sum constructors (like $+^{2}$ ), target subtyping says only that $+_{1}$ and +2 are subtypes of + ; this corresponds to datasort refinement systems, where every datasort is a subsort of a "top" datasort for the type being refined. Our typedirected translation (Section 5.2 transforms the gradual property of types into dynamic checks at the term level; rule TCast casts between sum constructors, and rule TMatchfail gives any type to matchfail, which represents the failure of a cast.

Our target language (Figure 11) has a standard call-by-value small-step semantics, extended with casts. Evaluation contexts $\mathcal{E}$ are terms with a hole [], where the hole represents a term in an evaluation position: if target term $M=\mathcal{E}\left[M_{0}\right]$, and $M_{0}$ reduceswritten $M_{0} \mapsto_{\mathrm{R}} M_{0}^{\prime}$-then the larger term $M$ steps to $\mathcal{E}\left[M_{0}^{\prime}\right]$.

The cast reduction rules represent the three relevant situations: (1) an upcast to a supertype succeeds ReduceUpcast); (2) a downcast from + to $+_{i}$ succeeds if $i$ matches the injection ReduceCastSuccess); (3) a downcast from + to $+_{i}$ fails, reducing to matchfail, if i doesn't match the injection ReduceCastFailure .

### 5.2 Type-Directed Translation $\hookrightarrow$

To translate source programs into target programs with explicit casts between sum types, we use a judgment $\Gamma \vdash e: A \hookrightarrow M$. Most of the rules (in Figure 12) follow the type assignment rules, with the addition of $\hookrightarrow M$. Given $e$ of type $A$, the rules produce a target term $M$ of type $T$ where $T$ is the translation of $A$, written $|\mathcal{A}|$.

| $\phi^{\prime} \leq \phi$ | Sum $\phi^{\prime}$ is a subsum of $\phi$ | $\mathrm{T}^{\prime} \leq \mathrm{T}$ | Target type $\mathrm{T}^{\prime}$ is a subtype of T |  |
| :---: | :---: | :---: | :---: | :---: |
| $\overline{\phi \leq \phi}$ | $\overline{+_{i} \leq+}$ | $\overline{U n i t \leq U n i t}$ | $\frac{\mathrm{~T}_{1}^{\prime} \leq \mathrm{T}_{1}}{\left(\mathrm{~T}_{1}^{\prime} \phi^{\prime} \mathrm{T}_{2}^{\prime}\right) \leq\left(\mathrm{T}_{1} \phi \mathrm{~T}_{2}\right)}$ | $\frac{\mathrm{T}_{1}^{\prime} \leq \mathrm{T}_{1}^{\prime}}{\left(\mathrm{T}_{1}^{\prime} \rightarrow \mathrm{T}_{2}^{\prime}\right) \leq\left(\mathrm{T}_{1} \rightarrow \mathrm{~T}_{2}\right)}$ |

$\Theta \vdash M: T$ Under context $\Theta$, target term $M$ has target type $T$

$$
\begin{aligned}
& \frac{\Theta(x)=T}{\Theta \vdash x: T} T V a r \\
& \frac{\Theta \vdash M: T^{\prime} \quad T^{\prime} \leq T}{\Theta \vdash M: T} T \text { ub } \\
& \frac{\Theta \vdash M:\left(\mathrm{T}_{1} \phi^{\prime} \mathrm{T}_{2}\right)}{\Theta \vdash\left\langle\phi \Leftarrow \phi^{\prime}\right\rangle M:\left(\mathrm{T}_{1} \phi \mathrm{~T}_{2}\right)} \text { TCast } \\
& \overline{\Theta \vdash \text { matchfail : } T} \text { TMatchfail } \overline{\Theta \vdash(): \text { Unit }} \text { TUnitIntro } \\
& \frac{\Theta \vdash M: T_{i}}{\Theta \vdash \operatorname{inj}_{i} M:\left(T_{1}+{ }_{i} T_{2}\right)} T+{ }_{i} \text { Intro } \\
& \frac{\Theta \vdash M_{0}: T_{1}+_{i} T_{2} \quad \Theta, x: T_{i} \vdash M: T}{\Theta \vdash \operatorname{case}\left(M_{0}, \operatorname{inj}_{i} x . M\right): T} T_{+} \text {Elim } \\
& \frac{\Theta, x: T_{1} \vdash M: T_{2}}{\Theta \vdash \lambda x \cdot M:\left(T_{1} \rightarrow T_{2}\right)} T \rightarrow \text { Intro } \\
& \frac{\Theta \vdash M_{1}: T^{\prime} \rightarrow T \quad \Theta \vdash M_{2}: T^{\prime}}{\Theta \vdash M_{1} M_{2}: T} T \rightarrow \text { Elim }
\end{aligned}
$$

Figure 10. Target subtyping and typing

| Evaluation contexts $\begin{aligned} \mathcal{E}::= & {[] } \\ & \mid \operatorname{inj}_{i} \mathcal{E} \\ & \left\|{\operatorname{case}\left(\mathcal{E}, \operatorname{inj}_{i} x . M\right)}\right\| \operatorname{case}^{\left(\mathcal{E}, \text { inj }_{1} x_{1} . M_{1}, \operatorname{inj}_{2} x_{2} . M_{2}\right)} \\ & \mid\left\langle\phi \Leftarrow \phi^{\prime}\right\rangle \mathcal{E} \\ & \left\|\mathcal{E} M_{2}\right\| W_{1} \mathcal{E} \end{aligned}$ | $M \mapsto_{\mathrm{R}} M^{\prime}$ Target term $M$ $\begin{array}{r} \left\langle+_{\mathrm{i}} \Leftarrow+\right\rangle(\mathrm{i} \\ \left\langle+_{\mathrm{k}} \Leftarrow \phi^{\prime}\right\rangle(\mathrm{i} \\ \mathrm{case}^{2} \mathrm{inj}_{\mathrm{i}} \mathrm{~W}, \mathrm{inj}^{2} \\ \text { case(inj} \mathrm{in}_{\mathrm{i}} \mathrm{~W}, \mathrm{inj}_{1} \mathrm{x}_{1} \cdot \mathrm{M}_{1}, \mathrm{inj}_{2} x \\ (\lambda x . \end{array}$ | ReduceUpcast <br> ReduceCastSuccess <br> ReduceCastFailure <br> ReduceCase1 <br> ReduceCase2 <br> Reduce $\beta$ |
| :---: | :---: | :---: |
| Target term $M$ steps to | $\frac{M \mapsto_{\mathrm{R}} M^{\prime}}{\mathcal{E}[M] \mapsto \mathcal{E}\left[M^{\prime}\right]} \text { StepContext }$ | epMatchfail |

Figure 11. Small-step semantics of the target language

This translation (Figure 12 top) maps the source sums + and $+{ }^{?}$ to the target sum + , and maps the other source sums to ${ }_{+}$.

We extend type assignment, rather than the bidirectional system, because translation should be independent of bidirectionality: Type assignment is stable under variations in the bidirectional "recipe", so if we decided to synthesize a type for (), we could leave the translation untouched. That said, an implementation would be based on a bidirectional version of the translation-replacing ":" with " $\Leftarrow$ " or " $\Rightarrow$ ", following Figure 7

The interesting translation rule is STCSub which inserts a coercion context $\mathcal{C}$. This context coerces between two directedconsistent types, so it composes up to three coercions (cf. Figure 57: from a more imprecise type to a less imprecise type, from that type to a supertype, and from the supertype to a more imprecise type.

Our coercion judgment $A^{\prime} \Rightarrow A \hookrightarrow \mathcal{C}$ produces a context $\mathcal{C}$, a target term containing a hole such that, if $M$ has type $T^{\prime}=\left|A^{\prime}\right|$, then $\mathcal{C}[M]$ has type $T=|A|$. Rule CoeUnit produces a hole, which behaves as the identity function. Rule $\mathrm{Coe} \rightarrow$ produces a function: given a hole [] filled by a function of type $\mathrm{T}_{1}^{\prime} \rightarrow \mathrm{T}_{2}^{\prime}$, it constructs $\lambda x . \mathcal{C}_{2}\left[[] \mathcal{C}_{1}[x]\right]$. This function has type $T_{1} \rightarrow T_{2}$ : it applies cast $\mathcal{C}_{1}$ to $x$, yielding a value of type $\mathrm{T}_{1}^{\prime}$. Applying the original function yields an $\mathrm{T}_{2}^{\prime}$, which cast $\mathcal{C}_{2}$ transforms into an $\mathrm{T}_{2}$.

Three rules generate coercions between sum types:CoeCase1L CoeCase1R and CoeCase2 The first two rules handle sums that are definitely a left injection, or definitely a right injection: we apply CoeCase1L whenever we are coercing from $A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}$ where $\delta^{\prime}$ is $+_{1}$ or $+\frac{?}{2}$, and CoeCase1R when $\delta^{\prime}$ is +2 or $+\frac{?}{2}$.

In CoeCase1L we recursively generate a coercion $\mathcal{C}_{1}$ from $A_{1}^{\prime}$, and a cast $\mathcal{C}_{3}$ from $\delta^{\prime}$. The conclusion generates a coercion by matching the given value (replacing []) against $\operatorname{inj}_{1} x_{1}$, constructing $\operatorname{inj}_{1}\left(\mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right)$, to which we apply $\mathcal{C}_{3}$. CoeCase1R is symmetric.

CoeCase2 handles the cases not covered by the previous two rules. In addition to doing the work of the previous two rules, it generates casts $\mathcal{C}_{1}^{\prime}$ and $\mathcal{C}_{2}^{\prime}$, applying them in each arm. According to STSumIntro an injection inj $j_{1}$ has a type whose sum constructor is $+{ }_{i}^{?}$, so CoeCase2 applies $\mathcal{C}_{1}^{\prime}$ which takes $+{ }_{i}^{?}$ to $\delta^{\prime}$. Similarly, the rule applies $\mathcal{C}_{2}^{\prime}$, which takes $+_{2}^{?}$ to $\delta^{\prime}$. Since CoeCase2 applies $\mathcal{C}_{3}$ (from $\delta^{\prime}$ to $\delta$ ) to the entire case, the result will be $\delta$.

### 5.3 Target Precision $\preccurlyeq$

We will prove that more precise source typings-differently annotated versions of the same source expression-produce more precise target terms. We will also prove that precision of the target terms is preserved by stepping, and that if a more precise target term converges (steps to a value), so does a less precise target term.

Sum translation $|\delta|=\phi \quad$ Type translation $|\mathcal{A}|=T \quad$ Typing context trans. $|\Gamma|=\Theta \quad$ Coercion contexts

Type translation $|\mathcal{A}|=\mathrm{T} \quad$ Typing context trans. $|\Gamma|=\Theta$

$$
\begin{array}{rlrl}
\mid \text { Unit } \mid & =\text { Unit } & |\cdot|= \\
\left|A_{1} \delta A_{2}\right| & =\left|A_{1}\right||\delta|\left|A_{2}\right| & & |\Gamma, x: A|=|\Gamma|, x:|A| \\
\left|A_{1} \rightarrow A_{2}\right| & =\left|A_{1}\right| \rightarrow\left|A_{2}\right| &
\end{array}
$$

Coercion contexts

$$
\mathcal{C}::=[]
$$

$\mid \operatorname{case}\left(\mathcal{C}\right.$, inj $\left._{\mathrm{i}} \mathrm{x} . \mathrm{Mi}_{\mathrm{i}}\right)$
$\mid \operatorname{case}\left(\mathcal{C}\right.$, inj $\left._{1} x_{1} \cdot M_{1}, \operatorname{inj}_{2} x_{2} . M_{2}\right)$
$\mid\left\langle\phi \Leftarrow \phi^{\prime}\right\rangle \mathcal{C}$
$\mid \lambda$ х. $\mathcal{C} \mid \mathcal{C} \mathrm{M}_{2}$
$\delta^{\prime} \Rightarrow \delta \hookrightarrow \mathcal{C}$ Coercion $\mathcal{C}$ coerces sum $\left|\delta^{\prime}\right|$ to sum $|\delta|$

$$
\frac{\left|\delta^{\prime}\right| \leq|\delta|}{\delta^{\prime} \Rightarrow \delta \hookrightarrow[]} \text { CoeSub }
$$

$$
\begin{aligned}
& A^{\prime} \Rightarrow A \hookrightarrow \mathcal{C} \text { Coercion } \mathcal{C} \text { coerces target type }\left|A^{\prime}\right| \text { to }|A|
\end{aligned}
$$

$\Gamma \vdash e: A \hookrightarrow M$ Under typing context $\Gamma$, expression $e$ has type $A$ and translates to target term $M$

$$
\begin{aligned}
& \frac{\Gamma(x)=A}{\Gamma \vdash x: A \hookrightarrow x} \text { STVar } \frac{\Gamma \vdash e: A^{\prime} \hookrightarrow M^{\prime} \quad \begin{array}{l}
A^{\prime} \leadsto A \\
A^{\prime} \Rightarrow A \hookrightarrow \mathcal{C}
\end{array}}{\Gamma \vdash e: A \hookrightarrow \mathcal{C}\left[M^{\prime}\right]} \operatorname{STCSub} \frac{\Gamma \vdash e: A \hookrightarrow M}{\Gamma \vdash(e:: A): A \hookrightarrow M} \text { STAnno } \frac{\Gamma \vdash(): \text { Unit } \hookrightarrow()}{\Gamma} \text { STUnitIntro } \\
& \frac{\Gamma \vdash e: A_{i} \hookrightarrow M}{\Gamma \vdash \operatorname{inj}_{i} e:\left(A_{1}+\frac{?}{i} A_{2}\right) \hookrightarrow \operatorname{inj}_{i} M} \text { STSumIntro } \\
& \frac{\Gamma \vdash e_{0}: A_{1}+{ }_{i}^{*} A_{2} \hookrightarrow M_{0} \quad \Gamma, x: A_{i} \vdash e: A \hookrightarrow M}{\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{i} x . e\right): A \hookrightarrow \operatorname{case}\left(M_{0}, \operatorname{inj}_{i} x . M\right)} \text { STSumElim1 } \frac{\Gamma \vdash e_{0}: A_{1}+A_{2} \hookrightarrow M_{0} \quad \Gamma, x_{2}: A_{2} \vdash e_{2}: A \hookrightarrow M_{2}}{\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{1} x_{1} \cdot e_{1}, \operatorname{inj}_{2} x_{2} \cdot e_{2}\right): A} \underset{\hookrightarrow \operatorname{case}\left(M_{0}, \operatorname{inj}_{1} x_{1} \cdot M_{1}, \operatorname{inj}_{2} x_{2} \cdot M_{2}\right)}{\hookrightarrow \cos } \text { STSumElim2 } \\
& \frac{\Gamma, x: A_{1} \vdash e: A_{2} \hookrightarrow M}{\Gamma \vdash \lambda x . e: A_{1} \rightarrow A_{2} \hookrightarrow \lambda x . M} S T \rightarrow \text { Intro } \quad \frac{\Gamma \vdash e_{1}: A_{1} \rightarrow A_{2} \hookrightarrow M_{1} \quad \Gamma \vdash e_{2}: A_{1} \hookrightarrow M_{2}}{\Gamma \vdash e_{1} e_{2}: A_{2} \hookrightarrow M_{1} M_{2}} \mathrm{ST} \rightarrow \text { Elim }
\end{aligned}
$$

Figure 12. Type-directed translation

Our relation, and the form of the result, were inspired by the approximation relation of Ahmed et al. (2011), as well as the term precision relation of Siek et al. (2015).

For source expressions, we defined $e^{\prime} \sqsubseteq e$ simply by applying $\sqsubseteq$ to the types in annotations. For target terms, we have no type precision relation; the target type system only has static sums, so $\mathrm{T}^{\prime} \sqsubseteq \mathrm{T}$ would degenerate to $\mathrm{T}^{\prime}=\mathrm{T}$. Instead, we define target precision $\preccurlyeq$ for terms only.

If $e^{\prime} \sqsubseteq e$, and these expressions translate to $M^{\prime}$ and $M$ respectively, we want to show $M^{\prime} \preccurlyeq M$. The difference between $e^{\prime}$ and $e$ is only in their annotations, so $M^{\prime}$ and $M$ must share a lot of structure-except that different annotations may lead to different casts. Thus, most of the rules in Figure 13 are homomorphic.

What about casts, which can step to matchfail? A static source typing is very precise, and the target term it produces never fails, so we might expect a more precisely typed term to "fail less"-but this would lead us astray. A better intuition is that imprecisely typed code "doesn't care", so it tends not to fail—while precisely typed code can fail, if it collides with imprecisely typed code. Therefore, terms with casts should be more precise than terms without. In addition, since casts can step to matchfail, and we want stepping to preserve precision, matchfail $\preccurlyeq M$ for any $M$.

Given two terms with casts $M^{\prime}=\left\langle\phi_{2}^{\prime} \Leftarrow \phi_{1}^{\prime}\right\rangle$ and $M=$ $\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle$, we will consider $M^{\prime}$ more precise than $M$ if the cast in $M^{\prime}$ is more precise: $\left\langle\phi_{2}^{\prime} \Leftarrow \phi_{1}^{\prime}\right\rangle \preccurlyeq\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle$. Let ac be a cast; it must be either a safe cast sc like $\langle+\Leftarrow+\rangle$ or $\left\langle+\Leftarrow+{ }_{1}\right\rangle$, a backward cast bc of the form $\left\langle+_{\mathrm{i}} \Leftarrow+\right\rangle$, or a (doomed) matchfailure cast $\mathrm{mc}-\left\langle+_{2} \Leftarrow+_{1}\right\rangle$ or $\left\langle+_{1} \Leftarrow+_{2}\right\rangle$. These are classified by the grammar in Figure 13

Equal casts should be equally precise, so rule Cast $\preccurlyeq$ Reff makes the relation $\mathrm{ac}^{\prime} \preccurlyeq$ ac reflexive. Following the idea that the more precisely typed term should "fail more", a safer cast should be less precise; this leads to Cast $\mathrm{M} \preccurlyeq \mathrm{B}, \mathrm{CastB} \preccurlyeq \mathrm{S}$ and CastM $\preccurlyeq S$

The other rules are subtle. They compare particular safe casts and/or backward casts, relying implicitly on typing. For example, the last rule says (with $\mathfrak{i}=1$ ) that $\langle+\Leftarrow+\rangle \preccurlyeq\left\langle+\Leftarrow+{ }_{1}\right\rangle$. We will ultimately need to show that if the cast on the left succeeds, so does the cast on the right. The left-hand cast is $\langle+\Leftarrow+\rangle$, which always succeeds. The right-hand cast succeeds if it is given $\mathrm{inj}_{1}$. If the value being cast is well-typed, then (by TCast) it will indeed have type ${ }_{1}$.

Finally, note that a more precise source typing may result in a one-armed case in a coercion, while the less precise typing results in a two-armed case. For example, $+^{?}$ is less precise than $+_{1}$;


Figure 13. Precision $\preccurlyeq$ on target terms
coercing $+_{1}$ to + results in one-armed case, and coercing $+^{?}$ to + results in a two-armed case. Hence, a one-armed case can be more precise than a two-armed case.

### 5.4 Metatheory

The target system satisfies preservation and progress:
Theorem 6 (Type preservation).
If $\cdot \vdash M: T$ and $M \mapsto M^{\prime}$ then $\cdot \vdash M^{\prime}: T$.
Theorem 7 (Progress).
If $\cdot \vdash M: T$ then either (a) $M$ is a value, or (b) there exists $M^{\prime}$ such that $M \mapsto M^{\prime}$, or (c) $M=$ matchfail.

By itself, the above progress statement leaves open the possibility that a well-typed target term $M$ will step to matchfail. However, if $M$ has no casts, it will not step to matchfail.
Theorem 8 (matchfail-freeness).
If $M$ is cast-free and matchfail-free and $M \mapsto M^{\prime}$ then $M^{\prime}$ is cast-free and matchfail-free.

For cast-free terms, combining Theorems 7 and 8 gives a version of progress without the possibility of match failure.
Corollary. If $M$ is cast-free and matchfail-free and $\cdot \vdash \mathrm{M}: \mathrm{T}$ then either (a) $M$ is a value, or (b) there exists $M^{\prime}$ such that $M \mapsto M^{\prime}$.

We also prove that the translation takes well-typed source programs to well-typed target programs. The theorem takes a type assignment derivation, but Theorem 2 can produce such a derivation from a bidirectional typing derivation.
Theorem 9 (Translation soundness).
If $\Gamma \vdash e: A$ then there exists $M$ such that $\Gamma \vdash e: A \hookrightarrow M$ and $|\Gamma| \vdash M:|A|$.

The proof relies on several lemmas, e.g. that the generated coercions $\mathcal{C}$ are well-typed; see the supplementary material.

A great advantage of static typing is that, for a suitable definition of "wrong", static programs don't go wrong. The theorem below proves that translating a static program yields a target term $M$ that has no casts; by Theorem $8, M$ will never step to matchfail.

Theorem 10 (Static derivations don't have match failures).
If $\Gamma^{\mathrm{S}} \vdash e^{\mathrm{S}} \Leftarrow \mathrm{A}^{\mathrm{S}}$ or $\Gamma^{\mathrm{S}} \vdash e^{\mathrm{s}} \Rightarrow A^{\mathrm{s}}$
then there exists $M$ such that $\Gamma^{\mathrm{S}} \vdash e^{\mathrm{S}}: A^{\mathrm{S}} \hookrightarrow M$
and $M$ is free of casts and matchfail.
Together, preservation and progress correspond to Theorem 3 (type safety) of Siek et al. (2015, p. 9). Their blame-subtyping Theorem 4 says that safe casts (casts from a subtype to a supertype) cannot be blamed (cannot fail); our translation does not insert safe casts at all, and our Theorem 10 shows that expressions without dynamic sums produce target terms without casts.

The remaining results concern precision. We show that more precise annotations translate to more precise terms, that target precision is preserved by stepping, and that if a target term converges, then a less precise version also converges.

We must note that the first of these results, Theorem 11 uses a modified version of the translation: one that always inserts casts, even safe ones; this simplifies part of the proof. In effect, the modified translation (Figure 21 in the appendix) does not have rule CoeSub and always uses rule CoeCast Similarly, we modify CoeCase1L and CoeCase1R to always insert casts within each arm, like $\mathcal{C}_{1}^{\prime}$ and $\mathcal{C}_{2}^{\prime}$ in CoeCase2 Since the only difference is the presence of casts that cannot fail, the terms generated by either translation must both step to the same value, or both generate matchfail.
Theorem 11 (Translation preserves precision).
Suppose $\Gamma^{\prime} \sqsubseteq \Gamma$ and $e^{\prime} \sqsubseteq e$.

1. If $\Gamma^{\prime} \vdash e^{\prime} \Leftarrow A^{\prime}$ and $\Gamma \vdash e \Leftarrow A$ and $A^{\prime} \sqsubseteq A$ then $\Gamma^{\prime} \vdash e^{\prime}: A^{\prime} \hookrightarrow M^{\prime}$ and $\Gamma \vdash e: A \hookrightarrow M$ where $M^{\prime} \preccurlyeq M$.
2. If $\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A^{\prime}$ and $\Gamma \vdash e \Rightarrow A$ then $\Gamma^{\prime} \vdash e^{\prime}: A^{\prime} \hookrightarrow M^{\prime}$ and $\Gamma \vdash e: A \hookrightarrow M$ where $A^{\prime} \sqsubseteq A$ and $M^{\prime} \preccurlyeq M$.

Theorem 12 (Stepping preserves precision).
If $\cdot \vdash M_{1}^{\prime}: T_{1}^{\prime}$ and $\cdot \vdash M_{1}: T_{1}$ and $M_{1}^{\prime} \preccurlyeq M_{1}$ and $M_{1}^{\prime} \mapsto M_{2}^{\prime}$ then either
(a) $M_{1}$ is a value and $M_{2}^{\prime} \preccurlyeq M_{1}$, or
(b) there exists $M_{2}$ such that $M_{1} \mapsto M_{2}$ and $M_{2}^{\prime} \preccurlyeq M_{2}$, or
(c) $M_{1}=$ matchfail and $M_{2}^{\prime} \preccurlyeq M_{1}$.

Definition 1. A closed term $M$ converges if $M \mapsto^{*} W$ for some value W , and diverges if the stepping sequence never terminates.

Note that matchfail neither converges nor diverges, and that divergence is not possible in our language.
Theorem 13 ( $\preccurlyeq$ respects convergence).
If $\mathrm{M}^{\prime} \preccurlyeq \mathrm{M}$ where $\cdot \vdash \mathrm{M}^{\prime}: \mathrm{T}^{\prime}$ and $\cdot \vdash \mathrm{M}: T$
and $M^{\prime}$ converges then $M$ also converges.
If $M^{\prime} \preccurlyeq M$, and they converge to injections $\operatorname{inj}_{i} W^{\prime}$ and $\operatorname{inj}_{k} W$, then Theorem 13 gives $\operatorname{inj}_{i} W^{\prime} \preccurlyeq \operatorname{inj}_{k} W$. By inversion on the definition of $\preccurlyeq$, we have $i=k$. Similar results would hold if $\preccurlyeq$ were extended for base types.

Together with Theorem 11 this means that if we translate two source expressions $e^{\prime} \sqsubseteq e$ to $M^{\prime}$ and $M$, and $M^{\prime}$ converges to a value of base type, $M$ will converge to the same value. This corresponds to Theorem 5 (gradual guarantee), part 2, of Siek et al. (2015).

## 6. Related Work

Sums and refinements. Sum types are well-established in a variety of programming languages, though practical languages tend to embed them within larger mechanisms: ML datatypes can encode sums, but also recursion. Refinement type systems, such as datasort refinements (Freeman and Pfenning 1991; Davies 2005) and indexed types (Xi and Pfenning 1999), have been built on these larger mechanisms. This gives a close connection to practice, but needs additional machinery such as constructor types and signatures. Such machinery is not central to our investigation; in contrast, we distill datasort refinements to one essential feature: distinguishing whether we have a left or right injection.

These systems often have a refinement relation $\sqsubset$ : if $\mathcal{A}$ is a sort (refined type) and $\tau$ is an unrefined type, $A \sqsubset \tau$ says that $A$ refines $\tau$. Both the symbol and the high-level concept resemble our relation $A^{\prime} \sqsubseteq A$, but the refinement relation is more rigid: it cannot compare two sorts, or two unrefined types, and it certainly cannot derive $\left(A_{1} \rightarrow A\right) \sqsubset\left(A_{1} \rightarrow \tau\right)$, where $\left(A_{1} \rightarrow \tau\right)$ mixes a refined type $A_{1}$ with an unrefined type $\tau$. Nonetheless, the covariance of this relation on function types-in contrast to subtyping, which must be contravariant-made us more confident that our precision relation should be covariant.

Koot and Hage (2015) formulate a constraint-based type system that analyzes pattern matches, using a characterization of data somewhat reminiscent of datasort refinements. Their system needs no type annotations, but is (necessarily) incomplete.

Gradual typing. Our approach to expressing uncertainty in a type system was inspired by gradual typing, introduced by Siek and Taha (2006), in which? (often written $\star$ ) is an uncertain type (it could be Int, a function type, or anything else). We confine uncertainty to refinement properties of sum types, making the effect on the overall type system less dramatic; still, several mechanisms of gradual typing appear in our work. For example, we also have precision relations on types and (through annotations) expressions.

Our directed consistency is somewhat similar to consistent subtyping for gradual object-based languages (Siek and Taha 2007). Consistent subtyping augments subsumption with consistent equality (roughly, gain and loss of precision) on either the subtype or supertype, but not both. Drawing on abstract interpretation, Garcia et al. (2016) give a different but equivalent formulation of consistent subtyping. In these systems, the underlying subtyping relation is defined over static types only. Allende et al. (2014) also have a notion of directed consistency, but the connection to our relation is less clear.

Siek et al. (2015) propose several criteria as desirable for gradual type systems. We prove properties that correspond to some of
their criteria: Theorems 5 and 15 correspond to the first parts of Theorems 1 and 2 of Siek et al. (2015), our Theorem 10 corresponds to their Theorem 4, our Theorem 4 corresponds to part 1 of their Theorem 5 (gradual guarantee), and our Theorems 11 and 13 corresponds to part 2 of their Theorem 5.

Some systems of gradual typing include a notion of blame (Wadler and Findler 2009), associating program labels to casts so that a failing cast "blames" some program location. It may be possible to incorporate blame into our approach; we omit it to focus on other issues.

We are not the first to apply ideas from gradual typing to lesstraditional areas: for example, Bañados Schwerter et al. (2014) develop a gradual effect system, and McDonell et al. (2016) develop a tool for moving between ADTs and more precise GADTs.
Bidirectional typing. Originating as folklore and first discussed explicitly by Pierce and Turner (1998), bidirectional typing has been used extensively in type systems for which full inference is undecidable or otherwise problematic (Freeman and Pfenning 1991; Coquand 1996; Xi and Pfenning 1999; Davies and Pfenning 2000; Pientka 2008). A strength of many bidirectional type systems, sometimes overlooked, is that they have some variety of subformula property. In some systems, this property serves to make type checking more feasible-for example, for Davies (2005) and Dunfield (2007), it controls the spread of intersection types. For Dunfield (2015), where evaluation order is implicit in terms and explicit in types, it prevents the spontaneous generation of by-name types; in our system, it prevents the spontaneous generation of gradual sum types.

The gradual type system of Garcia and Cimini (2015, p. 306) is not bidirectional, but enjoys a similar property: "dynamicity [the uncertain type ?] is introduced only via program annotations". However, their rules can be viewed as a bidirectional system that always synthesizes, except at annotations.

## 7. Future Work

We plan to implement the bidirectional type system, which will allow us to test whether our approach is practical. We are particularly interested in whether our formulation of precision, combined with the annotation discipline of bidirectional typing, strikes a good balance: the annotation burden should be reasonable, but imprecision should not appear out of nowhere. Also, it is unclear whether programmers would have any use for the sum types $+_{i}^{?}$ and $+_{i}^{*}$; if not, error messages should read "expected +1 or + "" rather than "expected $+{ }_{1}^{*}$ ", for example.

We would also like to enrich the language with intersection types, recursive types, and polymorphism. Intersection types are important for datasort refinements: for example, if we encode booleans as Unit + Unit, the datasorts True and False are Unit $+_{1}$ Unit and Unit $+_{2}$ Unit. Then negation should have type (True $\rightarrow$ False) $\cap$ (False $\rightarrow$ True). We also want to evaluate the run-time efficiency of coercions-a common concern in gradual type systems.

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## Appendix to "Sums of Uncertainty: Refinements go gradual" (POPL 2017)

## A. Dynamic System

$$
\begin{aligned}
& \text { Dynamic expressions } \quad e^{\mathrm{D}}::=()|x| \lambda x . e^{\mathrm{D}}\left|e_{1}^{\mathrm{D}} e_{2}^{\mathrm{D}}\right| \operatorname{inj}_{i} e^{\mathrm{D}} \mid\left(e^{\mathrm{D}}:: A^{\mathrm{D}}\right) \\
& \left|\operatorname{case}\left(e^{\mathrm{D}}, \mathrm{inj}_{1} \mathrm{x}_{1} . e_{1}^{\mathrm{D}}, \mathrm{inj}_{2} \mathrm{x}_{2} . e_{2}^{\mathrm{D}}\right)\right| \operatorname{case}\left(e^{\mathrm{D}}, \operatorname{inj}_{\mathrm{i}} \mathrm{x} . e_{\mathrm{i}}^{\mathrm{D}}\right) \\
& \text { Dynamic types } \\
& A^{D}::=\text { Unit }\left|A_{1}^{D}+{ }^{?} A_{2}^{D}\right| A_{1}^{D} \rightarrow A_{2}^{D} \\
& \text { Dynamic typing contexts } \quad \Gamma^{\mathrm{D}}::=\cdot \mid \Gamma^{\mathrm{D}}, x: A^{\mathrm{D}} \\
& \frac{\Gamma^{\mathrm{D}}(\mathrm{x})=A^{\mathrm{D}}}{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} x \Rightarrow A^{\mathrm{D}}} \mathrm{DVar} \quad \frac{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e^{\mathrm{D}} \Rightarrow A^{\mathrm{D}}}{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e^{\mathrm{D}} \Leftarrow A^{\mathrm{D}}} \text { DSub } \quad \frac{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e^{\mathrm{D}} \Leftarrow A^{\mathrm{D}}}{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}}\left(e^{\mathrm{D}}:: A^{\mathrm{D}}\right) \Rightarrow A^{\mathrm{D}}} \text { DAnno } \quad \quad \quad \frac{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}}() \Leftarrow \text { Unit }}{} \text { DUnitintro } \\
& \frac{\Gamma^{\mathrm{D}}, x: A_{1}^{\mathrm{D}} \vdash_{\mathrm{D}} e^{\mathrm{D}} \Leftarrow A_{2}^{\mathrm{D}}}{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} \lambda x . e^{\mathrm{D}} \Leftarrow A_{1}^{\mathrm{D}} \rightarrow A_{2}^{\mathrm{D}}} \mathrm{D} \rightarrow \operatorname{Intro} \frac{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e_{1}^{\mathrm{D}} \Rightarrow A_{1}^{\mathrm{D}} \rightarrow A_{2}^{\mathrm{D}} \Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e_{2}^{\mathrm{D}} \Leftarrow A_{1}^{\mathrm{D}}}{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e_{1}^{\mathrm{D}} e_{2}^{\mathrm{D}} \Rightarrow A_{2}^{\mathrm{D}}} \mathrm{D} \rightarrow \operatorname{Elim} \frac{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e^{\mathrm{D}} \Leftarrow A_{\mathrm{i}}^{\mathrm{D}}}{\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} \operatorname{inj}_{\mathrm{i}} e^{\mathrm{D}} \Leftarrow\left(A_{1}^{\mathrm{D}}+{ }^{?} A_{2}^{\mathrm{D}}\right)} \mathrm{D}+? \text { Intro }
\end{aligned}
$$

Figure 14. The dynamic system: the bidirectional system restricted to + ?
Figure 14 shows the syntax and typing rules for the dynamic system-the restriction of the bidirectional type system to the dynamic sum + ?

## B. Omitted Definitions

$e^{\prime} \sqsubseteq e$ Expression $e^{\prime}$ is more precise than $e$

$$
\begin{array}{rlccc}
\overline{() \sqsubseteq()} & \frac{e^{\prime} \sqsubseteq e}{x \sqsubseteq x} & \frac{e_{1}^{\prime} \sqsubseteq e_{1} e_{2}^{\prime} \sqsubseteq e_{2}}{\lambda x \cdot e^{\prime} \sqsubseteq \lambda x \cdot e} & \frac{e^{\prime} \sqsubseteq e}{e_{1}^{\prime} e_{2}^{\prime} \sqsubseteq e_{1} e_{2}} & \\
\left(\operatorname{inj}_{\mathrm{i}} e^{\prime}\right) \sqsubseteq\left(\operatorname{inj}_{\mathrm{i}} e\right) & \frac{e^{\prime} \sqsubseteq e}{\left(e^{\prime}:: A^{\prime}\right) \sqsubseteq(e:: A)} A^{\prime} \sqsubseteq A \\
& & & e^{\prime} \sqsubseteq e & e_{\mathrm{i}}^{\prime} \sqsubseteq e_{\mathrm{i}} \\
\hline \operatorname{case}\left(e^{\prime}, \operatorname{inj}_{1} x_{1} \cdot e_{1}^{\prime}, \operatorname{inj}_{2} x_{2} \cdot e_{2}^{\prime}\right) \sqsubseteq \operatorname{case}\left(e, \operatorname{inj}_{1} x_{1} \cdot e_{1}, \operatorname{inj}_{2} x_{2} \cdot e_{2}\right) & \left.e_{1}^{\prime} \sqsubseteq e_{1}, \operatorname{inj}_{i} x \cdot e_{i}^{\prime}\right) \sqsubseteq \operatorname{case}\left(e, \operatorname{inj}_{i} x \cdot e_{i}\right)
\end{array}
$$

$\Gamma^{\prime} \sqsubseteq \Gamma$ Typing context $\Gamma^{\prime}$ is more precise than $\Gamma$


$$
\frac{\Gamma^{\prime} \sqsubseteq \Gamma \quad A^{\prime} \sqsubseteq A}{\left(\Gamma^{\prime}, x: A^{\prime}\right) \sqsubseteq(\Gamma, x: A)}
$$

Figure 15. Precision on expressions and contexts
Several results involve precision of expressions and typing contexts, shown in Figure 15, these are the straightforward lifting of type precision (Figure 4).
$e^{\prime}=: e$ Expression $e^{\prime}$ is annotative-ly equivalent to $e$

$$
\begin{aligned}
& \overline{()=:()} \quad \overline{\chi=: \chi} \quad \frac{e^{\prime}=: e}{e^{\prime}=:(e:: A)} \\
& \frac{e^{\prime}=: e}{\lambda x . e^{\prime}=: \lambda x . e} \quad \frac{e_{1}^{\prime}=: e_{1} \quad e_{2}^{\prime}=: e_{2}}{e_{1}^{\prime} e_{2}^{\prime}=: e_{1} e_{2}} \quad \frac{e^{\prime}=: e}{\left(\operatorname{inj}_{i} e^{\prime}\right)=:\left(\operatorname{inj}_{i} e\right)} \quad \frac{e^{\prime}=: e \quad A^{\prime}=A}{\left(e^{\prime}:: A^{\prime}\right)=:(e:: A)} \\
& \frac{e^{\prime}=: e \quad e_{1}^{\prime}=: e_{1} \quad e_{2}^{\prime}=: e_{2}}{\operatorname{case}\left(e^{\prime}, \operatorname{inj}_{1} x_{1} \cdot e_{1}^{\prime}, \operatorname{inj}_{2} x_{2} \cdot e_{2}^{\prime}\right)=: \operatorname{case}\left(e, \operatorname{inj}_{1} x_{1} \cdot e_{1}, \operatorname{inj}_{2} \chi_{2} \cdot e_{2}\right)} \quad \frac{e^{\prime}=: e \quad e_{i}^{\prime}=: e_{i}}{\operatorname{case}\left(e^{\prime}, \operatorname{inj}_{i} x \cdot e_{i}^{\prime}\right)=: \operatorname{case}\left(e, \operatorname{inj}_{i} \chi \cdot e_{i}\right)}
\end{aligned}
$$

Figure 16. Annotation equivalence

## C. Differences from the Original Version

The paper that was submitted to POPL differs in two important ways from the final version.
No directed consistency. In the final version, ChkCSub SCSub etc. allow (a) gain of precision, (b) subtyping, and (c) loss of precision, formulated via directed consistency. In contrast, the original system had (in each system) two rules: one rule that allowed subtyping (exactly like a traditional subsumption rule), and one rule that allowed loss of precision. For example, the bidirectional system had

$$
\frac{\Gamma \vdash e \Rightarrow A^{\prime} \quad A^{\prime} \leq A}{\Gamma \vdash e \Leftarrow A} * * \text { ChkSub } \quad \frac{\Gamma \vdash e \Rightarrow A^{\prime} \quad A^{\prime} \sqsubseteq A}{\Gamma \vdash e \Leftarrow A} * * \text { ChkImp }
$$

These rules could not type the same expression without an extra annotation (to transition from the checking conclusion of one rule to the synthesizing conclusion of the other).

Moreover, there was no rule to gain precision. In a traditional gradual type system, this would be completely untenable: the point of the "unknown type" in a gradual system is that it can be downcasted to a static type. In the previous version of our system, programmers could write coercions "by hand":

$$
f:\left(A_{1}+{ }_{1} A_{2}\right) \rightarrow B, y:\left(A_{1}+{ }^{?} A_{2}\right) \vdash f\left(\operatorname{case}\left(y, \operatorname{inj}_{1} x \cdot x\right)\right) \Rightarrow B
$$

But this requires a change to the expression that goes beyond changing an annotation: the expression itself is being changed.
The lack of a way to gain precision, combined with the need for an extra annotation to use subtyping and loss of precision, meant that the varying precision property-Theorem 4 in the final version-did not hold. A weaker property-Theorem 14 below-did hold, but this property only provides that some expression $e_{j}$, which could be more imprecise than $e$, is well typed.
Different definition of imprecision. In Section [2 we explained why $+_{1}^{*} \sqsubseteq+{ }_{j}^{?}$ doesn't make sense. We also argued against $+{ }_{j}^{?} \sqsubseteq+{ }_{1}^{*}$, on the basis that in directed consistency SCSub one could gain precision from $+_{1}^{*}$ to $+{ }_{1}^{?}$, then use subtyping from $+?$ was no gain of precision, and even loss of precision could not be combined with subtyping (without extra annotation). Thus, we saw no clear argument against $+{ }_{i}^{?} \sqsubseteq+{ }_{1}^{*}$, and included it in the relation. However, in the absence of gain of precision, the only way the type system could use this was by moving from $+_{i}^{?}$ to $+_{i}^{*}$, which was also possible via subtyping.


Figure 17. Original, obsolete definition of precision

## C. 1 Original, weak version of varying precision

Theorem 4 does not hold for the original system. Instead, the following holds, where $e \sqsubseteq: e_{j}$ means that $e_{j}$ is a version of $e$ with moreimprecise annotations (like $\left.e^{\prime} \sqsubseteq e\right)$ and extra annotations. For example, $x \sqsubseteq:(x:: A)$.
Theorem 14 (Weak version of varying precision).

1. If $\Gamma^{\prime} \vdash e^{\prime} \Leftarrow A^{\prime}$ and $e^{\prime} \sqsubseteq e$ and $\Gamma^{\prime} \sqsubseteq \Gamma$
then there exist $e_{j}$ and $A$ such that $\Gamma \vdash e_{j} \Leftarrow A$ and $e \sqsubseteq: e_{j}$ and $A^{\prime} \sqsubseteq A$.
2. If $\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A^{\prime}$ and $e^{\prime} \sqsubseteq e$ and $\Gamma^{\prime} \sqsubseteq \Gamma$
then there exist $e_{j}$ and $A$
such that $\Gamma \vdash e_{j} \Rightarrow A$ and $e \sqsubseteq: e_{j}$ and $A^{\prime} \sqsubseteq A$.
Given $e^{\prime} \sqsubseteq e$, this weak version of varying precision yields some $e_{j}$ that may be more imprecise, $e^{\prime} \sqsubseteq e \sqsubseteq: e_{j}$. This is needed becausein the absence of ChkCSub which allows precision to be adjusted whenever subsumption is used-a more imprecise annotation may require changing other annotations to make them more imprecise. For example, suppose we are given

$$
\begin{aligned}
e^{\prime} & =((\lambda x \cdot(x:: B+2 B))::(B+2 B) \rightarrow(B+2 B)) \\
e & =\left(\left(\lambda x \cdot\left(x:: B+{ }^{2} B\right)\right)::(B+2 B) \rightarrow(B+2 B)\right)
\end{aligned}
$$

We can synthesize $A^{\prime}=(B+2 B) \rightarrow(B+2 B)$ for $e^{\prime}$, but not for $e$, because the inner annotation on $x$ makes the $\lambda$ fail to check against the outer annotation. But we can produce $e_{j}=\left(\left(\lambda x .\left(x:: B+{ }^{?} B\right)\right)::\left(B+{ }_{2} B\right) \rightarrow\left(B+{ }^{?} B\right)\right)$. Now the uses of $+^{?}$ match, and $e_{j}$ synthesizes $A=(B+2 B) \rightarrow\left(B+{ }^{?} B\right)$. The remaining +2 is okay, because of ${ }^{* *}$ ChkImp in $\left(x:: B+{ }^{?} B\right)$, we have $\Gamma(x)=B+2 B$, which is less imprecise than $B+{ }^{?} B$.

## D. Proofs

## D. 1 Source System

## D.1.1 Subtyping

Lemma 1 (Subtyping inversion).

1. If Unit $\leq A$ then $A=$ Unit.
2. If $A^{\prime} \leq$ Unit then $A^{\prime}=$ Unit.
3. If $A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \leq A$ then $A=A_{1} \delta A_{2}$ where $A_{1}^{\prime} \leq A_{1}$ and $A_{2}^{\prime} \leq A_{2}$ and $\delta^{\prime} \leq \delta$.
4. If $A^{\prime} \leq A_{1} \delta A_{2}$ then $A^{\prime}=A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}$ where $A_{1}^{\prime} \leq A_{1}$ and $A_{2}^{\prime} \leq A_{2}$ and $\delta^{\prime} \leq \delta$.
5. If $A_{1}^{\prime} \rightarrow A_{2}^{\prime} \leq A$ then $A=A_{1} \rightarrow A_{2}$ where $A_{1} \leq A_{1}^{\prime}$ and $A_{2}^{\prime} \leq A_{2}$.
6. If $A^{\prime} \leq A_{1} \rightarrow A_{2}$ then $A^{\prime}=A_{1}^{\prime} \rightarrow A_{2}^{\prime}$ where $A_{1} \leq A_{1}^{\prime}$ and $A_{2}^{\prime} \leq A_{2}$.

## Proof.

1. By case analysis on Unit $\leq A$.

- Case Unit $\leq$ Unit: Immediate that $A=$ Unit.

2. Symmetric to the previous statement, hence omitted.
3. By case analysis on $A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \leq A$.

- Case $A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \leq A_{1} \delta A_{2}$ : Immediate as $A=A_{1} \delta A_{2}$ and subderivations are $A_{1}^{\prime} \leq A_{1}$ and $A_{2}^{\prime} \leq A_{2}$ and $\delta^{\prime} \leq \delta$.

4. Symmetric to the previous statement, hence omitted.
5. By case analysis on $A_{1}^{\prime} \rightarrow A_{2}^{\prime} \leq A$.

- Case $A_{1}^{\prime} \rightarrow A_{2}^{\prime} \leq A_{1} \rightarrow A_{2}$ : Immediate as $A=A_{1} \rightarrow A_{2}$ and subderivations are $A_{1} \leq A_{1}^{\prime}$ and $A_{2}^{\prime} \leq A_{2}$.

6. Symmetric to the previous statement, hence omitted.

Lemma 2 (Reflexivity of subtyping).
For all types $A$, it is the case that $A \leq A$.
Proof. By induction on the structure of $A$.

- Case $A=$ Unit: By the definition of precision, $A \leq A$.
- Case $A=A_{1} \delta A_{2}$ : By the induction hypothesis, $A_{1} \leq A_{1}$ and $A_{2} \leq A_{2}$. By the reflexivity of subsum, $\delta \leq \delta$. Thus, by the definition of subtyping, $A \leq A$.
- Case $A=A_{2} \rightarrow A_{2}$ : By the induction hypothesis, $A_{1} \leq A_{1}$ and $A_{2} \leq A_{2}$. Thus, by the definition of subtyping, $A \leq A$.

Lemma 3 (Transitivity of subtyping).
If $A_{1} \leq A_{2}$ and $A_{2} \leq A_{3}$ then $A_{1} \leq A_{2}$

Proof. By induction on the structure of $\mathrm{A}_{2}$.

- Case $A_{2}=$ Unit:
$A_{1} \leq$ Unit $\quad$ Given
Unit $\leq A_{3} \quad$ Given
$A_{1}=$ Unit $\quad$ By Lemma 1 Subtyping inversion
$A_{3}=$ Unit By Lemma 1 Subtyping inversion)
Unit $\leq$ Unit By Lemma Reflexivity of subtyping
$A_{1} \leq A_{3} \quad$ Equivalent
- Case $A_{2}=A_{12} \delta_{2} A_{22}$ :
$A_{1} \leq A_{12} \delta_{2} A_{22} \quad$ Given
$A_{1}=A_{11} \delta_{1} A_{21} \quad$ By Lemma 1 Subtyping inversions
$A_{11} \leq A_{12}$
$A_{21} \leq A_{22} \quad$ "
$\delta_{1} \leq \delta_{2} \quad "$
$A_{12} \delta_{2} A_{22} \leq A_{3} \quad$ Given
$A_{3}=A_{13} \delta_{3} A_{23} \quad$ By Lemma 1 Subtyping inversion
$\begin{array}{ll}A_{12} \leq A_{13} & \prime \prime \\ A_{22} \leq A_{23} & \prime \prime \\ \delta_{2} \leq \delta_{3} & \prime \prime\end{array}$

$$
\begin{array}{rlrl}
A_{11} & \leq A_{13} & & \text { By the induction hypothesis } \\
A_{21} \leq A_{23} & & \text { By the induction hypothesis } \\
\delta_{1} & \leq \delta_{3} & & \text { By the transitivity of } \leq \\
A_{11} \delta_{1} A_{21} & \leq A_{13} \delta_{3} A_{23} & & \text { By the definition of } \leq \\
A_{1} & \leq A_{3} & & \text { Equivalent }
\end{array}
$$

- Case $A_{2}=A_{12} \rightarrow A_{22}:$

| $A_{1}$ | $\leq A_{12} \rightarrow A_{22}$ |  | Given |
| ---: | :--- | ---: | :--- |
| $A_{1}$ | $=A_{11} \rightarrow A_{21}$ |  | By Lemma Subtyping inversion |
| $A_{12}$ | $\leq A_{11}$ |  | $\prime \prime$ |
| $A_{21}$ | $\leq A_{22}$ |  |  |
| $A_{12} \rightarrow A_{22}$ | $\leq A_{3}$ |  | Given |
| $A_{3}$ | $=A_{13} \rightarrow A_{23}$ |  | By Lemma Subtyping inversion |
| $A_{13}$ | $\leq A_{12}$ |  | $\prime \prime$ |
| $A_{22}$ | $\leq A_{23}$ |  |  |
| $A_{13}$ | $\leq A_{11}$ |  | By the induction hypothesis |
| $A_{21}$ | $\leq A_{23}$ |  | By the induction hypothesis |
| $A_{11} \rightarrow A_{21}$ | $\leq A_{13} \rightarrow A_{23}$ |  | By the definition of $\leq$ |
| $A_{1}$ | $\leq A_{3}$ |  | Equivalent |

## D.1.2 Precision

Lemma 4 (Precision inversion).

1. If Unit $\sqsubseteq A$ then $A=$ Unit.
2. If $\mathrm{A}^{\prime} \sqsubseteq$ Unit then $\mathrm{A}^{\prime}=$ Unit.
3. If $A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \sqsubseteq A$ then $A=A_{1} \delta A_{2}$ where $A_{1}^{\prime} \sqsubseteq A_{1}$ and $A_{2}^{\prime} \sqsubseteq A_{2}$ and $\delta^{\prime} \sqsubseteq \delta$.
4. If $A^{\prime} \sqsubseteq A_{1} \delta A_{2}$ then $A^{\prime}=A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}$ where $A_{1}^{\prime} \sqsubseteq A_{1}$ and $A_{2}^{\prime} \sqsubseteq A_{2}$ and $\delta^{\prime} \sqsubseteq \delta$.
5. If $A_{1}^{\prime} \rightarrow A_{2}^{\prime} \sqsubseteq A$ then $A=A_{1} \rightarrow A_{2}$ where $A_{1}^{\prime} \sqsubseteq A_{1}$ and $A_{2}^{\prime} \sqsubseteq A_{2}$.
6. If $A^{\prime} \sqsubseteq A_{1} \rightarrow A_{2}$ then $A^{\prime}=A_{1}^{\prime} \rightarrow A_{2}^{\prime}$ where $A_{1}^{\prime} \sqsubseteq A_{1}$ and $A_{2}^{\prime} \sqsubseteq A_{2}$.

Proof.

1. By case analysis on Unit $\sqsubseteq A$.

- Case Unit $\sqsubseteq$ Unit: Immediate that $A=$ Unit.

2. Symmetric to the previous statement, hence omitted.
3. By case analysis on $A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \sqsubseteq A$.

- Case $A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \sqsubseteq A_{1} \delta A_{2}$ : Immediate as $A=A_{1} \delta A_{2}$ and subderivations are $A_{1}^{\prime} \sqsubseteq A_{1}$ and $A_{2}^{\prime} \sqsubseteq A_{2}$ and $\delta^{\prime} \sqsubseteq \delta$.

4. Symmetric to the previous statement, hence omitted.
5. By case analysis on $A_{1}^{\prime} \rightarrow A_{2}^{\prime} \sqsubseteq A$.

- Case $A_{1}^{\prime} \rightarrow A_{2}^{\prime} \sqsubseteq A_{1} \rightarrow A_{2}$ : Immediate as $A=A_{1} \rightarrow A_{2}$ and subderivations are $A_{1}^{\prime} \sqsubseteq A_{1}$ and $A_{2}^{\prime} \sqsubseteq A_{2}$.

6. Symmetric to the previous statement, hence omitted.

Lemma 5 (Reflexivity of precision).
For all types $A$, it is the case that $A \sqsubseteq A$.
Proof. By induction on the structure of A.

- Case $A=$ Unit: $\quad$ By the definition of precision, $A \sqsubseteq A$.
- Case $A=A_{1} \delta A_{2}$ : By the induction hypothesis, $A_{1} \sqsubseteq A_{1}$ and $A_{2} \sqsubseteq A_{2}$. By the reflexivity of precision on sums, $\delta \sqsubseteq \delta$. Thus, by the definition of subtyping, $A \sqsubseteq A$.
- Case $A=A_{2} \rightarrow A_{2}$ : By the induction hypothesis, $A_{1} \sqsubseteq A_{1}$ and $A_{2} \sqsubseteq A_{2}$. Thus, by the definition of subtyping, $A \sqsubseteq A$.

Lemma 6 (Transitivity of precision).
If $A_{1} \sqsubseteq A_{2}$ and $A_{2} \sqsubseteq A_{3}$ then $A_{1} \sqsubseteq A_{2}$.

Proof. By induction on the structure of $\mathrm{A}_{2}$.

- Case $A_{2}=$ Unit:



## D.1.3 Directed Consistency

Lemma 7 (Reflexivity of directed consistency).
For all types $A$, it is the case that $A \sim A$.
Proof. Immediate from Lemma 5 Reflexivity of precision, Lemma 2 Reflexivity of subtyping and rule DirConsU
Lemma 8 (Subtyping obeys directed consistency).
If $A \leq B$ then $A \leadsto B$.
Proof. By Lemma 5 Reflexivity of precision, $A \sqsubseteq A$ and $B \sqsubseteq B$. It is given that $A \leq B$. Therefore, by ruleDirConsU $A \sim B$.
Lemma 9 (Loss in precision obeys directed consistency).
If $\mathrm{A} \sqsubseteq \mathrm{B}$ then $\mathrm{A} \leadsto \mathrm{B}$.
Proof. By Lemma 5 Reflexivity of precision, $A \sqsubseteq A$. By Lemma 2 Reflexivity of subtyping, $A \leq A$. It is given that $A \sqsubseteq B$. Therefore, by rule DirConsU $A \sim B$.

Lemma 10 (Gain in precision obeys directed consistency).
If $\mathrm{A} \sqsubseteq \mathrm{B}$ then $\mathrm{B} \leadsto \mathrm{A}$.
Proof. It is given that $A \sqsubseteq B$. By Lemma 2 Reflexivity of subtyping, $A \leq A$. By Lemma 5 Reflexivity of precision , $A \sqsubseteq A$. Therefore, by rule DirConsU $B \leadsto \bar{A}$.
$A^{\prime} \simeq A$ Type $A^{\prime}$ is structurally equivalent to $A$

$$
\overline{\text { Unit } \simeq \text { Unit }} \quad \frac{A_{1}^{\prime} \simeq A_{1} A_{2}^{\prime} \simeq A_{2}}{\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right) \simeq\left(A_{1} \delta A_{2}\right)} \quad \frac{A_{1}^{\prime} \simeq A_{1} A_{2}^{\prime} \simeq A_{2}}{\left(A_{1}^{\prime} \rightarrow A_{2}^{\prime}\right) \simeq\left(A_{1} \rightarrow A_{2}\right)}
$$

## Figure 18. Source structural equivalence

## D.1.4 Structural Equivalence

Lemma 11 (Reflexivity of Structural Equivalence).
For all types $A$, it is the case that $A \simeq A$.
Proof. By induction on the structure of $A$. All cases are immediate by the induction hypothesis and the definition of $\simeq$.
Lemma 12 (Symmetry of Structural Equivalence).
If $A^{\prime} \simeq A$ then $A \simeq A^{\prime}$.
Proof. By structural induction on the derivation of $A^{\prime} \simeq A$. All cases are immediate by the induction hypothesis and the definition of $\simeq$.
Lemma 13 (Transitivity of Structural Equivalence).
If $A_{1} \simeq A_{2}$ and $A_{2} \simeq A_{3}$ then $A_{1} \simeq A_{3}$.
Proof. By induction on the structure of the type $A_{2}$. All cases are immediate from inversion on structural equivalence, the induction hypothesis, and the definition of $\simeq$.

Corollary 14 (Structural Equivalence is an equivalence relation).
The binary relation $\simeq$ on types is an equivalence relation.
Proof. Immediate from Lemma 11 (Reflexivity of Structural Equivalence, Lemma 12 Symmetry of Structural Equivalence, and Lemma 13 Transitivity of Structural Equivalence).

Lemma 15 (Subtyping obeys Structural Equivalence).
If $A^{\prime} \leq A$ then $A^{\prime} \simeq A$.
Proof. By induction on the structure of the derivation of $A^{\prime} \leq A$.

- Case Unit $\leq$ Unit: By definition of structural equivalence, Unit $\simeq$ Unit.
- Case

$$
\frac{A_{1}^{\prime} \leq A_{1} \quad A_{2}^{\prime} \leq A_{2} \quad \delta^{\prime} \leq \delta}{\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right) \leq\left(A_{1} \delta A_{2}\right)}
$$

| $A_{1}^{\prime} \leq A_{1}$ |  | Subderivation |
| ---: | :--- | ---: | :--- |
| $A_{2}^{\prime} \leq A_{2}$ |  | Subderivation |
| $A_{1}^{\prime} \simeq A_{1}$ |  | By the induction hypothesis |
| $A_{2}^{\prime} \simeq A_{2}$ |  | By the induction hypothesis |
| $A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \simeq A_{1} \delta A_{2}$ |  | By definition of $\simeq$ |

- Case
$\frac{A_{1} \leq A_{1}^{\prime} \quad A_{2}^{\prime} \leq A_{2}}{\left(A_{1}^{\prime} \rightarrow A_{2}^{\prime}\right) \leq\left(A_{1} \rightarrow A_{2}\right)}$
$A_{1} \leq A_{1}^{\prime} \quad$ Subderivation
$A_{2}^{\prime} \leq A_{2} \quad$ Subderivation
$A_{1} \simeq A_{1}^{\prime} \quad$ By the induction hypothesis
$A_{1}^{\prime} \simeq A_{1} \quad$ By Lemma 12 Symmetry of Structural Equivalence
$A_{2}^{\prime} \simeq A_{2} \quad$ By the induction hypothesis
$A_{1}^{\prime} \rightarrow A_{2}^{\prime} \simeq A_{1} \rightarrow A_{2} \quad$ By definition of $\simeq$
Lemma 16 (Precision obeys Structural Equivalence).
If $A^{\prime} \sqsubseteq A$ then $A^{\prime} \simeq A$.
Proof. By induction on the structure of the derivation of $A^{\prime} \sqsubseteq A$. All cases are immediate by the induction hypothesis and the definition of structural equivalence.

Lemma 17 (Directed consistency obeys Structural Equivalence).
If $A \sim B$ then $A \simeq B$.

Proof. It is given that $A \leadsto B$. By inversion on DirConsU there exist $A^{\prime}$ and $B^{\prime}$ such that $A^{\prime} \sqsubseteq A$ and $A^{\prime} \leq B^{\prime}$ and $B^{\prime} \sqsubseteq B$. By Lemma 16 Precision obeys Structural Equivalence, $A^{\prime} \simeq A$ and $B^{\prime} \simeq B$. By Lemma 11 Reflexivity of Structural Equivalence, $A \simeq A^{\prime}$. By Lemma 15 Subtyping obeys Structural Equivalence, $A^{\prime} \simeq B^{\prime}$. Therefore, by Lemma 13]Transitivity of Structural Equivalence), $A \simeq B$.

## D.1.5 Decidability

In this section, we write $\mathcal{J}$ decidable in proofs to indicate that the associated judgment form $\mathcal{J}$ is decidable.
$\delta^{\prime} \leq \delta$ Sum $\delta^{\prime}$ is a sub-sum of $\delta$

| $\overline{+\frac{?}{i} \leq+_{i}^{?}}$ | $\overline{+{ }_{i} \leq+^{?}}$ | $\overline{+{ }_{+} \leq+_{i}}$ | $\overline{+{ }_{i}^{?} \leq+_{k}^{*}}$ | $\overline{+{ }_{i}^{?} \leq+}$ | $\overline{+^{?} \leq+^{?}}$ | $\overline{+^{?} \leq+_{i}^{*}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\overline{+} \overline{+}$ | $\overline{+_{i} \leq+_{i}}$ | $\overline{+_{i} \leq+_{i}^{*}}$ | $\overline{+_{i} \leq+}$ | $\overline{+{ }_{i}^{*} \leq+_{i}^{*}}$ | $\overline{+_{i}^{*} \leq+}$ | $\overline{+}$ |

Figure 19. Reflexive, transitive closure of source subsum
Lemma 18 (Decidability of subsum).
Given $\delta^{\prime}$ and $\delta$, the judgment $\delta^{\prime} \leq \delta$ is decidable.
Proof. We present the reflexive, transitive closure of the subsum relation on source sums in Figure 19 We can view this relation as a finite set of ordered sums. Thus, the decidability of the subsum relation is equivalent to a membership check on this set.

Lemma 19 (Decidability of subtyping).
Given $A^{\prime}$ and $A$, the judgment $A^{\prime} \leq A$ is decidable.
Proof. By simultaneous induction on the structure of $A^{\prime}$ and $A$.
Proceed by case analysis on the head constructors of $A^{\prime}$ and $A$. Either they agree or they disagree.
If they disagree, then no rule can possibly derive $A^{\prime} \leq A$.
If they agree, then:

- Case $A^{\prime}=$ Unit and $A=$ Unit: By definition of subtyping, Unit $\leq$ Unit.
- Case $A^{\prime}=A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}$ and $A=A_{1} \delta A_{2}$ :

$$
\begin{array}{cl}
A_{1}^{\prime} \leq A_{1} \text { decidable } & \text { By the induction hypothesis } \\
A_{2}^{\prime} \leq A_{2} \text { decidable } & \text { By the induction hypothesis } \\
\delta^{\prime} \leq \delta \text { decidable } & \text { By Lemma (18) Decidability of subsum) } \\
A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \leq A_{1} \delta A_{2} \text { decidable } & \text { By decidability of premises }
\end{array}
$$

- Case $A^{\prime}=A_{1}^{\prime} \rightarrow A_{2}^{\prime}$ and $A=A_{1} \rightarrow A_{2}$ :
$A_{1} \leq A_{1}^{\prime}$ decidable $\quad$ By the induction hypothesis
$A_{2}^{\prime} \leq A_{2}$ decidable $\quad$ By the induction hypothesis
$A_{1}^{\prime} \rightarrow A_{2}^{\prime} \leq A_{1} \rightarrow A_{2}$ decidable By decidability of premises
$\delta^{\prime} \sqsubseteq \delta$ Sum $\delta^{\prime}$ is more precise than $\delta$

$$
\begin{array}{llllll}
\overline{+_{i} \sqsubseteq+_{i}} & \overline{+_{i} \sqsubseteq+_{i}^{?}} & \overline{+_{i} \sqsubseteq+_{i}^{*}} & \overline{+_{i} \sqsubseteq+^{?}} & \overline{+\sqsubseteq+} & \overline{+\sqsubseteq+^{?}} \\
\overline{+_{i}^{?} \sqsubseteq+_{i}^{?}} & \overline{+_{i}^{?} \sqsubseteq+^{?}} & & \overline{+_{i}^{*} \sqsubseteq+_{i}^{*}} & \overline{+_{i}^{*} \sqsubseteq+^{?}} & \\
\hline+^{?} \sqsubseteq++^{?}
\end{array}
$$

Figure 20. Reflexive, transitive closure of precision on sums
Lemma 20 (Decidability of precision on sums).
Given $\delta^{\prime}$ and $\delta$, the judgment $\delta^{\prime} \sqsubseteq \delta$ is decidable.
Proof. We present the reflexive, transitive closure of the precision relation on source sums in Figure 20 We could view this relation as a finite set of ordered sums. Thus, the decidability of the precision relation is equivalent to a membership check on this set. Therefore, given $\delta^{\prime}$ and $\delta$, check whether or not $\left(\delta^{\prime}, \delta\right) \in \sqsubseteq$.

Lemma 21 (Decidability of precision on types).
Given $A^{\prime}$ and $A$, the judgment $A^{\prime} \sqsubseteq A$ is decidable.

Proof. By simultaneous induction on the structure of $A^{\prime}$ and $A$.
Proceed by case analysis on the head constructors of $A^{\prime}$ and $A$. Either they agree or they disagree.
If they disagree, then no rule can possibly derive $A^{\prime} \sqsubseteq A$.
If they agree, then:

- Case $A^{\prime}=$ Unit and $A=$ Unit: By definition of precision, Unit $\sqsubseteq$ Unit and therefore derivablity is decidable.
- Case $A^{\prime}=A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}$ and $A=A_{1} \delta A_{2}$ :

$$
\begin{array}{cl}
A_{1}^{\prime} \sqsubseteq A_{1} \text { decidable } & \text { By the induction hypothesis } \\
A_{2}^{\prime} \sqsubseteq A_{2} \text { decidable } & \text { By the induction hypothesis } \\
\delta^{\prime} \sqsubseteq \delta \text { decidable } & \text { By Lemma 20 Decidability of precision on sums } \\
A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \sqsubseteq A_{1} \delta A_{2} \text { decidable } & \text { By decidability of premises }
\end{array}
$$

- Case $A^{\prime}=A_{1}^{\prime} \rightarrow A_{2}^{\prime}$ and $A=A_{1} \rightarrow A_{2}:$

| $A_{1}^{\prime} \sqsubseteq A_{1}$ decidable | By the induction hypothesis |
| :--- | :--- |
| $A_{2}^{\prime} \sqsubseteq A_{2}$ decidable | By the induction hypothesis |
| $A_{1}^{\prime} \rightarrow A_{2}^{\prime} \sqsubseteq A_{1} \rightarrow A_{2}$ decidable | By decidability of premises |

Lemma 22 (Decidability of directed consistency).
Given $A^{\prime}$ and $B^{\prime}$, the relation $A^{\prime} \leadsto B^{\prime}$ is decidable.
Proof. We have $A^{\prime} \leadsto B^{\prime}$ if and only if there exist $A$ and $B$ such that $A \sqsubseteq A^{\prime}$ and $A \leq B$ and $B \sqsubseteq B^{\prime}$. We are given $A^{\prime}$; there are only finitely many types such that $A \sqsubseteq A^{\prime}$. Each such $A$ has only finitely many supertypes, that is, types $B$ such that $A \leq B$. Since these two relations are decidable, $A^{\prime} \leadsto B^{\prime}$ is decidable.

Theorem 1 (Decidability of bidirectional typing).

1. Given $\Gamma$, $e$ and $A$, the judgment $\Gamma \vdash e \Leftarrow A$ is decidable.
2. Given $\Gamma$ and $e$, the judgment $\Gamma \vdash e \Rightarrow A$ is decidable.

Proof. By lexicographic induction on (1) the expression $e$, then on (2) the judgment form, with $\Rightarrow$ smaller than $\Leftarrow$.
In most rules, the expression gets smaller in all the premises: SynAnno Chk $\rightarrow$ Intro Syn $\rightarrow$ Elim ChkSumIntro ChkSumElim1 and ChkSumElim2

In ChkCSub the premise types the same expression but is a synthesizing judgment, which is smaller under our induction measure. By Lemma 22] the second premise of ChkCSub is decidable.

## D.1.6 Equivalence of type assignment and bidirectional system

Lemma 23 (All sums below +).
For all source sums $\delta$, it is the case that $\delta \leq+$.
Proof. By case analysis on $\delta$.

- Case $\delta=+_{i}^{*}$ : By the definition of subtyping, $+_{i}^{*} \leq+$.
- Case $\delta=+_{i}$ : By the definition of subtyping, $\boldsymbol{+}_{i} \leq+_{i}^{*}$. By the previous case, $+_{i}^{*} \leq+$. By the transitivity of subtyping, $+_{i} \leq+$.
- Case $\delta=+_{i}^{?}$ : By the definition of subtyping, $+_{i}^{?} \leq+_{i}$. By the previous case, $+_{i} \leq+$. By the transitivity of subtyping, $+_{i}^{?} \leq+$.
- Case $\delta=+^{?}$ : By the definition of subtyping, $+^{?} \leq+_{i}^{*}$. By the definition of subtyping, $+_{i}^{*} \leq+$. By the transitivity of subtyping, $+^{?} \leq+$.
- Case $\delta=+$ : By the reflexivity of subtyping, $+\leq+$.

Lemma 24 ( $\Rightarrow$ implies subsum).
If $\delta^{\prime} \Rightarrow \delta$ then $\delta^{\prime} \leq \delta$.
Proof. By case analysis on $\delta^{\prime} \Longrightarrow \delta$.

- Case $+_{i}^{?} \Longrightarrow+_{i}^{*}$ : By definition of subtyping, $+_{i}^{?} \leq+_{i}$. By definition of subtyping, $+_{i} \leq+_{i}^{*}$. By transitivity of subtyping, $+_{i}^{?} \leq+_{i}^{*}$.
- Case $+_{i} \Rightarrow+_{i}^{*}$ : By definition of subtyping, $+_{i} \leq+_{i}^{*}$.
- Case $+^{?} \Rightarrow+_{i}^{*}$ : By definition of subtyping, $+^{?} \leq+_{i}^{*}$.
- Case $+_{i}^{*} \Longrightarrow+_{i}^{*}$ : By reflexivity of subtyping, $+_{i}^{*} \leq+_{i}^{*}$.
- Case $\delta^{\prime} \Rightarrow+$ : By Lemma 23 All sums below + , $\delta^{\prime} \leq+$.

Theorem 2 (Bidirectional soundness).
If $\Gamma \vdash e \Leftarrow A$ or $\Gamma \vdash e \Rightarrow A$ then $\Gamma \vdash e: A$.
Proof. By induction on the structure of the given derivation.

- Case SynVar Apply rule SVar
- Case ChkCSub Use the induction hypothesis and apply ruleSCSub
- Case SynAnno Use the induction hypothesis, and apply rule SAnno
- Case ChkUnitIntro Apply ruleSUnitIntro
- Case
$\frac{\Gamma \vdash e_{0} \Leftarrow A_{i} \quad+_{i}^{?} \leq \delta}{\Gamma \vdash \operatorname{inj}_{i} e_{0} \Leftarrow\left(A_{1} \delta A_{2}\right)}$ ChkSumIntro
$\Gamma \vdash e_{0} \Leftarrow A_{i} \quad$ Subderivation
$\Gamma \vdash e_{0}: A_{i} \quad$ By the induction hypothesis
$\Gamma \vdash \operatorname{inj}_{i} e_{0}:\left(A_{1}+\frac{?}{i} A_{2}\right) \quad$ By rule SSumlntro
$A_{1} \leq A_{1} \quad$ By Lemma2 Reflexivity of subtyping
$A_{2} \leq A_{2} \quad$ By Lemma Reflexivity of subtyping
$+_{i}^{?} \leq \delta \quad$ Subderivation
$A_{1}+\frac{?}{i} A_{2} \leq A_{1} \delta A_{2} \quad$ By definition of $\leq$
$A_{1}+{ }_{i}^{?} A_{2} \leadsto A_{1} \delta A_{2} \quad$ By Lemma 8 (Subtyping obeys directed consistency)
$\Gamma \vdash \operatorname{inj}_{i} e_{0}:\left(A_{1} \delta A_{2}\right) \quad$ By ruleSCSub
- Case $\Gamma \vdash e_{0} \Rightarrow\left(A_{1} \delta A_{2}\right)$
$\frac{\delta \Rightarrow+_{i}^{*} \quad \Gamma, x: A_{i} \vdash e_{i} \Leftarrow A}{\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{i} x . e_{i}\right) \Leftarrow A}$ ChkSumElim1

| $\Gamma \vdash e_{0} \Rightarrow\left(A_{1} \delta A_{2}\right)$ | Subderivation |
| :---: | :---: |
| $\Gamma \vdash e_{0}:\left(A_{1} \delta A_{2}\right)$ | By the induction hypothesis |
| $\delta \Rightarrow+_{i}^{*}$ | Subderivation |
| $\delta \leq+_{i}^{*}$ | By Lemma $24 \Rightarrow$ implies subsum |
| $A_{1} \leq A_{1}$ | By Lemma 2 Reflexivity of subtyping |
| $A_{2} \leq A_{2}$ | By Lemma Reflexivity of subtyping |
| $A_{1} \delta A_{2} \leq A_{1}+_{i}^{*} A_{2}$ | By definition of $\leq$ |
| $A_{1} \delta A_{2} \sim A_{1}+_{i}^{*} A_{2}$ | By Lemma 8 (Subtyping obeys directed consistency) |
| $\Gamma \vdash e_{0}:\left(A_{1}+{ }_{i}^{*} A_{2}\right)$ | By rule SCSub |
| $\Gamma, \chi: \mathcal{A}_{\mathrm{i}} \vdash \mathrm{e}_{\mathrm{i}} \Leftarrow \mathrm{A}$ | Subderivation |
| $\Gamma, x: \mathcal{A}_{i} \vdash e_{i}: A$ | By the induction hypothesis |
| $\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{i}\right.$ x. $\left.e_{i}\right): A$ | By rule ChkSumElim1 |

- Case ChkSumElim2 Similar to the ChkSumElim1 case, hence omitted.
- Case Chk $\rightarrow$ Intro Use the induction hypothesis, and apply rule $S \rightarrow$ Intro
- Case Syn $\rightarrow$ Elim Use the induction hypothesis, and apply rule $S \rightarrow$ Elim

Lemma 25 (Reflexivity of annotation equivalence). For all expressions $e, e=: e$.

Proof. By induction on the structure of $e$.
All cases either hold directly by definition or by first using the induction hypothesis.

Lemma 26 (Synthesis also checks). If $\Gamma \vdash e \Rightarrow A$ then $\Gamma \vdash e \Leftarrow A$.
Proof. Apply rule ChkCSublas $A \sim A$ holds by Lemma 5 Reflexivity of precision.
Theorem 3 (Annotatability).
If $\Gamma \vdash e: A$ then there exist $e^{\prime}$ and $e^{\prime \prime}$ such that (1) $\Gamma \vdash e^{\prime} \Leftarrow \mathcal{A}$ where $e=: e^{\prime}$, and (2) $\Gamma \vdash e^{\prime \prime} \Rightarrow A$ where $e=: e^{\prime \prime}$.
Proof. By induction on the structure of the derivation of $\Gamma \vdash e: A$.

- Case
$\frac{\Gamma(x)=A}{\Gamma \vdash x: A} \overline{S V a r}$
$\Gamma(x)=A \quad$ Premise
* $\Gamma \vdash x \Rightarrow A \quad$ By rule SynVar
* $\quad \Gamma \vdash x \Leftarrow A \quad$ By Lemma 26 (Synthesis also checks)
* $\quad x=: x \quad$ By definition of $=$ :
- Case $\frac{\Gamma \vdash e: A^{\prime} \quad A^{\prime} \leadsto A}{\Gamma \vdash e: A}$ SCSub

|  | $\Gamma \vdash e: A^{\prime}$ | Subderivation |
| :---: | :---: | :---: |
| \% | $\begin{aligned} & \Gamma \vdash e^{\prime} \Rightarrow A^{\prime} \\ & e=: e^{\prime} \end{aligned}$ | By the induction hypothesis |
|  | $A^{\prime} \leadsto A$ | Subderivation |
| \% | $\Gamma \vdash e^{\prime} \Leftarrow A$ | By rule ChkCSub |
| \% | $\Gamma \vdash\left(e^{\prime}:: A\right) \Rightarrow A$ | By rule SynAnno |
| \% | $e=:\left(e^{\prime}:: ~\right.$ A $)$ | By definition of $=$ : |

- Case $\frac{\Gamma \vdash e_{0}: A}{\Gamma \vdash\left(e_{0}:: A\right): A}$ SAnno

| $\Gamma \vdash e_{0}: A$ |  |
| ---: | :--- |
| $\Gamma \vdash e_{0}^{\prime} \Leftarrow A$ | Subderivation |
| $e_{0}=: e_{0}^{\prime}$ | ${ }^{\prime \prime}$ |


|  | $\Gamma \vdash\left(e_{0}^{\prime}:: A\right) \Rightarrow A$ |
| :--- | :--- |
|  | By rule SynAnno |
|  | $\Gamma \vdash\left(e_{0}^{\prime},: A\right) \Leftarrow A$ | By Lemma 26, Synthesis also checks

- Case

$$
\overline{\Gamma \vdash(): \text { Unit }} \text { SUnitIntro }
$$

| $\Gamma \vdash() \Leftarrow$ Unit | By rule ChkUnitIntro |
| :--- | :--- |
| $\Gamma \vdash(()::$ Unit $) \Rightarrow$ Unit | By rule SynAnno |
| ()$=:()$ | By definition of $=:$ |
| ()$=:(()::$ Unit $)$ | By definition of $=:$ |

- Case

$$
\begin{aligned}
& \Gamma \vdash e_{0}: A_{i} \\
& \hline \Gamma \vdash \operatorname{inj}_{i} e_{0}:\left(A_{1}+\frac{?}{i} A_{2}\right) \\
& \Gamma \vdash e_{0}: A_{i} \\
& \Gamma \vdash e_{0}^{\prime} \Leftarrow A_{i} \\
& e_{0}=: e_{0}^{\prime} \\
& +{ }_{i}^{?} \leq+_{i}^{?} \\
& \Gamma \vdash \operatorname{inj}_{i} e_{0}^{\prime} \Leftarrow\left(A_{1}+{ }_{i}^{?} A_{2}\right) \\
& \Gamma \vdash\left(\operatorname{inj}_{i} e_{0}^{\prime}:: A_{1}+\frac{?}{i} A_{2}\right) \Rightarrow\left(A_{1}+A_{i}^{?} A_{2}\right) \\
& \operatorname{inj}_{i} e_{0}=: \operatorname{inj}_{i} e_{0}^{\prime} \\
& \operatorname{inj}_{i} e_{0}=:\left(\operatorname{inj}_{i} e_{0}^{\prime}:: A_{1}+{ }_{i}^{?} A_{2}\right)
\end{aligned}
$$

Subderivation
By the induction hypothesis

By definition of $\leq$
By rule ChkSumIntro
By rule SynAnno
By definition of =:
By definition of $=$ :

- Case
$\frac{\Gamma \vdash e_{0}: A_{1}+{ }_{i}^{*} A_{2} \quad \Gamma, x: A_{i} \vdash e_{i}: A}{\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{i} x . e_{i}\right): A}$ SSumElim1]

|  | $\begin{aligned} & \Gamma \vdash e_{0}: A_{1}++_{i}^{*} A_{2} \\ & \Gamma \vdash e_{0}^{\prime} \Rightarrow A_{1}+{ }_{i}^{*} A_{2} \\ & e_{0}=: e_{0}^{\prime} \end{aligned}$ | Subderivation <br> By the induction hypothesis /I |
| :---: | :---: | :---: |
|  | $\Gamma, x: A_{i} \vdash e_{i}: A$ | Subderivation |
|  | $\begin{gathered} \Gamma, x: \mathcal{A}_{\mathrm{i}} \vdash e_{\mathrm{i}}^{\prime} \Leftarrow A \\ e_{\mathrm{i}}=: e_{\mathrm{i}}^{\prime} \end{gathered}$ | By the induction hypothesis |
|  | $+_{i}^{*} \Rightarrow+_{i}^{*}$ | By definition of $\Rightarrow \Rightarrow$ |
| \% | $\Gamma \vdash \operatorname{case}\left(e_{0}^{\prime}, \operatorname{inj}_{i} \times . e_{i}^{\prime}\right) \Leftarrow A$ | By rule ChkSumElim1 |
| \% | $\Gamma \vdash\left(\operatorname{case}\left(e_{0}^{\prime}, \operatorname{inj}_{i} \chi . e_{i}^{\prime}\right):: A\right) \Rightarrow A$ | By rule SynAnno |
|  | $\operatorname{case}\left(e_{0}, \operatorname{inj}_{i} \chi . e_{i}\right)=: \operatorname{case}\left(e_{0}^{\prime}, \operatorname{inj}_{i} \chi . e_{i}^{\prime}\right)$ | By definition of $=$ : |
|  | $\operatorname{case}\left(e_{0}, \operatorname{inj}_{i} x . e_{i}\right)=:\left(\operatorname{case}\left(e_{0}^{\prime}, \operatorname{inj}_{i} x . e_{i}^{\prime}\right):: ~ A\right) ~$ | By definition of $=$ : |

- Case SSumElim2 Similar to the SSumElim1 case, hence omitted.
- Case $\frac{\Gamma, x: A_{1} \vdash e_{0}: A_{2}}{\Gamma \vdash \lambda x . e_{0}: A_{1} \rightarrow A_{2}}$ S $\rightarrow$ Intro



## D. 2 Typability under varying precision

Lemma 27 (Pointwise precision preserves domain). If $\Gamma^{\prime} \sqsubseteq \Gamma$ then $\operatorname{dom}\left(\Gamma^{\prime}\right)=\operatorname{dom}(\Gamma)$.

Proof. By induction on the structure of $\Gamma^{\prime} \sqsubseteq \Gamma$.

Lemma 28 (Context strengthening).
If $\Gamma, \mathrm{y}: A^{\prime} \vdash \mathrm{e}: A_{0}$ and $A \sqsubseteq A^{\prime}$ then $\Gamma, \mathrm{y}: A \vdash \mathrm{e}: A_{0}$.

Proof. By induction on the structure of the derivation of $\Gamma, y: A^{\prime} \vdash e: A_{0}$.

- Case

$$
\frac{\left(\Gamma, y: A^{\prime}\right)(e)=A_{0}}{\Gamma, y: A^{\prime} \vdash e: A_{0}} \text { SVar }
$$

Either $e=y$, or $e \neq y$.
In the first case:

$$
\begin{array}{cll}
\left(\Gamma, y: A^{\prime}\right)(y)=A_{0} & & \text { Premise } \\
A^{\prime}=A_{0} & \text { By definition } \\
\Gamma, y: A \vdash y: A & \text { By ruleSVar } \\
A \sqsubseteq A^{\prime} & \text { Given } \\
A \leadsto A^{\prime} & & \text { By Lemma } 9 \text { Loss in precision obeys directed consistency }) \\
\Gamma, y: A \vdash y: A^{\prime} & & \text { By ruleSCSub } \\
\Gamma, y: A \vdash e: A_{0} & & \text { By above equalities }
\end{array}
$$

In the second case:
$\Gamma, y: A \vdash e: A_{0} \quad B y r u l e S V a r$

- CaseSCSub Use the induction hypothesis and apply rule SCSub
- CaseSUnitIntro Immediate from rule SUnitIntro
- Case SSumIntro Use the induction hypothesis and apply rule SSumIntro
- Case SSumElim1 Use the induction hypothesis and apply rule SSumElim1
- CaseSSumElim2 Use the induction hypothesis and apply rule SSumElim2
- Case $S \rightarrow$ Intro Use the induction hypothesis and apply rule $S \rightarrow$ Intro
- Case $S \rightarrow$ Elim Use the induction hypothesis and apply rule $S \rightarrow$ Elim


## Corollary 29.

If $\Gamma^{\prime} \vdash e: \mathcal{A}$ and $\Gamma \sqsubseteq \Gamma^{\prime}$ then $\Gamma \vdash e: \mathcal{A}$.

Proof. By induction on the number of variables $x$ such that $x \in \operatorname{dom}\left(\Gamma^{\prime}\right)$ but $\Gamma^{\prime}(x) \neq \Gamma(x)$.
Note that we don't impose $x \in \operatorname{dom}(\Gamma)$ as $\operatorname{dom}(\Gamma)=\operatorname{dom}\left(\Gamma^{\prime}\right)$ by Lemma 27 Pointwise precision preserves domain.
If $\Gamma^{\prime}(x)=\Gamma(x)$ for all $x \in \operatorname{dom}\left(\Gamma^{\prime}\right)$, then $\Gamma=\Gamma^{\prime}$ so we already have the result.
Otherwise, use the induction hypothesis, and apply Lemma 28 Context strengthening.

Lemma 30 (Relating $+_{i}^{?}$-subsum and precision).
If $+_{i}^{?} \leq \delta^{\prime}$ and $\delta^{\prime} \sqsubseteq \delta$ then $+_{i}^{?} \leq \delta$.

Proof. Proceed by case analysis on $+{ }_{i}^{?} \leq \delta^{\prime}$.

- Case $+_{i}^{?} \leq+_{i}^{?}$ : From the definition of precision, either $\delta=+_{i}^{?}$ or $\delta=+^{?}$. In both cases, there exists a derivation for $+_{i}^{?} \leq \delta$.
- Case $+_{i}^{?} \leq+_{i}$ : From the definition of precision, either $\delta=+_{i}, \delta=+_{i}^{?}, \delta=+_{i}^{*}$ or $\delta=+^{?}$. In all cases, there exists a derivation for $+_{i}^{2} \leq \delta$.
- Case $+_{i}^{?} \leq+^{?}$ : From the definition of precision, $\delta=+^{?}$. We are given a derivation for $+_{i}^{?} \leq+^{?}$.
- Case $+_{i}^{?} \leq+_{k}^{*}$ : From the definition of precision, either $\delta=+_{k}^{*}$ or $\delta=+^{?}$. In both cases, there exists a derivation for $+_{i}^{?} \leq \delta$.
- Case $+_{i}^{?} \leq+$ : From the definition of precision, either $\delta=+$ or $\delta=+{ }^{?}$. In both cases, there exists a derivation for $+_{i}^{?} \leq \delta$.

Lemma 31 (Bidirectional sum precision).
If $\delta^{\prime} \Longrightarrow \delta_{1}$ and $\delta^{\prime} \sqsubseteq \delta$ then $\delta \Longrightarrow \delta_{1}$.

Proof. Proceed by case analysis on $\delta^{\prime} \nRightarrow \delta_{1}$.

- Case $+_{i}^{?} \Rightarrow+_{i}^{*}$ : From the definition of precision, either $\delta=+_{i}^{?}$ or $\delta=+^{?}$. In both cases, there exists a derivation for $\delta \Rightarrow+_{i}^{*}$.
- Case $+_{i} \Longrightarrow+_{i}^{*}$ : From the definition of precision, either $\delta=+_{i}, \delta=+_{i}^{\prime}, \delta=+_{i}^{*}$, or $\delta=+^{?}$. In all cases, there exists a derivations for $\delta \Rightarrow \boldsymbol{t}_{i}^{*}$.
- Case $+^{?} \Rightarrow+_{i}^{*}$ : From the definition of precision, $\delta=+^{?}$. We are given a derivation for $+^{?} \Rightarrow \boldsymbol{+}_{i}^{*}$.
- Case $\boldsymbol{+}_{i}^{*} \Longrightarrow \boldsymbol{+}_{i}^{*}$ : From the definition of precision, either $\delta=+_{i}^{*}$ or $\delta=\boldsymbol{+}^{?}$. In both cases, there exists a derivation for $\delta \Longrightarrow \boldsymbol{+}_{i}^{*}$.
- Case $\delta^{\prime} \Longrightarrow+$ : There exists a derivation for $\delta \Longrightarrow+$ for all $\delta$.

Theorem 4 (Varying precision of bidirectional typing).

1. If $\Gamma^{\prime} \vdash e^{\prime} \Leftarrow A^{\prime}$ and $e^{\prime} \sqsubseteq e$ and $\Gamma^{\prime} \sqsubseteq \Gamma$ and $A^{\prime} \sqsubseteq A$ then $\Gamma \vdash e \Leftarrow A$.
2. If $\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A^{\prime}$ and $e^{\prime} \sqsubseteq e$ and $\Gamma^{\prime} \sqsubseteq \Gamma$ then there exists $A$ such that $\Gamma \vdash e \Rightarrow A$ and $A^{\prime} \sqsubseteq A$.

Proof. By induction on the structure of the given derivation.

1. By case analysis on the rule concluding $\Gamma^{\prime} \vdash e^{\prime} \Leftarrow A^{\prime}$.

- Case

| () | $\sqsubseteq e$ |  | Given |
| ---: | :--- | ---: | :--- |
| $e$ | $=()$ |  | From definition of $\sqsubseteq$ |
| Unit | $\sqsubseteq A$ |  | Given |
| $A$ | $=$ Unit |  | By Lemma 4 Precision inversion |
| $\Gamma \vdash e \Leftarrow$ Unit |  | By rule ChkUnitlntro |  |

- Case $\frac{\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A_{0}^{\prime} \quad A_{0}^{\prime} \leadsto A^{\prime}}{\Gamma^{\prime} \vdash e^{\prime} \Leftarrow A^{\prime}}$ ChkCSub

| $\Gamma^{\prime} \vdash \mathrm{e}^{\prime} \Rightarrow A_{0}^{\prime}$ | Subderivation |
| :---: | :---: |
| $e^{\prime} \sqsubseteq \mathrm{e}$ | Given |
| $\Gamma^{\prime} \sqsubseteq \Gamma$ | Given |
| $\Gamma \vdash \mathrm{e} \Rightarrow \mathrm{A}_{0}$ | By the induction hypothesis |
| $A_{0}^{\prime} \sqsubseteq A_{0}$ |  |
| $A_{0}^{\prime} \sim A^{\prime}$ | Subderivation |
| $\mathrm{B}_{0}^{\prime} \sqsubseteq \mathrm{A}_{0}^{\prime}$ | By inversion on DirConsU |
| $\mathrm{B}_{0}^{\prime} \leq \mathrm{B}^{\prime}$ | " |
| $\mathrm{B}^{\prime} \sqsubseteq \mathrm{A}^{\prime}$ | " |
| $\mathrm{B}_{0}^{\prime} \sqsubseteq \mathrm{A}_{0}$ | By Lemma 6 Transitivity of precision |
| $A^{\prime} \sqsubseteq A$ | Given |
| $\mathrm{B}^{\prime} \sqsubseteq \mathrm{A}$ | By Lemma6(Transitivity of precision) |
| $A_{0} \sim A$ | By rule DirConsU |
| $\Gamma \vdash e \Leftarrow A$ | By rule ChkCSub |

- Case $\frac{\Gamma^{\prime}, x: A_{1}^{\prime} \vdash e_{0}^{\prime} \Leftarrow A_{2}^{\prime}}{\Gamma^{\prime} \vdash \underbrace{\lambda x \cdot e_{0}^{\prime}}_{e^{\prime}} \Leftarrow \underbrace{A_{1}^{\prime} \rightarrow A_{2}^{\prime}}_{A^{\prime}}}$ Chk $\rightarrow$ Intro

| $\begin{aligned} \lambda x . e_{0}^{\prime} & \sqsubseteq e \\ e & =\lambda x . e_{0} \\ e_{0}^{\prime} & \sqsubseteq e_{0} \end{aligned}$ | Given <br> From definition of $\sqsubseteq$ " |
| :---: | :---: |
| $A_{1}^{\prime} \rightarrow A_{2}^{\prime} \sqsubseteq A$ | Given |
| $A=A_{1} \rightarrow A_{2}$ | By Lemma4 Precision inversion |
| $A_{1}^{\prime} \sqsubseteq A_{1}$ | " |
| $A_{2}^{\prime} \sqsubseteq A_{2}$ | " |
| $\Gamma^{\prime} \sqsubseteq \Gamma$ | Given |
| $\Gamma^{\prime}, x: A_{1}^{\prime} \sqsubseteq \Gamma, x: A_{1}$ | By definition of $\sqsubseteq$ |
| $\Gamma^{\prime}, x: A_{1}^{\prime} \vdash e_{0}^{\prime} \Leftarrow A_{2}^{\prime}$ | Subderivation |
| $\Gamma, x: A_{1} \vdash e_{0} \Leftarrow A_{2}$ | By the induction hypothesis |
| $\Gamma \vdash \lambda x . e_{0} \Leftarrow A_{1} \rightarrow A_{2}$ | By rule Chk $\rightarrow$ Intro |

- Case $\frac{\Gamma^{\prime} \vdash e_{0}^{\prime} \Leftarrow A_{i}^{\prime} \quad+_{i}^{\prime} \leq \delta^{\prime}}{\Gamma^{\prime} \vdash \underbrace{i n j_{i} e_{0}^{\prime}}_{e^{\prime}}} \Leftarrow \underbrace{A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}}_{A^{\prime}}$ ChkSumIntro
$\operatorname{inj}_{\mathrm{i}} \mathrm{e}_{0}^{\prime} \sqsubseteq e$
$e=\operatorname{inj}_{i} e_{0}$
$e_{0}^{\prime} \sqsubseteq e_{0}$
$A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \sqsubseteq A$
$A=A_{1} \delta A_{2}$
$A_{i}^{\prime} \sqsubseteq A_{i}$
$\delta^{\prime} \sqsubseteq \delta$
$\Gamma^{\prime} \vdash e_{0}^{\prime} \Leftarrow A_{i}^{\prime}$
$\Gamma^{\prime} \sqsubseteq \Gamma$

$$
\Gamma \vdash e_{0} \Leftarrow A_{i}
$$

$$
\begin{aligned}
& +_{i}^{?} \leq \delta^{\prime} \\
& +_{i}^{?}<\delta \delta
\end{aligned}
$$

$$
\Gamma \vdash \operatorname{inj}_{i} e_{0} \Leftarrow\left(A_{1} \delta A_{2}\right)
$$

Given
From definition of $\sqsubseteq$ "

Given
By Lemma4 Precision inversion


Subderivation
Given
By the induction hypothesis
Subderivation
By Lemma 30 Relating $+_{i}^{?}$-subsum and precision
By rule ChkSumIntro

- Case $\Gamma^{\prime} \vdash e_{0}^{\prime} \Rightarrow A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}$

$$
\frac{\delta^{\prime} \Rightarrow+_{i}^{*}}{\Gamma^{\prime} \vdash \underbrace{\operatorname{case}\left(e_{0}^{\prime}, \operatorname{inj}_{i} x \cdot e_{i}^{\prime}\right)}_{e^{\prime}} \Leftarrow A^{\prime}} \text { ChkSumElim1 }
$$

$$
\begin{aligned}
& e^{\prime} \sqsubseteq e \\
& \text { Given } \\
& \text { Subderivation } \\
& \text { Given } \\
& \text { By the induction hypothesis } \\
& \text { From definition of } \sqsubseteq \\
& \text { " } \\
& \text { Subderivation } \\
& \text { By Lemma } 31 \text { Bidirectional sum precision } \\
& \Gamma^{\prime}, x: A_{i}^{\prime} \sqsubseteq \Gamma, x: A_{i} \\
& A^{\prime} \sqsubseteq A \\
& \text { By definition of } \sqsubseteq \\
& \text { Given } \\
& \text { Subderivation } \\
& \text { By the induction hypothesis } \\
& \text { By rule ChkSumElim1 }
\end{aligned}
$$

- Case $\Gamma^{\prime} \vdash e_{0}^{\prime} \Rightarrow A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \quad \Gamma^{\prime}, x_{1}: A_{1}^{\prime} \vdash e_{1}^{\prime} \Leftarrow A^{\prime}$
$\frac{\delta^{\prime} \Rightarrow+\quad \Gamma^{\prime}, x_{2}: A_{2}^{\prime} \vdash e_{2}^{\prime}}{\Gamma^{\prime} \vdash \underbrace{\prime} \underbrace{\operatorname{case}\left(e_{0}^{\prime}, \operatorname{inj}_{1} x_{1} \cdot e_{1}^{\prime}, \operatorname{inj}_{2} x_{2} . e_{2}^{\prime}\right)}_{e^{\prime}} \Leftarrow A^{\prime}}$ ChkSumElim2
$e^{\prime} \sqsubseteq e$
$e=\operatorname{case}\left(e_{0}, \operatorname{inj}_{1} x_{1} \cdot e_{1}, \operatorname{inj}_{2} x_{2} \cdot e_{2}\right)$
$e_{0}^{\prime} \sqsubseteq e_{0}$
$e_{1}^{\prime} \sqsubseteq e_{1}$
$e_{2}^{\prime} \sqsubseteq e_{2}$
$\Gamma^{\prime} \vdash e_{0}^{\prime} \Rightarrow A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}$ $\Gamma^{\prime} \sqsubseteq \Gamma$
$\Gamma \vdash e_{0} \Rightarrow A_{1} \delta A_{2}$
$A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \sqsubseteq A_{1} \delta A_{2}$
$A_{1}^{\prime} \sqsubseteq A_{1}$
$A_{2}^{\prime} \sqsubseteq A_{2}$
$\delta^{\prime} \sqsubseteq \delta$
$\delta^{\prime} \Longrightarrow+\quad$ Subderivation
$\delta \Rightarrow+$
$A^{\prime} \sqsubseteq A$
By Lemma 31 Bidirectional sum precision
Given
$\Gamma^{\prime}, x_{1}: A_{1}^{\prime} \sqsubseteq \Gamma, x_{1}: A_{1} \quad$ By definition of $\sqsubseteq$ $\Gamma^{\prime}, x_{1}: A_{1}^{\prime} \vdash e_{1}^{\prime} \Leftarrow A^{\prime} \quad$ Subderivation
$\Gamma, x_{1}: A_{1} \vdash e_{1} \Leftarrow A \quad$ By the induction hypothesis
$\Gamma^{\prime}, x_{2}: A_{2}^{\prime} \sqsubseteq \Gamma, x_{2}: A_{2} \quad$ By definition of $\sqsubseteq$
$\Gamma^{\prime}, x_{2}: A_{2}^{\prime} \vdash e_{2}^{\prime} \Leftarrow A^{\prime} \quad$ Subderivation
$\Gamma, x_{2}: A_{2} \vdash e_{2} \Leftarrow A \quad$ By the induction hypothesis
$\Gamma \vdash \operatorname{case}\left(e_{0}, \operatorname{inj}_{1} \mathrm{x}_{1} . e_{1}, \operatorname{inj}_{2} \mathrm{x}_{2} . e_{2}\right) \Leftarrow A \quad$ By rule ChkSumElim2

2. By case analysis on the rule concluding $\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A^{\prime}$.

- Case $\frac{\Gamma^{\prime}(x)=A^{\prime}}{\Gamma^{\prime} \vdash \underbrace{x}_{e^{\prime}} \Rightarrow A^{\prime}}$ SynVar

Let $A=\Gamma(x)$.

$$
\begin{align*}
& x \sqsubseteq e \quad \text { Given } \\
& e=x \quad \text { From definition of } \sqsubseteq \\
& \Gamma^{\prime}(x)=A^{\prime} \quad \text { Premise } \\
& \Gamma^{\prime} \sqsubseteq \Gamma \quad \text { Given } \\
& \Gamma^{\prime}(x) \sqsubseteq \Gamma(x) \quad \text { By definition of } \sqsubseteq \text { on contexts } \\
& A^{\prime} \sqsubseteq A \quad \text { Equivalent } \\
& \Gamma \vdash x \Rightarrow A \quad \text { By rule SynVar } \\
& \text { - Case } \\
& \frac{\Gamma^{\prime} \vdash e_{0}^{\prime} \Leftarrow A^{\prime}}{\Gamma^{\prime} \vdash \underbrace{\left(e_{0}^{\prime}:: A^{\prime}\right)}_{e^{\prime}} \Rightarrow A^{\prime}} \text { SynAnno } \\
& \left(e_{0}^{\prime}:: A^{\prime}\right) \sqsubseteq e \quad \text { Given } \\
& e=\left(e_{0}:: A_{0}\right) \quad \text { From definition of } \sqsubseteq \\
& e_{0}^{\prime} \sqsubseteq e_{0} \quad \text { " } \\
& A^{\prime} \sqsubseteq A \quad \text { " } \\
& \Gamma^{\prime} \vdash e_{0}^{\prime} \Leftarrow A^{\prime} \quad \text { Subderivation } \\
& \Gamma^{\prime} \sqsubseteq \Gamma \quad \text { Given } \\
& \Gamma \vdash \mathrm{e}_{0} \Leftarrow A \quad \text { By the induction hypothesis } \\
& \text { * } \\
& \Gamma \vdash\left(e_{0}:: A\right) \Rightarrow A \quad B y \text { rule SynAnno } \\
& \text { - Case } \\
& \frac{\Gamma^{\prime} \vdash e_{1}^{\prime} \Rightarrow A_{0}^{\prime} \rightarrow A^{\prime} \quad \Gamma^{\prime} \vdash e_{2}^{\prime} \Leftarrow A_{0}^{\prime}}{\Gamma^{\prime} \vdash \underbrace{e_{1}^{\prime} e_{2}^{\prime}}_{e^{\prime}} \Rightarrow A^{\prime}} \text { Syn } \rightarrow \text { Elim } \\
& e_{1}^{\prime} e_{2}^{\prime} \sqsubseteq e \quad \text { Given } \\
& e=e_{1} e_{2} \quad \text { From definition of } \sqsubseteq \\
& e_{1}^{\prime} \sqsubseteq e_{1} \\
& e_{2}^{\prime} \sqsubseteq e_{2} \\
& \Gamma^{\prime} \sqsubseteq \Gamma \quad \text { Given } \\
& \Gamma^{\prime} \vdash e_{1}^{\prime} \Rightarrow A_{0}^{\prime} \rightarrow A^{\prime} \\
& \text { Subderivation } \\
& \Gamma \vdash \mathrm{e}_{1} \Rightarrow \mathrm{~A}_{0} \rightarrow A \\
& \text { By the induction hypothesis } \\
& A_{0}^{\prime} \rightarrow A^{\prime} \sqsubseteq A_{0} \rightarrow A \\
& A_{0}^{\prime} \sqsubseteq A_{0} \quad \text { From definition of } \sqsubseteq \\
& \text { - } A^{\prime} \sqsubseteq A \\
& \Gamma^{\prime} \vdash e_{2}^{\prime} \Leftarrow A_{0}^{\prime} \\
& \text { Subderivation } \\
& \Gamma \vdash e_{2} \Leftarrow A_{0} \quad \text { By the induction hypothesis } \\
& \text { - } \Gamma \vdash e_{1} e_{2} \Rightarrow A
\end{align*}
$$

## D. 3 Properties of the Static System

Lemma 32 (Static looseness).
If $+_{i}^{?} \leq \delta^{s}$ then $+_{i} \leq_{s} \delta^{s}$.
Proof. By case analysis on $+{ }_{i}^{?} \leq \delta^{s}$.

- Case $+_{i}^{?} \leq+_{i}$ : By definition of static subtyping $\boldsymbol{+}_{i} \leq{ }_{s} \boldsymbol{+}_{i}$.
- Case $+_{i}^{?} \leq+$ : By definition of static subtyping $+_{i} \leq_{s}+$.

Lemma 33 (Static looseness, II).
If $\delta^{\mathrm{S}} \Rightarrow \boldsymbol{+}_{i}^{*}$ then $\delta^{\mathrm{S}}=+_{i}$.
Proof. By case analysis on $\delta^{s} \Longrightarrow+_{i}^{*}$.

- Case $\boldsymbol{+}_{i} \Rightarrow \boldsymbol{+}_{i}^{*}:$ It is the case that $\delta^{S}=\boldsymbol{+}_{i}$.

The following lemma states that static sums are the most precise and incomparable by the precision relation.
Lemma 34 (Precision for static sums).
If $\delta_{1} \sqsubseteq \delta_{2}^{\mathrm{S}}$ then $\delta_{1}=\delta_{2}^{\mathrm{S}}$.
Proof. Proceed by case analysis on $\delta_{2}^{\mathrm{S}}$.

- Case $\delta_{2}^{S}=+_{i}$ : By the definition of imprecision, $\delta_{1}=\boldsymbol{+}_{i}$ only.
- Case $\delta_{2}^{S}=+$ : By the definition of imprecision, $\delta_{1}=+$ only.

Lemma 35 (Precision for static types).
If $A_{1} \sqsubseteq A_{2}^{S}$ then $A_{1}=A_{2}^{S}$.
Proof. By induction on the structure of $A_{2}^{S}$.

- Case $A_{2}^{S}=$ Unit: By the definition of imprecision, $A_{1}^{S}=$ Unit only.
- Case $A_{2}^{\mathrm{S}}=A_{12}^{\mathrm{S}} \delta_{2}^{\mathrm{S}} A_{22}^{\mathrm{S}}$ :
$A_{1} \sqsubseteq A_{12}^{S} \delta_{2}^{S} A_{22}^{S} \quad$ Given
$A_{1}=A_{11} \delta_{1} A_{21} \quad$ From the definition of $\sqsubseteq$
$A_{11} \sqsubseteq A_{12}^{\mathrm{S}}$
"
$A_{21} \sqsubseteq A_{22}^{\mathrm{S}}$
$\delta_{1} \sqsubseteq \delta_{2}^{s}$

$$
\begin{aligned}
A_{11} & =A_{12}^{\mathrm{S}} & & \text { By the induction hypothesis } \\
A_{21} & =A_{22}^{\mathrm{S}} & & \text { By the induction hypothesis } \\
\delta_{1} & =\delta_{2}^{\mathrm{S}} & & \text { By Lemma } 34 \text { Precision for static sums } \\
A_{1} & =A_{2}^{\mathrm{S}} & & \text { By definition of }=
\end{aligned}
$$

- Case $A_{2}^{S}=A_{12}^{S} \rightarrow A_{22}^{S}$ : Similar to the previous case.

Lemma 36 (Equivalence for static subsum).

1. If $\delta_{1}^{\mathrm{S}} \leq_{\mathrm{s}} \delta_{2}^{\mathrm{S}}$ then $\delta_{1}^{\mathrm{S}} \leq \delta_{2}^{\mathrm{S}}$.
2. If $\delta_{1}^{S} \leq \delta_{2}^{S}$ then $\delta_{1}^{S} \leq_{S} \delta_{2}^{S}$.

Proof.

1. By case analysis on $\delta_{1}^{S} \leq_{S} \delta_{2}^{S}$.

- Case $\delta^{S} \leq_{S} \delta^{S}$ : By definition of subtyping, $\delta^{S} \leq \delta^{S}$.
- Case $+_{i} \leq{ }_{S}+$ : By definition of subtyping, $+_{i} \leq+_{i}^{*}$ and $+_{i}^{*} \leq+$. By transitivity of subtyping, $+_{i} \leq+$.

2. By case analysis on $\delta_{1}^{\mathrm{S}} \leq \delta_{2}^{\mathrm{S}}$.

- Case $\delta^{s} \leq \delta^{s}$ : By definition of static subtyping, $\delta^{s} \leq_{s} \delta^{s}$.
- Case $+_{i} \leq+$ : By definition of static subtyping, $+_{i} \leq s+$.

Lemma 37 (Equivalence for static subtyping).

1. If $A_{1}^{\mathrm{S}} \leq_{\mathrm{S}} A_{2}^{\mathrm{S}}$ then $A_{1}^{\mathrm{S}} \leq A_{2}^{\mathrm{S}}$.
2. If $A_{1}^{\mathrm{S}} \leq A_{2}^{\mathrm{S}}$ then $A_{1}^{\mathrm{S}} \leq_{\mathrm{S}} A_{2}^{\mathrm{S}}$.

Proof.

1. By induction on the structure of the derivation of $A_{1}^{\mathrm{S}} \leq_{S} A_{2}^{\mathrm{S}}$.

- Case Unit $\leq_{s}$ Unit: By definition of subtyping, Unit $\leq$ Unit.
- Case
$\frac{A_{11}^{\mathrm{S}} \leq_{\mathrm{S}} A_{12}^{\mathrm{S}} \quad A_{21}^{\mathrm{S}} \leq_{\mathrm{S}} A_{22}^{\mathrm{S}} \quad \delta_{1}^{\mathrm{S}} \leq_{\mathrm{S}} \delta_{2}^{\mathrm{S}}}{\left(A_{11}^{\mathrm{S}} \delta_{1}^{\mathrm{S}} A_{21}^{\mathrm{S}}\right) \leq_{\mathrm{S}}\left(A_{21}^{\mathrm{S}} \delta_{2}^{\mathrm{S}} A_{22}^{\mathrm{S}}\right)}$
$A_{11}^{S} \leq_{S} A_{12}^{S} \quad$ Subderivation
$A_{21}^{\mathrm{s}} \leq_{\mathrm{s}} A_{22}^{\mathrm{s}} \quad$ Subderivation
$\delta_{1}^{S} \leq_{S} \delta_{2}^{s} \quad$ Subderivation
$A_{11}^{\mathrm{s}} \leq A_{12}^{\mathrm{s}} \quad$ By the induction hypothesis
$A_{21}^{\mathrm{S}} \leq A_{22}^{\mathrm{S}} \quad$ By the induction hypothesis
$\delta_{1}^{\mathrm{s}} \leq \delta_{2}^{\mathrm{s}} \quad$ By Lemma 36 Equivalence for static subsum $A_{11}^{S} \delta_{1}^{S} A_{21}^{S} \leq A_{12}^{S} \delta_{2}^{S} A_{22}^{S} \quad$ By definition of $\leq$
- Case $\frac{A_{12}^{\mathrm{S}} \leq_{\mathrm{S}} A_{11}^{\mathrm{S}} \quad A_{21}^{\mathrm{S}} \leq_{\mathrm{s}} A_{22}^{\mathrm{S}}}{\left(A_{11}^{\mathrm{S}} \rightarrow A_{21}^{\mathrm{S}}\right) \leq_{\mathrm{s}}\left(A_{12}^{\mathrm{S}} \rightarrow A_{22}^{\mathrm{S}}\right)}$

Similar to the previous case.
2. By induction on the structure of the derivation of $A_{1}^{S} \leq A_{2}^{S}$.

- Case Unit $\leq$ Unit: By definition of subtyping, Unit $\leq_{\mathrm{s}}$ Unit.
- Case $\frac{A_{11}^{S} \leq A_{12}^{S} \quad A_{21}^{S} \leq A_{22}^{S} \quad \delta_{1}^{S} \leq \delta_{2}^{S}}{\left(A_{11}^{S} \delta_{1}^{S} A_{21}^{S}\right) \leq\left(A_{21}^{S} \delta_{2}^{S} A_{22}^{S}\right)}$

$$
\begin{array}{rlrl}
A_{11}^{\mathrm{S}} & \leq A_{12}^{\mathrm{S}} & & \text { Subderivation } \\
A_{21}^{\mathrm{S}} \leq A_{22}^{\mathrm{S}} & & \text { Subderivation } \\
\delta_{1}^{\mathrm{S}} \leq \delta_{2}^{\mathrm{S}} & & \text { Subderivation } \\
A_{11}^{\mathrm{S}} \leq_{\mathrm{S}} A_{12}^{\mathrm{S}} & & \text { By the induction hypothesis } \\
A_{21}^{\mathrm{S}} \leq \leq_{\mathrm{S}} A_{22}^{\mathrm{S}} & & \text { By the induction hypothesis } \\
\delta_{1}^{\mathrm{S}} \leq_{\mathrm{S}} \delta_{2}^{\mathrm{S}} & & \text { By Lemmab } 36 \text { Equivalence for static subsumb } \\
A_{11}^{\mathrm{S}} \delta_{1}^{\mathrm{S}} A_{21}^{\mathrm{S}} & \leq_{\mathrm{S}} A_{12}^{\mathrm{S}} \delta_{2}^{\mathrm{S}} A_{22}^{\mathrm{S}} & & \text { By definition of } \leq_{\mathrm{s}}
\end{array}
$$

- Case

$$
\frac{A_{12}^{\mathrm{S}} \leq A_{11}^{\mathrm{S}} \quad A_{21}^{\mathrm{S}} \leq A_{22}^{\mathrm{S}}}{\left(A_{11}^{\mathrm{S}} \rightarrow A_{21}^{\mathrm{S}}\right) \leq\left(A_{12}^{\mathrm{S}} \rightarrow A_{22}^{\mathrm{S}}\right)}
$$

Similar to the previous case.
Lemma 38 (Directed consistency for static types).
If $A_{1}^{s} \sim A_{2}^{s}$ then $A_{1}^{s} \leq A_{2}^{s}$.
Proof. It is given that $A_{1}^{s} \leadsto A_{2}^{s}$. By inversion on DirConsU there exist $A$ and $B$ such that $A \sqsubseteq A_{1}^{s}$ and $A \leq B$ and $B \sqsubseteq A_{2}^{s}$. By Lemma 35 Precision for static types, $A=A_{1}^{S}$ and $B=A_{2}^{S}$. Therefore, $A \leq B$ is equivalent to $A_{1}^{S} \leq A_{2}^{S}$.

Theorem 5 (Static soundness and completeness).

1. Soundness:
(a) If $\Gamma^{\mathrm{S}} \vdash_{\mathrm{s}} \mathrm{e}^{\mathrm{s}} \Leftarrow A^{\mathrm{s}}$ then $\Gamma^{\mathrm{S}} \vdash \mathrm{e}^{\mathrm{s}} \Leftarrow A^{\mathrm{s}}$
(b) If $\Gamma^{\mathrm{S}} \vdash_{\mathrm{S}} e^{\mathrm{S}} \Rightarrow A^{\mathrm{S}}$ then $\Gamma^{\mathrm{S}} \vdash e^{\mathrm{S}} \Rightarrow A^{\mathrm{S}}$.
2. Completeness:
(a) If $\Gamma^{\mathrm{S}} \vdash \mathrm{e}^{\mathrm{s}} \Leftarrow \mathrm{A}^{\mathrm{s}}$ then $\Gamma^{\mathrm{S}} \vdash_{\mathrm{s}} e^{\mathrm{s}} \Leftarrow \mathrm{A}^{\mathrm{S}}$.
(b) If $\Gamma^{\mathrm{S}} \vdash \mathrm{e}^{\mathrm{S}} \Rightarrow A^{\mathrm{S}}$ then $\Gamma^{\mathrm{S}} \vdash_{\mathrm{s}} e^{\mathrm{S}} \Rightarrow A^{\mathrm{S}}$.

## Proof.

1. By induction on the structure of the given derivation.

- Case StVar Apply ruleSynVar
- Case $\frac{\Gamma^{\mathrm{S}} \vdash_{\mathrm{s}} \mathrm{e}^{\mathrm{s}} \Rightarrow A_{0}^{\mathrm{s}} \quad A_{0}^{\mathrm{S}} \leq_{\mathrm{s}} A^{\mathrm{s}}}{\Gamma^{\mathrm{S}} \vdash_{\mathrm{s}} e^{\mathrm{S}} \Leftarrow A^{\mathrm{S}}}$ StSub

$$
\begin{array}{cl}
\Gamma^{\mathrm{s}} \vdash_{\mathrm{s}} e^{\mathrm{s}} \Rightarrow A_{0}^{\mathrm{s}} & \text { Subderivation } \\
A_{0}^{\mathrm{s}} \leq A^{\mathrm{s}} & \text { Subderivation } \\
\Gamma^{\mathrm{s}} \vdash e^{\mathrm{s}} \Rightarrow A_{0}^{\mathrm{s}} & \text { By the induction hypothesis } \\
A_{0}^{\mathrm{s}} \leq A^{\mathrm{s}} & \text { By Lemma, 37) (Equivalence for static subtyping) } \\
A_{0}^{\mathrm{s}} \sim A^{\mathrm{s}} & \text { By Lemma } 8 \text { (Subtyping obeys directed consistency) } \\
\Gamma^{\mathrm{s}} \vdash e^{\mathrm{s}} \Leftarrow A^{\mathrm{s}} & \text { By ruleChkCSub }
\end{array}
$$

- Case StAnno Use the induction hypothesis and apply rule SynAnno
- CaseStUnitIntro Apply rule ChkUnitIntro
- Case $\frac{\Gamma^{\mathrm{S}} \vdash_{\mathrm{s}} e_{i}^{\mathrm{S}} \Leftarrow A_{i}^{\mathrm{S}} \quad+_{i} \leq \delta^{\mathrm{s}}}{\Gamma^{\mathrm{S}} \vdash_{\mathrm{s}} \operatorname{inj}_{i} e_{\mathrm{i}}^{\mathrm{S}} \Leftarrow\left(A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}}\right)}$ StSumIntro

| $\Gamma^{\mathrm{S}} \vdash_{\mathrm{S}} \mathrm{e}_{\mathrm{i}}^{\mathrm{S}} \Leftarrow \mathrm{A}_{\mathrm{i}}^{\mathrm{S}}$ | Subderivation |
| :---: | :---: |
| $+_{i} \leq_{s} \delta^{s}$ | Subderivation |
| $\Gamma^{\mathbf{S}} \vdash e_{\mathrm{i}}^{\mathbf{S}} \Leftarrow \lambda_{\mathrm{i}}^{\mathbf{S}}$ | By the induction hypothesis |
| $+_{i}^{?} \leq+_{i}$ | By definition of $\leq$ |
| $+_{i} \leq \delta^{\text {S }}$ | By Lemma 36 Equivalence for static subsum |
| $+_{i}^{?} \leq \delta^{s}$ | By transitivity of $\leq$ |
| $\Gamma^{\mathbf{S}} \vdash \operatorname{inj}_{i} e_{i}^{S} \Leftarrow\left(A_{1}^{S} \delta^{\mathbf{S}} A_{2}^{S}\right)$ | By rule ChkSumIntro |

- Case StSumElim1 Use the induction hypothesis, the definition of $\Rightarrow$ and apply rule ChkSumElim1
- CaseStSumElim2 Use the induction hypothesis, the definition of $\Rightarrow$ and apply rule ChkSumElim2
- Case St $\rightarrow$ Intro Use the induction hypothesis and apply rule Chk $\rightarrow$ Intro
- Case St $\rightarrow$ Elim Use the induction hypothesis and apply rule Syn $\rightarrow$ Elim

2. By induction on the structure of the given derivation.

- Case SynVar Apply rule StVar
- Case $\frac{\Gamma^{\mathrm{S}} \vdash e^{\mathrm{s}} \Rightarrow A_{0}^{\mathrm{S}} \quad A_{0}^{\mathrm{S}} \leadsto A^{\mathrm{s}}}{\Gamma^{\mathrm{S}} \vdash e^{\mathrm{S}} \Leftarrow A^{\mathrm{s}}}$ ChkCSub

$$
\begin{array}{cl}
\Gamma^{\mathrm{s}} \vdash e^{\mathrm{s}} \Rightarrow A_{0}^{\mathrm{s}} & \text { Subderivation } \\
A_{0}^{\mathrm{s}} \leadsto A^{\mathrm{s}} & \text { Subderivation } \\
A_{0}^{\mathrm{s}} \leq A^{\mathrm{s}} & \text { By Lemma 38 (Directed consistency for static types } \\
\Gamma^{\mathrm{s}} \vdash_{\mathrm{s}} e^{\mathrm{s}} \Rightarrow A_{0}^{\mathrm{s}} & \text { By the induction hypothesis } \\
A_{0}^{\mathrm{s}} \leq_{\mathrm{s}} A^{\mathrm{s}} & \text { By Lemmat } 37 \text { Equivalence for static subtyping } \\
\Gamma^{\mathrm{s}} \vdash_{\mathrm{s}} e^{\mathrm{s}} \Leftarrow A^{\mathrm{s}} & \text { By rule StSub }
\end{array}
$$

- Case SynAnno Use the induction hypothesis and apply ruleStAnno
- Case ChkUnitIntro Apply ruleStUnitIntro
- Case $\frac{\Gamma^{\mathrm{S}} \vdash e_{i}^{\mathrm{S}} \Leftarrow A_{i}^{\mathrm{S}} \quad+_{i}^{?} \leq \delta^{\mathrm{S}}}{\Gamma^{\mathrm{S}} \vdash \mathrm{inj}_{\mathrm{i}} e_{\mathrm{i}}^{\mathrm{S}} \Leftarrow\left(A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}}\right)}$ ChkSumIntro

$$
\begin{array}{ll}
\Gamma^{S} \vdash e_{i}^{S} \Leftarrow A_{i}^{S} & \text { Subderivation } \\
+_{i}^{?} \leq \delta^{S} & \text { Subderivation } \\
\Gamma^{S} \vdash_{S}^{S} e_{i}^{S} \Leftarrow A_{i}^{S} & \text { By the induction hypothesis } \\
+_{i} \leq \delta_{S} \delta^{S} & \text { By Lemma 32 (Static looseness) } \\
\Gamma^{S} \vdash_{S} \operatorname{inj}_{i} e_{i}^{S} \Leftarrow\left(A_{1}^{S} \delta^{S} A_{2}^{S}\right) & \text { By rule StSumIntro }
\end{array}
$$

- Case $\Gamma^{\mathrm{S}} \vdash e_{0}^{\mathrm{S}} \Rightarrow\left(A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}}\right)$


$$
\begin{aligned}
& \Gamma^{\mathrm{S}} \vdash e_{0}^{\mathrm{S}} \Rightarrow\left(\mathcal{A}_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}}\right) \quad \text { Subderivation } \\
& \Gamma^{\mathrm{S}}, x: A_{i}^{\mathrm{S}} \vdash \mathrm{e}_{\mathrm{i}}^{\mathrm{S}} \Leftarrow A^{\mathrm{S}} \quad \text { Subderivation } \\
& \delta^{S} \Rightarrow+_{i}^{*} \quad \text { Subderivation } \\
& \Gamma^{\mathrm{S}} \vdash_{\mathrm{S}} e_{0}^{\mathrm{S}} \Rightarrow\left(A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}}\right) \quad \text { By the induction hypothesis } \\
& \Gamma^{\mathrm{S}}, x: A_{i}^{\mathrm{S}} \vdash_{\mathrm{s}} e_{i}^{\mathrm{s}} \Leftarrow A^{\mathrm{s}} \quad \text { By the induction hypothesis } \\
& \delta^{s}=+_{i} \quad \text { By Lemma33 Static looseness, II) } \\
& \Gamma^{\mathrm{S}} \vdash_{\mathrm{s}} \operatorname{case}\left(e_{0}^{\mathrm{S}}, \operatorname{inj}_{\mathrm{i}} \chi . e_{\mathrm{i}}^{\mathrm{S}}\right) \Leftarrow \AA^{\mathrm{S}} \quad \text { By rule StSumElim1 }
\end{aligned}
$$

- Case ChkSumElim2 Use the induction hypothesis, the definition of $\leq_{s}$ and apply rule StSumElim2
- Case Chk $\rightarrow$ Intro Use the induction hypothesis and apply rule $S t \rightarrow$ Intro
- Case Syn $\rightarrow$ Elim Use the induction hypothesis and apply rule St $\rightarrow$ Elim


## D. 4 Properties of the Dynamic System

Lemma 39 (Subtyping for dynamic types).
If $A_{1}^{\mathrm{D}} \leq A_{2}^{\mathrm{D}}$ then $A_{2}^{\mathrm{D}}=A_{1}^{\mathrm{D}}$.
Proof. By induction on the structure of $A_{1}^{\mathrm{D}}$.

- Case $A_{1}^{D}=$ Unit: By the definition of subtyping, $A_{2}^{D}=$ Unit only.
- Case $A_{1}^{\mathrm{D}}=A_{11}^{\mathrm{D}}+^{?} A_{21}^{\mathrm{D}}$ :

$$
\begin{array}{rlrl}
A_{11}^{D}+{ }^{?} A_{21}^{D} & \leq A_{2}^{D} & & \text { Given } \\
A_{2}^{D} & =A_{12}^{D}+? A_{22}^{D} & & \text { From the definition of } \leq \\
A_{11}^{D} \leq A_{12}^{D} & & \prime \prime \\
A_{21}^{D} \leq A_{22}^{D} & & \\
A_{12}^{D} & =A_{11}^{D} & & \text { By the induction hypothesis } \\
A_{22}^{D} & =A_{21}^{D} & & \text { By the induction hypothesis } \\
A_{2}^{D} & =A_{1}^{D} & & \text { By definition of }=
\end{array}
$$

- Case $A_{1}^{\mathrm{D}}=A_{11}^{\mathrm{D}} \rightarrow A_{21}^{\mathrm{D}}$ : $\quad$ Similar to the previous case.

Lemma 40 (Precision for dynamic types).
If $A_{1} \sqsubseteq A_{2}^{D}$ then $A_{1}=A_{2}^{D}$.

Proof. By induction on the structure of $A_{2}^{D}$.

- Case $A_{2}^{D}=$ Unit: By the definition of imprecision, $A_{1}=$ Unit only.
- Case $A_{2}^{\mathrm{D}}=A_{12}^{\mathrm{D}}+^{?} A_{22}^{\mathrm{D}}$ :

| $A_{1}$ | $\sqsubseteq A_{2}^{D}$ |  | Given |
| ---: | :--- | ---: | :--- |
| $A_{1}$ | $=A_{11}+{ }^{?} A_{21}$ |  | From the definition of $\sqsubseteq$ |
| $A_{11}$ | $\sqsubseteq A_{12}^{D}$ |  | $\prime \prime$ |
| $A_{21}$ | $\sqsubseteq A_{22}^{D}$ |  | $\prime \prime$ |

$A_{11}=A_{12}^{\mathrm{D}} \quad$ By the induction hypothesis
$A_{21}=A_{22}^{D} \quad$ By the induction hypothesis
$A_{1}=A_{2}^{D} \quad$ By definition of $=$

- Case $A_{1}^{\mathrm{D}}=A_{11}^{\mathrm{D}} \rightarrow A_{21}^{\mathrm{D}}$ : Similar to the previous case.

Lemma 41 (Directed consistency for dynamic types).
If $A_{1}^{\mathrm{D}} \leadsto A_{2}^{\mathrm{D}}$ then $A_{1}^{\mathrm{D}}=A_{2}^{\mathrm{D}}$.
Proof. It is given that $A_{1}^{D} \leadsto A_{2}^{D}$. By inversion on DirConsU there exist $A$ and $B$ such that $A \sqsubseteq A_{1}^{D}$ and $A \leq B$ and $B \sqsubseteq A_{2}^{D}$. By Lemma 40 Precision for dynamic types), $A=A_{1}^{\mathrm{D}}$ and $B=A_{2}^{\mathrm{D}}$. Therefore, $A \leq B$ is equivalent to $A_{1}^{\mathrm{D}} \leq A_{2}^{\mathrm{D}}$. By Lemma 39 (Subtyping for dynamic types), $A_{1}^{D}=A_{2}^{D}$.

Theorem 15 (Dynamic soundness and completeness).

1. (a) If $\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e^{\mathrm{D}} \Leftarrow A^{\mathrm{D}}$ then $\Gamma^{\mathrm{D}} \vdash \mathrm{e}^{\mathrm{D}} \Leftarrow A^{\mathrm{D}}$.
(b) If $\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e^{\mathrm{D}} \Rightarrow A^{\mathrm{D}}$ then $\Gamma^{\mathrm{D}} \vdash e^{\mathrm{D}} \Rightarrow A^{\mathrm{D}}$.
2. (a) If $\Gamma^{\mathrm{D}} \vdash e^{\mathrm{D}} \Leftarrow A^{\mathrm{D}}$ then $\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e^{\mathrm{D}} \Leftarrow A^{\mathrm{D}}$.
(b) If $\Gamma^{\mathrm{D}} \vdash e^{\mathrm{D}} \Rightarrow A^{\mathrm{D}}$ then $\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e^{\mathrm{D}} \Rightarrow A^{\mathrm{D}}$.

## Proof.

1. By induction on the structure of the given $\vdash_{\mathrm{D}}$-derivation.

- Case DVar Apply rule SynVar
- Case DSub Use the induction hypothesis, reflexivity of directed consistency, and apply rule ChkCSub
- Case DUnitlntro Apply rule ChkUnitIntro
- Case $\frac{\Gamma^{D} \vdash_{D} e_{i}^{D} \Leftarrow A_{i}^{D}}{\Gamma^{D} \vdash_{D} \operatorname{inj}_{i} e_{i}^{D} \Leftarrow\left(A_{1}^{D}+{ }^{?} A_{2}^{D}\right)} D+{ }^{\text {I } \operatorname{Intro}}$

$$
\begin{array}{ll}
\Gamma^{\mathrm{D}} \vdash_{\mathrm{D}} e_{i}^{\mathrm{D}} \Leftarrow A_{\mathrm{i}}^{\mathrm{D}} & \text { Subderivation } \\
\Gamma^{\mathrm{D}} \vdash e_{i}^{\mathrm{D}} \Leftarrow A_{\mathrm{i}}^{\mathrm{D}} & \text { By the induction hypothesis } \\
+_{i}^{?} \leq+^{?} & \text { By definition of } \leq \\
\Gamma^{\mathrm{D}} \vdash \operatorname{inj}_{\mathrm{i}} e_{i}^{\mathrm{D}} \Leftarrow\left(A_{1}^{\mathrm{D}}+{ }^{?} A_{2}^{\mathrm{D}}\right) & \text { By rule ChkSumIntro }
\end{array}
$$

- Case D+? Elim1 Use the induction hypothesis, the definition of $\Rightarrow$ and apply rule ChkSumElim1
- Case D+? ${ }^{?}$ Elim2 Use the induction hypothesis, the definition of $\Rightarrow$ and apply rule ChkSumElim2
- Case D $\rightarrow$ Intro Use the induction hypothesis and apply rule Chk $\rightarrow$ Intro
- Case D $\rightarrow$ Elim Use the induction hypothesis and apply rule Syn $\rightarrow$ Elim.

2. By induction on the structure of the given $\vdash$-derivation.

- Case SynVar Apply rule DVar
- Case $\frac{\Gamma^{D} \vdash e^{D} \Rightarrow A_{0}^{D} \quad A_{0}^{D} \leadsto A^{D}}{\Gamma^{D} \vdash e^{D} \Leftarrow A^{D}}$ ChkCSub

| $A_{0}^{D} \leadsto A^{D}$ | Subderivation |
| :--- | :--- |
| $A_{0}^{D}=A^{D}$ | By Lemma 41 Directed consistency for dynamic types |
| $\Gamma^{D} \vdash e^{D} \Rightarrow A_{0}^{D}$ | Subderivation |
| $\Gamma^{D} \vdash_{D} e^{D} \Rightarrow A_{0}^{D}$ | By the induction hypothesis |
| $\Gamma^{D} \vdash_{D} e^{D} \Rightarrow A^{D}$ | Equivalent |
| $\Gamma^{D} \vdash_{D} e^{D} \Leftarrow A^{D}$ | By rule DSub |

- Case SynAnno Use the induction hypothesis and apply rule DAnno
- Case ChkUnitIntro Apply rule DUnitIntro
- Case ChkSumlntro Use the induction hypothesis, and apply rule D+? Intro
- Case ChkSumElim1 Use the induction hypothesis, and apply rule D+ ${ }^{?}$ Elim1
- Case ChkSumElim2 Use the induction hypothesis, and apply rule D+? ${ }^{?}$ Elim2
- Case Chk $\rightarrow$ Intro Use the induction hypothesis and apply rule $D \rightarrow$ Intro
- Case Syn $\rightarrow$ Elim Use the induction hypothesis and apply rule D $\rightarrow$ Elim


## D. 5 Target System

## D.5.1 Subtyping

Lemma 42 (Subtyping inversion).

1. If $\mathrm{T}^{\prime} \leq$ Unit then $\mathrm{T}^{\prime}=$ Unit.
2. If Unit $\leq T$ then $T=$ Unit.
3. If $\mathrm{T}^{\prime} \leq \mathrm{T}_{1} \phi \mathrm{~T}_{2}$ then $\mathrm{T}^{\prime}=\mathrm{T}_{1}^{\prime} \phi^{\prime} \mathrm{T}_{2}^{\prime}$ where $\mathrm{T}_{1}^{\prime} \leq \mathrm{T}_{1}$ and $\mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{2}$ and $\phi^{\prime} \leq \phi$.
4. If $\mathrm{T}_{1}^{\prime} \phi^{\prime} \mathrm{T}_{2}^{\prime} \leq \mathrm{T}$ then $\mathrm{T}=\mathrm{T}_{1} \phi \mathrm{~T}_{2}$ where $\mathrm{T}_{1}^{\prime} \leq \mathrm{T}_{1}$ and $\mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{2}$ and $\phi^{\prime} \leq \phi$.
5. If $\mathrm{T}^{\prime} \leq \mathrm{T}_{1} \rightarrow \mathrm{~T}_{2}$ then $\mathrm{T}^{\prime}=\mathrm{T}_{1}^{\prime} \rightarrow \mathrm{T}_{2}^{\prime}$ where $\mathrm{T}_{1} \leq \mathrm{T}_{1}^{\prime}$ and $\mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{2}$
6. If $\mathrm{T}_{1}^{\prime} \rightarrow \mathrm{T}_{2}^{\prime} \leq \mathrm{T}$ then $\mathrm{T}=\mathrm{T}_{1} \rightarrow \mathrm{~T}_{2}$ where $\mathrm{T}_{1} \leq \mathrm{T}_{1}^{\prime}$ and $\mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{2}$.

Proof.

1. By case analysis on $T^{\prime} \leq$ Unit.

- Case Unit $\leq$ Unit: Immediate that $\mathrm{T}^{\prime}=$ Unit.

2. Symmetric to part 1 .
3. By case analysis on $T^{\prime} \leq T_{1} \phi T_{2}$.

- Case $T_{1}^{\prime} \phi^{\prime} T_{2}^{\prime} \leq T_{1} \phi T_{2}$ : Immediate as $T^{\prime}=T_{1}^{\prime} \phi^{\prime} T_{2}^{\prime}$ and subderivations are $T_{1}^{\prime} \leq T_{1}$ and $T_{2}^{\prime} \leq T_{2}$ and $\phi^{\prime} \leq \phi$.

4. Symmetric to part 3.
5. By case analysis on $\mathrm{T}^{\prime} \leq \mathrm{T}_{1} \rightarrow \mathrm{~T}_{2}$.

- Case $T_{1}^{\prime} \rightarrow T_{2}^{\prime} \leq T_{1} \rightarrow T_{2}$ : Immediate as $T^{\prime}=T_{1}^{\prime} \rightarrow T_{2}^{\prime}$ and subderivations are $T_{1} \leq T_{1}^{\prime}$ and $T_{2}^{\prime} \leq T_{2}$.

6. Symmetric to part 5.

Lemma 43 (Reflexivity of subtyping).
For all types T , it is the case that $\mathrm{T} \leq \mathrm{T}$.
Proof. By induction on the structure of T .

- Case $T=$ Unit: By the definition of subtyping, $T \leq T$.
- Case $T=T_{1} \phi T_{2}$ : By the induction hypothesis, $T_{1} \leq T_{1}$ and $T_{2} \leq T_{2}$. By the reflexivity of subsum, $\phi \leq \phi$. Thus, by the definition of subtyping, $\mathrm{T} \leq \mathrm{T}$.
- Case $T=T_{1} \rightarrow T_{2}$ : By the induction hypothesis, $T_{1} \leq T_{1}$ and $T_{2} \leq T_{2}$. Thus, by the definition of subtyping, $T \leq T$.

Lemma 44 (Transitivity of subtyping).
If $\mathrm{T}_{1} \leq \mathrm{T}_{2}$ and $\mathrm{T}_{2} \leq \mathrm{T}_{3}$ then $\mathrm{T}_{1} \leq \mathrm{T}_{2}$.
Proof. By induction on the structure of $\mathrm{T}_{2}$.

- Case $T_{2}=$ Unit:

| $T_{1}$ | $\leq$ Unit | Given |  |
| ---: | :--- | ---: | :--- |
| Unit | $\leq T_{3}$ | Given |  |
| $T_{1}$ | $=$ Unit | By Lemma 42 | Subtyping inversion |
| $T_{3}$ | $=$ Unit | By Lemma 42 | Subtyping inversion |
| Unit | $\leq$ Unit | By Lemma 43 | Reflexivity of subtyping |
| $\mathrm{T}_{1}$ | $\leq \mathrm{T}_{3}$ | Equivalent |  |

- Case $T_{2}=T_{12} \phi_{2} T_{22}$ :
$\mathrm{T}_{1} \leq \mathrm{T}_{12} \phi_{2} \mathrm{~T}_{22} \quad$ Given
$\mathrm{T}_{1}=\mathrm{T}_{11} \phi_{1} \mathrm{~T}_{21} \quad$ By Lemma 42 Subtyping inversion
$\mathrm{T}_{11} \leq \mathrm{T}_{12}$
"
$T_{21} \leq T_{22} \quad$ "
$\phi_{1} \leq \phi_{2} \quad "$
$\mathrm{T}_{12} \phi_{2} \mathrm{~T}_{22} \leq \mathrm{T}_{3} \quad$ Given
$T_{3}=T_{13} \phi_{3} T_{23} \quad$ By Lemma 42 Subtyping inversion)
$\mathrm{T}_{12} \leq \mathrm{T}_{13}$
$\mathrm{T}_{22} \leq \mathrm{T}_{23} \quad$ "
$\phi_{2} \leq \phi_{3} \quad \prime$

| $\mathrm{T}_{11}$ | $\leq \mathrm{T}_{13}$ |  | By the induction hypothesis |
| ---: | :--- | ---: | :--- |
| $\mathrm{T}_{21}$ | $\leq \mathrm{T}_{23}$ |  | By the induction hypothesis |
| $\phi_{1}$ | $\leq \phi_{3}$ |  | By the transitivity of $\leq$ |
| $\mathrm{T}_{11} \phi_{1} \mathrm{~T}_{21}$ | $\leq \mathrm{T}_{13} \phi_{3} \mathrm{~T}_{23}$ |  | By the definition of $\leq$ |
| $\mathrm{T}_{1}$ | $\leq \mathrm{T}_{3}$ |  | Equivalent |
| - Case $\mathrm{T}_{2}=\mathrm{T}_{12} \rightarrow \mathrm{~T}_{22}:$ |  |  |  |
| $\mathrm{T}_{1} \leq \mathrm{T}_{12} \rightarrow \mathrm{~T}_{22}$ |  | Given |  |
| $\mathrm{T}_{1}$ | $=\mathrm{T}_{11} \rightarrow \mathrm{~T}_{21}$ |  | By Lemma 42 Subtyping inversion) |
| $\mathrm{T}_{12}$ | $\leq \mathrm{T}_{11}$ |  | $\prime \prime$ |
| $\mathrm{~T}_{21}$ | $\leq \mathrm{T}_{22}$ |  |  |
| $\mathrm{~T}_{12} \rightarrow \mathrm{~T}_{22}$ | $\leq \mathrm{T}_{3}$ |  | Given |
| $\mathrm{T}_{3}$ | $=\mathrm{T}_{13} \rightarrow \mathrm{~T}_{23}$ |  | By Lemma 42 Subtyping inversion) |
| $\mathrm{T}_{13}$ | $\leq \mathrm{T}_{12}$ |  | "' |
| $\mathrm{T}_{22}$ | $\leq \mathrm{T}_{23}$ |  | " |
| $\mathrm{T}_{13}$ | $\leq \mathrm{T}_{11}$ |  | By the induction hypothesis |
| $\mathrm{T}_{21}$ | $\leq \mathrm{T}_{23}$ |  | By the induction hypothesis |
| $\mathrm{T}_{11} \rightarrow \mathrm{~T}_{21}$ | $\leq \mathrm{T}_{13} \rightarrow \mathrm{~T}_{23}$ |  | By the definition of $\leq$ |
| $\mathrm{T}_{1}$ | $\leq \mathrm{T}_{3}$ |  | Equivalent |

## Corollary 45 (Subtyping inversion).

1. If $\mathrm{T}_{1}^{\prime} \phi^{\prime} \mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{1} \phi \mathrm{~T}_{2}$ then $\mathrm{T}_{1}^{\prime} \leq \mathrm{T}_{1}$ and $\mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{2}$ and $\phi^{\prime} \leq \phi$.
2. If $\mathrm{T}_{1}^{\prime} \rightarrow \mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{1} \rightarrow \mathrm{~T}_{2}$ then $\mathrm{T}_{1} \leq \mathrm{T}_{1}^{\prime}$ and $\mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{2}$.

Proof.

1. Let $T^{\prime}=T_{1}^{\prime} \phi^{\prime} T_{2}^{\prime}$. We are given $T^{\prime} \leq T_{1} \phi T_{2}$. Therefore, by Lemma 42 Subtyping inversion, $T_{1}^{\prime} \leq T_{1}$ and $T_{2}^{\prime} \leq T_{2}$ and $\phi^{\prime} \leq \phi$.
2. Let $T^{\prime}=T_{1}^{\prime} \rightarrow T_{2}^{\prime}$. We are given $T^{\prime} \leq T_{1} \rightarrow T_{2}$. Therefore, by Lemma 42 Subtyping inversion, $T_{1} \leq T_{1}^{\prime}$ and $T_{2}^{\prime} \leq T_{2}$.

## D.5.2 Values

Lemma 46 (Value inversion).

1. If $\cdot \vdash \mathrm{W}: \mathrm{T}$ and $\mathrm{T} \leq\left(\mathrm{T}_{1}+\mathrm{T}_{2}\right)$ then $\mathrm{W}=\mathrm{inj}_{\mathrm{i}} \mathrm{W}_{\mathrm{i}}$ and $\cdot \vdash \mathrm{W}_{\mathrm{i}}: \mathrm{T}_{\mathrm{i}}$.

Moreover, if $\mathrm{T} \leq\left(\mathrm{T}_{1}+{ }_{k} \mathrm{~T}_{2}\right)$ then $\mathrm{i}=\mathrm{k}$.
2. If $\cdot \vdash W: T$ and $T \leq\left(T_{1} \rightarrow T_{2}\right)$ then $W=\lambda x$. $M$ and $\cdot, x: T_{1} \vdash M: T_{2}$.

Proof.

1. By induction on the structure of the derivation of $\cdot \vdash \mathrm{W}: \mathrm{T}$.

- CaseTVar Impossible because context $\Theta=$. is empty.
- Case $\frac{\qquad W: T^{\prime} \quad T^{\prime} \leq T}{\cdot \vdash W: T} T$ TSub

$$
\begin{array}{rlr}
\mathrm{T}^{\prime} & \leq \mathrm{T} & \\
\text { Subderivation } \\
\mathrm{T} \leq \mathrm{T}_{1}+\mathrm{T}_{2} & & \text { Given } \\
\mathrm{T}^{\prime} \leq \mathrm{T}_{1}+\mathrm{T}_{2} & & \text { By Lemma } 44 \text { Transitivity of subtyping). }
\end{array}
$$

Immediate from the induction hypothesis.

- Cases TCast TMatchfail Impossible because the subject term is not a value.
- CaseTUnitIntro Impossible because $T=$ Unit cannot be a subtype of $T_{1}+T_{2}$.
- Case

$$
\frac{\cdot \vdash W_{i}: T_{i}^{\prime}}{\cdot \vdash \underbrace{\operatorname{inj}_{i} W_{i}}_{W}: \underbrace{\left(\mathrm{T}_{1}^{\prime}+\mathrm{H}_{i} \mathrm{~T}_{2}^{\prime}\right)}_{\mathrm{T}}} \mathrm{T+i} \mathrm{\operatorname{lntro}}
$$

By the definition of values $W$, we know that $W_{i}$ is a value and $W=\operatorname{inj}_{i} W_{i}$.

$$
\begin{aligned}
\mathrm{T}_{1}^{\prime}++_{i} \mathrm{~T}_{2}^{\prime} & \leq \mathrm{T}_{1}+\mathrm{T}_{2} & & \text { Given } \\
\mathrm{T}_{\mathrm{i}}^{\prime} & \leq \mathrm{T}_{\mathrm{i}} & & \text { By Corollary45 } \\
\cdot & \vdash W_{i}: T_{i}^{\prime} & & \text { Subderivation } \\
\cdot & \vdash W_{i}: T_{\mathrm{i}} & & \text { By rule TSub } \\
\mathrm{T}_{1}^{\prime}++_{i} \mathrm{~T}_{2}^{\prime} & \leq \mathrm{T}_{1}+{ }_{k} \mathrm{~T}_{2} & & \text { Suppose } \\
+_{i} & \leq+_{k} & & \text { By Corollary } 45 \\
\mathfrak{i} & =k & & \text { From definition of } \leq
\end{aligned}
$$

- Cases T+ ${ }_{i}$ Elim, T+Elim Impossible because the subject term is not a value.
- Case $T \rightarrow$ Intro Impossible because $T=T_{1}^{\prime} \rightarrow T_{2}^{\prime}$ cannot be a subtype of $T_{1}+T_{2}$.
- Case $T \rightarrow$ Elim Impossible because the subject term is not a value.

2. By induction on the structure of the derivation of $\cdot \vdash W: T$.

- Case TVar Impossible because context $\Theta=$. is empty.
- Case $\frac{\qquad W: T^{\prime} \quad T^{\prime} \leq T}{\cdot \vdash W: T} T$ TSub

| $\mathrm{T}^{\prime}$ | $\leq \mathrm{T}$ |  | Subderivation |
| ---: | :--- | ---: | :--- |
| T | $\leq \mathrm{T}_{1} \rightarrow \mathrm{~T}_{2}$ |  | Given |
| $\mathrm{T}^{\prime}$ | $\leq \mathrm{T}_{1} \rightarrow \mathrm{~T}_{2}$ |  | By Lemma 44 Transitivity of subtyping). |

Immediate from the induction hypothesis.

- Cases TCast TMatchfail Impossible because the subject term is not a value.
- Case TUnitlntro Impossible because $T=$ Unit cannot be a subtype of $T_{1} \rightarrow T_{2}$.
- Case T+ Intro Impossible because $T=T_{1}^{\prime}+\frac{?}{i} T_{2}^{\prime}$ cannot be a subtype of $T_{1} \rightarrow T_{2}$.
- Cases T+ ${ }_{i}$ Elim, T+Elim Impossible because the subject term is not a value.
- Case

$$
\frac{\cdot, x: \mathrm{T}_{1}^{\prime} \vdash M: \mathrm{T}_{2}^{\prime}}{\cdot \vdash \underbrace{\lambda x \cdot M}_{W}: \underbrace{\left(\mathrm{T}_{1}^{\prime} \rightarrow \mathrm{T}_{2}^{\prime}\right)}_{\mathrm{T}}} \mathrm{~T} \rightarrow \operatorname{lntro}
$$

By the definition of values $W$, we know that $W=\lambda x$. $M$.

$$
\begin{array}{cl} 
& \mathrm{T}_{1}^{\prime} \rightarrow \mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{1} \rightarrow \mathrm{~T}_{2} \\
\mathrm{~T}_{1} \leq \mathrm{T}_{1}^{\prime} & \text { Given } \\
\mathrm{T}_{2}^{\prime} \leq \mathrm{T}_{2} & \text { By Corollary 45 } \\
\cdot, x: \mathrm{T}_{1}^{\prime} \vdash M: \mathrm{T}_{2}^{\prime} & \text { Subderivation } \\
\cdot, x: \mathrm{T}_{1} \vdash M: \mathrm{T}_{2}^{\prime} & \text { By Lemma } 50 \text { (Context Strengthening }) \\
\cdot, x: \mathrm{T}_{1} \vdash M: \mathrm{T}_{2} & \text { By rule TSub }
\end{array}
$$

- Case $T \rightarrow$ Elim Impossible because the subject term is not a value.

Corollary 47 (Target value inversion for $+_{i}$ ).
If $\cdot \vdash W:\left(T_{1}+T_{2}\right)$ then $W=\operatorname{inj}_{i} W_{i}$ and $\cdot \vdash W_{i}: T_{i}$.
Proof. Let $\mathrm{T}=\mathrm{T}_{1}+_{i} \mathrm{~T}_{2}$.

| $\mathrm{T}_{1} \leq \mathrm{T}_{1}$ | By Lemma 43 Reflexivity of subtyping |
| :---: | :---: |
| $\mathrm{T}_{2} \leq \mathrm{T}_{2}$ | By Lemma 43 Reflexivity of subtyping |
| $+_{i} \leq+$ | By definition of $\leq$ |
| $\mathrm{T} \leq \mathrm{T}_{1}+\mathrm{T}_{2}$ | By definition of $\leq$ |
| $\cdot \vdash W: T$ | Given |
| $\begin{aligned} & \qquad \vdash \mathrm{W}_{\mathrm{k}}: \mathrm{T}_{\mathrm{k}} \\ & \text { implies }(\mathrm{k}=\mathfrak{i}) \end{aligned}$ | " |

$$
\left(T \leq T_{1}+_{i} T_{2}\right) \text { implies }(k=i)
$$

$$
\quad W=\operatorname{inj}_{i} W_{i} \quad \text { Equivalent }
$$

$$
\begin{aligned}
\mathrm{T} & \leq \mathrm{T} & & \text { By Lemma } 43 \\
\mathrm{i} & =\mathrm{k} & & \text { Reflexivity of subtyping } \\
\mathrm{W} & =\operatorname{inj}_{\mathrm{i}} \mathrm{~W}_{\mathrm{i}} & & \text { Equivaliant } \\
\cdot & \vdash \mathrm{W}_{\mathrm{i}}: \mathrm{T}_{\mathrm{i}} & & \text { Equivalent }
\end{aligned}
$$

Corollary 48 (Target value inversion for + ).
If $\cdot \vdash W:\left(T_{1}+T_{2}\right)$ then $W=\operatorname{inj}_{i} W_{i}$ and $\cdot \vdash W_{i}: T_{i}$.
Proof. By Lemma 46 Value inversion with $\mathrm{T}=\mathrm{T}_{1}+\mathrm{T}_{2}$, using Lemma 43 Reflexivity of subtyping.

## Corollary 49.

If $\cdot \vdash W:\left(T_{1} \rightarrow T_{2}\right)$ then $W=\lambda x . M_{0}$ and $\cdot, x: T_{1} \vdash M_{0}: T_{2}$.
Proof. By Lemma (Value inversion) with $\mathrm{T}=\mathrm{T}_{1} \rightarrow \mathrm{~T}_{2}$, using Lemma Reflexivity of subtyping.

## D.5.3 Typing and Evaluation Contexts

Lemma 50 (Context Strengthening).
If $\Theta, \mathrm{y}: \mathrm{T}^{\prime} \vdash \mathrm{M}: \mathrm{T}_{0}$ and $\mathrm{T} \leq \mathrm{T}^{\prime}$ then $\Theta, \mathrm{y}: \mathrm{T} \vdash \mathrm{M}: \mathrm{T}_{0}$.
Proof. By induction on the structure of the derivation of $\Theta, y: \mathrm{T}^{\prime} \vdash \mathrm{M}: \mathrm{T}_{0}$.

- Case

$$
\frac{\left(\Theta, y: T^{\prime}\right)(M)=T_{0}}{\Theta, y: T^{\prime} \vdash M: T_{0}} T V a r
$$

Either $M=y$, or $M \neq y$.
In the first case:

$$
\begin{array}{rlrl}
\left(\Theta, y: T^{\prime}\right)(M) & =T_{0} & & \text { Premise } \\
T^{\prime}=T_{0} & & \text { By definition } \\
\Theta, y: T \vdash y: T & \text { By ruleTVar } \\
T \leq T^{\prime} & \text { Given } \\
\Theta, y: T \vdash y: T^{\prime} & \text { By ruleTSub } \\
\Theta, y: T \vdash M: T_{0} & & \text { By above equalities }
\end{array}
$$

In the second case:

$$
\Theta, y: T \vdash M: \mathrm{T}_{0} \quad \text { By rule } \mathrm{TVar}
$$

- CaseTSub Use the induction hypothesis and apply rule TSub
- Case TCast Use the induction hypothesis and apply rule TCast
- Case TMatchfail Immediate from rule TMatchfail
- CaseTUnitIntro Immediate from rule TUnitIntro
- Case $T+{ }_{i} \ln$ tro Use the induction hypothesis and apply rule $T+{ }_{i} \ln$ ntro
- Case T+ ${ }_{i}$ Elim Use the induction hypothesis and apply rule $T+{ }_{i}$ Elim
- Case T+Elim Use the induction hypothesis and apply rule T+Elim
- Case $T \rightarrow$ Intro Use the induction hypothesis and apply rule $T \rightarrow$ Intro
- Case T $\rightarrow$ Elim Use the induction hypothesis and apply rule $T \rightarrow$ Elim

Lemma 51 (Substitution).
If $\Theta, x: \mathrm{T}^{\prime} \vdash \mathrm{M}: \mathrm{T}$ and $\cdot \vdash \mathrm{W}: \mathrm{T}^{\prime}$ then $\Theta \vdash[\mathrm{W} / \mathrm{x}] \mathrm{M}: \mathrm{T}$.
Proof. By induction on the structure of the derivation of $\Theta, x: T^{\prime} \vdash M: T$.

- CaseTVar Use the definition of substitution, well-formedness of $\Theta$, and rule TVar
- Case TSub Use the induction hypothesis and apply rule TSub
- Case TCast Use the definition of substitution, the induction hypothesis and apply rule TCast
- Case TMatchfail Use the definition of substitution and apply rule TMatchfail
- CaseTUnitIntro Use the definition of substitution and apply rule TUnitIntro
- Case T+ ${ }_{i}$ Intro Use the definition of substitution, the induction hypothesis and apply rule $T+{ }_{i} \operatorname{Intro}$
- Case $T+{ }_{i}$ Elim Use the definition of substitution, the induction hypothesis and apply rule $T+{ }_{i}$ Elim
- Case T+Elim Use the definition of substitution, the induction hypothesis and apply rule T+Elim
- Case T $\rightarrow$ Intro Use the definition of substitution, the induction hypothesis and apply rule $T \rightarrow$ Intro
- Case $T \rightarrow$ Elim Use the definition of substitution, the induction hypothesis and apply rule $T \rightarrow$ Elim

Lemma 52 (Evaluation context typing).
If $\Theta \vdash \mathcal{E}\left[\mathrm{M}_{0}\right]: \mathrm{T}$ then there exists $\mathrm{T}_{0}$ such that $\Theta \vdash \mathrm{M}_{0}: \mathrm{T}_{0}$.
Proof. By induction on the structure of the derivation of $\Theta \vdash \mathcal{E}\left[M_{0}\right]: T$.

- CaseTVar Immediate as $\mathcal{E}\left[M_{0}\right]=M_{0}$, so $T_{0}=T$.
- Case TSub Immediate from the induction hypothesis.
- Case TCast Immediate from the induction hypothesis.
- Case TMatchfail Immediate as $\mathcal{E}\left[M_{0}\right]=M_{0}$, so $T_{0}=T$.
- CaseTUnitIntro Immediate as $\mathcal{E}\left[M_{0}\right]=M_{0}$, so $T_{0}=T$.
- Case $T+{ }_{i}$ Intro Immediate from the induction hypothesis.
- Case T+ ${ }_{i}$ Elim Immediate from the induction hypothesis.
- Case T+Elim Immediate from the induction hypothesis.
- Case $T \rightarrow$ Intro Immediate as $\mathcal{E}\left[M_{0}\right]=M_{0}$, so $T_{0}=T$.
- Case $T \rightarrow$ Elim Proceed by case analysis on $\mathcal{E}$. Each case is immediate from the induction hypothesis.

Lemma 53 (Evaluation context replacement).
If $\Theta \vdash \mathcal{E}\left[M_{0}\right]: T$ and $\Theta \vdash M_{0}: T_{0}$ and $\Theta \vdash M_{0}^{\prime}: T_{0}$ then $\Theta \vdash \mathcal{E}\left[M_{0}^{\prime}\right]: T$.
Proof. By induction on the structure of the derivation of $\Theta \vdash \mathcal{E}\left[M_{0}\right]: T$.

- Case TVar Immediate as $\mathcal{E}\left[M_{0}\right]=M_{0}$, so $T_{0}=T$ and $\mathcal{E}\left[M_{0}^{\prime}\right]=M_{0}^{\prime}$.
- Case TSub Use the induction hypothesis and apply rule TSub
- Case TCast Use the induction hypothesis and apply rule TCast
- Case TMatchfail Immediate as $\mathcal{E}\left[M_{0}\right]=M_{0}$, so $T_{0}=T$ and $\mathcal{E}\left[M_{0}^{\prime}\right]=M_{0}^{\prime}$.
- CaseTUnitIntro Immediate as $\mathcal{E}\left[M_{0}\right]=M_{0}$, so $T_{0}=T$ and $\mathcal{E}\left[M_{0}^{\prime}\right]=M_{0}^{\prime}$.
- Case $T+{ }_{i} \ln$ nro Use the induction hypothesis and apply rule $T+{ }_{i} \operatorname{Intro}$
- Case $T+{ }_{i}$ Elim Use the induction hypothesis and apply rule $T+{ }_{i}$ Elim
- Case T+Elim Use the induction hypothesis and apply rule T+Elim
- Case $T \rightarrow$ Intro Immediate as $\mathcal{E}\left[M_{0}\right]=M_{0}$, so $T_{0}=T$ and $\mathcal{E}\left[M_{0}^{\prime}\right]=M_{0}^{\prime}$.
- Case $T \rightarrow$ Elim Proceed by case analysis on $\mathcal{E}$. For each case, use the induction hypothesis and apply rule $T \rightarrow$ Elim


## D.5.4 Type Safety

Lemma 54 (Type preservation under reduction).
If $\cdot \vdash \mathrm{M}: \mathrm{T}$ and $\mathrm{M} \mapsto_{\mathrm{R}} \mathrm{M}^{\prime}$ then $\cdot \vdash \mathrm{M}^{\prime}: \mathrm{T}$.
Proof. By induction on the structure of the derivation of $\cdot \vdash \mathrm{M}: \mathrm{T}$.

- Case TVar Impossible because the context $\Theta=$. is empty.
- Case TSub Use the induction hypothesis and apply rule TSub
- Case

$$
\frac{\cdot \vdash M_{0}:\left(T_{1} \phi^{\prime} T_{2}\right)}{\cdot \vdash \underbrace{\left\langle\phi \Leftarrow \phi^{\prime}\right\rangle M_{0}}_{M}: \underbrace{\left(T_{1} \phi T_{2}\right)}_{T}} \text { TCast }
$$

Proceed by case analysis on $M \mapsto_{\mathrm{R}} M^{\prime}$.

- Case

$$
\frac{\phi^{\prime} \leq \phi}{\left\langle\phi \Leftarrow \phi^{\prime}\right\rangle \underbrace{W}_{M_{0}} \mapsto R \underbrace{W}_{M^{\prime}}} \text { ReduceUpcast }
$$

$\mathrm{T}_{1} \leq \mathrm{T}_{1} \quad$ By Lemma 43 Reflexivity of subtyping
$T_{2} \leq T_{2} \quad$ By Lemma Reflexivity of subtyping
$\phi^{\prime} \leq \phi \quad$ Given
$\left(T_{1} \phi^{\prime} T_{2}\right) \leq\left(T_{1} \phi T_{2}\right) \quad$ Definition of $\leq$
$\cdot \vdash W:\left(T_{1} \phi^{\prime} T_{2}\right) \quad$ Subderivation
$\cdot \vdash W:\left(T_{1} \phi T_{2}\right) \quad$ By rule TSub

- Case
$\overline{\left\langle\boldsymbol{+}_{\mathrm{i}} \Leftarrow+\right\rangle \underbrace{\left(\mathrm{inj}_{i} W\right)}_{M_{0}} \mapsto \mathrm{R} \underbrace{\mathrm{inj}_{\mathrm{i}} W}_{M^{\prime}}}$ ReduceCastSuccess

$$
\begin{array}{ll}
\cdot \vdash \operatorname{inj}_{i} W:\left(T_{1}+T_{2}\right) & \\
& \text { Subderivation } \\
\cdot \vdash W: T_{i} & \\
\cdot \vdash \operatorname{inj}_{i} W:\left(T_{1}+{ }_{i} T_{2}\right) & \\
\text { By Corollary } 48 \\
T+{ }_{i} \ln t r o
\end{array}
$$

- Case

$$
\frac{\phi^{\prime} \in\left\{\mathbf{+}_{i},+\right\} \quad i \neq k}{\left\langle\boldsymbol{+}_{k} \Leftarrow \phi^{\prime}\right\rangle \underbrace{\left(\operatorname{inj}_{i} W\right)}_{M_{0}} \mapsto_{\mathrm{R}} \underbrace{\text { matchfail }}_{M^{\prime}}} \text { ReduceCastFailure }
$$

$\cdot \vdash$ matchfail : T By rule TMatchfail

- Case TMatchfail Impossible because matchfail $\Vdash_{\mathrm{R}} M^{\prime}$ for any $M^{\prime}$.
- Case TUnitIntro Impossible because () $\vdash_{\mathrm{R}} M^{\prime}$ for any $M^{\prime}$.
- Case T+ ${ }_{i}$ Intro Impossible because $M=\operatorname{inj}_{i} M_{0} \nVdash_{R} M^{\prime}$ for any $M^{\prime}$.
- Case $\frac{. \vdash M_{0}: T_{1}+_{i} T_{2} \quad,, x: T_{i} \vdash M_{i}: T}{\cdot \vdash \underbrace{\operatorname{case}\left(M_{0}, \operatorname{inj}_{i} x . M_{i}\right)}_{M}: T} T+{ }_{i}$ Elim

Proceed by case analysis on $M \mapsto_{\mathrm{R}} M^{\prime}$.

- Case

$$
\begin{aligned}
& \overline{\operatorname{case}(\underbrace{\operatorname{inj}_{i} W}_{M_{0}}, \operatorname{inj}_{i} x . M_{i}) \mapsto R \underbrace{[W / x] M_{i}}_{M^{\prime}}} \text { ReduceCase1 } \\
& \cdot \vdash \operatorname{inj}_{i} W: T_{1}+T_{i} \quad \text { Subderivation } \\
& \cdot \vdash W: T_{i} \quad \text { By Corollary } 47 \\
& \cdot, x: T_{i} \vdash M_{i}: T \quad \text { Subderivation } \\
& \cdot \vdash[W / x] M_{i}: T \quad \text { By Lemma } 51 \text { Substitution }
\end{aligned}
$$

- Case T+Elim Similar to the T+ Elim case. Apply Corollary 48 instead of Corollary 47 when considering the ReduceCase2 case.
- Case $T \rightarrow$ Intro Impossible because $M=\lambda x . M_{0} \not \psi_{R} M^{\prime}$ for any $M^{\prime}$.
- Case $\frac{\qquad \vdash M_{1}: T^{\prime} \rightarrow T \quad \cdot \vdash M_{2}: T^{\prime}}{\cdot \vdash \underbrace{M_{1} M_{2}}_{M}: T} T \rightarrow$ Elim

Proceed by case analysis on $M \mapsto_{R} M^{\prime}$.

- Case

$$
\begin{aligned}
& \underbrace{\left(\lambda x . M_{0}\right)}_{M_{1}} \underbrace{W}_{M_{2}} \mapsto_{R} \underbrace{[W / x] M_{0}}_{M^{\prime}}
\end{aligned} \text { Reduce } \beta
$$

Theorem 6 (Type preservation).
If $\cdot \vdash \mathrm{M}: \mathrm{T}$ and $\mathrm{M} \mapsto \mathrm{M}^{\prime}$ then $\cdot \vdash \mathrm{M}^{\prime}: T$.
Proof. By case analysis on $M \mapsto M^{\prime}$.


- Case
$\frac{\mathcal{E} \neq[]}{\mathcal{E} \text { [matchfail] } \mapsto \text { matchfail } \text { StepMatchfail }}$
Immediate by TMatchfail
Theorem 7 (Progress).
If $\cdot \vdash \mathrm{M}: \mathrm{T}$ then either (a) $M$ is a value, or (b) there exists $M^{\prime}$ such that $M \mapsto M^{\prime}$, or (c) $M=$ matchfail.
Proof. By induction on the structure of the derivation of $\cdot \vdash \mathrm{M}: \mathrm{T}$.
- Case TVar Impossible, because the context $\Theta$ is empty.
- Case TSub Immediate by the induction hypothesis.
- Case TCast

We have $M=\left\langle\phi \Leftarrow \phi^{\prime}\right\rangle M_{0}$ and $T=T_{1} \phi T_{2}$ where $\cdot \vdash M_{0}:\left(T_{1} \phi^{\prime} T_{2}\right)$.
By the induction hypothesis, either $M_{0}$ is a value or there exists $M_{0}^{\prime}$ such that $M_{0} \mapsto M_{0}^{\prime}$.
In the first case, we need to consider all possible assignments to $\phi^{\prime}$ and $\phi$.
Suppose $\phi^{\prime} \leq \phi$, then $M \mapsto M_{0}$.
Suppose $\phi^{\prime}=+_{i}$ and $\phi=+_{k}$ where $i \neq k$, then $M_{0}=\operatorname{inj}_{i} W$ by Corollary 47 so $M \mapsto$ matchfail.

Suppose $\phi^{\prime}=+$ and $\phi=+_{i}$, then $M_{0}=\operatorname{inj}_{k} W$ by Corollary 48 Proceed by cases analysis on $i$, if $i=k$ then $M \mapsto M_{0}$, otherwise $M \mapsto$ matchfail.
In the second case, $\left\langle\phi \Leftarrow \phi^{\prime}\right\rangle M_{0} \mapsto\left\langle\phi \Leftarrow \phi^{\prime}\right\rangle M_{0}^{\prime}$.

- CaseTUnitIntro We have $M=()$, a value, which is alternative (a).
- Case TMatchfail We have $M=$ matchfail, which is alternative (c).
- Case T+i Intro

We have $M=\operatorname{inj}_{i} M_{0}$ and $T=T_{1}+{ }_{i}^{?} T_{2}$ where $\cdot \vdash M_{0}: T_{i}$.
By the induction hypothesis, either $M_{0}$ is a value or there exists $M_{0}^{\prime}$ such that $M_{0} \mapsto M_{0}^{\prime}$.
In the first case, $\operatorname{inj}_{i} M_{0}=M$ is a value.
In the second case, $\operatorname{inj}_{i} M_{0} \mapsto \operatorname{inj}_{i} M_{0}^{\prime}$.

- Case $T+{ }_{i}$ Elim

We have $M=\operatorname{case}\left(M_{0}, \operatorname{inj}_{i} x . M_{i}\right)$ where $\cdot \vdash M_{0}: T_{1}+_{i} T_{2}$ and $\cdot, x: T_{i} \vdash M_{i}: T$.
By the induction hypothesis, either $M_{0}$ is a value or there exists $M_{0}^{\prime}$ such that $M_{0} \mapsto M_{0}^{\prime}$.
In the first case, $M_{0}=\operatorname{inj}_{i} W$ by Corollary 47, so case $\left(M_{0}, \operatorname{inj}_{i} x . M_{i}\right) \mapsto[W / x] M_{i}$.
In the second case, case $\left(M_{0}, \operatorname{inj}_{i} \times . M_{i}\right) \mapsto \operatorname{case}\left(M_{0}^{\prime}\right.$, inj $_{i}$ x. $\left.M_{i}\right)$.

- Case T+Elim Similar to the T+ ${ }_{i}$ Elim case, using Corollary 48 instead of Corollary 47
- Case $T \rightarrow$ Intro We have $M=\lambda x . M_{0}$, a value.
- Case T $\rightarrow$ Elim

We have $M=M_{1} M_{2}$ where $\cdot \vdash M_{1}: T_{1} \rightarrow T_{2}$ and $\cdot \vdash M_{2}: T_{1}$.
By the induction hypothesis, either $M_{1}$ is a value or there exists $M_{1}^{\prime}$ such that $M_{1} \mapsto M_{1}^{\prime}$.
In the first case, $M_{1}=\lambda x . M_{0}$ by Corollary 49
By the induction hypothesis, either $M_{2}$ is a value or there exists $M_{2}^{\prime}$ such that $M_{2} \mapsto M_{2}^{\prime}$.
In the first subcase, $M_{2}$ is a value, so $\left(\lambda x . M_{0}\right) M_{2} \mapsto\left[M_{2} / x\right] M_{0}$.
In the second subcase, $\left(\lambda x . M_{0}\right) M_{2} \mapsto\left(\lambda x . M_{0}\right) M_{2}^{\prime}$.
In the second case, $M_{1} M_{2} \mapsto M_{1}^{\prime} M_{2}$.
Theorem 8 (matchfail-freeness).
If $M$ is cast-free and matchfail-free and $M \mapsto M^{\prime}$ then $M^{\prime}$ is cast-free and matchfail-free.
Proof. By induction on the derivation of $M \mapsto M^{\prime}$.
By the assumption that $M$ is matchfail-free, rule StepMatchfail is impossible. Therefore, the derivation is by StepContext with subderivation $M_{0} \mapsto_{\mathrm{R}} M_{0}^{\prime}$, where $M=\mathcal{E}\left[M_{0}\right]$ and $M^{\prime}=\mathcal{E}\left[M_{0}^{\prime}\right]$.

- Cases ReduceUpcast ReduceCastSuccess ReduceCastFailure In these cases, $M_{0}$ contains a cast, contradicting the assumption that $M=\overline{\mathcal{E}}\left[M_{0}\right]$ is cast-free. Hence, these cases are impossible.
- Case ReduceCase1

We have $M_{0}=\operatorname{case}\left(\mathrm{inj}_{\mathrm{i}} W\right.$, inj $_{\mathrm{i}}$ x. $\left.M_{i}\right)$ and $M_{0}^{\prime}=[W / x] M_{i}$.
Since $M_{0}$ is cast- and matchfail-free, its subterms $W$ and $M_{i}$ are cast- and matchfail-free.
Therefore, $[W / x] M_{i}$ is cast- and matchfail-free.

- Cases ReduceCase2 Reduce $\beta$ Similar to the ReduceCase1 case.


## D.5.5 Precision

Lemma 55 (Precision on values).
If $W^{\prime} \preccurlyeq M$ then $M=W$ for some value $M$.
Proof. By induction on the structure of the derivation of $W^{\prime} \preccurlyeq M$.

- Case () $\preccurlyeq M$ : From definition of $\preccurlyeq$, it is immediate that $M=()$, a value.
- Case $x \preccurlyeq M$ : From definition of $\preccurlyeq$, it is immediate that $M=x$, a value.
- Case $\lambda x . M_{0}^{\prime} \preccurlyeq M$ : From definition of $\preccurlyeq$, it is immediate that $M=\lambda x$. $M_{0}$, a value.
- Case $\operatorname{inj}_{i} W_{0}^{\prime} \preccurlyeq M$ : From definition of $\preccurlyeq, M=\operatorname{inj}_{i} M_{0}$ and $W_{0}^{\prime} \preccurlyeq M_{0}$. By the induction hypothesis, $M_{0}=W_{0}$ for some value $W_{0}$. Therefore, $M=\operatorname{inj}_{i} W_{0}$, a value.

Lemma 56 (Substitution preserves precision).
If $M^{\prime} \preccurlyeq M$ and $W^{\prime} \preccurlyeq W$ then $\left[W^{\prime} / x\right] M^{\prime} \preccurlyeq[W / x] M$.
Proof. By induction on the structure of the derivation of $\mathrm{M}^{\prime} \preccurlyeq M$. All cases are immediate by the induction hypothesis, the definition of substitution, and the definition of $\preccurlyeq$.

Lemma 57 (Precision inversion on evaluation contexts).
If $\mathcal{E}^{\prime}\left[M_{0}^{\prime}\right] \preccurlyeq M$ then there exists $\mathcal{E}$ and $M_{0}$ such that $M=\mathcal{E}\left[M_{0}\right]$ and $M_{0}^{\prime} \preccurlyeq M_{0}$.
Proof. Proceed by induction on the structure of $\mathcal{E}^{\prime}$.

- Case $\mathcal{E}^{\prime}=[]$ : Choose $\mathcal{E}=[]$ and $M_{0}=M$ then $M_{0}^{\prime} \preccurlyeq M_{0}$ is given.
- $\operatorname{Case} \mathcal{E}^{\prime}=\operatorname{inj}_{i} \mathcal{E}_{0}^{\prime}$ :

| $\mathcal{E}^{\prime}\left[M_{0}^{\prime}\right]$ | $\preccurlyeq M$ |  | Given |
| ---: | :--- | ---: | :--- |
| $\operatorname{inj}_{i} \mathcal{E}_{0}^{\prime}\left[M_{0}^{\prime}\right]$ | $\preccurlyeq M$ |  | By above equations |
| $M$ | $=\operatorname{inj}_{i} M_{i}$ |  | From the definition of $\preccurlyeq$ |
| $\mathcal{E}_{0}^{\prime}\left[M_{0}^{\prime}\right]$ | $\preccurlyeq M_{i}$ |  | $\prime \prime$ |
| $M_{i}$ | $=\mathcal{E}_{0}\left[M_{0}\right]$ |  | By the induction hypothesis |
| $M_{0}^{\prime}$ | $\preccurlyeq M_{0}$ |  |  |
| $M$ | $=\operatorname{inj}_{i} \mathcal{E}_{0}\left[M_{0}\right]$ |  | By above equations |

- $\operatorname{Case} \mathcal{E}^{\prime}=\operatorname{case}\left(\mathcal{E}_{0}^{\prime}, \operatorname{inj}_{i}\right.$ x. $\left.M_{\mathrm{i}}^{\prime}\right)$ : Similar to the $\mathcal{E}^{\prime}=\operatorname{inj}_{\mathrm{i}} \mathcal{E}_{0}^{\prime}$ case, hence omitted.
- Case $\mathcal{E}^{\prime}=\operatorname{case}\left(\mathcal{E}_{0}^{\prime}, \operatorname{inj} \chi_{1} \chi_{1} \cdot M_{1}^{\prime}, \operatorname{inj}_{2} \chi_{2} \cdot M_{2}^{\prime}\right):$ Similar to the $\mathcal{E}^{\prime}=\operatorname{inj} \mathcal{E}_{0}^{\prime}$ case, hence omitted.
- Case $\mathcal{E}^{\prime}=\left\langle\phi_{2}^{\prime} \Leftarrow \phi_{1}^{\prime}\right\rangle \mathcal{E}_{0}^{\prime}$ :

By inversion on $\left\langle\phi_{2}^{\prime} \Leftarrow \phi_{1}^{\prime}\right\rangle \mathcal{E}_{0}^{\prime}\left[M_{0}^{\prime}\right] \preccurlyeq M$, either $M=\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle M_{1}$ or $M \neq\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle M_{1}$. In the former case:

$$
\begin{array}{rlrl}
\mathcal{E}_{0}^{\prime}\left[M_{0}^{\prime}\right] & M_{1} & & \text { From the definition of } \preccurlyeq \\
M_{1} & =\mathcal{E}_{0}\left[M_{0}\right] & & \text { By the induction hypothesis } \\
M_{0}^{\prime} \preccurlyeq M_{0} & & \text { " } \\
M & =\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle \mathcal{E}_{0}\left[M_{0}\right] & & \text { By above equations }
\end{array}
$$

In the latter case:

$$
\begin{array}{cl}
\mathcal{E}_{0}^{\prime}\left[M_{0}^{\prime}\right] & \preccurlyeq M
\end{array} \quad \begin{aligned}
& \text { From the definition of } \preccurlyeq \\
& M
\end{aligned}=\mathcal{E}\left[M_{0}\right] \quad \text { By the induction hypothesis }
$$

- Case $\mathcal{E}^{\prime}=\mathcal{E}_{0}^{\prime} M_{2}^{\prime}$ :

| $\mathcal{E}^{\prime}\left[M_{0}^{\prime}\right]$ | $\preccurlyeq M$ |  | Given |
| ---: | :--- | ---: | :--- |
| $\mathcal{E}_{0}^{\prime}\left[M_{0}^{\prime}\right] M_{2}^{\prime}$ | $\preccurlyeq M$ |  | By above equations |
| $M$ | $=M_{1} M_{2}$ |  | From the definition of $\preccurlyeq$ |
| $\mathcal{E}_{0}^{\prime}\left[M_{0}^{\prime}\right]$ | $\preccurlyeq M_{1}$ |  |  |
| $M_{1}$ | $=\mathcal{E}_{0}\left[M_{0}\right]$ |  | By the induction hypothesis |
| $M_{0}^{\prime}$ | $\preccurlyeq M_{0}$ |  |  |
| $M$ | $=\mathcal{E}_{0}\left[M_{0}\right] M_{2}$ |  | By above equations |

- Case $\mathcal{E}^{\prime}=W_{1} \mathcal{E}_{0}^{\prime}$ :

| $\mathcal{E}^{\prime}\left[M_{0}^{\prime}\right]$ | $\preccurlyeq M$ |  | Given |
| ---: | :--- | ---: | :--- |
| $W_{1}^{\prime} \mathcal{E}_{0}^{\prime}\left[M_{0}^{\prime}\right]$ | $\preccurlyeq M$ |  | By above equations |
| $M$ | $=M_{1} M_{2}$ |  | From the definition of $\preccurlyeq$ |
| $\mathcal{E}_{0}^{\prime}\left[M_{0}^{\prime}\right]$ | $\preccurlyeq M_{2}$ |  | $\prime \prime$ |
| $M_{2}$ | $=\mathcal{E}_{0}\left[M_{0}\right]$ |  | By the induction hypothesis |
| $M_{0}^{\prime}$ | $\preccurlyeq M_{0}$ |  | " |
| $M$ | $=W_{1} \mathcal{E}_{0}\left[M_{0}\right]$ |  | By above equations |

Lemma 58 (Evaluation contexts preserve precision).
If $\mathcal{E}^{\prime}\left[M_{0}^{\prime}\right] \preccurlyeq \mathcal{E}\left[M_{0}\right]$ and $M_{0}^{\prime} \preccurlyeq M_{0}$ and $M_{1}^{\prime} \preccurlyeq M_{1}$ then $\mathcal{E}^{\prime}\left[M_{1}^{\prime}\right] \preccurlyeq \mathcal{E}\left[M_{1}\right]$.
Proof. By induction on the derivation of $\mathcal{E}^{\prime}\left[M_{0}^{\prime}\right] \preccurlyeq \mathcal{E}\left[M_{0}\right]$. All cases are straightforward, using the induction hypothesis and the definition of $\preccurlyeq$.

Lemma 59 (Reduction preserves precision).
If $\cdot \vdash M_{1}^{\prime}: T_{1}^{\prime}$ and $\cdot \vdash M_{1}: T_{1}$ and $M_{1}^{\prime} \preccurlyeq M_{1}$ and $M_{1}^{\prime} \mapsto_{R} M_{2}^{\prime}$ then either
(a) $M_{1}$ is a value and $M_{2}^{\prime} \preccurlyeq M_{1}$, or
(b) there exists $M_{2}$ such that $M_{1} \mapsto_{R} M_{2}$ and $M_{2}^{\prime} \preccurlyeq M_{2}$.

Proof. Proceed by case analysis on $M_{1}^{\prime} \mapsto_{R} M_{1}$.

- Case

$$
\underbrace{\underbrace{\left\langle\phi_{2}^{\prime} \Leftarrow \phi_{1}^{\prime}\right\rangle W^{\prime}}_{1} \mapsto_{\mathrm{R}} \underbrace{W^{\prime}}_{M_{2}^{\prime}}}_{M_{1}^{\prime}} \text { ReduceUpcast }
$$

Proceed by case analysis on $M_{1}^{\prime} \preccurlyeq M_{1}$.

- Case $\frac{W^{\prime} \preccurlyeq M \quad\left\langle\phi_{2}^{\prime} \Leftarrow \phi_{1}^{\prime}\right\rangle \preccurlyeq\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle}{\left\langle\phi_{2}^{\prime} \Leftarrow \phi_{1}^{\prime}\right\rangle W^{\prime} \preccurlyeq \underbrace{\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle M}_{M_{1}}}$

By Lemma 55 Precision on values, $M=W$ as $W^{\prime} \preccurlyeq M$. Since $\phi_{1}^{\prime} \leq \phi_{2}^{\prime}$, it is the case that $\left\langle\phi_{2}^{\prime} \Leftarrow \phi_{1}^{\prime}\right\rangle=\mathrm{sc}^{\prime}$.
Proceed by cases on the rule deriving $\mathrm{sc}^{\prime} \preccurlyeq\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle$.

- Case Cast $\preccurlyeq$ Refl| In this case, $\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle=$ sc $^{\prime}$. Since $\phi_{1} \leq \phi_{2}$ by rule ReduceUpcast it follows that $M_{1} \mapsto_{\mathrm{R}} \mathrm{W}$, and we already have $M_{2}^{\prime} \preccurlyeq M_{2}$.
- Cases Cast $\mathrm{M} \preccurlyeq \mathrm{B}$ Cast $\mathrm{B} \preccurlyeq \mathrm{S}$ Cast $\mathrm{M} \preccurlyeq$ S These rules do not have a safe cast on the left, so these cases are impossible.
- Case Rule deriving $\left\langle+_{i} \Leftarrow+_{i}\right\rangle \preccurlyeq\left\langle+_{i} \Leftarrow+\right\rangle$ :

In this case, $\mathrm{sc}^{\prime}=\left\langle\boldsymbol{+}_{\mathrm{i}} \Leftarrow \boldsymbol{+}_{\mathrm{i}}\right\rangle$ and $\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle=\left\langle\boldsymbol{+}_{\mathrm{i}} \Leftarrow+\right\rangle$.

| $\cdot \vdash\left\langle+_{i} \Leftarrow+_{i}\right\rangle W^{\prime}: \mathrm{T}^{\prime}$ | Given |
| :---: | :---: |
| $\cdot \vdash \mathrm{W}^{\prime}: \mathrm{T}_{11}^{\prime}+{ }_{\mathrm{i}} \mathrm{T}_{21}^{\prime}$ | By inversion on rule TCast |
| $\mathrm{W}^{\prime}=\mathrm{inj}_{\mathrm{i}} \mathrm{W}_{0}^{\prime}$ | By Corollary 47 |
| $W=\operatorname{inj}_{i} W_{0}$ | By inversion on ( $\mathrm{inj}_{\mathrm{i}} \mathrm{W}_{0}^{\prime}$ ) $\preccurlyeq \mathrm{W}$ |
| $M=\operatorname{inj}_{i} W_{0}$ | By equality |
| $M_{1} \mapsto_{\mathrm{R}}\left(\mathrm{inj}_{\mathrm{i}} \mathrm{W}_{0}\right)$ | By ReduceCastSuccess |
| $M_{2}^{\prime} \preccurlyeq M_{2}$ | By equality |

- Cases Remaining rules:

In the remaining rules, $\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle=$ sc. Thus, $\phi_{1} \leq \phi_{2}$.
By rule ReduceUpcast it follows that $M_{1} \mapsto_{\mathrm{R}}$ inj $\bar{W}_{\mathrm{i}}$ and it was already given that $M_{2}^{\prime} \preccurlyeq M_{2}$.

- Case

Case | $\frac{W^{\prime}}{} \preccurlyeq M_{1}$ |  |
| :--- | :--- |
| $\left\langle\phi_{2}^{\prime} \Leftarrow \phi_{1}^{\prime}\right\rangle W^{\prime} \preccurlyeq M_{1}$ |  |
|  | $W^{\prime} \preccurlyeq M_{1} \quad$ Subderivation |
| $M_{1}=W$ | By Lemma 55 (Precision on values) |
| $M_{2}^{\prime} \preccurlyeq M_{1} \quad$ By above equations |  |

- Case
$\underbrace{\left\langle++_{i} \Leftarrow+\right\rangle \operatorname{inj}_{i} W^{\prime}}_{M_{1}^{\prime}} \mapsto_{\mathrm{R}} \underbrace{\operatorname{inj}_{i} W^{\prime}}_{M_{2}^{\prime}}$ ReduceCastSuccess
Proceed by case analysis on $M_{1}^{\prime} \preccurlyeq M_{1}$.
- Case

$$
\frac{\operatorname{inj}_{\mathrm{i}} W^{\prime} \preccurlyeq M \quad\left\langle+_{i} \Leftarrow+\right\rangle \preccurlyeq\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle}{\left\langle+_{i} \Leftarrow+\right\rangle \operatorname{inj}_{\mathrm{i}} W^{\prime} \preccurlyeq \underbrace{\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle M}_{M_{1}}}
$$

Inversion on inj $i_{i} W^{\prime} \preccurlyeq M$ gives $M=\operatorname{inj}_{i} M_{0}$ and $W^{\prime} \preccurlyeq M_{0}$.
By Lemma 55 Precision on values, $M_{0}=W$.
Since $\left\langle+_{\mathrm{i}} \Leftarrow+\right\rangle$ is a backward cast $\mathrm{bc}^{\prime}$, to derive $\mathrm{bc}^{\prime} \preccurlyeq\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle$, we either used Cast $\preccurlyeq \operatorname{Refl}$ or CastB $\preccurlyeq S$
In the former case, we have $\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle=\mathrm{bc}^{\prime}$. By rule ReduceCastSuccess we have $M_{1} \mapsto_{\mathrm{R}} M$, and we already have $M_{2}^{\prime} \preccurlyeq M_{2}$. In the latter case, we have $\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle=$ sc. By definition of being a safe cast, $\phi_{1} \leq \phi_{2}$. Therefore, by rule ReduceUpcast we have $M_{1} \mapsto_{\mathrm{R}} M$, and we already have $M_{2}^{\prime} \preccurlyeq M_{2}$.

## - Case

$$
\frac{\operatorname{inj}_{i} W^{\prime} \preccurlyeq M_{1}}{\left\langle\boldsymbol{t}_{\mathrm{i}} \Leftarrow+\right\rangle \operatorname{inj}_{\mathrm{i}} W^{\prime} \preccurlyeq M_{1}}
$$

$$
\begin{array}{cl}
\operatorname{inj}_{\mathrm{i}} W^{\prime} \preccurlyeq M_{1} & \text { Subderivation } \\
M_{1}=W & \text { By Lemma }[55 \text { (Precision on values) } \\
M_{2}^{\prime} \preccurlyeq M_{1} & \text { By above equations }
\end{array}
$$

- Case

$$
\underbrace{\frac{\phi^{\prime} \in\left\{+_{i},+\right\}}{\left\langle+_{\mathrm{k}} \Leftarrow \phi^{\prime}\right\rangle_{\mathrm{i}} \mathrm{inj}_{\mathrm{i}} W^{\prime}} \mapsto_{\mathrm{R}} \underbrace{\text { matchfail }}_{M_{2}^{\prime}}}_{M_{1}^{\prime}} \text { ReduceCastFailure }
$$

Proceed by case analysis on $M_{1}^{\prime} \preccurlyeq M_{1}$.

- Case $\frac{\operatorname{inj}_{i} W^{\prime} \preccurlyeq M \quad\left\langle+_{k} \Leftarrow \phi^{\prime}\right\rangle \preccurlyeq\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle}{\left\langle+_{k} \Leftarrow \phi^{\prime}\right\rangle \operatorname{inj}_{\mathrm{i}} W^{\prime} \preccurlyeq \underbrace{\left\langle\phi_{2} \Leftarrow \phi_{1}\right\rangle M}_{M_{1}}}$

Since $\cdot \vdash M_{1}: T_{1}$ and $M_{1}$ is not a value nor is it matchfail, by Theorem 7 there exists $M_{2}$ such that $M_{1} \mapsto M_{2}$. By definition, $M_{2}^{\prime}=$ matchfail $\preccurlyeq M_{2}$.

- Case

$$
\frac{\operatorname{inj}_{\mathrm{i}} W^{\prime} \preccurlyeq M_{1}}{\left\langle+_{\mathrm{k}} \Leftarrow \phi^{\prime}\right\rangle \operatorname{inj}_{\mathrm{i}} W^{\prime} \preccurlyeq M_{1}}
$$

$$
\begin{array}{ccl}
\operatorname{inj}_{i} W^{\prime} \preccurlyeq M_{1} & \text { Subderivation } \\
M_{1}=W & \text { By Lemma } 55 \text { Precision on values } \\
M_{2}^{\prime} \preccurlyeq M_{1} & \text { By definition of } \preccurlyeq
\end{array}
$$

- Case

$$
\overline{\underbrace{\operatorname{case}\left(\operatorname{inj}_{\mathrm{i}} W^{\prime}, \operatorname{inj}_{\mathrm{i}} \times . M_{i}^{\prime}\right)}_{M_{1}^{\prime}} \mapsto_{\mathrm{R}} \underbrace{\left[W^{\prime} / x\right] M_{i}^{\prime}}_{M_{2}^{\prime}}} \text { ReduceCase1 }
$$

Proceed by inversion on case $\left(\operatorname{inj}_{i} W^{\prime}, \operatorname{inj}_{i} \times . M_{i}^{\prime}\right) \preccurlyeq M_{1}$.
In the first case, $M_{1}=\operatorname{case}\left(M, \operatorname{inj}_{i} \chi . M_{i}\right)$ :

$$
\begin{array}{rlrl}
\operatorname{inj}_{i} W^{\prime} \preccurlyeq M & \text { From definition of } \preccurlyeq \\
M_{i}^{\prime} \preccurlyeq M_{i} & \prime \prime \\
M & =\operatorname{inj}_{\mathrm{i}} M_{0} & \text { From definition of } \preccurlyeq \\
W^{\prime} \preccurlyeq M_{0} & \prime \prime \\
M_{0} & =W & & \text { By Lemma } 55 \\
W^{\prime} \preccurlyeq W & & \text { By above equations } \\
M_{1} & =\operatorname{case}\left(\text { inj }_{i} W, \operatorname{inj}_{i} x . M_{i}\right) & \text { By above equations } \\
M_{1} \mapsto_{\mathrm{R}} \underbrace{[W / x] M_{i}}_{M_{2}} & & \text { By rule ReduceCase1 } \\
{\left[W^{\prime} / x\right] M_{i}^{\prime} \preccurlyeq[W / x] M_{i}} & & \\
& \text { By Lemma } 56 \text { Substitution preserves precision }
\end{array}
$$

In the second case, $M_{1}=\operatorname{case}\left(M, \operatorname{inj}_{1} x_{1} \cdot M_{11}, \operatorname{inj}_{2} x_{2} \cdot M_{21}\right)$ :

$$
\begin{aligned}
& \operatorname{inj}_{i} W^{\prime} \preccurlyeq M \\
& M_{i}^{\prime} \preccurlyeq M_{i 1} \\
& M=\operatorname{inj}_{\mathrm{i}} M_{0} \\
& W^{\prime} \preccurlyeq M_{0} \\
& M_{0}=W \\
& W^{\prime} \preccurlyeq W \\
& \text { From definition of } \preccurlyeq \\
& \text { " } \\
& \text { From definition of } \preccurlyeq \\
& \text { " } \\
& \text { By Lemma } 55 \text { Precision on values } \\
& \text { By above equations } \\
& M_{1}=\operatorname{case}\left(\text { inj }_{i} W, \text { inj }_{1} x_{1} \cdot M_{11}, \text { inj }_{2} x_{2} \cdot M_{21}\right) \\
& M_{1} \mapsto_{R} \underbrace{\left[W / x_{i}\right] M_{i 1}}_{M_{2}} \\
& 1 \\
& {\left[W^{\prime} / x\right] M_{i}^{\prime} \preccurlyeq\left[W / x_{i}\right] M_{i 1}} \\
& \text { By above equations } \\
& \text { By rule ReduceCase2 } \\
& \text { By Lemma } 56 \text { Substitution preserves precision) }
\end{aligned}
$$

- Case

$$
\underbrace{\overline{\operatorname{case}\left(\operatorname{inj}_{\mathrm{i}} W^{\prime}, \operatorname{inj}_{1} x_{1} \cdot M_{11}^{\prime}, \operatorname{inj}_{2} x_{2} \cdot M_{21}^{\prime}\right)} \mapsto R \underbrace{\left[W^{\prime} / x_{i}\right] M_{i 1}^{\prime}}_{M_{2}^{\prime}}}_{M_{1}^{\prime}} \text { ReduceCase2] }
$$

| $M_{1}^{\prime} \preccurlyeq M_{1}$ | Given |
| ---: | :--- |
| $M_{1}=\operatorname{case}\left(M, \operatorname{inj}_{1} x_{1} \cdot M_{11}\right.$, inj $\left._{2} x_{2} \cdot M_{21}\right)$ | From definition of $\preccurlyeq$ |
| $\operatorname{inj}_{i} W^{\prime} \preccurlyeq M$ | $\prime \prime$ |
| $M_{11}^{\prime} \preccurlyeq M_{11}$ | $\prime \prime$ |
| $M_{21}^{\prime} \preccurlyeq M_{21}$ | $\prime \prime$ |
| $M=\operatorname{inj}_{i} M_{0}$ | From definition of $\preccurlyeq$ |
| $W^{\prime} \preccurlyeq M_{0}$ | $\prime \prime$ |
| $M_{0}=W$ | By Lemma 55 Precision on values) |
| $W^{\prime} \preccurlyeq W$ | By above equations |


|  | $M_{1}=\operatorname{case}^{\text {(inj }}{ }_{i} W$ | By above equations |
| :---: | :---: | :---: |
| - | $M_{1} \mapsto_{R} \underbrace{\left[W / x_{i}\right] M_{i 1}}$ | By rule ReduceCase2 |
|  | $\left.{ }_{\text {i }}\right] M_{i 1}^{\prime} \preccurlyeq \stackrel{M_{2}}{\left[W / x_{i}\right] M_{i 1}}$ | By Lemma 56 Substitution preserves precision, |

- Case
$\underbrace{\left(\lambda x . M_{0}^{\prime}\right) W^{\prime}}_{M_{1}^{\prime}} \mapsto_{R} \underbrace{\left[W^{\prime} / x\right] M_{0}^{\prime}}_{M_{2}^{\prime}}$ Reduce $\beta$
$\left(\lambda x . M_{0}^{\prime}\right) W^{\prime} \preccurlyeq M_{1} \quad$ Given
$M_{1}=M_{11} M_{21} \quad$ From definition of $\preccurlyeq$ $\lambda x . M_{0}^{\prime} \preccurlyeq M_{11} \quad$ "
$W^{\prime} \preccurlyeq M_{21} \quad \prime$ $M_{11}=\lambda x . M_{0} \quad$ From definition of $\preccurlyeq$
$M_{0}^{\prime} \preccurlyeq M_{0}$
$M_{21}=W \quad$ By Lemma 55 Precision on values
$W^{\prime} \preccurlyeq W \quad$ By above equations
$M_{1}=\left(\lambda x . M_{0}\right) W \quad$ By above equations
* 

$\left(\lambda x . M_{0}\right) W \mapsto_{R} \underbrace{[W / x] M_{0}} \quad$ By rule Reduce $\beta$
(W $\left.\quad W^{\prime} / x\right] M_{0} \preccurlyeq[W / x] M$
By Lemma 56 Substitution preserves precision)
Theorem 12 (Stepping preserves precision).
If $\cdot \vdash M_{1}^{\prime}: T_{1}^{\prime}$ and $\cdot \vdash M_{1}: T_{1}$ and $M_{1}^{\prime} \preccurlyeq M_{1}$ and $M_{1}^{\prime} \mapsto M_{2}^{\prime}$ then either
(a) $M_{1}$ is a value and $M_{2}^{\prime} \preccurlyeq M_{1}$, or
(b) there exists $M_{2}$ such that $M_{1} \mapsto M_{2}$ and $M_{2}^{\prime} \preccurlyeq M_{2}$, or
(c) $M_{1}=$ matchfail and $M_{2}^{\prime} \preccurlyeq M_{1}$.

Proof. Proceed by case analysis on $M_{1}^{\prime} \mapsto M_{2}^{\prime}$.

- Case

$$
\frac{M_{0_{1}^{\prime}}^{\prime} \mapsto_{\mathrm{R}} M_{02}^{\prime}}{\underbrace{\mathcal{E}^{\prime}\left[M_{o}^{\prime}\right]}_{M_{1}^{\prime}} \mapsto \underbrace{\mathcal{E}^{\prime}\left[M_{02}^{\prime}\right]}_{M_{2}^{\prime}}} \text { StepContext }
$$

$$
\begin{array}{rlrl}
\mathcal{E}^{\prime}\left[M_{01}^{\prime}\right] & \preccurlyeq M_{1} & & \text { Given } \\
M_{1} & =\mathcal{E}\left[M_{01}\right] & & \text { By Lemma } 57 \text { (Precision inversion on evaluation contexts) } \\
M_{01}^{\prime} & \preccurlyeq M_{01} & & \\
& \cdot \vdash \mathcal{E}^{\prime}\left[M_{01}^{\prime}\right]: T_{1}^{\prime} & & \text { Given } \\
& \cdot \vdash \mathcal{E}\left[M_{01}\right]: T_{1} & & \text { Given } \\
& \cdot \vdash M_{01}^{\prime}: T_{01}^{\prime} & & \text { By Lemma } 52 \text { Evaluation context typing } \\
& \cdot \vdash M_{01}: T_{01} & & \text { By Lemma } 52 \text { Evaluation context typing } \\
M_{01}^{\prime} & \mapsto & M_{02}^{\prime} & \\
\text { Given }
\end{array}
$$

Proceed by case analysis on the result of applying Lemma 59 Reduction preserves precision .
In the first case, $M_{01}$ is a value and $M_{02}^{\prime} \preccurlyeq M_{01}$. Since $M_{01}^{\prime} \preccurlyeq M_{01}$ and $\mathcal{E}^{\prime}\left[M_{01}^{\prime}\right] \preccurlyeq \mathcal{E}\left[M_{01}\right]$, by Lemma [58] (Evaluation contexts preserve precision) it follows that $\mathcal{E}^{\prime}\left[\mathrm{M}_{02}^{\prime}\right] \preccurlyeq \mathcal{E}\left[\mathrm{M}_{01}\right]$. This is alternative (a).
In the second case, $M_{01} \mapsto_{\mathrm{R}} M_{02}$ and $M_{02}^{\prime} \preccurlyeq M_{02}$. Therefore, by rule StepContext it follows $M_{1} \mapsto \mathcal{E}\left[M_{02}\right]$. Since $M_{01}^{\prime} \preccurlyeq M_{01}$ and $\mathcal{E}^{\prime}\left[M_{01}^{\prime}\right] \preccurlyeq \mathcal{E}\left[M_{01}\right]$, by Lemma 58 Evaluation contexts preserve precision) it follows that $\mathcal{E}^{\prime}\left[M_{02}^{\prime}\right] \preccurlyeq \mathcal{E}\left[M_{02}\right]$. This is alternative (b).

- Case

Since $\cdot \vdash M_{1}: T_{1}$, by Theorem 7 it follows that either $M_{1}$ is a value, or there exists $M_{2}$ such that $M_{1} \mapsto M_{2}$, or $M_{1}=$ matchfail. In the first case, $M_{2}^{\prime}=$ matchfail $\preccurlyeq M_{1}$ by definition of $\preccurlyeq$, which is alternative (a).
In the second case, $M_{2}^{\prime}=$ matchfail $\preccurlyeq M_{2}$ by definition of $\preccurlyeq$, which is alternative (b).
In the first case, $M_{2}^{\prime}=$ matchfail $\preccurlyeq$ matchfail $=M_{1}$ by definition of $\preccurlyeq$, which is alternative (c).
Theorem 13 ( $\preccurlyeq$ respects convergence).
If $M^{\prime} \preccurlyeq M$ where $\cdot \vdash M^{\prime}: T^{\prime}$ and $\cdot \vdash M: T$
and $M^{\prime}$ converges then $M$ also converges.

Proof. It is given that $M^{\prime}$ converges. By Definition 1 there exists a value $W^{\prime}$ such that $M^{\prime} \mapsto^{*} W^{\prime}$. Proceed by induction on the number of steps in $M^{\prime} \mapsto^{*} W^{\prime}$.

If $M^{\prime}=W^{\prime}$ then $W^{\prime} \preccurlyeq M$. By Lemma 55 Precision on values, $M=W$ for some value $W$. Therefore, $M$ converges as well.
Otherwise, $M^{\prime}$ takes at least one step, that is, $M^{\prime} \mapsto M_{0}^{\prime} \mapsto^{*} W^{\prime}$. Then $M_{0}^{\prime}$ must also converge, with $M_{0}^{\prime} \mapsto{ }^{*} W^{\prime}$ in fewer steps than $M^{\prime} \mapsto^{*} W^{\prime}$. Since $M^{\prime} \mapsto M_{0}^{\prime}$, proceed by case analysis on the result of applying Theorem 12

- In the first case (a), $M$ is a value, so $M$ converges.
- In the second case (b), there exists $M_{0}$ such that $M \mapsto M_{0}$ and $M_{0}^{\prime} \preccurlyeq M_{0}$. By Theorem6, $\cdot \stackrel{M_{0}^{\prime}}{ }: T^{\prime}$ and similarly $\cdot \vdash M_{0}: T$. By the induction hypothesis, $M_{0}$ converges. Since $M_{0}$ converges, $M$ must also converge to the same value.
- In the third case (c), $M=$ matchfail and $M_{0}^{\prime} \preccurlyeq M$. By inversion on $M_{0}^{\prime} \preccurlyeq$ matchfail it follows that $M_{0}^{\prime}=$ matchfail. But we know that $M_{0}^{\prime}$ converges, a contradiction. Hence, this case is impossible.


## D. 6 Translation

## D.6.1 Soundness

Theorem 16 (Sum Translation soundness).
Given $\delta^{\prime}$ and $\delta$, there exists $\mathcal{C}$ such that $\delta^{\prime} \Rightarrow \delta \hookrightarrow \mathcal{C}$.
Moreover, if $\Theta \vdash M:\left(T_{1}\left|\delta^{\prime}\right| T_{2}\right)$ then $\Theta \vdash \mathcal{C}[M]:\left(T_{1}|\delta| T_{2}\right)$.
Proof. Proceed by case analysis on whether $\left|\delta^{\prime}\right| \leq|\delta|$.

- Case $\left|\delta^{\prime}\right| \leq|\delta|:$

| $\mathrm{T}_{1} \leq \mathrm{T}_{1}$ | By Lemma 43 Reflexivity of subtyping |
| :---: | :---: |
| $\mathrm{T}_{2} \leq \mathrm{T}_{2}$ | By Lemma 43 Reflexivity of subtyping, |
| $\left\|\delta^{\prime}\right\| \leq\|\delta\|$ | Given |
| $\left(\mathrm{T}_{1}\left\|\delta^{\prime}\right\| \mathrm{T}_{2}\right) \leq\left(\mathrm{T}_{1}\|\delta\| \mathrm{T}_{2}\right)$ | By definition of $\leq$ |
| $\delta^{\prime} \Rightarrow \delta \hookrightarrow[]$ | By rule CoeSub |
| $\Theta \vdash \mathrm{M}:\left(\mathrm{T}_{1}\left\|\delta^{\prime}\right\| \mathrm{T}_{2}\right)$ | Suppose |
| $\Theta \vdash M:\left(T_{1}\|\delta\| \mathrm{T}_{2}\right)$ | By rule TSub |
| $\mathcal{C}[\mathrm{M}]=\mathrm{M}$ | By definition |
| $\Theta \vdash \mathcal{C}[M]:\left(\mathrm{T}_{1}\|\delta\| \mathrm{T}_{2}\right)$ | By above equations |

- Case $\left|\delta^{\prime}\right| \not \leq|\delta|:$

| $\left\|\delta^{\prime}\right\|$ | $\leq\|\delta\|$ |  | Given |
| ---: | :--- | ---: | :--- |
| $\left\|\delta^{\prime}\right\|$ | $\Rightarrow\|\delta\| \hookrightarrow\langle \| \delta\|\Leftarrow\| \delta^{\prime}\| \rangle[]$ |  | By rule CoeCast |
| $\Theta \vdash M:\left(T_{1}\left\|\delta^{\prime}\right\| T_{2}\right)$ |  | Suppose |  |
| $\Theta \vdash$ | $\vdash\langle \| \delta\|\Leftarrow\| \delta^{\prime}\| \rangle M:\left(\mathrm{T}_{1}\|\delta\| \mathrm{T}_{2}\right)$ |  | By rule TCast |
| $\mathcal{C}[M]$ | $=\langle \| \delta\|\Leftarrow\| \delta^{\prime}\| \rangle M$ |  | By definition |
| $\Theta$ | $\vdash \mathcal{C}[M]:\left(\mathrm{T}_{1}\|\delta\| \mathrm{T}_{2}\right)$ |  | By above equations |

Theorem 17 (Type translation soundness).
If $A^{\prime} \simeq A$ then there exists $\mathcal{C}$ such that $A^{\prime} \Rightarrow A \hookrightarrow \mathcal{C}$.
Moreover, if $\Theta \vdash M:\left|A^{\prime}\right|$ then $\Theta \vdash \mathcal{C}[M]:|A|$.
Proof. By induction on the structure of the derivation of $A^{\prime} \simeq A$.

- Case Unit $\simeq$ Unit:

| Unit | $\Rightarrow$ Unit $\hookrightarrow[]$ |  | By rule CoeUnit |
| ---: | :--- | ---: | :--- |
| $\Theta \vdash M: \mid$ Unit $\mid$ |  | Suppose |  |
| $\Theta \vdash \mathcal{C}[M]: \mid$ Unit $\mid$ |  | By definition of $\mathcal{C}$ |  |

- Case

$$
\frac{A_{1}^{\prime} \simeq A_{1} \quad A_{2}^{\prime} \simeq A_{2}}{\left(A_{A^{\prime}}^{\prime} \delta^{\prime} A_{2}^{\prime}\right)} \simeq \underbrace{\left(A_{1} \delta A_{2}\right)}_{A}
$$

Proceed by case analysis on the definition of $\delta^{\prime}$.
In the first case, suppose $\delta^{\prime} \in\left\{+{ }_{i}^{?},+{ }_{1}\right\}$.

```
    |\mathcal{A}
    |A2'| \leq|A ' | By Lemma 43,Reflexivity of subtyping
    |\mp@subsup{\delta}{}{\prime}}|\leq+1\quad By definition of \leq
|\mp@subsup{A}{1}{\prime}|}|\mp@subsup{\delta}{}{\prime}||\mp@subsup{A}{2}{\prime}|\leq|\mp@subsup{A}{1}{\prime}|+\mp@subsup{}{1}{}|\mp@subsup{A}{2}{\prime}|\quad\mathrm{ By definition of }
    \Theta\vdashM:|(A, (A \delta' }\mp@subsup{A}{2}{\prime})|\quad\mathrm{ Suppose
    \Theta\vdashM:(|A⿱㇒⿴囗⿱一一夊心}|||\mp@subsup{\delta}{}{\prime}||\mp@subsup{A}{2}{\prime}|)\quad By definition of type translation
    \Theta\vdashM:(|\mp@subsup{A}{1}{\prime}|+\mp@subsup{}{1}{}|\mp@subsup{A}{2}{\prime}|)
    |A | | \leq |A |
    |A2|}\leq|\mp@subsup{A}{2}{}
        +1}\leq|\mp@subsup{\delta}{}{\prime}
|\mp@subsup{A}{1}{}|+\mp@subsup{}{1}{}|\mp@subsup{A}{2}{}|\leq|\mp@subsup{A}{1}{}||\mp@subsup{\delta}{}{\prime}||\mp@subsup{A}{2}{}|
\Theta, \mp@subsup{x}{1}{}:|\mp@subsup{A}{1}{\prime}|\vdash\mp@subsup{x}{1}{\prime}:|\mp@subsup{A}{1}{\prime}|
            A1}\simeq\mp@subsup{A}{1}{\prime
    A
\Theta, \mp@subsup{x}{1}{}:|\mp@subsup{\mathcal{A}}{1}{\prime}|\vdash\mp@subsup{\mathcal{C}}{1}{}[\mp@subsup{x}{1}{}]:|\mp@subsup{\mathcal{A}}{1}{}|
\Theta, \mp@subsup{x}{1}{}:|\mp@subsup{A}{1}{\prime}|\vdash\mp@subsup{\operatorname{inj}}{1}{}\mp@subsup{\mathcal{C}}{1}{}[\mp@subsup{x}{1}{}]:(|\mp@subsup{A}{1}{}|+\mp@subsup{+}{1}{}|\mp@subsup{A}{2}{}|)\quad\mathrm{ By rule T+ilntro}
\Theta, \mp@subsup{x}{1}{}:|\mp@subsup{A}{1}{\prime}|\vdash\mp@subsup{\operatorname{inj}}{1}{}\mp@subsup{\mathcal{C}}{1}{}[\mp@subsup{x}{1}{}]:(|\mp@subsup{A}{1}{}||\mp@subsup{\delta}{}{\prime}||\mp@subsup{A}{2}{}|)\quad\mathrm{ By rule TSub}
\Theta\vdash\operatorname{case(M,inj ( }\mp@subsup{x}{1}{}.\mp@subsup{\operatorname{inj}}{1}{}\mp@subsup{\mathcal{C}}{1}{}[\mp@subsup{x}{1}{}]):(|\mp@subsup{A}{1}{}||\mp@subsup{\delta}{}{\prime}||\mp@subsup{A}{2}{}|)\quad\mathrm{ By rule T+}\mp@subsup{}{i}{}Elim
\delta'}=>\delta\hookrightarrow\mp@subsup{\mathcal{C}}{3}{
    By Theorem 16
    \Theta\vdash\mathcal{C}
    \Theta\vdash, \underbrace
(A'\delta}\mp@subsup{\delta}{}{\prime}\mp@subsup{A}{2}{\prime})=>(\mp@subsup{A}{1}{}\delta\mp@subsup{A}{2}{})\hookrightarrow\mp@subsup{\underbrace}{\mathcal{C}}{\mp@subsup{\mathcal{C}}{3}{}[\operatorname{case}([],\mp@subsup{\textrm{inj}}{1}{}\mp@subsup{x}{1}{}\cdot\mp@subsup{\textrm{inj}}{1}{}\mp@subsup{\mathcal{C}}{1}{}[\mp@subsup{x}{1}{}])]}\quad\mathrm{ By rule CoeCase1L
```

In the second case，suppose $\delta^{\prime} \in\left\{+\frac{?}{2},+_{2}\right\}$ ．Symmetric to the previous case，hence omitted． In the last case，suppose $\delta^{\prime} \in\left\{+^{?},+_{1}^{*},+_{2}^{*},+\right\}$ ．

| $\left\|\mathcal{A}_{1}^{\prime}\right\| \leq\left\|\mathcal{A}_{1}^{\prime}\right\| \quad$ By Lemma 43 R | By Lemma 43 Reflexivity of subtyping |  |
| :---: | :---: | :---: |
| $\left\|A_{2}^{\prime}\right\| \leq\left\|A_{2}^{\prime}\right\| \quad$ By Lemma 43 Re | By Lemma 43 Reflexivity of subtyping） |  |
| $\left\|\delta^{\prime}\right\| \leq+\quad$ By definition of $\leq$ | By definition of $\leq$ |  |
| $\left\|A_{1}^{\prime}\right\|\left\|\delta^{\prime}\right\|\left\|A_{2}^{\prime}\right\| \leq\left\|A_{1}^{\prime}\right\|+\left\|A_{2}^{\prime}\right\| \quad$ By definition of $\leq$ | By definition of $\leq$ |  |
| $\Theta \vdash \mathrm{M}:\left\|\left(\mathcal{A}_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right)\right\|$ | $\left.\lambda_{2}^{\prime}\right) \mid$ Suppose |  |
| $\Theta \vdash M:\left(\left\|A_{1}^{\prime}\right\|\left\|\delta^{\prime}\right\|\left\|A_{2}^{\prime}\right\|\right)$ | ｜$\left.A_{2}^{\prime} \mid\right) \quad$ By definition of type translation |  |
| $\Theta \vdash M:\left(\left\|A_{1}^{\prime}\right\|+\left\|A_{2}^{\prime}\right\|\right)$ | A 2 I）By rule TSub |  |
| $\Theta, x_{1}:\left\|A_{1}^{\prime}\right\| \vdash x_{1}:\left\|\mathcal{A}_{1}^{\prime}\right\|$ | By rule TVar |  |
| $A_{1}^{\prime} \simeq A_{1}$ | Subderivation |  |
| $A_{1}^{\prime} \Rightarrow A_{1} \hookrightarrow \mathcal{C}_{1}$ | By the induction hypothesis |  |
| $\Theta, \mathrm{x}_{1}:\left\|\mathcal{A}_{1}^{\prime}\right\| \vdash \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]:\left\|\mathcal{A}_{1}\right\|$ |  |  |
| $\Theta, \mathrm{x}_{1}:\left\|\mathcal{A}_{1}^{\prime}\right\| \vdash \operatorname{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]:\left(\left\|A_{1}\right\|+{ }_{1}\left\|\mathrm{~A}_{2}\right\|\right)$ | ：$\left(A_{1}\left\|+{ }_{1}\right\| A_{2} \mid\right) \quad$ By rule $T+{ }_{\text {i }} \operatorname{lntro}$ |  |
| $+{ }_{i}^{\prime} \Rightarrow \delta^{\prime} \hookrightarrow \mathcal{C}_{1}^{\prime}$ | By Theorem 16 |  |
| $\Theta, \mathrm{x}_{1}:\left\|\mathcal{A}_{1}^{\prime}\right\| \vdash \mathcal{C}_{1}^{\prime}\left[\mathrm{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right]:\left(\left\|A_{1}\right\|\left\|\delta^{\prime}\right\|\left\|\mathrm{A}_{2}\right\|\right)$ | ］］：$\left(\left\|A_{1}\right\|\left\|\delta^{\prime}\right\|\left\|A_{2}\right\|\right) \quad "$ |  |
| $\Theta, x_{2}:\left\|A_{2}^{\prime}\right\| \vdash x_{2}:\left\|\mathcal{A}_{2}^{\prime}\right\|$ | By rule TVar |  |
| $A_{2}^{\prime} \simeq A_{2}$ | Subderivation |  |
| $A_{2}^{\prime} \Rightarrow A_{2} \hookrightarrow \mathcal{C}_{2}$ | By the induction hypothesis |  |
| $\Theta, x_{2}:\left\|\mathcal{A}_{2}^{\prime}\right\| \vdash \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]:\left\|\mathcal{A}_{2}\right\|$ |  |  |
| $\Theta, \mathrm{x}_{2}:\left\|\mathrm{A}_{2}^{\prime}\right\| \vdash \operatorname{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]:\left(\left\|\mathrm{A}_{1}\right\|+{ }_{2}\left\|\mathrm{~A}_{2}\right\|\right)$ | $\left(\left\|A_{1}\right\|+2\left\|A_{2}\right\|\right) \quad$ By rule T＋intro |  |
| $+_{2}^{?} \Rightarrow \delta^{\prime} \hookrightarrow \mathcal{C}_{2}^{\prime}$ | By Theorem 16 |  |
| $\Theta, \mathrm{x}_{2}:\left\|\mathcal{A}_{2}^{\prime}\right\| \vdash \mathcal{C}_{2}^{\prime}\left[\mathrm{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]\right]:\left(\left\|A_{1}\right\|\left\|\delta^{\prime}\right\|\left\|A_{2}\right\|\right)$ | ］］：$\left(\left\|A_{1}\right\|\left\|\delta^{\prime}\right\|\left\|A_{2}\right\|\right) \quad "$ |  |
| $\Theta \vdash \operatorname{case}\left(M, \operatorname{inj}_{1} \mathrm{x}_{1} \cdot \mathcal{C}_{1}^{\prime}\left[\mathrm{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right], \mathrm{inj}_{2} \mathrm{x}_{2}\right.$. | $\left.\left.\left.\mathrm{nj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right], \mathrm{inj}_{2} \mathrm{x}_{2} \cdot \mathcal{C}_{2}^{\prime}\left[\mathrm{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]\right]\right)\right]:\left(\left\|A_{1}\right\|\left\|\delta^{\prime}\right\|\left\|A_{2}\right\|\right)$ | By rule T＋Elim |
| $\delta^{\prime} \Rightarrow \delta \hookrightarrow \mathcal{C}_{3}$ |  | By Theorem 16 |
| $\Theta \vdash \mathcal{C}_{3}\left[\operatorname{case}\left(M, \mathrm{inj}_{1} \mathrm{x}_{1} \cdot \mathcal{C}_{1}^{\prime}\left[\mathrm{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right], \mathrm{inj}_{2}\right.\right.$ | $\left.\left.\left.{ }_{1}^{\prime}\left[\operatorname{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right], \operatorname{inj}_{2} \mathrm{x}_{2} . \mathcal{C}_{2}^{\prime}\left[\mathrm{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]\right]\right)\right]\right]:\left(\left\|\mathcal{A}_{1}\right\|\|\delta\|\left\|\mathcal{A}_{2}\right\|\right)$ |  |
| $\Theta \vdash \underbrace{\mathcal{C}_{3}\left[\operatorname{case}\left(M, \mathrm{inj}_{1} \mathrm{x}_{1} \cdot \mathcal{C}_{1}^{\prime}\left[\mathrm{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right], \mathrm{inj}_{2}\right.\right.}$ | $\left.\left.\left.\left.{ }^{\prime}\left[\operatorname{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right], \operatorname{inj}_{2} \mathrm{x}_{2} \cdot \mathcal{C}_{2}^{\prime}\left[\mathrm{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]\right]\right)\right]\right]: \mid A_{1} \delta A_{2}\right) \mid$ | By definition |

$$
\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right) \Rightarrow\left(A_{1} \delta A_{2}\right) \hookrightarrow \underbrace{\left.\mathcal{C}_{3}\left[\operatorname{case}\left([], \operatorname{inj}_{1} x_{1} \cdot \mathcal{C}_{1}^{\prime}\left[\operatorname{inj}_{1} \mathcal{C}_{1}\left[x_{1}\right]\right], \operatorname{inj}_{2} x_{2} \cdot \mathcal{C}_{2}^{\prime}\left[\operatorname{inj}_{2} \mathcal{C}_{2}\left[x_{2}\right]\right]\right)\right]\right]}_{\mathcal{C}} \quad \text { By rule CoeCase2 }
$$

- Case $\frac{A_{1}^{\prime} \simeq A_{1}}{\underbrace{\left(A_{1}^{\prime} \rightarrow A_{2}^{\prime}\right)}_{A^{\prime}} \simeq \underbrace{\left(A_{1}^{\prime} \simeq A_{2}\right)}_{A}}$

| $\Theta, x:\left\|A_{1}\right\| \vdash x:\left\|A_{1}\right\|$ | By rule TVar |
| :---: | :---: |
| $A_{1}^{\prime} \simeq A_{1}$ | Subderivation |
| $A_{1} \simeq A_{1}^{\prime}$ | By Lemma 12 Symmetry of Structural Equivalence |
| $A_{1} \Rightarrow A_{1}^{\prime} \hookrightarrow \mathcal{C}_{1}$ | By the induction hypothesis |
| $\Theta, x:\left\|A_{1}\right\| \vdash \mathcal{C}_{1}[x]:\left\|A_{1}^{\prime}\right\|$ |  |
| $\Theta \vdash M:\left\|\left(A_{1}^{\prime} \rightarrow A_{2}^{\prime}\right)\right\|$ | Suppose |
| $\Theta \vdash M:\left(\left\|A_{1}^{\prime}\right\| \rightarrow\left\|\mathcal{A}_{2}^{\prime}\right\|\right)$ | By definition of type translation |
| $\Theta, \mathrm{x}:\left\|\mathrm{A}_{1}\right\| \vdash \mathrm{M} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]:\left\|\mathcal{A}_{2}^{\prime}\right\|$ | By rule T $\rightarrow$ Elim |
| $A_{2}^{\prime} \simeq A_{2}$ | Subderivation |
| $A_{2}^{\prime} \Rightarrow A_{2} \hookrightarrow \mathcal{C}_{2}$ | By the induction hypothesis |
| $\Theta, x:\left\|A_{1}\right\| \vdash \mathcal{C}_{2}\left[M \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right]:\left\|A_{2}\right\|$ |  |
| $\Theta \vdash \lambda$. $\mathcal{C}_{2}\left[M \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right]:\left(\left\|\mathcal{A}_{1}\right\| \rightarrow\left\|\mathcal{A}_{2}\right\|\right)$ | By rule $T \rightarrow$ Intro |
| $\Theta \vdash \underbrace{\lambda x . \mathcal{C}_{2}\left[M \mathcal{C}_{1}\left[x_{1}\right]\right]}:\left\|\left(A_{1} \rightarrow A_{2}\right)\right\|$ | By definition of type translation |
| $\mathcal{C}[\mathrm{M}]$ |  |
| $\left(A_{1}^{\prime} \rightarrow A_{2}^{\prime}\right) \Rightarrow\left(A_{1} \rightarrow A_{2}\right) \hookrightarrow \underbrace{\lambda x \cdot \mathcal{C}_{2}\left[[] \mathcal{C}_{1}[x]\right]}_{\mathcal{C}}$ | By rule $\mathrm{Coe} \rightarrow$ |

Theorem 9 (Translation soundness).
If $\Gamma \vdash e: A$ then there exists $M$ such that $\Gamma \vdash e: A \hookrightarrow M$ and $|\Gamma| \vdash M:|A|$.
Proof. By induction on the structure of the derivation of $\Gamma \vdash e: A$.

- CaseSVar Apply rules STVar andTVar
- Case


| $\Gamma$ | $\vdash e: A^{\prime}$ |  | Subderivation |
| ---: | :--- | ---: | :--- |
| $\Gamma \vdash e: A^{\prime} \hookrightarrow M^{\prime}$ |  | By the induction hypothesis |  |
| $\|\Gamma\| \vdash M^{\prime}:\left\|A^{\prime}\right\|$ |  | $\prime \prime$ |  |
| $A^{\prime} \leadsto A$ |  | Given |  |
| $A^{\prime} \simeq A$ |  | By Lemma 17 (Directed consistency obeys Structural Equivalence |  |
| $A^{\prime} \Rightarrow A \hookrightarrow \mathcal{C}$ |  | By Theorem 17 |  |
| $\|\Gamma\| \vdash \mathcal{C}\left[M^{\prime}\right]:\|A\|$ |  |  |  |
| $\Gamma \vdash e: A \hookrightarrow \mathcal{C}\left[M^{\prime}\right]$ |  | By ruleSTCSub |  |

- CaseSAnno Use the induction hypothesis and apply rule STAnno
- CaseSUnitIntro Apply rules STUnitIntro and TUnitIntro
- Case SSumlntro Use the induction hypothesis and apply rules STSumIntro and T+ilntro
- Case SSumElim1 Use the induction hypothesis and apply rules STSumElim1 and T+iElim
- CaseSSumElim2 Use the induction hypothesis and apply rules STSumElim2 and T+Elim.
- Case $S \rightarrow$ Intro Use the induction hypothesis and apply rules $S T \rightarrow$ Intro and $T \rightarrow$ Intro
- Case $S \rightarrow$ Elim Use the induction hypothesis and apply rules $S T \rightarrow$ Elim and $T \rightarrow$ Elim


## D.6.2 Precision

Theorem 11 depends on Lemma 60 Cast insertion preserves precision, which uses a modified version of the translation that always inserts casts, even safe ones. In effect, the modified translation does not have rule CoeSubland always uses rule CoeCast (Figure 12). It also inserts safe casts $\mathcal{C}_{1}^{\prime}$ and $\mathcal{C}_{2}^{\prime}$, similar to CoeCase2 in rules "CoeCase1L and "CoeCase1R See Figure 21
Lemma 60 (Cast insertion preserves precision).
If $\delta_{1}^{\prime} \Rightarrow \delta_{2}^{\prime} \hookrightarrow \mathcal{C}^{\prime}$ and $\delta_{1} \Rightarrow \delta_{2} \hookrightarrow \mathcal{C}$
and $\delta_{1}^{\prime} \sqsubseteq \delta_{1}$ and $\delta_{2}^{\prime} \sqsubseteq \delta_{2}$ and $M^{\prime} \preccurlyeq M$
then $\mathcal{C}^{\prime}\left[M^{\prime}\right] \preccurlyeq \mathcal{C}[M]$.
$\delta^{\prime} \Rightarrow \delta \hookrightarrow \mathcal{C}$ Coercion $\mathcal{C}$ coerces sum $\left|\delta^{\prime}\right|$ to sum $|\delta|$

$$
\frac{\mid \delta^{\prime}}{\delta^{\prime} \Rightarrow \delta \hookrightarrow\langle | \delta|\Leftarrow| \delta^{\prime}| \rangle[]} * \text { CoeCast }
$$

$A^{\prime} \Rightarrow A \hookrightarrow \mathcal{C}$ Coercion $\mathcal{C}$ coerces target type $\left|A^{\prime}\right|$ to $|A|$

$$
\begin{aligned}
& \frac{A_{1} \Rightarrow A_{1}^{\prime} \hookrightarrow \mathcal{C}_{1} \quad A_{2}^{\prime} \Rightarrow A_{2} \hookrightarrow \mathcal{C}_{2}}{\left(A_{1}^{\prime} \rightarrow A_{2}^{\prime}\right) \Rightarrow\left(A_{1} \rightarrow A_{2}\right) \hookrightarrow \lambda x . \mathcal{C}_{2}\left[[] \mathcal{C}_{1}[x]\right]} \text { Coe } \rightarrow
\end{aligned}
$$

$$
\begin{aligned}
& \hookrightarrow \mathcal{C}_{3}\left[\operatorname { c a s e } ( [ ] , \operatorname { i n j } _ { 1 } \mathrm { x } _ { 1 } \cdot \mathcal { C } _ { 1 } ^ { \prime } [ \mathrm { inj } _ { 1 } \mathcal { C } _ { 1 } [ \mathrm { x } _ { 1 } ] ) ] \quad \hookrightarrow \mathcal { C } _ { 3 } \left[\operatorname{case}\left([], \mathrm{inj}_{2} \mathrm{x}_{2} \cdot \mathcal{C}_{2}^{\prime}\left[\operatorname{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]\right)\right]\right.\right. \\
& +{ }_{i}^{\prime} \Rightarrow \delta^{\prime} \hookrightarrow \mathcal{C}_{1}^{\prime} \quad+{ }_{2}^{\prime} \Rightarrow \delta^{\prime} \hookrightarrow \mathcal{C}_{2}^{\prime} \\
& \frac{\delta^{\prime} \in\left\{+^{?},+_{1}^{*},+_{2}^{*},+\right\}}{\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right) \Rightarrow\left(A_{1} \delta A_{2}\right) \hookrightarrow \mathcal{C}_{3}\left[\operatorname{case}\left([], \operatorname{inj}_{1} x_{1} \cdot \mathcal{C}_{1}^{\prime}\left[\operatorname{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right], \operatorname{inj}_{2} \mathrm{x}_{2} \cdot \mathcal{C}_{2}^{\prime}\left[\operatorname{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]\right]\right)\right]} \text { CoeCase2 }
\end{aligned}
$$

Figure 21. Part of the type-directed translation, modified to insert safe casts; differences highlighted

Proof. Note the following reasons for arriving at the result.
(a) If the translated sums are equal, that is, $\left|\delta_{1}^{\prime}\right|=\left|\delta_{1}\right|$ and $\left|\delta_{2}^{\prime}\right|=\left|\delta_{2}\right|$, we have $\mathcal{C}^{\prime}=\mathcal{C}$. (Casts are unique; in this context, this is immediate because we are using a translation that generates casts even if they are safe, so there is only one rule, CoeCast that derives the judgment.) Then the result follows from $M^{\prime} \preccurlyeq M$ and the definition of $\preccurlyeq$.
(b) If $\mathcal{C}^{\prime}=\left\langle\delta_{2}^{\prime} \Leftarrow\right| \delta_{1}^{\prime}| \rangle[]$ and $\mathcal{C}=\langle | \delta_{2}|\Leftarrow| \delta_{1}| \rangle[]$ and $\left\langle\delta_{2}^{\prime} \Leftarrow \delta_{1}^{\prime}\right\rangle \preccurlyeq\left\langle\delta_{2} \Leftarrow \delta_{1}\right\rangle$ then $\mathcal{C}^{\prime}\left[M^{\prime}\right] \preccurlyeq \mathcal{C}[M]$ by definition of $\preccurlyeq$ as $M^{\prime} \preccurlyeq M$.

Proceed by case analysis on $\delta_{1}^{\prime} \sqsubseteq \delta_{1}$ based on the reflexive, transitive closure of precision on sums.

- Cases $\boldsymbol{+}_{i} \sqsubseteq+_{i},+_{i} \sqsubseteq+_{i}^{?},+_{i} \sqsubseteq+_{i}^{*},+_{i}^{?} \sqsubseteq+_{i}^{?},+_{i}^{*} \sqsubseteq+_{i}^{*}$ : In these cases, $\left|\delta_{1}^{\prime}\right|=\left|\delta_{1}\right|=+_{i}$.

Proceed by case analysis on $\overline{\delta_{2}^{\prime}} \sqsubseteq \delta_{2}$.

- Cases $+_{i} \sqsubseteq+_{i},+_{i} \sqsubseteq++_{i}^{?},+_{i} \sqsubseteq+_{i}^{*},+_{i}^{?} \sqsubseteq+_{i}^{?},+_{i}^{*} \sqsubseteq+_{i}^{*}$ :

Here, $\left|\delta_{2}^{\prime}\right|=\left|\delta_{2}\right|=+{ }_{i}$.
The translated sums are equal: go to (a) above.

- Cases $\boldsymbol{+}_{i} \sqsubseteq+^{?},+_{i}^{?} \sqsubseteq+^{?},+_{i}^{*} \sqsubseteq+^{?}$ :

Here, $\left|\delta_{2}^{\prime}\right|=+_{i}$ and $\left|\delta_{2}\right|=+$.
We have $\left\langle\boldsymbol{+}_{\mathfrak{i}} \Leftarrow \boldsymbol{+}_{\mathrm{i}}\right\rangle \preccurlyeq\left\langle+\boldsymbol{+}_{\boldsymbol{+}}\right\rangle$. Go to (b).

- Cases $+\sqsubseteq+,+\sqsubseteq+^{?},+^{?} \sqsubseteq+^{?}$ :

Here, $\left|\delta_{2}^{\prime} \overline{=}=\left|\delta_{2}\right|=+\right.$. Go to (a) above.

- Cases $+_{k} \sqsubseteq+_{k},+_{k} \sqsubseteq+_{k}^{?},+_{k} \sqsubseteq+_{k}^{*},+_{k}^{?} \sqsubseteq+_{k}^{?},+_{k}^{*} \sqsubseteq+_{k}^{*}$ :

Here, $\left|\delta_{2}^{\prime}\right|=\left|\delta_{2}\right|=+_{k}$. Go to (a).

- Cases $+_{k} \sqsubseteq+^{?},+_{k}^{2} \sqsubseteq+^{?},+_{k}^{*} \sqsubseteq+^{?}$ :

Here, $\left|\delta_{2}^{\prime}\right|=\boldsymbol{+}_{k}$ and $\left|\delta_{2}\right|=+$.
We have $\left\langle+_{\mathrm{k}} \Leftarrow+_{i}\right\rangle \preccurlyeq\left\langle+\Leftarrow+_{i}\right\rangle$. Go to (b).

- Cases $+_{i} \sqsubseteq+^{?},+_{i}^{?} \sqsubseteq+^{?},+_{i}^{*} \sqsubseteq+^{?}$ : In these cases, $\left|\delta_{1}^{\prime}\right|=+_{i}$ and $\left|\delta_{1}\right|=+$. Proceed by case analysis on $\delta_{2}^{\prime} \sqsubseteq \delta_{2}$.
- Cases $\boldsymbol{+}_{i} \sqsubseteq+_{i},+_{i} \sqsubseteq+_{i}^{?},+_{i} \sqsubseteq+_{i}^{*},+_{i}^{?} \sqsubseteq+_{i}^{?},+_{i}^{*} \sqsubseteq+_{i}^{*}$ : We have $\left\langle+_{i} \Leftarrow+_{i}\right\rangle \preccurlyeq\left\langle+_{i} \Leftarrow+\right\rangle$. Go to (b).
- Cases $+_{i} \sqsubseteq+^{?},+_{i}^{?} \sqsubseteq+^{?},+_{i}^{*} \sqsubseteq+^{?}$ : We have $\left\langle+_{i} \Leftarrow+_{i}\right\rangle \preccurlyeq\langle+\Leftarrow+\rangle$. Go to (b).
- Cases $\boldsymbol{+} \sqsubseteq+,+\sqsubseteq+^{?}, \boldsymbol{+}^{?} \sqsubseteq \boldsymbol{+}^{?}$ : We have $\left\langle\boldsymbol{+} \Leftarrow \boldsymbol{+}_{\mathrm{i}}\right\rangle \preccurlyeq\langle+\Leftarrow+\rangle$. Go to (b).

- Cases $\boldsymbol{+}_{k} \sqsubseteq+^{?},+_{k}^{?} \sqsubseteq+^{?},+_{k}^{*} \sqsubseteq+^{?}$ : We have $\left\langle\boldsymbol{+}_{k} \Leftarrow \boldsymbol{+}_{\mathrm{i}}\right\rangle \preccurlyeq\langle+\Leftarrow+\rangle$. Go to (b).
- Cases $+\sqsubseteq+,+\sqsubseteq+^{?},+^{?} \sqsubseteq+^{?}$ : In these cases, $\left|\delta_{1}^{\prime}\right|=\left|\delta_{1}\right|=+$. Proceed by case analysis on $\delta_{2}^{\prime} \sqsubseteq \delta_{2}$.
- Cases $\boldsymbol{+}_{i} \sqsubseteq+_{i},+_{i} \sqsubseteq+_{i}^{?},+_{i} \sqsubseteq+_{i}^{*},+_{i}^{?} \sqsubseteq+_{i}^{?},+_{i}^{*} \sqsubseteq+_{i}^{*}$ : Here, $\left|\delta_{2}^{\prime}\right|=\left|\delta_{2}\right|=\boldsymbol{+}_{i}$. Go to (a).
- Cases $\boldsymbol{+}_{\mathrm{i}} \sqsubseteq \boldsymbol{+}^{?}, \boldsymbol{+}_{\mathrm{i}}^{?} \sqsubseteq \boldsymbol{+}^{?}, \boldsymbol{+}_{\mathrm{i}}^{*} \sqsubseteq \boldsymbol{+}^{?}$ : We have $\left\langle\boldsymbol{+}_{\mathrm{i}} \Leftarrow \boldsymbol{+}\right\rangle \preccurlyeq\langle\boldsymbol{+} \Leftarrow+\rangle$. Go to (b).
- Cases $\boldsymbol{+} \sqsubseteq+,+\sqsubseteq+^{?},+^{?} \sqsubseteq+^{?}$ : Here, $\left|\delta_{2}^{\prime}\right|=\left|\delta_{2}\right|=+$. Go to (a).
- Cases $\boldsymbol{+}_{\mathrm{k}} \sqsubseteq+_{\mathrm{k}},+_{\mathrm{k}} \sqsubseteq+_{\mathrm{k}}^{?},+_{\mathrm{k}} \sqsubseteq+_{\mathrm{k}}^{*},+_{\mathrm{k}}^{?} \sqsubseteq+_{\mathrm{k}}^{?},+_{\mathrm{k}}^{*} \sqsubseteq+_{\mathrm{k}}^{*}$ : Here, $\left|\delta_{2}^{\prime}\right|=\left|\delta_{2}\right|=\boldsymbol{+}_{\mathrm{k}}$. Go to (a).
- Cases $\boldsymbol{+}_{\mathrm{k}} \sqsubseteq+^{?}, \boldsymbol{+}_{\mathrm{k}} \sqsubseteq+^{?},+_{\mathrm{k}}^{*} \sqsubseteq+^{?}$ : We have $\left\langle\boldsymbol{+}_{\mathrm{k}} \Leftarrow+\right\rangle \preccurlyeq\langle+\Leftarrow+\rangle$. Go to (b).

Lemma 61 (Coercion preserves precision).
If $A_{1}^{\prime} \Rightarrow A_{2}^{\prime} \hookrightarrow \mathcal{C}^{\prime}$ and $A_{1} \Rightarrow A_{2} \hookrightarrow \mathcal{C}$
and $A_{1}^{\prime} \sqsubseteq A_{1}$ and $A_{2}^{\prime} \sqsubseteq A_{2}$ and $M^{\prime} \preccurlyeq M$
then $\mathcal{C}^{\prime}\left[\bar{M}^{\prime}\right] \preccurlyeq \mathcal{C}[M]$.

Proof. By induction on the structure of the derivation of $A_{1}^{\prime} \Rightarrow A_{2}^{\prime} \hookrightarrow \mathcal{C}^{\prime}$.

- Case
$\overline{\text { Unit } \Rightarrow \text { Unit } \hookrightarrow[]}$ CoeUnit

| Unit | $\sqsubseteq A_{1}$ |  | Given |
| ---: | :--- | ---: | :--- |
| Unit | $\sqsubseteq A_{2}$ |  | Given |
| $A_{1}$ | $=$ Unit |  | By Lemma 4 (Precision inversion) |
| $A_{2}$ | $=$ Unit |  | By Lemma 4 (Precision inversion $)$ |
| Unit | $\Rightarrow$ Unit $\hookrightarrow \mathcal{C}$ |  | Given |
| $\mathcal{C}$ | $=[]$ |  | By inversion on CoeUnit |
| $M^{\prime}$ | $\preccurlyeq M$ |  | Given |
| $\mathcal{C}^{\prime}\left[M^{\prime}\right]$ | $\preccurlyeq \mathcal{C}[M]$ |  | By definition of $\mathcal{C}^{\prime}$ and $\mathcal{C}$ |

- Case $\frac{A_{12}^{\prime} \Rightarrow A_{11}^{\prime} \hookrightarrow \mathcal{C}_{1}^{\prime} \quad A_{21}^{\prime} \Rightarrow A_{22}^{\prime} \hookrightarrow \mathcal{C}_{2}^{\prime}}{\left(A_{11}^{\prime} \rightarrow A_{21}^{\prime}\right) \Rightarrow\left(A_{12}^{\prime} \rightarrow A_{22}^{\prime}\right) \hookrightarrow \lambda x . \mathcal{C}_{2}^{\prime}\left[[] \mathcal{C}_{1}^{\prime}[x]\right]} C$ Coe $\rightarrow$

| $A_{11}^{\prime} \rightarrow A_{21}^{\prime}$ | $\sqsubseteq A_{1}$ |  | Given |
| ---: | :--- | ---: | :--- |
| $A_{1}$ | $=A_{11} \rightarrow A_{21}$ |  | By Lemma4(Precision inversion) |
| $A_{11}^{\prime} \sqsubseteq A_{11}$ |  | $\prime \prime$ |  |
| $A_{21}^{\prime}$ | $\sqsubseteq A_{21}$ |  |  |
| $A_{12}^{\prime} \rightarrow A_{22}^{\prime}$ | $\sqsubseteq A_{2}$ |  | Given |
| $A_{2}$ | $=A_{12} \rightarrow A_{22}$ |  | By Lemma4 (Precision inversion) |
| $A_{12}^{\prime} \sqsubseteq A_{12}$ |  | $\prime \prime$ |  |
| $A_{22}^{\prime} \sqsubseteq A_{22}$ |  | $\prime \prime$ |  |


| $\left(A_{11} \rightarrow A_{21}\right)$ | $\Rightarrow\left(A_{12} \rightarrow A_{22}\right) \hookrightarrow \mathcal{C}$ |  |  |
| ---: | :--- | ---: | :--- |
| $A_{12}$ | $\Rightarrow A_{11} \hookrightarrow \mathcal{C}_{1}$ |  | Given |
| $A_{21}$ | $\Rightarrow A_{22} \hookrightarrow \mathcal{C}_{2}$ |  | By inversion on Coe $\rightarrow$ |
| $\mathcal{C}$ | $=\lambda x . \mathcal{C}_{2}\left[[] \mathcal{C}_{1}[x]\right]$ |  | $\prime \prime$ |
| $x$ | $\preccurlyeq x$ |  |  |
| $A_{12}^{\prime}$ | $\Rightarrow A_{11}^{\prime} \hookrightarrow \mathcal{C}_{1}^{\prime}$ |  | Sy definition of $\preccurlyeq$ |
| $\mathcal{C}_{1}^{\prime}[x]$ | $\preccurlyeq \mathcal{C}_{1}[x]$ |  | By the induction hypothesis |
| $M^{\prime}$ | $\preccurlyeq M$ |  | Given |
| $M^{\prime} \mathcal{C}_{1}^{\prime}[x]$ | $\preccurlyeq \mathcal{C}_{1}[x]$ |  | By definition of $\preccurlyeq$ |
| $A_{21}^{\prime}$ | $\Rightarrow A_{22}^{\prime} \hookrightarrow \mathcal{C}_{2}^{\prime}$ |  | Subderivation |
| $\mathcal{C}_{2}^{\prime}\left[M^{\prime} \mathcal{C}_{1}^{\prime}[x]\right]$ | $\preccurlyeq \mathcal{C}_{2}\left[M \mathcal{C}_{1}[x]\right]$ |  | By the induction hypothesis |
| $\lambda x . \mathcal{C}_{2}^{\prime}\left[M^{\prime} \mathcal{C}_{1}^{\prime}[x]\right]$ | $\preccurlyeq \lambda x . \mathcal{C}_{2}\left[M \mathcal{C}_{1}[x]\right]$ |  | By definition of $\preccurlyeq$ |

- Case

$$
\begin{aligned}
& \begin{array}{cl}
\delta_{1}^{\prime} \in\left\{+{ }_{1}^{?},+{ }_{1}\right\} & +\stackrel{i}{\prime} \Rightarrow \delta_{1}^{\prime} \hookrightarrow \mathcal{C}_{11}^{\prime} \\
A_{11}^{\prime} \Rightarrow A_{12}^{\prime} \hookrightarrow \mathcal{C}_{1}^{\prime} & \delta_{1}^{\prime} \Rightarrow \delta_{2}^{\prime} \hookrightarrow \mathcal{C}_{3}^{\prime} \\
\left(A_{11}^{\prime} \delta_{1}^{\prime} A_{21}^{\prime}\right) \Rightarrow\left(A_{12}^{\prime} \delta_{2}^{\prime} A_{22}^{\prime}\right) & \text { CoeCase1L }
\end{array} \\
& \hookrightarrow \mathcal{C}_{3}^{\prime}\left[\operatorname{case}\left([], \operatorname{inj}_{1} \mathrm{x}_{1} \cdot \mathcal{C}_{11}^{\prime}\left[\operatorname{inj}_{1} \mathcal{C}_{1}^{\prime}\left[\mathrm{x}_{1}\right]\right]\right)\right]
\end{aligned}
$$

| $A_{12}^{\prime} \delta_{2}^{\prime} A_{22}^{\prime} \sqsubseteq A_{2}$ |  | Given |  |
| ---: | :--- | ---: | :--- |
| $A_{2}$ | $=A_{12} \delta_{2} A_{22}$ |  | By Lemma 4 (Precision inversion $)$ |
| $A_{12}^{\prime} \sqsubseteq A_{12}$ | $\prime \prime$ |  |  |
| $A_{22}^{\prime} \sqsubseteq A_{22}$ | $\prime \prime$ |  |  |
| $\delta_{2}^{\prime} \sqsubseteq \delta_{2}$ | $\prime \prime$ |  |  |
| $A_{11}^{\prime} \delta_{1}^{\prime} A_{21}^{\prime} \sqsubseteq A_{1}$ |  | Given |  |
| $A_{1}=A_{11} \delta_{1} A_{21}$ |  | By Lemma 4 (Precision inversion) |  |
| $A_{11}^{\prime} \sqsubseteq A_{11}$ | $\prime \prime$ |  |  |
| $A_{21}^{\prime} \sqsubseteq A_{21}$ | $\prime \prime$ |  |  |
| $\delta_{1}^{\prime} \sqsubseteq \delta_{1}$ | $\prime \prime$ |  |  |

Since $\delta_{1}^{\prime} \in\left\{\mathbf{+}_{1}^{?},+_{1}\right\}$ and $\delta_{1}^{\prime} \sqsubseteq \delta_{1}$, by definition of $\sqsubseteq$ it follows that $\delta_{1} \in\left\{\boldsymbol{+}_{1}^{?},+_{1},+_{1}^{*},+^{?}\right\}$ as well. Consider the case when $\delta_{1} \in\left\{+{ }_{\mathrm{i}}^{\mathrm{i}},+{ }_{1}\right\}$.

$$
\begin{aligned}
& \left(A_{11} \delta_{1} A_{21}\right) \Rightarrow\left(A_{12} \delta_{2} A_{22}\right) \hookrightarrow \mathcal{C} \\
& A_{11} \Rightarrow A_{12} \hookrightarrow \mathcal{C}_{1} \\
& +{ }_{i} \Rightarrow \delta_{1} \hookrightarrow \mathcal{C}_{11} \\
& \delta_{1} \Rightarrow \delta_{2} \hookrightarrow \mathcal{C}_{3} \\
& \mathcal{C}=\mathcal{C}_{3}\left[\operatorname{case}\left([], \operatorname{inj}_{1} x_{1} \cdot \mathcal{C}_{11}\left[\operatorname{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right]\right)\right] \\
& \mathrm{x}_{1} \preccurlyeq \mathrm{x}_{1} \\
& A_{11}^{\prime} \Rightarrow A_{12}^{\prime} \hookrightarrow \mathcal{C}_{1}^{\prime} \\
& \mathcal{C}_{1}^{\prime}\left[\mathrm{x}_{1}\right] \preccurlyeq \mathcal{C}_{1}\left[\mathrm{x}_{1}\right] \\
& \operatorname{inj}_{1} \mathcal{C}_{1}^{\prime}\left[\mathrm{x}_{1}\right] \preccurlyeq \operatorname{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right] \\
& +{ }_{i}^{\prime} \Rightarrow \delta_{1}^{\prime} \hookrightarrow \mathcal{C}_{11}^{\prime} \\
& +{ }_{i} \sqsubseteq+{ }_{i}^{2} \\
& \underbrace{\mathcal{C}_{11}^{\prime}\left[\mathrm{inj}_{1} \mathcal{C}_{1}^{\prime}\left[\mathrm{x}_{1}\right]\right]}_{M_{1}^{\prime}} \preccurlyeq \underbrace{\mathcal{C}_{11}\left[\mathrm{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right]}_{M_{1}} \\
& M^{\prime} \preccurlyeq M \\
& \underbrace{\operatorname{case}\left(M^{\prime}, \operatorname{inj}_{1} x_{1} \cdot M_{1}^{\prime}\right)}_{M_{0}^{\prime}} \preccurlyeq \underbrace{\operatorname{case}\left(M, \operatorname{inj}_{1} x_{1} \cdot M_{1}\right)}_{M_{0}} \\
& \delta_{1}^{\prime} \Rightarrow \delta_{2}^{\prime} \hookrightarrow \mathcal{C}_{3}^{\prime} \\
& \mathcal{C}_{3}^{\prime}\left[\mathrm{M}_{0}^{\prime}\right] \preccurlyeq \mathcal{C}_{3}\left[\mathrm{M}_{0}\right] \\
& \text { Given } \\
& \text { By inversion on CoeCase1L } \\
& \text { " } \\
& \text { " } \\
& \text { " } \\
& \text { By definition of } \preccurlyeq \\
& \text { Subderivation } \\
& \text { By the induction hypothesis } \\
& \text { By definition of } \preccurlyeq \\
& \text { Subderivation } \\
& \text { By definition of } \sqsubseteq \\
& \text { By Lemma } 60 \text { Cast insertion preserves precision } \\
& \text { Given } \\
& \text { By definition of } \preccurlyeq \\
& \text { Subderivation } \\
& \text { By Lemma } 60 \text { Cast insertion preserves precision }
\end{aligned}
$$

Consider the case when $\delta_{1} \in\left\{+{ }_{1}^{*},+^{?}\right\}$.

```
\(\left(A_{11} \delta_{1} A_{21}\right) \Rightarrow\left(A_{12} \delta_{2} A_{22}\right) \hookrightarrow \mathcal{C} \quad\) Given
        \(A_{11} \Rightarrow A_{12} \hookrightarrow \mathcal{C}_{1} \quad\) By inversion on CoeCase2
        \(A_{21} \Rightarrow A_{22} \hookrightarrow \mathcal{C}_{2}\)
        \(\delta_{1} \Rightarrow \delta_{2} \hookrightarrow \mathcal{C}_{3} \quad\) "
        \(+{ }_{i}^{i} \Rightarrow \delta_{1} \hookrightarrow \mathcal{C}_{11} \quad\) "
        \(+\frac{?}{2} \Rightarrow \delta_{1} \hookrightarrow \mathcal{C}_{21} \quad\) "
            \(\mathcal{C}=\mathcal{C}_{3}\left[\operatorname{case}\left([], \operatorname{inj}_{1} \mathrm{x}_{1} \cdot \mathcal{C}_{11}\left[\mathrm{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right], \operatorname{inj}_{2} \mathrm{x}_{2} \cdot \mathcal{C}_{21}\left[\mathrm{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]\right]\right)\right]\)
            \(\chi_{1} \preccurlyeq \chi_{1}\)
        \(A_{11}^{\prime} \Rightarrow A_{12}^{\prime} \hookrightarrow \mathcal{C}_{1}^{\prime}\)
        \(\mathcal{C}_{1}^{\prime}\left[\mathrm{x}_{1}\right] \preccurlyeq \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\)
    \(\operatorname{inj}_{1} \mathcal{C}_{1}^{\prime}\left[\mathrm{x}_{1}\right] \preccurlyeq \operatorname{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\)
            \(+{ }_{i}^{\prime} \Rightarrow \delta_{1}^{\prime} \hookrightarrow \mathcal{C}_{11}^{\prime}\)
            \(+{ }_{1}^{?} \sqsubseteq+{ }_{i}^{?}\)
\(\underbrace{\mathcal{C}_{11}^{\prime}\left[\mathrm{inj}_{1} \mathcal{C}_{1}^{\prime}\left[\mathrm{x}_{1}\right]\right]}_{M_{1}^{\prime}} \preccurlyeq \underbrace{\mathcal{C}_{11}\left[\mathrm{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right]}_{M_{1}}\)
            \(M^{\prime} \preccurlyeq M\)
\(\underbrace{\operatorname{case}\left(M^{\prime}, \operatorname{inj}_{1} x_{1} \cdot M_{1}^{\prime}\right)}_{M_{0}^{\prime}} \preccurlyeq \underbrace{\operatorname{case}\left(M, \operatorname{inj}_{1} x_{1} \cdot M_{1}, \operatorname{inj}_{2} x_{2} \cdot \mathcal{C}_{21}\left[\operatorname{inj}_{2} \mathcal{C}_{2}\left[x_{2}\right]\right]\right)}_{M_{0}} \quad\) By definition of \(\preccurlyeq\)
            \(\delta_{1}^{\prime} \Rightarrow \delta_{2}^{\prime} \hookrightarrow \mathcal{C}_{3}^{\prime}\)
            \(\mathcal{C}_{3}^{\prime}\left[M_{0}^{\prime}\right] \preccurlyeq \mathcal{C}_{3}\left[M_{0}\right]\)
        Subderivation
                            By Lemma 60 Cast insertion preserves precision)
```

- Case CoeCase1R Symmetric to the CoeCase1L case.


Since $\delta_{1}^{\prime} \in\left\{\boldsymbol{+}^{?},+_{1}^{*},+_{2}^{*}, \boldsymbol{+}\right\}$ and $\delta_{1}^{\prime} \sqsubseteq \delta_{1}$, by definition of $\sqsubseteq$ it follows that $\delta_{1} \in\left\{\boldsymbol{+}^{?},+_{1}^{*},+_{2}^{*},+\right\}$ as well.

$$
\begin{array}{rlrl}
\left(A_{11} \delta_{1} A_{21}\right) & \Rightarrow\left(A_{12} \delta_{2} A_{22}\right) \hookrightarrow \mathcal{C} & & \text { Given } \\
A_{11} & \Rightarrow A_{12} \hookrightarrow \mathcal{C}_{1} & & \text { By inversion on CoeCase2 } \\
A_{21} & \Rightarrow A_{22} \hookrightarrow \mathcal{C}_{2} & & \prime \prime \\
\delta_{1} & \Rightarrow \delta_{2} \hookrightarrow \mathcal{C}_{3} & & \prime \prime \\
+\stackrel{2}{1} & \Rightarrow \delta_{1} \hookrightarrow \mathcal{C}_{11} & & \prime \prime \\
+\stackrel{2}{2} & \Rightarrow \delta_{1} \hookrightarrow \mathcal{C}_{21} & & \prime \prime \\
\mathcal{C}=\mathcal{C}_{3}\left[\operatorname{case}\left([], \text { inj }_{1} x_{1} \cdot \mathcal{C}_{11}\left[\mathrm{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right], \text { inj }_{2} \mathrm{x}_{2} \cdot \mathcal{C}_{21}\left[\mathrm{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]\right]\right)\right]
\end{array}
$$

$M^{\prime} \preccurlyeq M$
$\underbrace{\operatorname{case}\left(M^{\prime}, \operatorname{inj}_{1} x_{1} \cdot M_{1}^{\prime}, \operatorname{inj}_{2} x_{2} \cdot M_{2}^{\prime}\right)}_{M_{0}^{\prime}} \preccurlyeq \underbrace{\operatorname{case}\left(M, \operatorname{inj}_{1} x_{1} \cdot M_{1}, \operatorname{inj}_{2} x_{2} \cdot M_{2}\right)}_{M_{0}} \quad$ By definition of $\preccurlyeq$

$$
\begin{aligned}
\delta_{1}^{\prime} \Rightarrow \delta_{2}^{\prime} \hookrightarrow \mathcal{C}_{3}^{\prime} & \text { Subderivation } \\
\mathcal{C}_{3}^{\prime}\left[M_{0}^{\prime}\right] \preccurlyeq \mathcal{C}_{3}\left[M_{0}\right] & \text { By Lemma } 60 \text { Cast insertion preserves precision) }
\end{aligned}
$$

Theorem 11 (Translation preserves precision).
Suppose $\Gamma^{\prime} \sqsubseteq \Gamma$ and $e^{\prime} \sqsubseteq e$.

1. If $\Gamma^{\prime} \vdash e^{\prime} \Leftarrow A^{\prime}$ and $\Gamma \vdash e \Leftarrow A$ and $A^{\prime} \sqsubseteq A$ then
$\Gamma^{\prime} \vdash e^{\prime}: A^{\prime} \hookrightarrow M^{\prime}$ and $\Gamma \vdash e: A \hookrightarrow M$ where $M^{\prime} \preccurlyeq M$.
2. If $\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A^{\prime}$ and $\Gamma \vdash e \Rightarrow A$ then $\Gamma^{\prime} \vdash e^{\prime}: A^{\prime} \hookrightarrow M^{\prime}$
and $\Gamma \vdash e: A \hookrightarrow M$ where $A^{\prime} \sqsubseteq A$ and $M^{\prime} \preccurlyeq M$.
Proof. By induction on the structure of the derivation of $\Gamma^{\prime} \vdash e^{\prime} \Leftarrow A^{\prime}$ (part 1) or $\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A^{\prime}$ (part 2).

- Case $\frac{\Gamma^{\prime}(x)=A^{\prime}}{\Gamma^{\prime} \vdash x \Rightarrow A^{\prime}}$ SynVar

|  | $x \sqsubseteq e$ | Given |
| :---: | :---: | :---: |
|  | $e=x$ | From definition of $\sqsubseteq$ |
|  | $\Gamma \vdash x \Leftarrow A$ | Given |
|  | $\Gamma(x)=A$ | By inversion on SynVar |
|  | $\Gamma^{\prime}(x)=A^{\prime}$ | Premise |
| 5 | $\Gamma^{\prime} \vdash x: A^{\prime} \hookrightarrow x$ | By ruleSTVar |
| \% | $\Gamma \vdash x: A \hookrightarrow x$ | By ruleSTVar |
| \% | $\chi \preccurlyeq x$ | By definition of $\preccurlyeq$ |

- Case $\frac{\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A_{0}^{\prime} \quad A_{0}^{\prime} \leadsto A^{\prime}}{\Gamma^{\prime} \vdash e^{\prime} \Leftarrow A^{\prime}}$ ChkCSub

By inversion on $\Gamma \vdash e \Leftarrow A$, ruleChkCSub was applied.

| $\Gamma \vdash e \Rightarrow A_{0}$ | By inversion on ChkCSub |
| :---: | :---: |
| $A_{0} \sim A$ |  |
| $\Gamma^{\prime} \vdash e^{\prime} \Rightarrow A_{0}^{\prime}$ | Subderivation |
| $\Gamma^{\prime} \vdash e^{\prime}: A_{0}^{\prime} \hookrightarrow M_{0}^{\prime}$ | By the induction hypothesis |
| $\Gamma \vdash e: A_{0} \hookrightarrow M_{0}$ |  |
| $A_{0}^{\prime} \sqsubseteq A_{0}$ | " |
| $M_{0}^{\prime} \preccurlyeq M_{0}$ | " |
| $A_{0}^{\prime} \sim A^{\prime}$ | Subderivation |
| $A_{0}^{\prime} \simeq A^{\prime}$ | By Lemma 17 Directed consistency obeys Structural Equivalence |
| $A_{0} \simeq A$ | By Lemma 17 Directed consistency obeys Structural Equivalence) |
| $A_{0}^{\prime} \Rightarrow A^{\prime} \hookrightarrow \mathcal{C}^{\prime}$ | By Theorem 17 |
| $A_{0} \Rightarrow A \hookrightarrow C$ | By Theorem 17 |
| $A^{\prime} \sqsubseteq A$ | Given |
| $\mathcal{C}^{\prime}\left[M_{0}^{\prime}\right] \preccurlyeq \mathcal{C}\left[M_{0}\right]$ | By Lemma 61 (Coercion preserves precision) |
| $\Gamma^{\prime} \vdash \mathrm{e}^{\prime}: A^{\prime} \hookrightarrow \mathcal{C}^{\prime}\left[M_{0}^{\prime}\right]$ | By rule STCSub |
| $\Gamma \vdash e: A \hookrightarrow \mathcal{C}\left[\mathrm{M}_{0}\right]$ | By ruleSTCSub |

- Case
$\frac{\Gamma^{\prime} \vdash e_{0}^{\prime} \Leftarrow A^{\prime}}{\Gamma^{\prime} \vdash\left(e_{0}^{\prime}:: A^{\prime}\right) \Rightarrow A^{\prime}}$ SynAnno
$\left(e_{0}^{\prime}:: A^{\prime}\right) \sqsubseteq e$

$$
e=\left(e_{0}:: A\right)
$$

$$
e_{0}^{\prime} \sqsubseteq e_{0}
$$

$$
\Gamma \vdash\left(e_{0}:: A\right) \Rightarrow A
$$

$\Gamma^{\prime} \sqsubseteq \Gamma$
$\Gamma^{\prime} \vdash e_{0}^{\prime} \Leftarrow A^{\prime}$
$\Gamma \vdash e_{0}: A \hookrightarrow M$
$M^{\prime} \preccurlyeq M$
$\Gamma^{\prime} \vdash\left(e_{0}^{\prime}:: A^{\prime}\right): A^{\prime} \hookrightarrow M^{\prime}$
Given
From definition of $\sqsubseteq$
"

$$
A^{\prime} \sqsubseteq A
$$

"
Given

$$
\Gamma \vdash e_{0} \Leftarrow A
$$

$\Gamma^{\prime} \vdash e_{0}^{\prime}: A^{\prime} \hookrightarrow M^{\prime}$
$\Gamma \vdash\left(e_{0}:: A\right): A \hookrightarrow M$
By inversion on rule SynAnno
Given
Subderivation
By the induction hypothesis
"
"
By rule STAnno
By ruleSTAnno

- Case
$\overline{\Gamma^{\prime} \vdash() \Leftarrow \text { Unit ChkUnitIntro }}$

| () $\sqsubseteq e$ | Given |
| :---: | :---: |
| $e=()$ | From definition of $\sqsubseteq$ |
| $\Gamma \vdash() \Leftarrow A$ | Given |
| A $=$ Unit | By inversion on ChkUnitIntro |
| $\Gamma^{\prime} \vdash()$ : Unit $\hookrightarrow()$ | By ruleSTUnitIntro |
| $\Gamma \vdash():$ Unit $\hookrightarrow()$ | By ruleSTUnitIntro |
| () $\preccurlyeq()$ | By definition of $\preccurlyeq$ |

- Case
$\frac{\Gamma^{\prime} \vdash e_{0}^{\prime} \Leftarrow A_{i}^{\prime} \quad+_{i}^{?} \leq \delta^{\prime}}{\Gamma^{\prime} \vdash\left(\mathrm{inj}_{\mathrm{i}} e_{0}^{\prime}\right) \Leftarrow\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right)}$ ChkSumIntro

| $\begin{aligned} \operatorname{inj}_{i} e_{0}^{\prime} & \sqsubseteq e \\ e & =\operatorname{inj}_{i} e_{0} \\ e_{0}^{\prime} & \sqsubseteq e_{0} \end{aligned}$ |  | Given <br> From definition of " |
| :---: | :---: | :---: |
| $\Gamma \vdash\left(\operatorname{inj}_{i} e_{0}\right) \Leftarrow A$ |  | Given |
| $\Gamma \vdash \mathrm{e}_{0} \Leftarrow \mathrm{~A}_{\mathrm{i}}$ |  | By inversion on ChkSumIntro |
| $A=A_{1} \delta A_{2}$ |  |  |
| $+_{i}^{?} \leq \delta$ |  | " |
| $A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \sqsubseteq A_{1} \delta A_{2}$ |  | Given |
| $A_{1}^{\prime} \sqsubseteq A_{1}$ |  | From definition of $\sqsubseteq$ |
| $A_{2}^{\prime} \sqsubseteq A_{2}$ |  |  |
| $+_{i}^{?} \sqsubseteq+{ }_{i}^{\text {? }}$ |  | By definition of $\sqsubseteq$ |
| $A_{1}^{\prime}+{ }_{i}^{?} A_{2}^{\prime} \sqsubseteq A_{1}+\frac{?}{i} A_{2}$ |  | By definition of $\sqsubseteq$ |
| $\Gamma^{\prime} \vdash e_{0}^{\prime} \Leftarrow A_{i}^{\prime}$ |  | Subderivation |
| $\Gamma^{\prime} \vdash e_{0}^{\prime}: A_{i}^{\prime} \hookrightarrow M_{0}^{\prime}$ |  | By the induction hypothesis |
| $\Gamma \vdash e_{0}: A_{i} \hookrightarrow M_{0}$ |  | " |
| $M_{0}^{\prime} \preccurlyeq M_{0}$ |  | " |
| $\Gamma^{\prime} \vdash\left(\operatorname{inj}_{i} e_{0}^{\prime}\right):\left(A_{1}^{\prime}+\frac{?}{i} A_{2}^{\prime}\right) \hookrightarrow\left(\operatorname{inj}_{i} M_{0}^{\prime}\right)$ |  | By rule STSumIntro |
| $\Gamma \vdash\left(\operatorname{inj}_{i} e_{0}\right):\left(A_{1}+\frac{?}{i} A_{2}\right) \hookrightarrow\left(\operatorname{inj}_{i} M_{0}\right)$ |  | By ruleSTSumintro |
| $\operatorname{inj}_{i} M_{0}^{\prime} \preccurlyeq \operatorname{inj}_{i} M_{0}$ |  | By definition of $\preccurlyeq$ |
| $A_{1}^{\prime} \leq A_{1}^{\prime}$ | By Lemma 2 Reflexivit | of subtyping, |
| $A_{1} \leq A_{1}$ | By Lemma 2 Reflexivit | of subtyping |
| $A_{2}^{\prime} \leq A_{2}^{\prime}$ | By Lemma 2 Reflexivit | of subtyping |
| $A_{2} \leq A_{2}$ | By Lemma2 Reflexivit | of subtyping) |
| $A_{1}^{\prime}+{ }_{i}^{\prime} A_{2}^{\prime} \leq A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}$ | By definition of $\leq$ |  |
| $A_{1}+{ }_{i}^{\prime} A_{2} \leq A_{1} \delta A_{2}$ | By definition of $\leq$ |  |
| $A_{1}^{\prime}+{ }_{i}^{?} A_{2}^{\prime} \sim A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}$ | By Lemma 8 Subtyping | obeys directed consistency) |
| $A_{1}+{ }_{i}^{?} A_{2} \sim A_{1} \delta A_{2}$ | By Lemma 8 Subtyping | obeys directed consistency) |

$A_{1}+A_{2} \leadsto A_{1} \delta A_{2} \quad$ By Lemma Subtyping obeys directed consistency)

$$
\begin{aligned}
A_{1}^{\prime}+\frac{?}{i} A_{2}^{\prime} & \simeq A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \\
A_{1}+\frac{2}{i} A_{2} & \simeq A_{1} \delta A_{2} \\
A_{1}^{\prime}+\frac{1}{i} A_{2}^{\prime} & \Rightarrow A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime} \hookrightarrow \mathcal{C}^{\prime} \\
A_{1}+{ }_{i}^{\prime} A_{2} & \Rightarrow A_{1} \delta A_{2} \hookrightarrow \mathcal{C} \\
\Gamma^{\prime} & \vdash\left(\text { inj }_{i} e_{0}^{\prime}\right):\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right) \hookrightarrow \mathcal{C}^{\prime}\left[\text { inj }_{i} M_{0}^{\prime}\right] \\
\Gamma & \vdash\left(\operatorname{inj}_{i} e_{0}\right):\left(A_{1} \delta A_{2}\right) \hookrightarrow \mathcal{C}\left[\mathrm{inj}_{i} M_{0}\right] \\
\mathcal{C}^{\prime}\left[\mathrm{inj}_{i} M_{0}^{\prime}\right] & \preccurlyeq \mathcal{C}\left[\mathrm{inj}_{i} M_{0}\right]
\end{aligned}
$$

By Lemma 17 Directed consistency obeys Structural Equivalence
By Lemma 17 Directed consistency obeys Structural Equivalence
By Theorem 17
By Theorem 17
By rule STCSub
By rule STCSub
By Lemma 61 Coercion preserves precision

- Case $\Gamma^{\prime} \vdash e_{0}^{\prime} \Rightarrow\left(A_{1}^{\prime} \delta^{\prime} A_{2}^{\prime}\right)$

$$
\frac{\delta^{\prime} \Rightarrow+_{i}^{*}}{\Gamma^{\prime} \vdash \operatorname{case}\left(e_{0}^{\prime}, \operatorname{inj}_{i} x . e_{i}^{\prime}\right) \Leftarrow A^{\prime}} \text { ChkSumElim1 }
$$



- Case ChkSumElim2 Similar to the ChkSumElim1 case, hence omitted.
- Case

$$
\frac{\Gamma^{\prime}, x: A_{1}^{\prime} \vdash e_{0}^{\prime} \Leftarrow A_{2}^{\prime}}{\Gamma^{\prime} \vdash\left(\lambda x \cdot e_{0}^{\prime}\right) \Leftarrow\left(A_{1}^{\prime} \rightarrow A_{2}^{\prime}\right)} \text { Chk } \rightarrow \text { Intro }
$$



## D. 7 Static programs don't go wrong

We write $\Gamma{ }_{V}$ for $\Gamma$ restricted to the set of variables $V$.
Theorem 18 (Static programs don't go wrong).
If $\Gamma \vdash e \Leftarrow A$ by a static derivation then $\left.\Gamma\right|_{F V(e)} \vdash e: A \hookrightarrow M$ and, for all $M^{\prime}$ such that $M \mapsto^{*} M^{\prime}$, it is the case that $M^{\prime}$ free.
Proof. Apply Theorem 19 and Theorem 10 to show M free.
The result follows by induction on the number of steps in $M \mapsto^{*} M^{\prime}$, using Theorem 8

## D.7.1 Static derivations

Definition 2. We say that a derivation of $\Gamma \vdash e \Leftarrow A$ or $\Gamma \vdash e \Rightarrow A$ is a static derivation if, for all subderivations deriving checking or synthesis judgments, the types checked or synthesized are static.

Note. If a derivation is static, then all of its subderivations must be static.
Lemma 62 (Context thinning).
If $\mathrm{y} \notin F V(\mathrm{e})$ then:

1. If $\Gamma, y: A^{\prime} \vdash e \Leftarrow A$ then $\Gamma \vdash e \Leftarrow A$.
2. If $\Gamma, y: A^{\prime} \vdash e \Rightarrow A$ then $\Gamma \vdash e \Rightarrow A$.

Proof. By induction on the structure of the given derivation.

- Case $\frac{\left(\Gamma, y: A^{\prime}\right)(x)=A}{\Gamma, y: A^{\prime} \vdash x \Rightarrow A}$ SynVar

$$
y \neq x \quad \text { Since } y \notin F V(x)
$$

$\left(\Gamma, y: A^{\prime}\right)(x)=A \quad$ Premise
$\Gamma(x)=A \quad$ By definition
$\Gamma \vdash x \Rightarrow A \quad$ By rule SynVar

- CaseChkCSub Use the induction hypothesis and apply rule ChkCSub
- Case SynAnno Use the induction hypothesis, and apply rule SynAnno
- Case ChkUnitIntro Apply rule ChkUnitIntro
- Case ChkSumIntro Use the induction hypothesis, the definition of $F V(-)$, and apply rule ChkSumlntro
- CaseChkSumElim1 Use the induction hypothesis, the definition of $F V(-)$, and apply rule ChkSumElim1
- Case ChkSumElim2 Use the induction hypothesis, the definition of $F V(-)$, and apply rule ChkSumElim2
- CaseChk $\rightarrow$ Intro Use the induction hypothesis, the definition of $F V(-)$, and apply rule Chk $\rightarrow$ Intro
- Case Syn $\rightarrow$ Elim Use the induction hypothesis, the definition of $F V(-)$, and apply rule Syn $\rightarrow$ Elim

Corollary 63 (Context support).

1. If $\Gamma \vdash \mathrm{e} \Leftarrow A$ then $\left.\Gamma\right|_{F V(e)} \vdash \mathrm{e} \Leftarrow A$.
2. If $\Gamma \vdash e \Rightarrow A$ then $\left.\Gamma\right|_{F V(e)} \vdash \mathrm{e} \Rightarrow A$.

Proof. By induction on $|\operatorname{dom}(\Gamma) \backslash F V(e)|$.
If dom $(\Gamma)=F V(e)$, then $\Gamma=\left.\Gamma\right|_{F V(e)}$ so we already have the result.
Otherwise, use the induction hypothesis, and apply Lemma 62 (Context thinning).
Theorem 19 (Static subformula).

1. If $\Gamma \vdash e \Leftarrow A$ by a static derivation then $\Gamma^{\mathrm{S}} \vdash \mathrm{e}^{\mathrm{S}} \Leftarrow A^{\mathrm{S}}$ where $\Gamma^{\mathrm{S}}=\left.\Gamma\right|_{F V(e)}, e^{\mathrm{s}}=e$, and $A^{\mathrm{S}}=A$.
2. If $\Gamma \vdash e \Rightarrow A$ by a static derivation then $\Gamma^{S} \vdash e^{S} \Rightarrow A^{S}$ where $\Gamma^{S}=\left.\Gamma\right|_{F V(e)}, e^{S}=e$, and $A^{S}=A$.

Proof. By induction on the height of the given derivation.
Since $\Gamma \vdash e \Leftarrow A$ and $\Gamma \vdash e \Rightarrow A$ by static derivations, all occurrences of types in checking and synthesizing positions are static, including $A$. Therefore, $A^{S}=A$ already holds.

Applying Corollary 63 individually to $\Gamma \vdash e \Leftarrow A^{\mathrm{S}}$ and $\Gamma \vdash e \Rightarrow A^{\mathrm{S}}$ produces the derivations $\left.\Gamma\right|_{F V(e)} \vdash e \Leftarrow A^{\mathrm{S}}$ and $\left.\Gamma\right|_{F V(e)} \vdash e \Rightarrow A^{\mathrm{S}}$ respectively.

Note that $\left.\Gamma\right|_{F V(e)} \vdash e \Leftarrow A^{\mathrm{S}}$ and $\left.\Gamma\right|_{F V(e)} \vdash e \Rightarrow A^{\mathrm{S}}$ are also static derivations.
All cases are then immediate by the induction hypothesis and applying the relevant rule.

## D.7.2 Static translations are free of casts and match failures

Notation. We write $M$ free to denote that the target term $M$ contains no casts or matchfails.
Lemma 64 (Subsums don't need casts).

1. If $+{ }_{i}^{?} \leq \delta^{S}$ and $+_{i}^{?} \Rightarrow \delta^{S} \hookrightarrow \mathcal{C}$ then $\mathcal{C}=[]$.
2. If $+_{i} \leq+_{i}^{*}$ and $+_{i} \Rightarrow+_{i}^{*} \hookrightarrow \mathcal{C}$ then $\mathcal{C}=[]$.

Proof.

1. From definition of subtyping, it is either the case that $\delta^{S}=+_{i}$ or $\delta^{S}=+$. In both cases, by definition of subtyping, $\left|+_{i}^{?}\right|=+_{i} \leq\left|\delta^{S}\right|$. By inversion on $+_{i}^{?} \Rightarrow \delta^{S} \hookrightarrow \mathcal{C}$, either rule CoeSub or CoeCast was applied. If rule CoeCast was applied then $+_{i} \not \leq\left|\delta^{S}\right|$, a contradiction. If rule CoeSub was applied, then indeed $\mathcal{C}=[]$.
2. By definition of subtyping, $\left|+_{i}\right|=+_{i} \leq+_{i}=\left|+_{i}^{*}\right|$. By inversion on $+_{i} \Rightarrow+_{i}^{*} \hookrightarrow \mathcal{C}$, either rule CoeSub or CoeCast was applied. If rule CoeCast was applied then $+_{i} \not \subset+_{i}$, a contradiction. If rule CoeSub was applied, then indeed $\mathcal{C}=[]$.

Lemma 65 (Gradual sums in static don't need casts).

1. If $A_{11}^{\mathrm{S}}++_{i}^{?} A_{21}^{\mathrm{S}} \leq A_{12}^{\mathrm{S}} \delta^{\mathrm{S}} A_{22}^{\mathrm{S}}$ and $A_{11}^{\mathrm{S}}++_{i}^{?} A_{21}^{\mathrm{S}} \Rightarrow A_{12}^{\mathrm{S}} \delta^{\mathrm{S}} A_{22}^{\mathrm{S}} \hookrightarrow \mathcal{C}$ and $M$ free then $\mathcal{C}[M]$ free.
2. If $A_{11}^{\mathrm{S}}++_{i} A_{21}^{\mathrm{S}} \leq A_{12}^{\mathrm{S}}+_{i}^{*} A_{22}^{\mathrm{S}}$ and $A_{11}^{\mathrm{S}}+_{i} A_{21}^{\mathrm{S}} \Rightarrow A_{12}^{\mathrm{S}}+_{i}^{*} A_{22}^{\mathrm{S}} \hookrightarrow \mathcal{C}$ and $M$ free then $\mathcal{C}[M]$ free.

## Proof.

1. $A_{11}^{\mathrm{S}}+{ }_{i}^{?} A_{21}^{\mathrm{S}} \Rightarrow A_{12}^{\mathrm{S}} \delta^{\mathrm{S}} A_{22}^{\mathrm{S}} \hookrightarrow \mathcal{C}$

$$
A_{\mathrm{ii}}^{\mathrm{s}} \Rightarrow A_{\mathrm{i} 2}^{\mathrm{s}} \hookrightarrow \mathcal{C}_{\mathrm{i}}
$$

$$
+\frac{\delta^{\mathrm{s}}}{} \rightarrow \mathcal{C}_{3}
$$

$$
\mathcal{C}=\mathcal{C}_{3}\left[\operatorname{case}\left([], \operatorname{inj}_{i} x_{i} \cdot \operatorname{inj}_{i} \mathcal{C}_{i}\left[x_{i}\right]\right)\right]
$$

$A_{11}^{S}+{ }_{i}^{?} A_{21}^{S} \leq A_{12}^{S} \delta^{S} A_{22}^{S}$
$A_{i 1}^{\mathrm{s}} \leq A_{\mathrm{i} 2}^{\mathrm{s}}$
$+{ }_{i}^{?} \leq \delta^{s}$
$\mathcal{C}_{3}=[]$
$M$ free
$x_{i}$ free
$\mathcal{C}_{i}\left[x_{i}\right]$ free
inj $_{i} \mathcal{C}_{i}\left[x_{i}\right]$ free
$\operatorname{case}\left(M, \operatorname{inj}_{i} x_{i} . \operatorname{inj}_{i} \mathcal{C}_{i}\left[x_{i}\right]\right)$ free
$\mathcal{C}_{3}\left[\operatorname{case}\left(M, \operatorname{inj}_{i} x_{i} . \operatorname{inj}_{i} \mathcal{C}_{i}\left[x_{i}\right]\right)\right]$ free
2. Similar to the proof for the previous statement, hence omitted.

Lemma 66 (Static sums don't need casts).
If $\delta_{0}^{\mathrm{S}} \leq \delta^{\mathrm{S}}$ and $\delta_{0}^{\mathrm{S}} \Rightarrow \delta^{\mathrm{S}} \hookrightarrow \mathcal{C}$ then $\mathcal{C}=[]$.

Proof. By definition of sum translation, $\left|\delta_{0}^{\mathrm{S}}\right|=\delta_{0}^{\mathrm{S}}$ and $\left|\delta^{\mathrm{S}}\right|=\delta^{\mathrm{S}}$. Therefore, $\left|\delta_{0}^{\mathrm{S}}\right| \leq\left|\delta^{\mathrm{S}}\right|$. By inversion on $\delta_{0}^{\mathrm{S}} \Rightarrow \delta^{\mathrm{S}} \hookrightarrow \mathcal{C}$, either rule CoeSub or CoeCast was applied. If rule CoeCast was applied then $\left|\delta_{0}^{\mathrm{S}}\right| \not \leq\left|\delta^{\mathrm{S}}\right|$, a contradiction. If rule CoeSub was applied, then indeed $\mathcal{C}=[]$.

Lemma 67 (Static subtypes don't need casts).
If $A_{0}^{\mathrm{S}} \leq A^{\mathrm{S}}$ and $A_{0}^{\mathrm{S}} \Rightarrow A^{\mathrm{S}} \hookrightarrow \mathcal{C}$ then $\mathcal{C}[M]$ free for any $M$ free.

Proof. By induction on the structure of the derivation of $A_{0}^{S} \Rightarrow A^{S} \hookrightarrow \mathcal{C}$.

- Case CoeUnit Immediate by the definition of $\mathcal{C}=[]$.
- Case $\frac{A_{12}^{\mathrm{S}} \Rightarrow A_{11}^{\mathrm{S}} \hookrightarrow \mathcal{C}_{1} \quad A_{21}^{\mathrm{S}} \Rightarrow A_{22}^{\mathrm{S}} \hookrightarrow \mathcal{C}_{2}}{\left(\mathrm{~A}_{11}^{\mathrm{S}} \rightarrow A_{21}^{\mathrm{S}}\right) \Rightarrow\left(A_{12}^{\mathrm{S}} \rightarrow A_{22}^{\mathrm{S}}\right) \hookrightarrow \lambda x . \mathcal{C}_{2}\left[[] \mathcal{C}_{1}[\mathrm{x}]\right]}$ Coe $\rightarrow$

$$
A_{11}^{\mathrm{S}} \rightarrow A_{21}^{\mathrm{S}} \leq A_{12}^{\mathrm{S}} \rightarrow A_{22}^{\mathrm{S}} \quad \text { Given }
$$

$A_{12}^{\mathrm{S}} \leq A_{11}^{\mathrm{S}}$
$A_{21}^{\mathrm{S}} \leq A_{22}^{\mathrm{S}}$

$A_{12}^{\mathrm{s}} \Rightarrow$| free |
| :--- |
| $A_{11}^{\mathrm{s}} \hookrightarrow \mathcal{C}_{1}$ |
| $\mathcal{C}_{1}[\mathrm{x}]$ free |

$M$ free
$M \mathcal{C}_{1}[x]$ free $\quad$ By the definition of free
$A_{21}^{\mathrm{S}} \Rightarrow A_{22}^{\mathrm{S}} \hookrightarrow \mathcal{C}_{2} \quad$ Subderivation
$\mathcal{C}_{2}\left[M \mathcal{C}_{1}[x]\right]$ free $\quad$ By the induction hypothesis
$\lambda x . \mathcal{C}_{2}\left[M \mathcal{C}_{1}[x]\right]$ free By the definition of free

- Case

$$
\left.\begin{array}{rl}
\frac{A_{11}^{S}}{\left(A_{11}^{\mathrm{S}}+1 A_{21}^{\mathrm{S}}\right)} & \left.\Rightarrow A_{12}^{\mathrm{S}} \hookrightarrow \mathcal{C}_{1} \quad+_{1} \Rightarrow \AA_{12}^{\mathrm{S}} \delta^{\mathrm{S}} A_{22}^{\mathrm{S}}\right) \\
& \hookrightarrow \mathcal{C}_{3}\left[\operatorname{case}\left([], \mathrm{inj}_{1} x_{1} \cdot \operatorname{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right]\right)\right]
\end{array} \text { CoeCase1L }\right]
$$

$$
\begin{aligned}
& A_{11}^{S}+{ }_{1} A_{21}^{S} \leq A_{12}^{S} \delta^{S} A_{22}^{S} \\
& A_{11}^{\mathrm{s}} \leq A_{12}^{\mathrm{S}} \\
& A_{21}^{\mathrm{S}} \leq A_{22}^{\mathrm{S}} \\
& +1 \leq \delta^{S} \\
& +_{1} \Rightarrow \delta^{\mathrm{S}} \hookrightarrow \mathcal{C}_{3} \\
& \mathcal{C}_{3}=[] \\
& \mathrm{x}_{1} \text { free } \\
& A_{11}^{\mathrm{S}} \Rightarrow A_{12}^{\mathrm{S}} \hookrightarrow \mathcal{C}_{1} \\
& \mathcal{C}_{1}\left[\mathrm{x}_{1}\right] \text { free } \\
& \operatorname{inj}_{1} \mathcal{C}_{1}\left[\mathrm{x}_{1}\right] \text { free } \\
& M \text { free } \\
& \operatorname{case}\left(M, \operatorname{inj}_{1} x_{1} \cdot \mathcal{C}_{1}\left[x_{1}\right]\right) \text { free } \\
& \mathcal{C}_{3}\left[\operatorname{case}\left(M, \operatorname{inj}_{1} x_{1} \cdot \mathcal{C}_{1}\left[x_{1}\right]\right)\right] \text { free } \\
& \text { Given } \\
& \text { By Lemma } 1 \text { Subtyping inversion) } \\
& \text { " } \\
& \text { " } \\
& \text { Subderivation } \\
& \text { By Lemma 66 Static sums don't need casts } \\
& \text { By the definition of free } \\
& \text { Subderivation } \\
& \text { By the induction hypothesis } \\
& \text { By the definition of free } \\
& \text { Suppose } \\
& \text { By the definition of free } \\
& \text { By the definition of } \mathcal{C}_{3}
\end{aligned}
$$

- Case CoeCase1R Symmetric to the CoeCase1L case, hence omitted.
- Case
$A_{11}^{\mathrm{S}}+A_{21}^{\mathrm{S}} \leq A_{12}^{\mathrm{S}} \delta^{\mathrm{S}} A_{22}^{\mathrm{S}} \quad$ Given $A_{11}^{\mathrm{s}} \leq A_{12}^{\mathrm{s}} \quad$ By Lemma 1 Subtyping inversion $A_{21}^{\mathrm{S}} \leq A_{22}^{\mathrm{S}}$
$+\leq \delta^{\mathrm{s}} \quad "$
$+\stackrel{?}{i} \Rightarrow+\hookrightarrow \mathcal{C}_{1}^{\prime} \quad$ Subderivation
$+{ }_{2}^{\prime} \Rightarrow+\hookrightarrow \mathcal{C}_{2}^{\prime} \quad$ Subderivation
$+\Rightarrow \delta^{\mathrm{S}} \hookrightarrow \mathcal{C}_{3} \quad$ Subderivation
$\mathcal{C}_{1}^{\prime}=[] \quad$ By inversion on CoeSub
$\mathcal{C}_{2}^{\prime}=[] \quad$ By inversion on CoeSub
$\mathcal{C}_{3}=[] \quad$ By Lemma 66 Static sums don't need casts
$\begin{array}{ll} & x_{1} \text { free } \\ A_{11}^{s} \Rightarrow A_{12}^{\mathrm{S}} \hookrightarrow \mathcal{C}_{1} & \\ \mathcal{C}_{1} & \text { By the definition of free } \\ \text { Subderivation }\end{array}$
$\mathcal{C}_{1}\left[x_{1}\right]$ free $\quad$ By the induction hypothesis
$\operatorname{inj}_{1} \mathcal{C}_{1}\left[x_{1}\right]$ free $\quad$ By the definition of free
$\mathcal{C}_{1}^{\prime}\left[\operatorname{inj}_{1} \mathcal{C}_{1}\left[x_{1}\right]\right]$ free $\quad$ By the definition of $\mathcal{C}_{1}^{\prime}$

|  | $x_{2}$ free |
| :--- | :--- |
| $A_{21}^{S} \Rightarrow A_{22}^{\mathrm{S}} \hookrightarrow \mathcal{C}_{2}$ | By the definition of free |
| Subderivation |  |

$\mathcal{C}_{2}\left[\mathrm{x}_{2}\right]$ free $\quad$ By the induction hypothesis
$\operatorname{inj}_{2} \mathcal{C}_{2}\left[x_{2}\right]$ free $\quad$ By the definition of free
$\mathcal{C}_{2}^{\prime}\left[\mathrm{inj}_{2} \mathcal{C}_{2}\left[\mathrm{x}_{2}\right]\right]$ free By the definition of $\mathcal{C}_{2}^{\prime}$

```
M free
```




Suppose
By the definition of free
By definition of $\mathcal{C}_{3}$

Theorem 10 (Static derivations don't have match failures).
If $\Gamma^{\mathrm{S}} \vdash e^{\mathrm{s}} \Leftarrow \mathcal{A}^{\mathrm{s}}$ or $\Gamma^{\mathrm{S}} \vdash e^{\mathrm{s}} \Rightarrow A^{\mathrm{s}}$
then there exists $M$ such that $\Gamma^{\mathrm{s}} \vdash e^{\mathrm{s}}: \AA^{\mathrm{s}} \hookrightarrow M$
and $M$ is free of casts and matchfail.
Proof. By induction on the structure of the given derivation.

- Case SynVar Apply rule STVar $M=x$ is free of casts and matchfail.
- Case $\frac{\Gamma^{\mathrm{s}} \vdash e^{\mathrm{s}} \Rightarrow A_{0}^{\mathrm{S}} \quad A_{0}^{\mathrm{S}} \leadsto A^{\mathrm{s}}}{\Gamma^{\mathrm{S}} \vdash e^{\mathrm{S}} \Leftarrow A^{\mathrm{S}}}$ ChkCSub

- CaseSynAnno Use the induction hypothesis, the definition of free, and apply rule STAnno
- Case ChkUnitIntro Apply ruleSTUnitIntro $M=()$ is free of casts and matchfail.
- Case

- Case $\Gamma^{\mathrm{S}} \vdash \mathrm{e}_{0}^{\mathrm{S}} \Rightarrow\left(\mathcal{A}_{1}^{\mathrm{S}} \delta^{\mathrm{S}} \mathcal{A}_{2}^{\mathrm{S}}\right)$

$\delta^{S} \Longrightarrow+_{i}^{*} \quad$ Subderivation
$\delta^{S}=+_{i} \quad$ By Lemma 33 Static looseness, II
$A_{1}^{S} \leq A_{1}^{s}$
By Lemma 2 Reflexivity of subtyping
$A_{2}^{\mathrm{s}} \leq A_{2}^{\mathrm{s}} \quad$ By Lemma Reflexivity of subtyping
$\delta^{s} \leq+_{i}^{*} \quad$ By definition of $\leq$
$A_{1}^{S} \delta^{S} A_{2}^{S} \leq A_{1}^{S}+_{i}^{*} A_{2}^{S} \quad$ By definition of $\leq$
$A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}} \leadsto A_{1}^{\mathrm{S}}+_{i}^{*} A_{2}^{\mathrm{S}} \quad$ By Lemma 8 (Subtyping obeys directed consistency)
$A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}} \simeq A_{1}^{\mathrm{S}}+_{i}^{*} A_{2}^{\mathrm{S}} \quad$ By Lemma 17 (Directed consistency obeys Structural Equivalence
$A_{1}^{S} \delta^{S} A_{2}^{S} \Rightarrow A_{1}^{S}+{ }_{i}^{*} A_{2}^{S} \hookrightarrow \mathcal{C} \quad$ By Theorem 17
$\Gamma^{\mathrm{S}} \vdash e_{0}^{\mathrm{S}} \Rightarrow\left(A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}}\right) \quad$ Subderivation
 $M_{0}$ free
$\Gamma^{\mathrm{S}} \vdash \mathrm{e}_{\mathrm{O}}^{\mathrm{S}}:\left(A_{1}^{\mathrm{S}}+_{i}^{*} A_{2}^{\mathrm{S}}\right) \hookrightarrow \mathcal{C}\left[\mathrm{M}_{0}\right] \quad$ By rule STCSub
$\mathcal{C}\left[M_{0}\right]$ free
$\Gamma^{\mathrm{S}}, x: A_{i}^{\mathrm{S}} \vdash \mathrm{e}_{\mathrm{i}}^{\mathrm{S}} \Leftarrow A^{\mathrm{s}} \quad$ Subderivation
By Lemma 65 Gradual sums in static don't need casts
$\Gamma^{\mathrm{S}}, x: A_{i}^{\mathrm{S}} \vdash e_{i}^{\mathrm{S}}: A^{\mathrm{S}} \hookrightarrow M_{i} \quad$ By the induction hypothesis
$M_{i}$ free
- $\Gamma^{S} \vdash \operatorname{case}\left(e_{0}^{S}, \operatorname{inj}_{i} \chi . e_{i}^{S}\right): A^{S} \hookrightarrow \operatorname{case}\left(\mathcal{C}\left[M_{0}\right], \operatorname{inj}_{i} \chi . M_{i}\right) \quad$ By ruleSTSumElim1
- $\operatorname{case}\left(\mathcal{C}\left[M_{0}\right], \operatorname{inj}_{i} x . M_{i}\right)$ free By definition of free
- Case $\Gamma^{\mathrm{S}} \vdash e_{0}^{\mathrm{S}} \Rightarrow\left(A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}}\right) \quad \Gamma^{\mathrm{S}}, \mathrm{x}_{1}: A_{1}^{\mathrm{S}} \vdash e_{1}^{\mathrm{S}} \Leftarrow A^{\mathrm{S}}$
$\frac{\delta^{\mathrm{S}} \Rightarrow+\quad \Gamma^{\mathrm{S}}, \mathrm{x}_{2}: A_{2}^{\mathrm{S}} \vdash \mathrm{e}_{2}^{\mathrm{S}} \Leftarrow A^{\mathrm{S}}}{\Gamma^{\mathrm{S}} \vdash \operatorname{case}\left(e_{0}^{\mathrm{S}}, \operatorname{inj}_{1} \mathrm{x}_{1} . e_{1}^{\mathrm{S}}, \mathrm{inj}_{2} \mathrm{x}_{2} . e_{2}^{\mathrm{S}}\right) \Leftarrow A^{\mathrm{S}}}$ ChkSumElim2
$\begin{array}{rlrl}A_{1}^{S} & \leq A_{1}^{S} & & \text { By Lemma } 2 \text { Reflexivity of subtyping } \\ A_{2}^{S} \leq A_{2}^{S} & & \text { By Lemma } 2 \text { Reflexivity of subtyping } \\ \delta^{S} & \leq+ & & \text { By Lemma } 23 \text { All sums below + }\end{array}$
$A_{1}^{S} \delta^{s} A_{2}^{S} \leq A_{1}^{S}+A_{2}^{S}$
By definition of $\leq$
$A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}} \leadsto A_{1}^{\mathrm{S}}+A_{2}^{\mathrm{S}} \quad$ By Lemma 8 (Subtyping obeys directed consistency)
$A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}} \simeq A_{1}^{\mathrm{S}}+A_{2}^{\mathrm{S}} \quad$ By Lemma 17 (Directed consistency obeys Structural Equivalence)
$A_{1}^{\mathrm{S}} \delta^{\mathrm{S}} A_{2}^{\mathrm{S}} \Rightarrow A_{1}^{\mathrm{S}}+A_{2}^{\mathrm{S}} \hookrightarrow \mathcal{C} \quad$ By Theorem 17
$\Gamma^{\mathrm{S}} \vdash \mathrm{e}_{0}^{\mathrm{S}} \Rightarrow\left(\mathrm{A}_{1}^{\mathrm{S}} \delta^{\mathrm{S}} \mathrm{A}_{2}^{\mathrm{S}}\right) \quad$ Subderivation
$\Gamma^{\mathrm{S}} \vdash \mathrm{e}_{0}^{\mathrm{S}}:\left(A_{1}^{\mathrm{S}} \delta^{\mathrm{s}} A_{2}^{\mathrm{S}}\right) \hookrightarrow \mathrm{M}_{0} \quad$ By the induction hypothesis
$M_{0}$ free
$\Gamma^{\mathrm{S}} \vdash e_{0}^{\mathrm{S}}:\left(A_{1}^{\mathrm{S}}+A_{2}^{\mathrm{S}}\right) \hookrightarrow \mathcal{C}\left[\mathrm{M}_{0}\right] \quad$ By ruleSTCSub
$\mathcal{C}\left[M_{0}\right]$ free By Lemma 67](Static subtypes don't need casts)
$\Gamma^{\mathrm{s}}, \mathrm{x}_{1}: A_{1}^{\mathrm{s}} \vdash e_{1}^{\mathrm{s}} \Leftarrow A^{\mathrm{s}} \quad$ Subderivation
$\Gamma^{\mathrm{S}}, x_{1}: A_{1}^{\mathrm{S}} \vdash e_{1}^{\mathrm{S}}: A^{\mathrm{S}} \hookrightarrow M_{1} \quad \underset{\not /}{\text { By }}$ the induction hypothesis
$M_{1}$ free
$\Gamma^{\mathrm{s}}, \mathrm{x}_{2}: A_{2}^{\mathrm{s}} \vdash \mathrm{e}_{2}^{\mathrm{s}} \Leftarrow A^{\mathrm{s}} \quad$ Subderivation
$\Gamma^{\mathrm{s}}, x_{2}: A_{2}^{\mathrm{s}} \vdash e_{2}^{\mathrm{s}}: A^{\mathrm{s}} \hookrightarrow M_{2} \quad \underset{\text { " }}{\text { By the induction hypothesis }}$
$M_{2}$ free
* $\Gamma^{S} \vdash \operatorname{case}\left(e_{0}^{S}, \operatorname{inj}_{1} x_{1} \cdot e_{1}^{S}, \operatorname{inj}_{2} x_{2} \cdot e_{2}^{S}\right): A^{S} \hookrightarrow \operatorname{case}\left(\mathcal{C}\left[M_{0}\right], \operatorname{inj}_{1} x_{1} \cdot M_{1}, \operatorname{inj}_{2} x_{2} \cdot M_{2}\right) \quad$ By ruleSTSumElim2
- $\operatorname{case}\left(\mathcal{C}\left[M_{0}\right], \operatorname{inj}_{1} x_{1} \cdot M_{1}, \operatorname{inj}_{2} x_{2} \cdot M_{2}\right)$ free By definition of free
- Case Chk $\rightarrow$ Intro Use the induction hypothesis, the definition of free, and apply rule $S T \rightarrow \ln$ tro
- Case Syn $\rightarrow$ Elim Use the induction hypothesis, the definition of free, and apply rule ST $\rightarrow$ Elim

