Sums of Uncertainty: Refinements Go Gradual

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Abstract

A long-standing shortcoming of statically typed functional languages is that type checking does not rule out pattern-matching failures (run-time match exceptions). Refinement types distinguish different values of datatypes; if a program annotated with refinements passes type checking, pattern-matching failures become impossible. Unfortunately, refinement is a monolithic property of a type, exacerbating the difficulty of adding refinement types to nontrivial programs.

Gradual typing has explored how to incrementally move between static typing and dynamic typing. We develop a type system of *gradual sums* that combines refinement with imprecision. Then, we develop a bidirectional version of the type system, which rules out excessive imprecision, and give a type-directed translation to a target language with explicit casts. We prove that the static sublanguage cannot have match failures, that a well-typed program remains well-typed if its type annotations are made less precise, and that making annotations less precise causes target programs to fail later. Several of these results correspond to criteria for gradual typing given by Siek et al. (2015).

Categories and Subject Descriptors F.3.3 [Mathematical Logic and Formal Languages]: Studies of Program Constructs—Type structure

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1. Introduction

A central feature of statically typed functional languages is pattern matching over user-defined datatypes that combine several fundamental constructs: sum types (for example, an element of a bool datatype can be *either* True or False), recursive types (such as lists), and polymorphic types. The aspect of ML datatypes that corresponds to sum types is the focus of this paper.

Static typing is said to catch run-time errors—at least, errors that would manifest in a dynamically typed language as *tag check failures*, such as subtracting a string from a number. Using the venerable encoding of dynamic typing as injections into a datatype Dynamic (Abadi et al. 1991), these tag check failures become errors raised in the "fall-through" arm of a case expression over Dynamic. The impossibility of such errors is a convincing argument in favour of static typing.

Yet Standard ML programmers frequently write code that is essentially the same as the scorned operations on Dynamic—and that has the same unfortunate risk of run-time errors. The definition of SML (Milner et al. 1997) requires compilers to accept *nonexhaustive* case expressions, which do not cover all the possible instances of the datatype. A nonexhaustive case expression is isomorphic to an implicit tag check over Dynamic: the non-error case is the only one written out explicitly, while an error case is inserted by the sneaky compiler.

In fairness, the definition encourages compilers to warn about nonexhaustive case expressions. But this only causes programmers to write their own "raise Match" arms, even when the fall-through case is impossible because of an invariant known by the programmer. This leads to verbose code. In response, Freeman and Pfenning (1991) developed datasort refinements that can encode many invariants about datatypes, allowing compilers to accept "nonexhaustive" case expressions when they are known to cover all *possible* cases. For case analyses of refined types, the nonexhaustiveness *warning* becomes a nonexhaustiveness *error*, which the programmer should solve by declaring and using refinements of the datatype.

Unfortunately, this approach is all-or-nothing: either a type is refined and the compiler rejects a nonexhaustive match over it, or the type is not refined and the compiler issues a noncommittal warning. In practice, programmers may want to migrate code written with unrefined types to code that uses refined types; doing this in a single pass over a nontrivial program is extremely difficult. Instead, programmers should be able to add type annotations gradually. This was essentially the motivation for gradual typing (Siek and Taha 2006), except that, where they contemplated migration from dynamically typed code to statically typed code, we are interested in migration from code that is statically typed (modulo nonexhaustiveness) to code that is more statically typed.

Gradual typing is about the possibility of uncertainty: in some cases, one knows exactly what type one has; in other cases, one does not even know whether something is an integer. In this paper, we always know whether something is an integer (or a function, etc.); uncertainty is possible, but only about sum types. This is like the uncertainty of SML datatypes, with one key difference: we allow SML-style uncertainty *and* refinement-style certainty.

As an example, consider a red-black tree library that passes the SML type checker, but does not use refinement types. Datasort refinements can express the colour invariant, which says that every red node's children must be black. By reasoning about how the library functions should work, a programmer can add annotations that say when the colour invariant should hold, which the refinement type checker will verify. With gradual refinements, this reasoning can be done gradually and in tandem with testing. In fact, the programmer could start by annotating a single function r. If all test cases use r in accordance with its refinement type annotation, the programmer gains confidence that the annotation is correct; if any tests violate the annotation, then either the annotation is wrong,

or there is a bug somewhere else. Thus, the more precise invariants guaranteed by refinements can be verified piecemeal.

Contributions. We make the following contributions:

- We define a type assignment system of *gradual sums* that includes both static *refinement sums* and *dynamic sums*. Programs, and even individual types, can be partly static and partly dynamic. However, this system does not readily yield an algorithm, and it allows typing derivations that are *gratuitously* dynamic (more dynamic than indicated by the programmer's type annotations), which give rise to gratuitous run-time errors.
- We define a bidirectional type system that is easy to implement and suppresses gratuitous dynamism, and prove that it corresponds to the type assignment system. We also prove that a well-typed program remains well-typed if its type annotations are made less precise (more dynamic).
- We define a type-directed translation to a target language with explicit casts. We prove that, given one program with two sets of type annotations (one more precise than the other), the more precisely typed one "fails earlier": either they produce the same result, or they both fail, or the more precisely typed program fails earlier. (For technical reasons, part of this result uses a slightly different version of the translation.)
- We define static and dynamic fragments of the source type system. The static fragment is related to classic datasort refinement type systems; the dynamic fragment is related to Standard ML. We prove that translating a program in the static fragment yields a program that cannot raise Match.

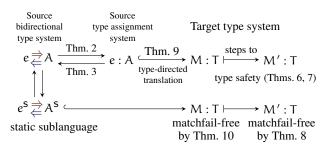


Figure 1. Some key results

Figure 1 depicts some of the results: source programs e are translated to target terms M, which step to M', preserving typing; source programs e^S with only static types are translated to target terms with no match failures.

For space reasons, lemmas, proofs, and a few definitions can be found in the supplementary material.

2. Overview

We define a type system that has one of the essential capabilities of datasort refinements: the types can express the knowledge that a value is a *particular* alternative of a datatype; for example, that a value is not simply a list—either Nil or Cons(...)—but specifically Cons(...). We represent this knowledge through sum types, not through the usual form of datasort refinements, but that is not the important difference.

• Like conventional datatype systems and datasort refinement systems, we can express that a value is either inj₁ e₁ where e₁ has type A₁ or inj₂ e₂ where e₂ has type A₂. Like datasort refinement systems, we only allow an exhaustive (two-armed) case expression over such a type: if we don't know which

injection it is, the programmer must handle both cases. This is a standard sum type $A_1 + A_2$.

- Like datasort refinement systems, we can express that a value must be a particular injection. We use a *subscript sum* $A_1+_kA_2$ for the type of the kth injection into A_1+A_2 . For example, $\operatorname{inj}_2\operatorname{True}$ has type $\operatorname{Int}+_2\operatorname{Bool}$, but $\operatorname{inj}_1 5$ has type $\operatorname{Int}+_1\operatorname{Bool}$. Also like datasort refinement systems, we allow case expressions over such types to have just one arm, because we know which injection we have; there is no need to handle an impossible case.
- Like conventional datatype systems, but unlike datasort refinement systems, we can also express that we don't know which injection we have, but want to allow nonexhaustive matches: the dynamic sum A₁ +² A₂ can be deconstructed by a one-armed case expression. If, at run time, the specified arm does not match the scrutinee, it is a run-time error.

The three sum types +, +₁, and +₂ are essentially a datasort refinement system. Following datasort refinement systems, A_1 +₁ A_2 and A_1 +₂ A_2 are subtypes of A_1 + A_2 .

We can also make $+^2$ a subtype of +: the only elimination form permitted for + is a two-armed case, which is always safe. But $+^2$ must not be a subtype of $+_1$ and $+_2$, because $+^2$ contains both left and right injections; through subsumption, we could use a one-armed case on the left injection inj_1 to eliminate a value of type $+_2$, which would fail at run time.

This yields the following subtype relation:

$$A_1 + A_2$$
 $A_1 + A_2$
 $A_1 + A_2$
 $A_1 + A_2$
 $A_1 + A_2$

For brevity, we can omit A_1 and A_2 from the diagram.



Comparison to datasort refinements. Our type $A_1 + A_2$ corresponds to the *top datasort* of a datatype—the datasort that contains all the values of that datatype. A case expression on + must provide two arms, one for each injection.

Our type $A_1 +_1 A_2$ corresponds to a datasort that includes exactly the values of the form $c_1(\nu_1)$ where $\nu_1:A_1$; similarly, $A_1 +_2 A_2$ corresponds to a datasort whose values are $c_2(\nu_2)$ where $\nu_2:A_2$.

In contrast, our type $A_1 + {}^2 A_2$ corresponds to the *unrefined* datatype. In datasort refinement systems, unrefined datatypes are part of the unrefined type system; the top datasort for a datatype contains the same values as the unrefined datatype, and is often notated in exactly the same way—but the unrefined datatype is not usable as a datasort. In contrast, both + and + 2 are types in our system. Moreover, they can be freely combined.

2.1 Developing Typing and Subtyping

Verificationists and pragmatists. In the *verificationist* approach to type theory, followed by Gentzen (1934) and Martin-Löf (1996), introduction forms are taken as the definition of a type; for example, a boolean type is defined by its constructors True and False. The elimination forms are secondary. In the *pragmatist* approach considered by Dummett (1991) and Zeilberger (2009), elimination forms are taken as the definition, and the introduction forms are

secondary. For example, a boolean type is defined primarily by its elimination form (say, an if-then-else expression).

In our setting, neither strict verificationism nor strict pragmatism seems adequate. Verificationism serves refinements well: the introduction rules directly express the intuition that refinements identify subsets of values. But introduction rules alone cannot distinguish $A_1 + A_2$ and $A_1 +^2 A_2$, because they have identical sets of inhabiting values (namely, all $\ln j_1 v_1$ and $\ln j_2 v_2$ such that $v_1 : A_1$ and $v_2 : A_2$). The difference must lie in the elimination forms: only a two-armed case can eliminate +, while +² can be eliminated by a two-armed case or a one-armed case (since the point is to allow nonexhaustive matches). To start from a better-understood foundation, we begin with the introduction rules.

Designing a type system can require trading off simplicity in one set of rules for complexity in another. We choose to minimize the number of typing rules, even though it leads to more complicated subtyping.

Introduction rules. Sum types need introduction forms. Since $+_1$ should contain only left injections, and $+_2$ should contain only right injections, we could have a rule

$$\frac{\Gamma \vdash e : A_k}{\Gamma \vdash (\operatorname{inj}_k e) : (A_1 +_k A_2)} +_k \operatorname{Intro}$$

(This rule is really two rules, one for $(inj_1 \nu)$ with a premise $\Gamma \vdash e : A_1$ and one for $(inj_2 \nu)$ with a premise $\Gamma \vdash e : A_2$.)

Combined with subsumption, this rule gives the desired inhabitants to +, that is, both left and right injections. However, it does not add any inhabitants to +², so we could add another rule:

$$\frac{\Gamma \vdash e : A_k}{\Gamma \vdash (\operatorname{inj}_k e) : (A_1 + A_2)} + \operatorname{Intro}$$

This goes against our goal of minimizing the number of typing rules: now there are two rules that type $inj_k e$ directly, that is, without using subsumption. The types $+_k$ (given by $+_k$ Intro) and $+^2$ (given by +Intro) are not in a subtyping relation with each other—neither is a subtype of the other. Hence, neither rule encompasses the other, and both are required.

We can avoid this nondeterminism by adding more sum types. By placing the additional sum types at the bottom of the subtyping relation, we can write a single introduction rule that will (through subsumption) populate all of our types with the desired injections.



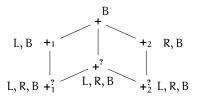
Now, we need only one introduction rule:

$$\frac{\Gamma \vdash e : A_k}{\Gamma \vdash (\operatorname{inj}_k e) : (A_1 +_k^? A_2)} +_k^? \operatorname{Intro}$$

We can think of $+\frac{1}{1}$ and $+\frac{2}{2}$ as "innate" types: when an injection inj_k is created, it has type $+\frac{2}{k}$. Through subtyping, we can interpret $+\frac{2}{k}$ as +k, or as the dynamic sum $+\frac{2}{k}$.

Elimination rules. To design the elimination rules, it is helpful to annotate the subtyping diagram with the elimination forms that each type should allow. We write L for a one-armed case expression

on the left injection (inj₁), R for a one-armed case on the right injection (inj₂), and B for a two-armed case.



According to this diagram, all types support a two-armed case expression B. The types $+_1$ and $+_1^2$ are inhabited only by inj_1 , so they support the left one-armed case L; similarly, $+_2$ and $+_2^2$ support the right one-armed case R. However, $+_1^2$ and $+_2^2$ are subtypes of $+_1^2$, so by subsumption they also support the "wrong" one-armed cases. The dynamic sum $+_1^2$ supports all three eliminations, with the risk of failing at run time.

Handling the two-armed case expression is straightforward: all the sum types support that elimination form, and all the sum types are subtypes of +, so we can write a single rule that types the scrutinee with +. Given $e: (A_1 \varphi A_2)$ where φ is any of our sum types, subsumption can be used to derive $e: (A_1 + A_2)$.

$$\frac{\Gamma, x_1 : A_1 \vdash e_1 : B}{\Gamma, x_2 : A_2 \vdash e_2 : B} + \text{Elim}$$

$$\frac{\Gamma \vdash e : (A_1 + A_2)}{\Gamma \vdash \mathsf{case}(e, \mathsf{inj}_1 \ x_1.e_1, \mathsf{inj}_2 \ x_2.e_2) : B} + \text{Elim}$$

One-armed case expressions are more troublesome. Consider a left one-armed case, which matches only values of the form $\operatorname{inj}_1 \nu$. Any subtype of $+_1$ will work, so we can write a rule that handles $+_1$ and $+_1^2$ (and symmetrically, $+_2$ and $+_2^2$). However, $+^2$ should support a left one-armed case, but $+^2$ is not a subtype of $+_1$, leading us to a second rule that handles $+^2$.

Since +² supports one-armed cases, it violates a type-theoretic principle: the introduction and elimination rules of a logical connective should be in *harmony*—that is, they should be *locally sound* (Dummett 1991) and *locally complete* (Pfenning and Davies 2001). Local soundness holds when the elimination rules are not more powerful than the introduction rules. Consider some standard rules for pairs:

$$\frac{\Gamma \vdash e_1 : A_1 \qquad \Gamma \vdash e_2 : A_2}{\Gamma \vdash (e_1, e_2) : (A_1 \times A_2)} \qquad \frac{\Gamma \vdash e : (A_1 \times A_2)}{\Gamma \vdash (\mathtt{proj}_k \ e) : A_k}$$

These rules are locally sound: given something of type $(A_1 \times A_2)$, projection can only extract things of type A_1 and A_2 .

Dually, local completeness says that the elimination rules can extract all the information used in the introduction rules. (For a concise explanation of harmony, see Pfenning (2009).)

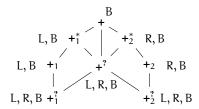
When the Curry–Howard correspondence holds, a type is inhabited iff the corresponding proposition is provable. Consider the following derivation (eliding empty contexts):

$$\frac{e: A_1}{(\text{inj}_1 e): (A_1 +_1^2 A_2)} \cdot (A_1 +_1^2 A_2) \le (A_1 +_1^2 A_2)}{(\text{inj}_1 e): (A_1 +_1^2 A_2)} \times A_2 \vdash x: A_2$$

$$case(\text{inj}_1 e, \text{inj}_2 x.x): A_2$$

By constructing $\operatorname{inj}_1 e$, we have shown that A_1 is inhabited. By subsumption, $\operatorname{inj}_1 e$ has type $A_1 +^2 A_2$. An elimination rule for $+^2$ must permit a one-armed case on the second injection, ostensibly having type A_2 . Simply returning x as the result of the case should show that the proposition corresponding to A_2 is provable. But we never constructed something of type A_2 , so $+^2$ does not satisfy local soundness.

As we did for the introduction forms, a single elimination rule *can* suffice: we just need more sum types. For the introduction forms, we added types at the bottom of the subtyping relation. Since eliminations should behave dually, we will add types at (or, at least, near) the *top* of the subtyping relation.



The types $+_1^*$ and $+_2^*$ support exactly the same eliminations as the subscript sums $+_1$ and $+_2$, but unlike the subscript sums, they are supertypes of the dynamic sum $+_1^*$.

Then the single elimination rule for one-armed cases is

$$\frac{\Gamma \vdash e : (A_1 +_k^* A_2) \qquad \Gamma, x : A_k \vdash e_k : B}{\Gamma \vdash \mathsf{case}(e, \mathsf{inj}_k \ x.e_k) : B} +_k^* \mathsf{Elim}$$

We could simplify the diagram slightly by removing the edge from $+^2$ to +, since we now have an alternate routing via the $+^*_k$ types.

The high-water mark. Have we added enough sum types? We believe so. First, the additional types (beyond +, $+_1$, $+_2$ and $+^2$) are motivated by limiting the number of typing rules. Second, there seem to be no other types that could be useful. Consider the following table:

| | elimination forms supported | | | |
|---------------------------------------|-----------------------------|----------|----------|-------------|
| inhabitants | B only | B and L | B and R | B, L, and R |
| inj ₁ | note (a) | +1 | note (b) | +1 |
| inj_2 | note (a) | note (b) | +2 | +2 |
| inj ₁ and inj ₂ | + | +*1 | +*2 | +? |

In the spaces marked "note (a)", such a type would pointlessly restrict the possible elimination forms: the top left space would be a type that could only be eliminated by a two-armed case ("B only"), but was inhabited only by left injections inj₁.

In the spaces marked "note (b)", such a type would allow onearmed cases that *always* fail: a left one-armed case L on inj₂, or a right one-armed case R on inj₁. We provide +² to give programmers the freedom to use one-armed cases that may fail; it seems pointless to give them one-armed cases that are *guaranteed* to fail.

If anything, we may have more sum types than we want in practice: having fewer typing rules is good, but showing $+_1^*$ or $+_2^?$ in a compiler error message seems unhelpful.

2.2 Developing Precision

Our ultimate goal is a language in which precisely typed code and imprecisely typed code can coexist. In precisely typed code, the impossibility of match failures is a consequence of typing. In imprecisely typed code, bugs may lead to match failures, but imprecisely typed code can be correct: a one-armed case expression may be exhaustive in practice, thanks to some invariant not expressed through the type system.

The approach to typing and subtyping, developed above, already permits some forms of coexistence. For example, if a function f expects a sum type + and we have some x of type +², we can

pass x to f. In the derivation below, $\Gamma = f: (A_1 + A_2) \rightarrow B, x: (A_1 +^2 A_2)$.

$$\frac{\Gamma \vdash r: (A_1 + A_2) \rightarrow B}{\Gamma \vdash r: A_1 + A_2} \frac{\Gamma \vdash x: A_1 + A_2}{\Gamma \vdash x: A_1 + A_2}$$
$$\Gamma \vdash fx: B$$

What about the reverse situation? Suppose a function g from the imprecisely typed part of the program expects $+^2$, and we want to pass something of type +. This is possible, but annoying: we have to use a two-armed case to decompose the sum, and immediately rebuild it at type $+^2$. Here, $\Gamma = g: (A_1 +^2 A_2) \rightarrow B_1 y: (A_1 + A_2)$.

$$\frac{\Gamma, x_1 : A_1 \vdash \operatorname{inj}_1 x_1 : (A_1 +^? A_2)}{\Gamma, x_2 : A_2 \vdash \operatorname{inj}_2 x_2 : (A_1 +^? A_2)}$$

$$\frac{\Gamma \vdash g \left(\operatorname{case}(y, \operatorname{inj}_1 x_1, \operatorname{inj}_2 x_2, \operatorname{inj}_2 x_2) \right) : B}{\Gamma}$$

To support directly calling imprecise code from precise code, we develop *precision relations* on sum constructors and types. These relations are inspired by precision relations developed in gradual typing, e.g. Siek and Vachharajani (2008) and Garcia et al. (2016), where ? (or \star) is an unknown, and thus very imprecise, type.

Our static sums +, +₁, +₂ are precise in the sense that the "reach" of their information is known. If we have a closed value ν of type $A_1 + A_2$, the type system "knows" only that ν is either a left or right injection, with no further information. So the type system rejects a one-armed case on ν .

On the other hand, the dynamic sum +² is *imprecise*. Some programs that use +² will have run-time match failures, but some programs that use +² will *not* have such failures, even some that use one-armed cases. If such one-armed cases always succeed, it is because the program follows invariants that are not expressed in the types—but which may be known by the programmer.

the types—but which may be known by the programmer.

So we would expect + to be more precise than +?, notated $+ \sqsubseteq +$? (which can also be read "+ is less imprecise than +?"). What about $+_1$ and $+_2$? They should be more precise than +?; indeed, +? should be more *imprecise* than everything else. How do $+_1$ and + compare? It is true that $+_1$ has fewer inhabitants than +, but precision is not subtyping. All the static sums have the same degree of *certainty*: they are equally certain about different propositions (being a left injection, being a right injection, or being either). Thus, we will put $+_1$, $+_2$ and + together at the bottom of the precision relation \sqsubseteq (they are the *least imprecise*), with +? at the top:



What properties should precision have? In gradual typing, an important property of precision is that a program should remain well-typed when type annotations are made *less* precise. In the limit, we should be able to replace all static sums in annotations with $+^2$. We call this property *varying precision*; it is part of the "gradual guarantee" of Siek et al. (2015). (Making annotations *more* precise does not necessarily preserve typing: for example, changing a $+^2$ annotation on inj₂() to $+_1$.)

This property reinforces the intuition that $+^2$ should be at the top: this is what lets us substitute $+^2$ for more-precise sums. Dually, the static sums should be at the bottom: replacing a sum with a static sum should not, in general, preserve typing.

With this property in mind, how precise are $+^2_1$ and $+^*_1$, which we

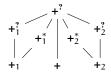
With this property in mind, how precise are $+_1^2$ and $+_1^*$, which we put in to reduce the number of typing rules? It doesn't make sense to "mix subscripts": moving between $+_2$ and $+_1^2$ in an annotation, or between $+_1$ to $+_2^*$, never preserves typing. Types with 1 subscripts should stay on the left of the edge from + to $+_1^2$, and 2 subscripts should stay on the right.

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Hence, we will place $+_1^2$ and $+_1^*$ left of the vertical edge (from + to $+^2$), and $+_2^*$ and $+_2^*$ right of the vertical edge.

Moving to a less precise type should not lose inhabitants, because the lost inhabitants will become ill-typed. Suppose we put $+_1^*$ below $+_1^2$, making $+_1^*$ more precise. The sum $+_1^*$ contains both left and right injections (by the above subtyping relation, $+_2^2 \le +_1^*$), meaning that $+_1^*$ has *more* inhabitants than $+_1^2$. Therefore, we should not have $+_1^* \sqsubseteq +_1^2$.

should not have $+_1^* \sqsubseteq +_1^2$. The reverse, where $+_1^2 \sqsubseteq +_1^*$, is more plausible but would have unfortunate consequences (discussed at the end of this section). So we have no edge between $+_1^2$ and $+_1^*$.



Lifting this relation \sqsubseteq on sum constructors to sum *types* is straightforward: if $\delta' \sqsubseteq \delta$ then $(A_1' \delta' A_2') \sqsubseteq (A_1 \delta A_2)$, provided $A_1' \sqsubseteq A_1$ and $A_2' \sqsubseteq A_2$. For function types, we diverge from subtyping: precision is covariant in the codomain *and* in the domain. This is consistent with precision in gradual typing, e.g. Siek and Vachharajani (2008) and Garcia et al. (2016), and with the refinement relations of Freeman (1994, p. 31) and Davies (2005).

Can we use this relation to type the above example g y, where we want to pass a value of type + to a function expecting something of $+^2$ type? Subtyping is internalized through a subsumption rule (the rule on the left); we extend the rule to allow *loss of precision*: in addition to moving from A to a supertype B, we can move from B to a less-precise B'.

$$\frac{\Gamma \vdash e : A \quad A \leq B}{\Gamma \vdash e : B} \text{ sub.} \quad \frac{\Gamma \vdash e : A \quad A \leq B \quad B \sqsubseteq B'}{\Gamma \vdash e : B'} \text{ sub.+loss}$$

Imprecision is fundamentally unsound: Using $B \sqsubseteq B'$, we move from a precise type (containing, say, + and $+_2$) to an imprecise type containing $+^2$. Above, we showed that $+^2$ does not satisfy local soundness. The purpose of the $B \sqsubseteq B'$ premise is to allow more-precisely-typed code to interface with less-precisely-typed code. However, a type checker that lost precision wherever possible would behave like a type checker for a system that only had $+^2$.

In addition to losing precision after subtyping, we allow *gaining* precision *before* subtyping:

$$\frac{\Gamma \vdash e : A' \quad A \sqsubseteq A' \quad A \leq B \quad B \sqsubseteq B'}{\Gamma \vdash e : B'} \text{ gain+sub.+loss}$$

Gaining precision is clearly unsound: $A \sqsubseteq A'$ allows moving from $+^2$ to $+_1$ or $+_2$. While unsound, this is needed for the property of varying precision: the typing of a single part of a program can become more or less precise, independent of the typing of the rest of the program.

We compose the three premises—gaining precision $A \sqsubseteq A'$, subtyping $A \leq B$, and losing precision $B \sqsubseteq B'$ —into a relation $A' \leadsto B'$, called *directed consistency*.

With this relation, allowing $+_1^2 \sqsubseteq +_1^*$ would nearly erase the distinction between $+_1^*$ and $+_2^*$: first, $+_1^2 \sqsubseteq +_1^*$; second, $+_1^2 \le +_2^*$; third, $+_2^* \sqsubseteq +_2^*$. (An earlier version of our system did allow $+_1^2 \sqsubseteq +_1^*$ —see Appendix C.)

Ideally, we should apply imprecision only when the programmer intends it. This goal motivates the bidirectional system in Section 4.

3. Source Type System

Figure 2. Source syntax

The syntax of the source language is in Figure 2. Here, and throughout the paper, i ranges over 1 and 2. The symbol δ ranges over the sum constructors: + is the standard (static) sum, +₁ and +₂ are subscript sums denoting the ith injections, and +² is the gradual or dynamic sum. The final sum constructors, +²_i and +^{*}_i, are motivated by the desire to have the smallest number of introduction and elimination rules, as described in Section 2.

Source expressions are the unit (), variables x, abstraction $\lambda x. e$ and application $e_1 e_2$, sum injection $\inf_i e$, annotation (or ascription) (e :: A), a two-armed case that eliminates +, and a one-armed case that eliminates $+_i^*$.

Types A and B are Unit, sums A δ B, and functions A \rightarrow B. Typing contexts Γ are unordered sets of typings x:A, where the x are assumed to be distinct.

3.1 Subtyping and Precision

Figure 3 gives the rules for a *subsum* judgment on sum constructors, written $\delta' \leq \delta$. These rules follow the diagram in Section 2. The subtyping rule for sum types uses the subsum judgment. As is standard, the subtyping rule for functions is contravariant in the domain $(A_1 \leq A_1')$ and covariant in the codomain $(A_2' \leq A_2)$.

Precision on sum constructors (top of Figure 4) corresponds to the diagram from Section 2. On function types, precision is covariant in the domain, as discussed above.

In both subtyping and precision (for types), reflexivity and transitivity are admissible rules. Including transitivity rules would be fine on paper, but hard to implement since the middle type must be guessed. (The relations on sum constructors are a small finite set, so we do include transitivity rules; for an implementation, we would take the transitive closure.)

Subtyping and precision compose to form the directed consistency relation, which has a single rule, DirConsU, in Figure 5. The "U" in the name comes from the depiction to the right of the rule. Since precision is reflexive, DirConsU includes all pairs of types that are related by subtyping.

3.2 Typing Rules

5

Typing rules for the source language are shown in Figure 6. The rule for variables, SVar, is standard. Rules SAnno and SUnitIntro are standard, as are the rules S→Intro and S→Elim for functions.

Rule SCSub is a *consistent subsumption* rule: if e has type A' and A' is directed consistent (Figure 5) with A, then e has type A.

The rules for sums (SSumIntro, SSumElim1, SSumElim2) were developed in Section 2.1.

4. Bidirectional Source Typing

Motivation. The type assignment system of Section 3 includes all the sensible sum types, along with subtyping and precision. By itself, the consistent subsumption rule SCSub makes type inference, and even type-checking, nontrivial: we should apply SCSub only

$$\begin{array}{c|c} \hline \delta' \leq \delta & \operatorname{Sum} \delta' \text{ is a subsum of } \delta \\ \hline \hline \delta \leq \delta & \overline{+_i^2 \leq +^2} & \overline{+^2 \leq +_i^*} & \overline{+_i^2 \leq +_i} \\ \hline \\ \overline{+_i \leq +_i^*} & \overline{+_i^* \leq +} & \overline{\delta' \leq \delta_1} & \delta_1 \leq \delta \\ \hline A' \leq A & \operatorname{Type} \ A' \text{ is a subtype of } A \\ \hline \hline \\ \overline{Unit \leq Unit} & \underline{A'_1 \leq A_1} & A'_2 \leq A_2 & \delta' \leq \delta \\ \hline \\ \underline{A_1 \leq A'_1} & A'_2 \leq A_2 \\ \hline \\ \underline{A_1 \leq A'_1} & A'_2 \leq A_2 \\ \hline \\ \overline{(A'_1 \rightarrow A'_2) \leq (A_1 \rightarrow A_2)} \\ \hline \end{array}$$

Figure 3. Source subtyping

$$\begin{array}{|c|c|c|c|}\hline \delta'\sqsubseteq\delta & \text{Sum }\delta' \text{ is more precise than }\delta \\ \hline \delta\sqsubseteq\delta & \overline{+_i\sqsubseteq+_i^2} & \overline{+_i\sqsubseteq+_i^*} & \overline{+_i^*\sqsubseteq+_i^2} & \overline{+_i^2\sqsubseteq+_i^2} & \overline{+_i^2\sqsubseteq+_i^2} \\ & \underline{\delta'\sqsubseteq\delta_1} & \delta_1\sqsubseteq\delta \\ \hline \delta'\sqsubseteq\delta \\ \hline A'\sqsubseteq A & \text{Type }A' \text{ is more precise than }A \\ \hline \hline Unit\sqsubseteq \text{Unit} & \underline{A'_1\sqsubseteq A_1} & \underline{A'_2\sqsubseteq A_2} & \underline{\delta'\sqsubseteq\delta} \\ \hline \underline{A'_1\sqsubseteq A_1} & \underline{A'_2\sqsubseteq A_2} & \underline{\delta'\sqsubseteq\delta} \\ \hline \underline{A'_1\sqsubseteq A_1} & \underline{A'_2\sqsubseteq A_2} \\ \hline \underline{(A'_1\to A'_2)\sqsubseteq (A_1\to A_2)} \\ \hline \end{array}$$

Figure 4. Precision

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline A' \leadsto B' & \text{Type A}' \text{ is directed consistent} \\ & \text{with B}' & & A' & B' \\ \hline A \sqsubseteq A' & A \leq B & B \sqsubseteq B' \\ \hline A' \leadsto B' & & A \leq B \end{array}$$

Figure 5. Directed consistency

where necessary. This problem arises even with ordinary subsumption (subtyping, without changes of precision), which "forgets" that e has a smaller type. Allowing changes of precision makes the problem worse: loss of precision "forgets" that e has a more precise type, while gain of precision may add a downcast that fails at run time.

Such algorithmic difficulties could, perhaps, be resolved through careful design; the real problem with the type assignment system is that it types too many programs. Since SCSub is always applicable, any expression meant to be typed using only + could be typed using +² instead.

A related problem is that our elimination rules for sums, while elegant, are excessively permissive: since $+\frac{2}{2}$ is a subtype of $+^*_1$, an expression of type $+^2_2$ can be eliminated with a left-arm case—even though such an elimination is *guaranteed* to cause a match failure at run time. Since this is a consequence of the subtyping part of SCSub, it wouldn't help to remove the changes of precision from directed consistency.

Figure 6. Source typing

We solve all of these problems via a bidirectional version of the system. In many settings, bidirectional typing has been chosen to overcome fundamental limitations of type inference, such as undecidability of inference for object-oriented subtyping (Pierce and Turner 1998), dependent types (Xi and Pfenning 1999; Pientka and Dunfield 2010) and first-class polymorphism (Dunfield and Krishnaswami 2013). It can also be motivated by better localization of type error messages. Our motivation is different: we want to stop the type-checker from doing certain things *unless* the programmer has signalled that they really want to do those things. Programmers signal their intent through type annotations, which are propagated through the bidirectional typing rules.

In Section 4.3, we show that the bidirectional system is sound and complete (under annotation) with respect to the type assignment system of Section 3.

Checking and synthesis. Bidirectional typing splits typing into two judgments. The checking judgment $\Gamma \vdash e \Leftarrow A$ is read "e checks against type A"; the synthesis judgment $\Gamma \vdash e \Rightarrow A$ is read "e synthesizes type A". Both judgments can be interpreted as saying that e has type A; the difference is that in checking, the type A is already known, while synthesis infers A from the available information (Γ and e). The type in the checking judgment "flows" from some type annotation, either directly or (usually) indirectly.

An important advantage of the bidirectional system is a kind of subformula property (Gentzen 1934; Prawitz 1965). In our case, this property says that in a derivation of $\Gamma \vdash e \Rightarrow A$, every type synthesized or checked against is derived from types found in Γ and e. For $\Gamma \vdash e \Leftarrow A$, every such type is derived from Γ , e, and A. Consequently, dynamic sums cannot appear out of nowhere: they result only from type annotations. We exploit this property in, for example, the proof of Theorem 5.

From type assignment rules to bidirectional rules. As is often the case with bidirectional type systems, our bidirectional rules will strongly resemble our type assignment rules. In general, we construct a bidirectional rule by replacing ":" with "\(= \)" or "\(= \)". The main question is when to use checking, and when to use synthesis. Checking is more powerful than synthesis; for a premise, we generally prefer to make it a checking judgment, but a checking conclusion may increase the number of required type annotations.

Figure 7. Bidirectional typing (source)

For the most part, we follow the recipe of Davies and Pfenning (2000); Dunfield and Pfenning (2004): introduction rules check, and elimination rules synthesize. More precisely, the judgment that includes the relevant connective—the *principal judgment*—should check for an introduction rule, and synthesize for an elimination rule.

Doing this step naturally determines the directions of many other judgments. For example, in rule Syn \rightarrow Elim, the principal judgment is the first premise $\Gamma \vdash e_1 \Rightarrow (A_1 \rightarrow A_2)$. Since the type in a synthesis judgment is output, deriving this premise tells us what A_1 is, enabling us to make the second premise a checking judgment. The premise also tells us what A_2 is—so we can make the conclusion a synthesis judgment. Consequently, applications $e_1 e_2$ will synthesize a type, without any local annotation, whenever the function e_1 synthesizes. In rule Chk \rightarrow Intro, not following the recipe—by making the conclusion synthesize, $\Gamma \vdash \lambda x. e \Rightarrow (A_1 \rightarrow A_2)$ —means that we don't know A_1 , and cannot construct the context Γ , α : α in the premise. (It may be possible to design a more complicated system in which α in α deferent type system.)

Rule ChkSumIntro says that inj_1 e checks against $A_1 \delta A_2$, where δ is any sum above $+^2_1$ —that is, any sum constructor *except* $+^2_2$ and $+_2$. This is a checking rule for two reasons. First, it is an introduction form, so according to the recipe its principal judgment (the conclusion) should check. Second, the simplest synthesizing rule would synthesize $A_1 +^2_1 A_2$. But that is a subtype of $A_1 +^2 A_2$, introducing a possibly undesired dynamic sum.

In the (one-armed) elimination rule SSumElim1, the principal judgment is the premise $\Gamma \vdash e_0 : A_1 +_i^* A_2$. Following the recipe, the corresponding premise of ChkSumElim1 synthesizes. It

would be unfortunate to require it to synthesize *exactly* $A_1 +_i^* A_2$: assuming programmers mostly write type annotations using $+_1$, $+_2$, + and $+^2$, virtually no expressions will synthesize $+_i^*$. On the other hand, checking e_0 against $A_1 +_i^* A_2$ would be too permissive: if we have a left one-armed case $case(e_0, inj_1 x.e)$, we would accept e_0 of type $+_2^2$, even though $+_2^2$ is a *right* injection, guaranteeing a runtime failure. Instead, we require that e_0 synthesize $A_1 \delta A_2$ where $\delta \Longrightarrow +_i^*$. The judgment $\delta \Longrightarrow +_1^*$ is derivable when δ is $+_1^2$, $+_1$, $+_2^2$ or $+_1^*$.

For consistency with ChkSumElim1, our two-armed elimination rule ChkSumElim2 has a similar structure (with an additional premise for the second arm) and also uses the \Rightarrow judgment; however, $\delta \Rightarrow$ + is *always* derivable, because a two-armed case is safe for every sum constructor. We include this premise anyway, to highlight the two rules' similarity.

Several rules are not tied to specific type connectives. An assumption x : A in Γ could be read "x synthesizes A", so SynVar synthesizes its type. Rule SynAnno synthesizes the type given in an annotation (e :: A), provided e checks against A. Following earlier bidirectional systems (Davies and Pfenning 2000; Dunfield and Pfenning 2004), the subsumption rule has a checking conclusion and a synthesizing premise. The checking conclusion ensures that subsumption, which loses information, is applied only with the programmer's consent: the type being checked against is derived from a type annotation. The synthesizing premise ensures that we "make progress" as we move from the goal $e \leftarrow A$ to the subgoal $e \Rightarrow A'$: we cannot use ChkCSub as the concluding rule of its own premise. In addition to subtyping and change of precision, ChkCSub with A = A' (using reflexivity) allows us to use a derivation of $\Gamma \vdash e \Rightarrow A$ where we need a derivation of $\Gamma \vdash e \Leftarrow A$. For example, applying a function to a variable requires this rule: SynVar synthesizes, but Syn→Elim has a checking premise.

Complexity. Typing in the bidirectional system takes polynomial time. With one exception, the bidirectional rules are in one-to-one correspondence with syntactic forms. The exception is ChkCSub, which can be used to check any synthesizing form. So bidirectional typing is syntax-directed in a slightly looser sense than the usual one: For each pair of a syntactic form and a direction (checking or synthesis), exactly one rule applies; if that rule is ChkCSub, then exactly one rule applies to derive its synthesizing premise. Thus, the size of a derivation (if one exists) is, at most, twice the size of the expression.

Variations on a theme. Several checking rules could be supplemented with a synthesizing rule, or (in the case of ChkUnitIntro) replaced. A synthesizing version of ChkSumIntro, however, would be problematic: while we might synthesize the sum constructor $+_i$, synthesizing e for A_i tells us only one component of the sum. Our system enjoys uniqueness of synthesis: given Γ and e, e synthesizes (at most) one type. Synthesizing the other component of the sum would synthesize an infinite number of types. Moreover, a direct implementation would need to guess the other component.

A synthesizing version of ChkSumElim1 would be straightforward; for ChkSumElim2, we could synthesize $e_1 \Rightarrow B_1$ and $e_2 \Rightarrow B_2$ and synthesize their join $B_1 \vee B_2$ in the conclusion.

Except for ChkUnitIntro, all of these variations—while perhaps convenient in practice—would make the system larger and more complicated. This paper presents a core calculus; we leave exploration of such variations to future work.

4.1 Static System

7

Two restricted versions of the bidirectional system are of interest. The first is a *static* system: a simply typed λ -calculus with sums and refinements over sums, without any dynamic sums. The syntax (Figure 8) is the same as the source language, except for δ^S which

Figure 8. The static system: the bidirectional system restricted to +, +₁, +₂

8

can only be +, $+_1$, or $+_2$. We follow the bidirectional system in deriving rules for sub-sum, subtyping, and typing; the judgments are decorated with S for "static". The interesting difference is in the typing rules for sums: the introduction rule checks that the sum is above $+_i$ (instead of $+_i^2$), and the one-arm elimination StSumElim1 checks that the sum is below $+_i$ (instead of $+_i^*$), that is, the sum is exactly $+_i$.

4.2 Dynamic System

The static system omits dynamic sums; the dynamic system's only sum is the dynamic sum +². Since one-armed cases are allowed on type +², this corresponds to datatypes in Standard ML. The metavariables and judgments are decorated with D for "dynamic". For space reasons, the definition of this system is in the supplementary material (Appendix A).

4.3 Metatheory

The bidirectional system is decidable. The $\delta' \leq \delta$ judgment is immediately decidable (taking the transitive closure of the rules), and the $A' \leq A$ judgment is decidable because each rule moves from larger type expressions to smaller ones. The same holds for \sqsubseteq , so directed consistency is decidable. The argument for the typing rules is slightly more interesting, as ChkCSub is a *stationary* rule (the premise and conclusion type the same expression). However, since this rule moves from checking to synthesis, and no stationary rule moves from synthesis to checking (in SynAnno, the expression becomes smaller), decidability holds.

Theorem 1 (Decidability of bidirectional typing).

- 1. Given Γ , e and A, the judgment $\Gamma \vdash e \Leftarrow A$ is decidable.
- 2. Given Γ and e, the judgment $\Gamma \vdash e \Rightarrow A$ is decidable.

The bidirectional system is sound with respect to the type assignment system: if ϵ is well-typed in the bidirectional system, it is well-typed in the type assignment system. (Proofs can be found in the supplementary material.)

Theorem 2 (Bidirectional soundness). *If* $\Gamma \vdash e \Leftarrow A$ *or* $\Gamma \vdash e \Rightarrow A$ *then* $\Gamma \vdash e : A$.

The bidirectional system is also complete: given e:A in the type assignment system, it is always possible to add annotations that make e well-typed in the bidirectional system. We write e=e' when e' is the same as e except that e' may have extra annotations.

Theorem 3 (Annotatability).

If $\Gamma \vdash e : A$ then there exist e' and e'' such that (1) $\Gamma \vdash e' \Leftarrow A$ where e =: e', and (2) $\Gamma \vdash e'' \Rightarrow A$ where e =: e''.

We also show that bidirectional typing derivations are robust under imprecision: if $e' \Leftarrow A'$, replacing annotations in e' with more imprecise types preserves typing. This corresponds to part 1 of the *gradual guarantee* of Siek et al. (2015, Theorem 5 on p. 11). An example illustrating this theorem's significance appears below in Section 4.4.

First, $\Gamma' \sqsubseteq \Gamma$ is defined pointwise. Second, let $e' \sqsubseteq e$ if, for each annotation $(e'_0 :: A')$ in e', there is a corresponding annotation $(e_0 :: A)$ in e where $A' \sqsubseteq A$. (For full inductive definitions, see Figures 15 and 16 in the supplementary material.)

Theorem 4 (Varying precision of bidirectional typing).

- 1. If $\Gamma' \vdash e' \Leftarrow A'$ and $e' \sqsubseteq e$ and $\Gamma' \sqsubseteq \Gamma$ and $A' \sqsubseteq A$ then $\Gamma \vdash e \Leftarrow A$.
- 2. If $\Gamma' \vdash e' \Rightarrow A'$ and $e' \sqsubseteq e$ and $\Gamma' \sqsubseteq \Gamma$ then there exists A such that $\Gamma \vdash e \Rightarrow A$ and $A' \sqsubseteq A$.

The nonempty context is needed for the proof cases for rules whose premises add to Γ' , such as ChkSumElim1.

An earlier version of the system, which did not allow gain of precision, has a weaker property: in that system, the given expression e is not necessarily typable, but there exists some "even more imprecise" expression e_j that is typable. See Theorem 14 in Appendix C.

Static system. As the static system is essentially a restriction of the bidirectional system, it is easy to turn a derivation in the static system into a derivation in the bidirectional system; this is the first part of the following theorem.

Completeness is more interesting: Given a bidirectional derivation whose *conclusion* is static—that is, the context Γ , expression e, and type A are within the restricted static grammar—we can build a derivation in the static system. This holds because of a subformula property: if there are no dynamic sums in Γ , e and A, then dynamic sums cannot appear anywhere in the bidirectional derivation.

Theorem 5 (Static soundness and completeness).

1. Soundness:

(a) If
$$\Gamma^{S} \vdash_{S} e^{S} \Leftarrow A^{S}$$
 then $\Gamma^{S} \vdash e^{S} \Leftarrow A^{S}$
(b) If $\Gamma^{S} \vdash_{S} e^{S} \Rightarrow A^{S}$ then $\Gamma^{S} \vdash e^{S} \Rightarrow A^{S}$.

2. Completeness:

(a) If
$$\Gamma^{S} \vdash e^{S} \Leftarrow A^{S}$$
 then $\Gamma^{S} \vdash_{S} e^{S} \Leftarrow A^{S}$.
(b) If $\Gamma^{S} \vdash e^{S} \Rightarrow A^{S}$ then $\Gamma^{S} \vdash_{S} e^{S} \Rightarrow A^{S}$.

This theorem directly corresponds to part 1 of Theorem 1 of Siek et al. (2015, p. 9) for "fully annotated" expressions. In that work, an expression is fully annotated if it has no gradual type annotations. In our system, expressions without annotations are static.

A corresponding theorem holds for the dynamic system and, in turn, corresponds to part 1 of Theorem 2 of Siek et al. (2015, p. 9). This is a rough correspondence: in our bidirectional system, dynamism is restricted to sum types and arises only through annotations. See Theorem 15 in the appendix.

4.4 Example

To see why Theorem 4 matters, consider the following example. Suppose we want to transform a program that uses dynamic sums into one that uses static sums. The program has a function f of type (Unit $+^2$ Int) \rightarrow Int, which is called with an argument x of type Unit $+^2$ Int.

let
$$f = (\lambda y, \dots) :: (Unit +^{2} Int) \rightarrow Int in \dots$$

let $x = e_{x} :: (Unit +^{2} Int) in$

(We assume that e_x is a checking form that needs an annotation; if e_x synthesizes (Unit $+^2$ Int), the annotation could be removed.) The programmer realizes that f only works with a right injection (perhaps its body is a one-armed case on inj_2), and that x should always be a right injection.

let
$$f = (\lambda y, \dots) :: (Unit +_2 Int) \rightarrow Int in \dots$$

let $x = e_x :: (Unit +_2 Int) in$
 $f x$

If this program type-checks and contains no remaining dynamic sum annotations, we know that f and x actually satisfy their annotations, and that the application f x will not cause any match or cast failures. Theorem 4 says that the annotations can be changed *one* at a time: the program with $+^2$ in the type of f but $+_2$ in the type of x is well-typed, as is the program with $+_2$ in the type of f but $+^2$ in the type of x:

let
$$f = (\lambda y. \cdots) :: (Unit +2 Int) \rightarrow Int in \dots$$

let $x = e_x :: (Unit +2 Int) in f x$

When synthesizing the type of f x, we use ChkCSub to gain precision in x:

$$\frac{\Gamma \vdash x \Rightarrow (\mathsf{Unit} +^{?} \mathsf{Int})}{(\mathsf{Unit} +_{2} \mathsf{Int}) \to \mathsf{Int}} \frac{(\mathsf{Unit} +^{?} \mathsf{Int}) \leadsto (\mathsf{Unit} +_{2} \mathsf{Int})}{\Gamma \vdash x \Leftrightarrow (\mathsf{Unit} +_{2} \mathsf{Int})} \frac{\mathsf{ChkCSub}}{\mathsf{Syn} \to \mathsf{Elim}}$$

A precise annotation that differs from the correct one, such as $Unit +_1 Int$ on x, may cause an error—either at type-checking time, or at run time. But a precise annotation that is correct will not cause an error, and constitutes a step towards a completely static program.

5. Target Language and Translation

5.1 Target Syntax and Semantics

Figure 9. Target syntax

Our target language is a statically typed λ -calculus with static sum types and a cast construct. The syntax is shown in Figure 9. We write M for target terms (expressions), W for values, and T for target types. The target sum constructors are all the static sum types from the source language: +, +₁, and +₂. In addition, we have a cast construct $\langle \varphi_2 \Leftarrow \varphi_1 \rangle M$, which casts from sum φ_1 to φ_2 . A failing cast, such as $\langle +_2 \Leftarrow + \rangle (\text{inj}_1 \ ())$, steps to the error term matchfail.

Much of the target type system (Figure 10) follows the source type assignment system, if that system were restricted to static sum types. Since the target lacks any dynamic sum constructors (like +²), target subtyping says only that +1 and +2 are subtypes of +; this corresponds to datasort refinement systems, where every datasort is a subsort of a "top" datasort for the type being refined. Our type-directed translation (Section 5.2) transforms the gradual property of types into dynamic checks at the term level; rule TCast casts between sum constructors, and rule TMatchfail gives any type to matchfail, which represents the failure of a cast.

Our target language (Figure 11) has a standard call-by-value small-step semantics, extended with casts. Evaluation contexts \mathcal{E} are terms with a hole [], where the hole represents a term in an evaluation position: if target term $M = \mathcal{E}[M_0]$, and M_0 reduces—written $M_0 \mapsto_R M'_0$ —then the larger term M steps to $\mathcal{E}[M'_0]$.

The cast reduction rules represent the three relevant situations: (1) an *upcast* to a supertype succeeds (ReduceUpcast); (2) a downcast from + to $+_i$ succeeds if i matches the injection (ReduceCastSuccess); (3) a downcast from + to $+_i$ fails, reducing to matchfail, if i doesn't match the injection (ReduceCastFailure).

$\textbf{5.2} \quad \textbf{Type-Directed Translation} \hookrightarrow$

9

To translate source programs into target programs with explicit casts between sum types, we use a judgment $\Gamma \vdash e : A \hookrightarrow M$. Most of the rules (in Figure 12) follow the type assignment rules, with the addition of $\hookrightarrow M$. Given e of type A, the rules produce a target term M of type T where T is the translation of A, written |A|.

Figure 10. Target subtyping and typing

Figure 11. Small-step semantics of the target language

This translation (Figure 12, top) maps the source sums + and +² to the target sum +, and maps the other source sums to +₁.

We extend type assignment, rather than the bidirectional system, because translation should be independent of bidirectionality: Type assignment is stable under variations in the bidirectional "recipe", so if we decided to synthesize a type for (), we could leave the translation untouched. That said, an implementation would be based on a bidirectional version of the translation—replacing ":" with "\(=\)" or "\(\Rightarrow\)", following Figure 7.

The interesting translation rule is STCSub, which inserts a coercion context C. This context coerces between two directed-consistent types, so it composes up to three coercions (cf. Figure 5): from a more imprecise type to a less imprecise type, from that type to a supertype, and from the supertype to a more imprecise type.

Our coercion judgment $A'\Rightarrow A\hookrightarrow \mathcal{C}$ produces a context \mathcal{C} , a target term containing a hole such that, if M has type T'=|A'|, then $\mathcal{C}[M]$ has type T=|A|. Rule CoeUnit produces a hole, which behaves as the identity function. Rule Coe \rightarrow produces a function: given a hole [] filled by a function of type $T_1'\to T_2'$, it constructs $\lambda x.\,\mathcal{C}_2[[]\,\mathcal{C}_1[x]]$. This function has type $T_1\to T_2$: it applies cast \mathcal{C}_1 to x, yielding a value of type T_1' . Applying the original function yields an T_2' , which cast \mathcal{C}_2 transforms into an T_2 .

Three rules generate coercions between sum types: CoeCase1L, CoeCase1R, and CoeCase2. The first two rules handle sums that are definitely a left injection, or definitely a right injection: we apply CoeCase1L whenever we are coercing from $A_1' \delta' A_2'$ where δ' is $+_1$ or $+_1^2$, and CoeCase1R when δ' is $+_2$ or $+_2^2$.

In CoeCase1L, we recursively generate a coercion C_1 from A'_1 , and a cast C_3 from δ' . The conclusion generates a coercion by matching the given value (replacing []) against $\inf_1 x_1$, constructing $\inf_1 (C_1[x_1])$, to which we apply C_3 . CoeCase1R is symmetric.

CoeCase2 handles the cases not covered by the previous two rules. In addition to doing the work of the previous two rules, it generates casts \mathcal{C}_1' and \mathcal{C}_2' , applying them in each arm. According to STSumIntro, an injection inj_1 has a type whose sum constructor is $+_1^2$, so CoeCase2 applies \mathcal{C}_1' which takes $+_1^2$ to δ' . Similarly, the rule applies \mathcal{C}_2' , which takes $+_2^2$ to δ' . Since CoeCase2 applies \mathcal{C}_3 (from δ' to δ) to the entire case, the result will be δ .

5.3 Target Precision ≼

We will prove that more precise source typings—differently annotated versions of the same source expression—produce more precise target terms. We will also prove that precision of the target terms is preserved by stepping, and that if a more precise target term converges (steps to a value), so does a less precise target term.

$$\begin{array}{c} \text{Sum translation} |\delta| = \varphi \\ |+| = |+^2| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = + \\ |+| = |+| = |+| = |+| = + \\ |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |+| = |$$

Figure 12. Type-directed translation

Our relation, and the form of the result, were inspired by the approximation relation of Ahmed et al. (2011), as well as the term precision relation of Siek et al. (2015).

For source expressions, we defined $e' \sqsubseteq e$ simply by applying \sqsubseteq to the types in annotations. For target terms, we have no type precision relation; the target type system only has static sums, so $T' \sqsubseteq T$ would degenerate to T' = T. Instead, we define target precision \preccurlyeq for terms only.

If $e' \sqsubseteq e$, and these expressions translate to M' and M respectively, we want to show M' \preccurlyeq M. The difference between e' and e is only in their annotations, so M' and M must share a lot of structure—except that different annotations may lead to different casts. Thus, most of the rules in Figure 13 are homomorphic.

What about casts, which can step to matchfail? A static source typing is very precise, and the target term it produces never fails, so we might expect a more precisely typed term to "fail less"—but this would lead us astray. A better intuition is that imprecisely typed code "doesn't care", so it tends *not* to fail—while precisely typed code *can* fail, if it collides with imprecisely typed code. Therefore, terms with casts should be *more* precise than terms without. In addition, since casts can step to matchfail, and we want stepping to preserve precision, matchfail $\leq M$ for any M.

Given two terms with casts $M' = \langle \varphi_2' \Leftarrow \varphi_1' \rangle$ and $M = \langle \varphi_2 \Leftarrow \varphi_1 \rangle$, we will consider M' more precise than M if the cast in M' is more precise: $\langle \varphi_2' \Leftarrow \varphi_1' \rangle \preccurlyeq \langle \varphi_2 \Leftarrow \varphi_1 \rangle$. Let ac be a cast; it must be either a safe cast sc like $\langle + \Leftarrow + \rangle$ or $\langle + \Leftarrow +_1 \rangle$, a backward cast bc of the form $\langle +_1 \Leftarrow +_2 \rangle$. These are classified by the grammar in Figure 13.

Equal casts should be equally precise, so rule Cast \preccurlyeq Refl makes the relation ac' \preccurlyeq ac reflexive. Following the idea that the more precisely typed term should "fail more", a safer cast should be *less* precise; this leads to CastM \preccurlyeq B, CastB \preccurlyeq S, and CastM \preccurlyeq S.

The other rules are subtle. They compare *particular* safe casts and/or backward casts, relying implicitly on typing. For example, the last rule says (with i=1) that $\langle + \Leftarrow + \rangle \preccurlyeq \langle + \Leftarrow +_1 \rangle$. We will ultimately need to show that if the cast on the left succeeds, so does the cast on the right. The left-hand cast is $\langle + \Leftarrow + \rangle$, which always succeeds. The right-hand cast succeeds if it is given $\inf_{j=1}^{\infty} I_j$. If the value being cast is well-typed, then (by TCast) it will indeed have type $+_1$.

Finally, note that a more precise source typing may result in a one-armed case in a coercion, while the less precise typing results in a two-armed case. For example, +² is less precise than +₁;

Figure 13. Precision ≼ on target terms

coercing $+_1$ to + results in one-armed case, and coercing $+^2$ to + results in a two-armed case. Hence, a one-armed case can be more precise than a two-armed case.

5.4 Metatheory

The target system satisfies preservation and progress:

Theorem 6 (Type preservation).

If $\cdot \vdash M : T$ and $M \mapsto M'$ then $\cdot \vdash M' : T$.

Theorem 7 (Progress).

If $\cdot \vdash M : T$ then either (a) M is a value, or (b) there exists M' such that $M \mapsto M'$, or (c) M = matchfail.

By itself, the above progress statement leaves open the possibility that a well-typed target term M will step to matchfail. However, if M has no casts, it will not step to matchfail.

Theorem 8 (matchfail-freeness).

If M is cast-free and matchfail-free and $M \mapsto M'$ then M' is cast-free and matchfail-free.

For cast-free terms, combining Theorems 7 and 8 gives a version of progress without the possibility of match failure.

Corollary. If M is cast-free and matchfail-free and $\cdot \vdash M : T$ then either (a) M is a value, or (b) there exists M' such that $M \mapsto M'$.

We also prove that the translation takes well-typed source programs to well-typed target programs. The theorem takes a type assignment derivation, but Theorem 2 can produce such a derivation from a bidirectional typing derivation.

Theorem 9 (Translation soundness).

If $\Gamma \vdash e : A$ then there exists M such that $\Gamma \vdash e : A \hookrightarrow M$ and $|\Gamma| \vdash M : |A|$.

The proof relies on several lemmas, e.g. that the generated coercions C are well-typed; see the supplementary material.

A great advantage of static typing is that, for a suitable definition of "wrong", static programs don't go wrong. The theorem below proves that translating a static program yields a target term M that has no casts; by Theorem 8, M will never step to matchfail.

Theorem 10 (Static derivations don't have match failures). If $\Gamma^{S} \vdash e^{S} \Leftarrow A^{S}$ or $\Gamma^{S} \vdash e^{S} \Rightarrow A^{S}$ then there exists M such that $\Gamma^{S} \vdash e^{S} : A^{S} \hookrightarrow M$ and M is free of casts and matchfail.

Together, preservation and progress correspond to Theorem 3 (type safety) of Siek et al. (2015, p. 9). Their *blame-subtyping* Theorem 4 says that safe casts (casts from a subtype to a supertype) cannot be blamed (cannot fail); our translation does not insert safe casts at all, and our Theorem 10 shows that expressions without dynamic sums produce target terms without casts.

The remaining results concern precision. We show that more precise annotations translate to more precise terms, that target precision is preserved by stepping, and that if a target term converges, then a less precise version also converges.

We must note that the first of these results, Theorem 11, uses a modified version of the translation: one that always inserts casts, even safe ones; this simplifies part of the proof. In effect, the modified translation (Figure 21 in the appendix) does not have rule CoeSub and always uses rule CoeCast. Similarly, we modify CoeCase1L and CoeCase1R to always insert casts within each arm, like \mathcal{C}_1' and \mathcal{C}_2' in CoeCase2. Since the only difference is the presence of casts that cannot fail, the terms generated by either translation must both step to the same value, or both generate matchfail.

Theorem 11 (Translation preserves precision). Suppose $\Gamma' \sqsubseteq \Gamma$ and $e' \sqsubseteq e$.

1. If
$$\Gamma' \vdash e' \Leftarrow A'$$
 and $\Gamma \vdash e \Leftarrow A$ and $A' \sqsubseteq A$ then $\Gamma' \vdash e' : A' \hookrightarrow M'$ and $\Gamma \vdash e : A \hookrightarrow M$ where $M' \preccurlyeq M$.
2. If $\Gamma' \vdash e' \Rightarrow A'$ and $\Gamma \vdash e \Rightarrow A$ then $\Gamma' \vdash e' : A' \hookrightarrow M'$ and $\Gamma \vdash e : A \hookrightarrow M$ where $A' \sqsubseteq A$ and $M' \preccurlyeq M$.

Theorem 12 (Stepping preserves precision).

If $\cdot \vdash M_1': T_1'$ and $\cdot \vdash M_1: T_1$ and $M_1' \preccurlyeq M_1$ and $M_1' \mapsto M_2'$ then either

(a) M_1 is a value and $M_2' \leq M_1$, or

(b) there exists M_2 such that $M_1 \mapsto M_2$ and $M_2' \leq M_2$, or

(c) $M_1 = \text{matchfail} \text{ and } M_2' \leq M_1$.

12

Definition 1. A closed term M converges if $M \mapsto^* W$ for some value W, and diverges if the stepping sequence never terminates.

Note that matchfail neither converges nor diverges, and that divergence is not possible in our language.

Theorem 13 (\preccurlyeq respects convergence). If $M' \preccurlyeq M$ where $\cdot \vdash M' : T'$ and $\cdot \vdash M : T$ and M' converges then M also converges.

If $M' \preccurlyeq M$, and they converge to injections $\operatorname{inj}_i W'$ and $\operatorname{inj}_k W$, then Theorem 13 gives $\operatorname{inj}_i W' \preccurlyeq \operatorname{inj}_k W$. By inversion on the definition of \preccurlyeq , we have $\mathfrak{i} = k$. Similar results would hold if \preccurlyeq were extended for base types.

Together with Theorem 11, this means that if we translate two source expressions $e' \sqsubseteq e$ to M' and M, and M' converges to a value of base type, M will converge to the same value. This corresponds to Theorem 5 (gradual guarantee), part 2, of Siek et al. (2015).

6. Related Work

Sums and refinements. Sum types are well-established in a variety of programming languages, though practical languages tend to embed them within larger mechanisms: ML datatypes can encode sums, but also recursion. Refinement type systems, such as datasort refinements (Freeman and Pfenning 1991; Davies 2005) and indexed types (Xi and Pfenning 1999), have been built on these larger mechanisms. This gives a close connection to practice, but needs additional machinery such as constructor types and signatures. Such machinery is not central to our investigation; in contrast, we distill datasort refinements to one essential feature: distinguishing whether we have a left or right injection.

These systems often have a refinement relation \sqsubseteq : if A is a sort (refined type) and τ is an unrefined type, $A \sqsubseteq \tau$ says that A refines τ . Both the symbol and the high-level concept resemble our relation $A' \sqsubseteq A$, but the refinement relation is more rigid: it cannot compare two sorts, or two unrefined types, and it certainly cannot derive $(A_1 \to A) \sqsubseteq (A_1 \to \tau)$, where $(A_1 \to \tau)$ mixes a refined type A_1 with an unrefined type τ . Nonetheless, the covariance of this relation on function types—in contrast to subtyping, which must be contravariant—made us more confident that our precision relation should be covariant.

Koot and Hage (2015) formulate a constraint-based type system that analyzes pattern matches, using a characterization of data somewhat reminiscent of datasort refinements. Their system needs no type annotations, but is (necessarily) incomplete.

Gradual typing. Our approach to expressing uncertainty in a type system was inspired by gradual typing, introduced by Siek and Taha (2006), in which? (often written \star) is an uncertain type (it could be Int, a function type, or anything else). We confine uncertainty to refinement properties of sum types, making the effect on the overall type system less dramatic; still, several mechanisms of gradual typing appear in our work. For example, we also have precision relations on types and (through annotations) expressions.

Our directed consistency is somewhat similar to consistent subtyping for gradual object-based languages (Siek and Taha 2007). Consistent subtyping augments subsumption with consistent equality (roughly, gain *and* loss of precision) on either the subtype or supertype, but not both. Drawing on abstract interpretation, Garcia et al. (2016) give a different but equivalent formulation of consistent subtyping. In these systems, the underlying subtyping relation is defined over static types only. Allende et al. (2014) also have a notion of directed consistency, but the connection to our relation is less clear.

Siek et al. (2015) propose several criteria as desirable for gradual type systems. We prove properties that correspond to some of their criteria: Theorems 5 and 15 correspond to the first parts of Theorems 1 and 2 of Siek et al. (2015), our Theorem 10 corresponds to their Theorem 4, our Theorem 4 corresponds to part 1 of their Theorem 5 (gradual guarantee), and our Theorems 11 and 13 corresponds to part 2 of their Theorem 5.

Some systems of gradual typing include a notion of *blame* (Wadler and Findler 2009), associating program labels to casts so that a failing cast "blames" some program location. It may be possible to incorporate blame into our approach; we omit it to focus on other issues.

We are not the first to apply ideas from gradual typing to less-traditional areas: for example, Bañados Schwerter et al. (2014) develop a gradual effect system, and McDonell et al. (2016) develop a tool for moving between ADTs and more precise GADTs.

Bidirectional typing. Originating as folklore and first discussed explicitly by Pierce and Turner (1998), bidirectional typing has been used extensively in type systems for which full inference is undecidable or otherwise problematic (Freeman and Pfenning 1991; Coquand 1996; Xi and Pfenning 1999; Davies and Pfenning 2000; Pientka 2008). A strength of many bidirectional type systems, sometimes overlooked, is that they have some variety of subformula property. In some systems, this property serves to make type checking more feasible—for example, for Davies (2005) and Dunfield (2007), it controls the spread of intersection types. For Dunfield (2015), where evaluation order is implicit in terms and explicit in types, it prevents the spontaneous generation of by-name types; in our system, it prevents the spontaneous generation of gradual sum types.

The gradual type system of Garcia and Cimini (2015, p. 306) is not bidirectional, but enjoys a similar property: "dynamicity [the uncertain type?] is introduced only via program annotations". However, their rules can be viewed as a bidirectional system that always synthesizes, except at annotations.

7. Future Work

We plan to implement the bidirectional type system, which will allow us to test whether our approach is practical. We are particularly interested in whether our formulation of precision, combined with the annotation discipline of bidirectional typing, strikes a good balance: the annotation burden should be reasonable, but imprecision should not appear out of nowhere. Also, it is unclear whether programmers would have any use for the sum types $+^2_i$ and $+^*_i$; if not, error messages should read "expected $+_1$ or $+^2$ " rather than "expected $+^*_1$ ", for example.

We would also like to enrich the language with intersection types, recursive types, and polymorphism. Intersection types are important for datasort refinements: for example, if we encode booleans as Unit + Unit, the datasorts True and False are Unit +1 Unit and Unit +2 Unit. Then negation should have type (True \rightarrow False) \cap (False \rightarrow True). We also want to evaluate the run-time efficiency of coercions—a common concern in gradual type systems.

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Appendix to "Sums of Uncertainty: Refinements go gradual" (POPL 2017)

A. Dynamic System

$$\begin{array}{ll} \text{Dynamic expressions} & e^D ::= () \mid x \mid \lambda x. \, e^D \mid e_1^D \, e_2^D \mid \text{inj}_i \, e^D \mid (e^D :: A^D) \\ & \mid \text{case}(e^D, \text{inj}_1 \, x_1. e_1^D, \text{inj}_2 \, x_2. e_2^D) \mid \text{case}(e^D, \text{inj}_i \, x. e_i^D) \\ \\ \text{Dynamic types} & A^D ::= \text{Unit} \mid A_1^D +^? \, A_2^D \mid A_1^D \rightarrow A_2^D \\ \\ \text{Dynamic typing contexts} & \Gamma^D ::= \cdot \mid \Gamma^D, x : A^D \\ \end{array}$$

$$\begin{array}{c|c} \Gamma^D \vdash_D e^D \Leftarrow A^D & \text{Under typing context } \Gamma^D, \text{ expression } e^D \text{ checks against type } A^D \\ \Gamma^D \vdash_D e^D \Rightarrow A^D & \text{Under typing context } \Gamma^D, \text{ expression } e^D \text{ synthesizes type } A^D \end{array}$$

$$\begin{array}{c} \Gamma^D \vdash_D e^D \Leftarrow A^D \\ \Gamma^D \vdash_D e^D \Rightarrow A^D \end{array} \ \, \text{Under typing context Γ^D, expression e^D checks against type A^D \\ \hline \Gamma^D \vdash_D e^D \Rightarrow A^D \end{array} \ \, \text{Under typing context Γ^D, expression e^D synthesizes type A^D \\ \hline \frac{\Gamma^D(x) = A^D}{\Gamma^D \vdash_D x \Rightarrow A^D} \ \, \text{DVar} \qquad \frac{\Gamma^D \vdash_D e^D \Rightarrow A^D}{\Gamma^D \vdash_D e^D \Leftrightarrow A^D} \ \, \text{DSub} \qquad \frac{\Gamma^D \vdash_D e^D \Leftrightarrow A^D}{\Gamma^D \vdash_D (e^D :: A^D) \Rightarrow A^D} \ \, \text{DAnno} \qquad \frac{\Gamma^D \vdash_D () \Leftrightarrow \text{Unit}}{\Gamma^D \vdash_D () \Leftrightarrow \text{Unit}} \ \, \text{DUnitIntro} \\ \hline \frac{\Gamma^D, x : A^D_1 \vdash_D e^D \Leftrightarrow A^D_2}{\Gamma^D \vdash_D \lambda x. \, e^D \Leftrightarrow A^D_1 \rightarrow A^D_2} \ \, \text{D} \rightarrow \text{Intro} \qquad \frac{\Gamma^D \vdash_D e^D_1 \Rightarrow A^D_1 \rightarrow A^D_2}{\Gamma^D \vdash_D e^D_1 \Leftrightarrow A^D_2 \rightarrow A^D_2} \ \, \text{D} \rightarrow \text{Elim} \qquad \frac{\Gamma^D \vdash_D e^D \Leftrightarrow A^D_1}{\Gamma^D \vdash_D \text{inj}_i \, e^D \Leftrightarrow (A^D_1 + ^2 A^D_2)} \ \, \text{D} \rightarrow ^2 \text{Intro} \\ \hline \frac{\Gamma^D \vdash_D e^D_0 \Rightarrow A^D_1 + ^2 A^D_2}{\Gamma^D \vdash_D \text{case}(e^D_0, \text{inj}_i \, x. e^D) \Leftrightarrow A^D} \ \, \text{D} \rightarrow ^2 \text{Elim} 1 \qquad \frac{\Gamma^D \vdash_D e^D_0 \Rightarrow A^D_1 + ^2 A^D_2}{\Gamma^D \vdash_D \text{case}(e^D_0, \text{inj}_i \, x. e^D) \Leftrightarrow A^D} \ \, \text{D} \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline \Gamma^D \vdash_D \text{case}(e^D, \text{inj}_i \, x. e^D) \Leftrightarrow A^D \qquad D \rightarrow ^2 \text{Elim} 2 \\ \hline$$

Figure 14. The dynamic system: the bidirectional system restricted to +

Figure 14 shows the syntax and typing rules for the dynamic system—the restriction of the bidirectional type system to the dynamic sum +².

Omitted Definitions

 $e' \sqsubseteq e$ | Expression e' is more precise than e

$$\frac{e'\sqsubseteq e}{\lambda x. e'\sqsubseteq \lambda x. e} \qquad \frac{e'_1\sqsubseteq e_1}{e'_1e'_2\sqsubseteq e_2} \qquad \frac{e'_2\sqsubseteq e_2}{e'_1e'_2\sqsubseteq e_1} \qquad \frac{e'\sqsubseteq e}{(\operatorname{inj}_ie')\sqsubseteq (\operatorname{inj}_ie)} \qquad \frac{e'\sqsubseteq e \qquad A'\sqsubseteq A}{(e'::A')\sqsubseteq (e::A)}$$

$$\frac{e'\sqsubseteq e \qquad e'_1\sqsubseteq e_1 \qquad e'_2\sqsubseteq e_2}{\operatorname{case}(e',\operatorname{inj}_1x_1.e'_1,\operatorname{inj}_2x_2.e'_2)\sqsubseteq \operatorname{case}(e,\operatorname{inj}_1x_1.e_1,\operatorname{inj}_2x_2.e_2)} \qquad \frac{e'\sqsubseteq e \qquad e'_1\sqsubseteq e}{\operatorname{case}(e',\operatorname{inj}_1x.e'_1)\sqsubseteq \operatorname{case}(e,\operatorname{inj}_1x.e_1)}$$

 $\Gamma' \sqsubseteq \Gamma$ Typing context Γ' is more precise than Γ

$$\frac{\Gamma' \sqsubseteq \Gamma \qquad A' \sqsubseteq A}{(\Gamma', x : A') \sqsubseteq (\Gamma, x : A)}$$

Figure 15. Precision on expressions and contexts

Several results involve precision of expressions and typing contexts, shown in Figure 15; these are the straightforward lifting of type precision (Figure 4).

e' =: e | Expression e' is annotative-ly equivalent to e

$$\frac{e' =: e}{\lambda x. e' =: \lambda x. e} \qquad \frac{e' =: e}{e' =: (e :: A)}$$

$$\frac{e' =: e}{\lambda x. e' =: \lambda x. e} \qquad \frac{e'_1 =: e_1 \quad e'_2 =: e_2}{e'_1 e'_2 =: e_1 e_2} \qquad \frac{e' =: e}{(inj_i e') =: (inj_i e)} \qquad \frac{e' =: e \quad A' = A}{(e' :: A') =: (e :: A)}$$

$$\frac{e' =: e \quad e'_1 =: e_1 \quad e'_2 =: e_2}{case(e', inj_1 x_1. e'_1, inj_2 x_2. e'_2) =: case(e, inj_1 x_1. e_1, inj_2 x_2. e_2)} \qquad \frac{e' =: e \quad e'_i =: e_i}{case(e', inj_i x. e'_i) =: case(e, inj_i x. e_i)}$$

Figure 16. Annotation equivalence

C. Differences from the Original Version

The paper that was submitted to POPL differs in two important ways from the final version.

No directed consistency. In the final version, ChkCSub, SCSub, etc. allow (a) gain of precision, (b) subtyping, and (c) loss of precision, formulated via directed consistency. In contrast, the original system had (in each system) two rules: one rule that allowed subtyping (exactly like a traditional subsumption rule), and one rule that allowed loss of precision. For example, the bidirectional system had

$$\frac{\Gamma \vdash e \Rightarrow A' \qquad A' \leq A}{\Gamma \vdash e \Leftarrow A} **\mathsf{ChkSub} \qquad \frac{\Gamma \vdash e \Rightarrow A' \qquad A' \sqsubseteq A}{\Gamma \vdash e \Leftarrow A} **\mathsf{ChkImp}$$

These rules could not type the same expression without an extra annotation (to transition from the checking conclusion of one rule to the synthesizing conclusion of the other).

Moreover, there was no rule to gain precision. In a traditional gradual type system, this would be completely untenable: the point of the "unknown type" in a gradual system is that it can be downcasted to a static type. In the previous version of our system, programmers could write coercions "by hand":

$$f: (A_1 +_1 A_2) \rightarrow B, y: (A_1 +^? A_2) \vdash f(case(y, inj_1 x.x)) \Rightarrow B$$

But this requires a change to the expression that goes beyond changing an annotation: the expression itself is being changed.

The lack of a way to gain precision, combined with the need for an extra annotation to use subtyping *and* loss of precision, meant that the varying precision property—Theorem 4 in the final version—did not hold. A weaker property—Theorem 14, below—did hold, but this property only provides that some expression e_i , which could be more imprecise than e_i is well typed.

Different definition of imprecision. In Section 2, we explained why $+_1^* \sqsubseteq +_1^2$ doesn't make sense. We also argued against $+_1^2 \sqsubseteq +_1^*$, on the basis that in directed consistency (SCSub) one could gain precision from $+_1^*$ to $+_1^2$, then use subtyping from $+_1^2$ to $+_2^*$. In the old system, there was no gain of precision, and even loss of precision could not be combined with subtyping (without extra annotation). Thus, we saw no clear argument against $+_1^2 \sqsubseteq +_1^*$, and included it in the relation. However, in the absence of gain of precision, the only way the type system could use this was by moving from $+_1^2$ to $+_1^*$, which was also possible via subtyping.

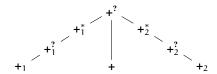


Figure 17. Original, obsolete definition of precision

C.1 Original, weak version of varying precision

Theorem 4 does not hold for the original system. Instead, the following holds, where $e \sqsubseteq : e_j$ means that e_j is a version of e with more-imprecise annotations (like $e' \sqsubseteq e$) and extra annotations. For example, $x \sqsubseteq : (x :: A)$.

Theorem 14 (Weak version of varying precision).

If Γ' ⊢ e' ← A' and e' ⊆ e and Γ' ⊆ Γ
 then there exist e; and A such that Γ ⊢ e; ← A and e ⊆: e; and A' ⊆ A.
 If Γ' ⊢ e' ⇒ A' and e' ⊆ e and Γ' ⊆ Γ
 then there exist e; and A
 such that Γ ⊢ e; ⇒ A and e ⊆: e; and A' ⊆ A.

Given $e' \sqsubseteq e$, this weak version of varying precision yields some e_j that may be more imprecise, $e' \sqsubseteq e \sqsubseteq e_j$. This is needed because—in the absence of ChkCSub, which allows precision to be adjusted whenever subsumption is used—a more imprecise annotation may require changing other annotations to make them more imprecise. For example, suppose we are given

$$\begin{array}{l} e' \ = \ \left((\lambda x. \, (x :: B +_2 B)) :: (B +_2 B) \rightarrow (B +_2 B) \right) \\ e \ = \ \left((\lambda x. \, (x :: B \stackrel{?}{+^2} B)) :: (B +_2 B) \rightarrow (B +_2 B) \right) \end{array}$$

We can synthesize $A' = (B +_2 B) \to (B +_2 B)$ for e', but not for e, because the inner annotation on x makes the λ fail to check against the outer annotation. But we can produce $e_j = ((\lambda x. (x :: B +^2 B)) :: (B +_2 B) \to (B +^2 B))$. Now the uses of $+^2$ match, and e_j synthesizes $A = (B +_2 B) \to (B +^2 B)$. The remaining $+_2$ is okay, because of **ChkImp: in $(x :: B +^2 B)$, we have $\Gamma(x) = B +_2 B$, which is less imprecise than $B +^2 B$.

D. Proofs

D.1 Source System

D.1.1 Subtyping

Lemma 1 (Subtyping inversion).

- 1. If Unit \leq A then A = Unit.
- 2. If $A' \leq Unit$ then A' = Unit.
- 3. If $A_1' \delta' A_2' \leq A$ then $A = A_1 \delta A_2$ where $A_1' \leq A_1$ and $A_2' \leq A_2$ and $\delta' \leq \delta$.
- 4. If $A' \leq A_1 \delta A_2$ then $A' = A'_1 \delta' A'_2$ where $A'_1 \leq A_1$ and $A'_2 \leq A_2$ and $\delta' \leq \delta$.
- 5. If $A_1' \to A_2' \le A$ then $A = A_1 \to A_2$ where $A_1 \le A_1'$ and $A_2' \le A_2$.
- 6. If $A' \leq A_1 \rightarrow A_2$ then $A' = A'_1 \rightarrow A'_2$ where $A_1 \leq A'_1$ and $A'_2 \leq A_2$.

Proof.

- 1. By case analysis on Unit $\leq A$.
 - Case Unit \leq Unit: Immediate that A = Unit.
- 2. Symmetric to the previous statement, hence omitted.
- 3. By case analysis on $A_1' \delta' A_2' \leq A$.
 - Case $A_1' \delta' A_2' \le A_1 \delta A_2$: Immediate as $A = A_1 \delta A_2$ and subderivations are $A_1' \le A_1$ and $A_2' \le A_2$ and $\delta' \le \delta$.
- 4. Symmetric to the previous statement, hence omitted.
- 5. By case analysis on $A'_1 \rightarrow A'_2 \leq A$.
 - Case $A_1' \to A_2' \le A_1 \to A_2$: Immediate as $A = A_1 \to A_2$ and subderivations are $A_1 \le A_1'$ and $A_2' \le A_2$.
- 6. Symmetric to the previous statement, hence omitted.

Lemma 2 (Reflexivity of subtyping).

For all types A, it is the case that $A \leq A$.

Proof. By induction on the structure of A.

- Case A = Unit: By the definition of precision, $A \le A$.
- Case $A = A_1 \delta A_2$: By the induction hypothesis, $A_1 \leq A_1$ and $A_2 \leq A_2$. By the reflexivity of subsum, $\delta \leq \delta$. Thus, by the definition of subtyping, $A \leq A$.
- Case $A = A_2 \rightarrow A_2$: By the induction hypothesis, $A_1 \le A_1$ and $A_2 \le A_2$. Thus, by the definition of subtyping, $A \le A$.

Lemma 3 (Transitivity of subtyping).

If $A_1 \leq A_2$ and $A_2 \leq A_3$ then $A_1 \leq A_2$

Proof. By induction on the structure of A_2 .

• Case $A_2 = Unit$:

$$A_1 \leq \mathsf{Unit} \quad \mathsf{Given}$$

$$\mathsf{Unit} \leq A_3 \qquad \mathsf{Given}$$

 $A_1 = Unit$ By Lemma 1 (Subtyping inversion)

 $A_3 = Unit$ By Lemma 1 (Subtyping inversion)

 $Unit \leq Unit$ By Lemma 2 (Reflexivity of subtyping)

 $A_1 \leq A_3$ Equivalent

• Case $A_2 = A_{12} \delta_2 A_{22}$:

$$A_1 \le A_{12} \, \delta_2 \, A_{22}$$
 Given

$$A_1 = A_{11} \delta_1 A_{21}$$
 By Lemma 1 (Subtyping inversion)

″

$$A_{11} \leq A_{12}$$
 "

$$A_{21} \leq A_{22} \qquad \qquad "$$

$$\begin{array}{c} \lambda_{21} \leq \lambda_{22} \\ \delta_1 \leq \delta_2 \end{array}$$

$$A_{12} \, \delta_2 \, A_{22} \le A_3 \qquad \qquad \text{Given}$$

$$A_3 = A_{13} \delta_3 A_{23}$$
 By Lemma 1 (Subtyping inversion)

$$A_{12} \leq A_{13} \qquad "$$

$$A_{22} \leq A_{22} \qquad "$$

$$A_{22} \le A_{23}$$

$$\delta_2 \leq \lambda_{23}$$

$$\delta_2 \leq \delta_3$$

17 2020/9/22

D.1.2 Precision

Lemma 4 (Precision inversion).

- 1. If Unit $\sqsubseteq A$ then A = Unit.
- 2. If $A' \sqsubseteq Unit then A' = Unit$.
- 3. If $A_1' \delta' A_2' \sqsubseteq A$ then $A = A_1 \delta A_2$ where $A_1' \sqsubseteq A_1$ and $A_2' \sqsubseteq A_2$ and $\delta' \sqsubseteq \delta$.
- 4. If $A' \sqsubseteq A_1 \delta A_2$ then $A' = A'_1 \delta' A'_2$ where $A'_1 \sqsubseteq A_1$ and $A'_2 \sqsubseteq A_2$ and $\delta' \sqsubseteq \delta$.
- 5. If $A_1' \to A_2' \sqsubseteq A$ then $A = A_1 \to A_2$ where $A_1' \sqsubseteq A_1$ and $A_2' \sqsubseteq A_2$.
- 6. If $A' \sqsubseteq A_1 \rightarrow A_2$ then $A' = A'_1 \rightarrow A'_2$ where $A'_1 \sqsubseteq A_1$ and $A'_2 \sqsubseteq A_2$.

Proof.

- 1. By case analysis on Unit $\sqsubseteq A$.
 - Case Unit \sqsubseteq Unit: Immediate that A = Unit.
- 2. Symmetric to the previous statement, hence omitted.
- 3. By case analysis on $A_1' \delta' A_2' \sqsubseteq A$.
 - Case $A_1' \delta' A_2' \sqsubseteq A_1 \delta A_2$: Immediate as $A = A_1 \delta A_2$ and subderivations are $A_1' \sqsubseteq A_1$ and $A_2' \sqsubseteq A_2$ and $\delta' \sqsubseteq \delta$.
- 4. Symmetric to the previous statement, hence omitted.
- 5. By case analysis on $A'_1 \rightarrow A'_2 \sqsubseteq A$.
 - Case $A_1' \to A_2' \sqsubseteq A_1 \to A_2$: Immediate as $A = A_1 \to A_2$ and subderivations are $A_1' \sqsubseteq A_1$ and $A_2' \sqsubseteq A_2$.
- 6. Symmetric to the previous statement, hence omitted.

Lemma 5 (Reflexivity of precision).

For all types A, it is the case that $A \sqsubseteq A$.

Proof. By induction on the structure of A.

- Case A = Unit: By the definition of precision, $A \sqsubseteq A$.
- Case $A = A_1 \delta A_2$: By the induction hypothesis, $A_1 \sqsubseteq A_1$ and $A_2 \sqsubseteq A_2$. By the reflexivity of precision on sums, $\delta \sqsubseteq \delta$. Thus, by the definition of subtyping, $A \sqsubseteq A$.
- Case $A = A_2 \rightarrow A_2$: By the induction hypothesis, $A_1 \sqsubseteq A_1$ and $A_2 \sqsubseteq A_2$. Thus, by the definition of subtyping, $A \sqsubseteq A$.

Lemma 6 (Transitivity of precision).

If $A_1 \sqsubseteq A_2$ and $A_2 \sqsubseteq A_3$ then $A_1 \sqsubseteq A_2$.

Proof. By induction on the structure of A_2 .

• Case $A_2 = Unit$:

18 2020/9/22

```
A_1 \sqsubseteq Unit Given
      Unit \Box A_3
                           Given
        A_1 = \mathsf{Unit}
                          By Lemma 4 (Precision inversion)
        A_3 = \mathsf{Unit}
                          By Lemma 4 (Precision inversion)
      Unit \sqsubseteq Unit
                          By Lemma 5 (Reflexivity of precision)
        A_1 \sqsubseteq A_3
                           Equivalent
  • Case A_2 = A_{12} \delta_2 A_{22}:
                 A_1 \sqsubseteq A_{12} \, \delta_2 \, A_{22}
                                             Given
                  A_1 = A_{11} \, \delta_1 \, A_{21}
                                             By Lemma 4 (Precision inversion)
                A_{11} \sqsubseteq A_{12}
                                              "
                A_{21} \sqsubseteq A_{22}
                                              "
                  \delta_1 \sqsubseteq \delta_2
       A_{12} \delta_2 A_{22} \sqsubseteq A_3
                                             Given
                  A_3 = A_{13} \, \delta_3 \, A_{23}
                                             By Lemma 4 (Precision inversion)
                A_{12} \sqsubseteq A_{13}
                                              "
                A_{22} \sqsubseteq A_{23}
                  \delta_2 \sqsubseteq \delta_3
                A_{11} \sqsubseteq A_{13}
                                             By the induction hypothesis
                A_{21} \sqsubseteq A_{23}
                                             By the induction hypothesis
                  \delta_1 \sqsubseteq \delta_3
                                             By transitivity of \Box
       A_{11} \delta_1 A_{21} \sqsubseteq A_{13} \delta_3 A_{23}
                                            By the definition of \Box
                  A_1 \sqsubseteq A_3
                                             Equivalent
  • Case A_2 = A_{12} \to A_{22}:
                  A_1 \sqsubseteq A_{12} \rightarrow A_{22}
                                               Given
                  A_1 = A_{11} \rightarrow A_{21}
                                               By Lemma 4 (Precision inversion)
                 A_{11} \sqsubseteq A_{12}
                 A_{21} \sqsubseteq A_{22}
       A_{12} \rightarrow A_{22} \sqsubseteq A_3
                                               Given
                  A_3 = A_{13} \to A_{23}
                                               By Lemma 4 (Precision inversion)
                 A_{12} \sqsubset A_{13}
                 A_{22} \sqsubset A_{23}
                 A_{11} \sqsubseteq A_{13}
                                               By the induction hypothesis
                 A_{21} \sqsubseteq A_{23}
                                               By the induction hypothesis
       A_{11} \rightarrow A_{21} \sqsubseteq A_{13} \rightarrow A_{23}
                                               By the definition of \Box
                  A_1 \sqsubseteq A_3
                                               Equivalent
                                                                                                                                                                                 П
D.1.3 Directed Consistency
Lemma 7 (Reflexivity of directed consistency).
For all types A, it is the case that A \rightsquigarrow A.
Proof. Immediate from Lemma 5 (Reflexivity of precision), Lemma 2 (Reflexivity of subtyping) and rule DirConsU.
                                                                                                                                                                                 Lemma 8 (Subtyping obeys directed consistency).
If A \leq B then A \rightsquigarrow B.
Proof. By Lemma 5 (Reflexivity of precision), A \subseteq A and B \subseteq B. It is given that A < B. Therefore, by rule DirConsU, A \rightsquigarrow B.
                                                                                                                                                                                 Lemma 9 (Loss in precision obeys directed consistency).
If A \subseteq B then A \rightsquigarrow B.
Proof. By Lemma 5 (Reflexivity of precision), A \subseteq A. By Lemma 2 (Reflexivity of subtyping), A \leq A. It is given that A \subseteq B. Therefore,
by rule DirConsU, A \sim B.
Lemma 10 (Gain in precision obeys directed consistency).
If A \sqsubseteq B then B \rightsquigarrow A.
Proof. It is given that A \sqsubseteq B. By Lemma 2 (Reflexivity of subtyping), A \le A. By Lemma 5 (Reflexivity of precision), A \sqsubseteq A. Therefore,
```

by rule DirConsU, $B \rightsquigarrow A$.

 $\mathsf{A}'\simeq\mathsf{A}$ | Type A' is structurally equivalent to A

 $Unit \simeq Unit$

$$\frac{A_1' \simeq A_1 \qquad A_2' \simeq A_2}{(A_1' \delta' A_2') \simeq (A_1 \delta A_2)}$$

$$\frac{A_1^\prime \simeq A_1 \qquad A_2^\prime \simeq A_2}{(A_1^\prime \to A_2^\prime) \simeq (A_1 \to A_2)}$$

Figure 18. Source structural equivalence

D.1.4 Structural Equivalence

Lemma 11 (Reflexivity of Structural Equivalence).

For all types A, it is the case that $A \simeq A$.

Proof. By induction on the structure of A. All cases are immediate by the induction hypothesis and the definition of \simeq .

Lemma 12 (Symmetry of Structural Equivalence).

If $A' \simeq A$ then $A \simeq A'$.

Proof. By structural induction on the derivation of $A' \simeq A$. All cases are immediate by the induction hypothesis and the definition of \simeq . \square

Lemma 13 (Transitivity of Structural Equivalence).

If $A_1 \simeq A_2$ and $A_2 \simeq A_3$ then $A_1 \simeq A_3$.

Proof. By induction on the structure of the type A₂. All cases are immediate from inversion on structural equivalence, the induction hypothesis, and the definition of \simeq .

Corollary 14 (Structural Equivalence is an equivalence relation).

The binary relation \simeq on types is an equivalence relation.

Proof. Immediate from Lemma 11 (Reflexivity of Structural Equivalence), Lemma 12 (Symmetry of Structural Equivalence), and Lemma 13 (Transitivity of Structural Equivalence).

Lemma 15 (Subtyping obeys Structural Equivalence).

If A' < A then $A' \simeq A$.

Proof. By induction on the structure of the derivation of $A' \leq A$.

- Case Unit < Unit: By definition of structural equivalence, Unit ≥ Unit.
- $A'_1 \le A_1$ $A'_2 \le A_2$ $A'_1 \simeq A_1$ Subderivation Subderivation By the induction hypothesis $A_2' \simeq A_2$ By the induction h $A_1' \delta' A_2' \simeq A_1 \delta A_2$ By definition of \simeq By the induction hypothesis

• Case
$$\frac{A_1 \leq A_1' \qquad A_2' \leq A_2}{(A_1' \rightarrow A_2') \leq (A_1 \rightarrow A_2)}$$

 $A_1 < A_1'$ Subderivation $A_2' \leq A_2$ Subderivation

By the induction hypothesis

By Lemma 12 (Symmetry of Structural Equivalence)

By the induction hypothesis

 $A_1 \simeq A_1'$ By the induction hy $A_1' \simeq A_1$ By Lemma 12 (Syn $A_2' \simeq A_2$ By the induction hy $A_1' \to A_2' \simeq A_1 \to A_2$ By definition of \simeq

Lemma 16 (Precision obeys Structural Equivalence).

If $A' \sqsubseteq A$ then $A' \simeq A$.

Proof. By induction on the structure of the derivation of $A' \sqsubseteq A$. All cases are immediate by the induction hypothesis and the definition of structural equivalence.

Lemma 17 (Directed consistency obeys Structural Equivalence).

If $A \rightsquigarrow B$ then $A \simeq B$.

Proof. It is given that $A \leadsto B$. By inversion on DirConsU, there exist A' and B' such that $A' \sqsubseteq A$ and $A' \le B'$ and $B' \sqsubseteq B$. By Lemma 16 (Precision obeys Structural Equivalence), $A' \simeq A$ and $B' \simeq B$. By Lemma 11 (Reflexivity of Structural Equivalence), $A \simeq A'$. By Lemma 15 (Subtyping obeys Structural Equivalence), $A' \simeq B'$. Therefore, by Lemma 13 (Transitivity of Structural Equivalence), $A \simeq B$.

D.1.5 Decidability

In this section, we write $\mathcal J$ decidable in proofs to indicate that the associated judgment form $\mathcal J$ is decidable.

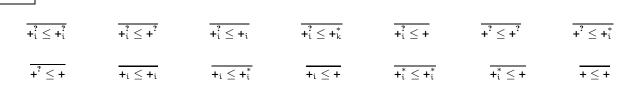


Figure 19. Reflexive, transitive closure of source subsum

Lemma 18 (Decidability of subsum).

 $\leq \delta$ Sum δ' is a sub-sum of δ

Given δ' and δ , the judgment $\delta' \leq \delta$ is decidable.

Proof. We present the reflexive, transitive closure of the subsum relation on source sums in Figure 19. We can view this relation as a finite set of ordered sums. Thus, the decidability of the subsum relation is equivalent to a membership check on this set.

Lemma 19 (Decidability of subtyping).

Given A' and A, the judgment $A' \leq A$ is decidable.

Proof. By simultaneous induction on the structure of A' and A.

Proceed by case analysis on the head constructors of A' and A. Either they agree or they disagree.

If they disagree, then no rule can possibly derive A' < A.

If they agree, then:

- Case A' = Unit and A = Unit: By definition of subtyping, $Unit \leq Unit$.
- Case $A' = A_1' \delta' A_2'$ and $A = A_1 \delta A_2$:

 $A_1' \le A_1$ decidable By the induction hypothesis $A_2' \le A_2$ decidable By the induction hypothesis

 $\delta' \leq \delta$ decidable By Lemma 18 (Decidability of subsum)

 $A_1' \delta' A_2' \leq A_1 \delta A_2$ decidable By decidability of premises

• Case $A' = A'_1 \rightarrow A'_2$ and $A = A_1 \rightarrow A_2$:

 $\begin{array}{ll} A_1 \leq A_1' \text{ decidable} & By \text{ the induction hypothesis} \\ A_2' \leq A_2 \text{ decidable} & By \text{ the induction hypothesis} \\ A_1' \rightarrow A_2' \leq A_1 \rightarrow A_2 \text{ decidable} & By \text{ decidability of premises} \end{array}$

 $\delta' \sqsubseteq \delta$ Sum δ' is more precise than δ

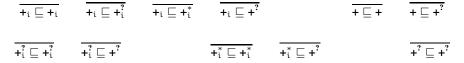


Figure 20. Reflexive, transitive closure of precision on sums

Lemma 20 (Decidability of precision on sums).

Given δ' and δ , the judgment $\delta' \sqsubseteq \delta$ is decidable.

Proof. We present the reflexive, transitive closure of the precision relation on source sums in Figure 20. We could view this relation as a finite set of ordered sums. Thus, the decidability of the precision relation is equivalent to a membership check on this set. Therefore, given δ' and δ , check whether or not $(\delta', \delta) \in \sqsubseteq$.

Lemma 21 (Decidability of precision on types).

Given A' and A, the judgment $A' \sqsubseteq A$ is decidable.

21 2020/9/22

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Proof. By simultaneous induction on the structure of A' and A.
```

Proceed by case analysis on the head constructors of A' and A. Either they agree or they disagree.

If they disagree, then no rule can possibly derive $A' \sqsubseteq A$.

If they agree, then:

- Case A' = Unit and A = Unit: By definition of precision, $Unit \sqsubseteq Unit$ and therefore derivability is decidable.
- Case $A' = A'_1 \delta' A'_2$ and $A = A_1 \delta A_2$:

 $A_1' \sqsubseteq A_1 \text{ decidable} \\ A_2' \sqsubseteq A_2 \text{ decidable} \\ \delta' \sqsubseteq \delta \text{ decidable} \\ A_1' \delta' A_2' \sqsubseteq A_1 \delta A_2 \text{ decidable} \\ By \text{ the induction hypothesis} \\ By \text{ Lemma 20 (Decidability of precision on sums)} \\ By \text{ decidability of premises}$

• Case $A' = A'_1 \rightarrow A'_2$ and $A = A_1 \rightarrow A_2$:

 $A_1' \sqsubseteq A_1$ decidable By the induction hypothesis $A_2' \sqsubseteq A_2$ decidable By the induction hypothesis $A_1' \to A_2' \sqsubseteq A_1 \to A_2$ decidable By decidability of premises

Lemma 22 (Decidability of directed consistency).

Given A' and B', the relation A' \sim B' is decidable.

Proof. We have $A' \rightsquigarrow B'$ if and only if there exist A and B such that $A \sqsubset A'$ and A < B and $B \sqsubset B'$. We are given A'; there are only finitely many types such that $A \sqsubseteq A'$. Each such A has only finitely many supertypes, that is, types B such that $\tilde{A} \leq B$. Since these two relations are decidable, $A' \rightsquigarrow B'$ is decidable.

Theorem 1 (Decidability of bidirectional typing).

- 1. Given Γ , e and A, the judgment $\Gamma \vdash e \Leftarrow A$ is decidable.
- 2. Given Γ and e, the judgment $\Gamma \vdash e \Rightarrow A$ is decidable.

Proof. By lexicographic induction on (1) the expression e, then on (2) the judgment form, with \Rightarrow smaller than \Leftarrow .

In most rules, the expression gets smaller in all the premises: SynAnno, Chk→Intro, Syn→Elim, ChkSumIntro, ChkSumElim1, and ChkSumElim2.

In ChkCSub, the premise types the same expression but is a synthesizing judgment, which is smaller under our induction measure. By Lemma 22, the second premise of ChkCSub is decidable.

D.1.6 Equivalence of type assignment and bidirectional system

Lemma 23 (All sums below +).

For all source sums δ , it is the case that $\delta \leq +$.

Proof. By case analysis on δ .

- Case $\delta = +_i^*$: By the definition of subtyping, $+_i^* \le +$.
- Case $\delta = +_i$: By the definition of subtyping, $+_i \le +_i^*$. By the previous case, $+_i^* \le +$. By the transitivity of subtyping, $+_i \le +$.
- Case $\delta = +\frac{?}{i}$: By the definition of subtyping, $+\frac{?}{i} \le +i$. By the previous case, $+i \le +i$. By the transitivity of subtyping, $+\frac{?}{i} \le +i$.
- Case $\delta = +^{?}$: By the definition of subtyping, $+^{?} \le +^{*}_{i}$. By the definition of subtyping, $+^{*}_{i} \le +$. By the transitivity of subtyping, $+^{?} \le +$.
- Case $\delta = +$: By the reflexivity of subtyping, $+ \le +$.

Lemma 24 (\Rightarrow) implies subsum).

If $\delta' \Rightarrow \delta$ then $\delta' \leq \delta$.

Proof. By case analysis on $\delta' \Rightarrow \delta$.

- Case $+_i^* \implies +_i^*$: By definition of subtyping, $+_i^* \le +_i$. By definition of subtyping, $+_i \le +_i^*$. By transitivity of subtyping, $+_i^* \le +_i^*$.
- Case $+_i \Rightarrow +_i^*$: By definition of subtyping, $+_i \le +_i^*$.
- Case $+^? \Rightarrow +_i^*$: By definition of subtyping, $+^? \le +_i^*$.
- Case $+_i^* \Rightarrow +_i^*$: By reflexivity of subtyping, $+_i^* \le +_i^*$.
- Case $\delta' \Longrightarrow$ +: By Lemma 23 (All sums below +), $\delta' \le$ +.

Theorem 2 (Bidirectional soundness).

If $\Gamma \vdash e \Leftarrow A$ or $\Gamma \vdash e \Rightarrow A$ then $\Gamma \vdash e : A$.

Proof. By induction on the structure of the given derivation.

- Case SynVar: Apply rule SVar.
- Case ChkCSub: Use the induction hypothesis and apply rule SCSub.

2020/9/22 22

- Case SynAnno: Use the induction hypothesis, and apply rule SAnno.
- Case ChkUnitIntro: Apply rule SUnitIntro.

• Case $\Gamma \vdash e_0 \Rightarrow (A_1 \, \delta \, A_2)$

$$\frac{\delta \Rightarrow +_i^*}{\Gamma \vdash \mathsf{case}(e_0, \mathsf{inj}_i \, x. e_i)} \stackrel{\Gamma}{\leftarrow} A \qquad \mathsf{ChkSumElim1}$$

$$\Gamma \vdash e_0 \Rightarrow (A_1 \, \delta \, A_2) \qquad \mathsf{Subderivation}$$

$$\Gamma \vdash e_0 : (A_1 \, \delta \, A_2) \qquad \mathsf{By the induction hypothesis}$$

$$\delta \Rightarrow +_i^* \qquad \mathsf{Subderivation}$$

$$\delta \leq +_i^* \qquad \mathsf{By Lemma 24} \ (\Rightarrow \mathsf{implies subsum})$$

$$A_1 \leq A_1 \qquad \mathsf{By Lemma 2} \ (\mathsf{Reflexivity of subtyping})$$

$$A_2 \leq A_2 \qquad \mathsf{By Lemma 2} \ (\mathsf{Reflexivity of subtyping})$$

$$A_1 \, \delta \, A_2 \leq A_1 +_i^* A_2 \qquad \mathsf{By Lemma 2} \ (\mathsf{Reflexivity of subtyping})$$

$$A_1 \, \delta \, A_2 \leq A_1 +_i^* A_2 \qquad \mathsf{By Lemma 8} \ (\mathsf{Subtyping obeys directed consistency})$$

$$\Gamma \vdash e_0 : (A_1 +_i^* A_2) \qquad \mathsf{By rule SCSub}$$

$$\Gamma, x : A_i \vdash e_i \in A \qquad \mathsf{Subderivation}$$

$$\Gamma, x : A_i \vdash e_i : A \qquad \mathsf{By the induction hypothesis}$$

$$\Gamma \vdash \mathsf{case}(e_0, \mathsf{inj}_i \, x. e_i) : A \qquad \mathsf{By rule ChkSumElim1}$$

- Case ChkSumElim2: Similar to the ChkSumElim1 case, hence omitted.
- ullet Case Chk \rightarrow Intro: Use the induction hypothesis, and apply rule $S\rightarrow$ Intro.
- Case Syn \rightarrow Elim: Use the induction hypothesis, and apply rule S \rightarrow Elim.

Lemma 25 (Reflexivity of annotation equivalence). For all expressions e, e =: e.

Proof. By induction on the structure of e.

All cases either hold directly by definition or by first using the induction hypothesis.

Lemma 26 (Synthesis also checks). If $\Gamma \vdash e \Rightarrow A$ then $\Gamma \vdash e \Leftarrow A$.

Proof. Apply rule ChkCSub as $A \rightsquigarrow A$ holds by Lemma 5 (Reflexivity of precision).

Theorem 3 (Annotatability).

If $\Gamma \vdash e : A$ then there exist e' and e'' such that (1) $\Gamma \vdash e' \Leftarrow A$ where e =: e', and (2) $\Gamma \vdash e'' \Rightarrow A$ where e =: e''.

Proof. By induction on the structure of the derivation of $\Gamma \vdash e : A$.

• Case
$$\frac{\Gamma(x) = A}{\Gamma \vdash x : A}$$
 SVar $\Gamma(x) = A$ Premise $\Gamma(x) = A$ By rule SynVar $\Gamma \vdash x \Rightarrow A$ By Lemma 26 (Synthesis also checks) $\Gamma \vdash x \Rightarrow A$ By definition of $\Gamma \vdash x \Rightarrow A$ By definition of $\Gamma \vdash x \Rightarrow A$

23 2020/9/22

• Case
$$\begin{array}{c} \Gamma \vdash e : A' \\ Ry the induction hypothesis \\ \blacksquare \quad e = e' \\ R' \quad By the induction hypothesis \\ \blacksquare \quad \Gamma \vdash e' \Leftrightarrow A \\ Ry rule ChkCSub \\ \blacksquare \quad \Gamma \vdash e' \Leftrightarrow A \\ Ry rule CynAnno \\ \blacksquare \quad \Gamma \vdash e_0 : A \\ \Gamma \vdash (e_0 : A) \Rightarrow A \\ Ry rule CynAnno \\ \blacksquare \quad \Gamma \vdash e_0 : A \\ \Gamma \vdash (e_0 : A) \Rightarrow A \\ Ry rule CynAnno \\ \blacksquare \quad \Gamma \vdash e_0 : A \\ \Gamma \vdash (e_0 : A) \Rightarrow A \\ Ry rule ChkUsub \\ Ry rule ChkUnithon for \\ Ry rule ChkUnithon of =: \\ Ry rule ChkSuminton of =: \\ Ry rule ChkSuminto$$

D.2 Typability under varying precision

Lemma 27 (Pointwise precision preserves domain).

If $\Gamma' \sqsubseteq \Gamma$ then $dom(\Gamma') = dom(\Gamma)$.

Proof. By induction on the structure of $\Gamma' \sqsubseteq \Gamma$.

Lemma 28 (Context strengthening).

If $\Gamma, y : A' \vdash e : A_0$ and $A \sqsubseteq A'$ then $\Gamma, y : A \vdash e : A_0$.

Proof. By induction on the structure of the derivation of Γ , $y : A' \vdash e : A_0$.

$$\begin{tabular}{ll} \bullet & \pmb{Case} \\ \hline & \frac{(\Gamma, y: A')(e) = A_0}{\Gamma, y: A' \vdash e: A_0} \ \mbox{SVar} \\ \hline \end{tabular}$$

Either e = y, or $e \neq y$.

In the first case:

$$\begin{array}{ll} (\Gamma,y:A')(y) = A_0 & \text{Premise} \\ A' = A_0 & \text{By definition} \\ \Gamma,y:A \vdash y:A & \text{By rule SVar} \\ A \sqsubseteq A' & \text{Given} \\ A \leadsto A' & \text{By Lemma 9 (Loss in precision obeys directed consistency)} \\ \Gamma,y:A \vdash y:A' & \text{By rule SCSub} \\ \Gamma,y:A \vdash e:A_0 & \text{By above equalities} \end{array}$$

In the second case:

$$\Gamma, y : A \vdash e : A_0$$
 By rule SVar

- Case SCSub: Use the induction hypothesis and apply rule SCSub.
- Case SUnitIntro: Immediate from rule SUnitIntro.
- Case SSumIntro: Use the induction hypothesis and apply rule SSumIntro.
- Case SSumElim1: Use the induction hypothesis and apply rule SSumElim1.
- Case SSumElim2: Use the induction hypothesis and apply rule SSumElim2.
- Case S→Intro: Use the induction hypothesis and apply rule S→Intro.
- Case S→Elim: Use the induction hypothesis and apply rule S→Elim.

Corollary 29.

If
$$\Gamma' \vdash e : A$$
 and $\Gamma \sqsubseteq \Gamma'$ then $\Gamma \vdash e : A$.

25 2020/9/22

Proof. By induction on the number of variables x such that $x \in dom(\Gamma')$ but $\Gamma'(x) \neq \Gamma(x)$.

Note that we don't impose $x \in dom(\Gamma)$ as $dom(\Gamma) = dom(\Gamma')$ by Lemma 27 (Pointwise precision preserves domain).

If $\Gamma'(x) = \Gamma(x)$ for all $x \in \text{dom}(\Gamma')$, then $\Gamma = \Gamma'$ so we already have the result.

Otherwise, use the induction hypothesis, and apply Lemma 28 (Context strengthening).

Lemma 30 (Relating +?-subsum and precision).

If
$$+_i^? \le \delta'$$
 and $\delta' \sqsubseteq \delta$ then $+_i^? \le \delta$.

Proof. Proceed by case analysis on $+_i^? \le \delta'$.

- Case $+_i^2 \le +_i^2$: From the definition of precision, either $\delta = +_i^2$ or $\delta = +^2$. In both cases, there exists a derivation for $+_i^2 \le \delta$.
- Case $+_i^2 \le +_i$: From the definition of precision, either $\delta = +_i$, $\delta = +_i^2$, $\delta = +_i^2$ or $\delta = +^2$. In all cases, there exists a derivation for $+_i^2 \le \delta$.
- Case $+_i^? \le +_i^?$: From the definition of precision, $\delta = +_i^?$. We are given a derivation for $+_i^? \le +_i^?$.
- Case $+_i^? \le +_k^*$: From the definition of precision, either $\delta = +_k^*$ or $\delta = +^?$. In both cases, there exists a derivation for $+_i^? \le \delta$.
- Case $+_i^2 \le +$: From the definition of precision, either $\delta = +$ or $\delta = +^2$. In both cases, there exists a derivation for $+_i^2 \le \delta$.

Lemma 31 (Bidirectional sum precision).

If $\delta' \Rightarrow \delta_1$ and $\delta' \sqsubseteq \delta$ then $\delta \Rightarrow \delta_1$.

Proof. Proceed by case analysis on $\delta' \Rightarrow \delta_1$.

- Case $+_i^2 \Rightarrow +_i^*$: From the definition of precision, either $\delta = +_i^2$ or $\delta = +_i^2$. In both cases, there exists a derivation for $\delta \Rightarrow +_i^*$.
- Case $+_i \implies +_i^*$: From the definition of precision, either $\delta = +_i$, $\delta = +_i^2$, or $\delta = +_i^*$. In all cases, there exists a derivations for $\delta \implies +_i^*$.
- Case $+^? \implies +_i^*$: From the definition of precision, $\delta = +^?$. We are given a derivation for $+^? \implies +_i^*$.
- Case $+_i^* \implies +_i^*$: From the definition of precision, either $\delta = +_i^*$ or $\delta = +_i^*$. In both cases, there exists a derivation for $\delta \implies +_i^*$.
- Case $\delta' \Longrightarrow +$: There exists a derivation for $\delta \Longrightarrow +$ for all δ .

Theorem 4 (Varying precision of bidirectional typing).

- 1. If $\Gamma' \vdash e' \Leftarrow A'$ and $e' \sqsubseteq e$ and $\Gamma' \sqsubseteq \Gamma$ and $A' \sqsubseteq A$ then $\Gamma \vdash e \Leftarrow A$.
- 2. If $\Gamma' \vdash e' \Rightarrow A'$ and $e' \sqsubseteq e$ and $\Gamma' \sqsubseteq \Gamma$ then there exists A such that $\Gamma \vdash e \Rightarrow A$ and $A' \sqsubseteq A$.

Proof. By induction on the structure of the given derivation.

- 1. By case analysis on the rule concluding $\Gamma' \vdash e' \Leftarrow A'$.
 - Case

$$\overline{\Gamma' \vdash \underbrace{()}_{e'} \Leftarrow \underbrace{\mathsf{Unit}}_{A'}} \ \mathsf{ChkUnitIntro}$$

() ⊑ e Giver

e = () From definition of \Box

Unit $\sqsubseteq A$ Given

A = Unit By Lemma 4 (Precision inversion)

 $\Gamma \vdash e \Leftarrow \mathsf{Unit}$ By rule $\mathsf{ChkUnitIntro}$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma' \vdash e' \Rightarrow A_0' \qquad A_0' \rightsquigarrow A'}{\Gamma' \vdash e' \Leftarrow A'} \ \, \textbf{ChkCSub}$$

26 2020/9/22

```
\Gamma' \vdash e' \Rightarrow A'_0
                                                  Subderivation
            \begin{array}{c} e' \sqsubseteq e \\ \Gamma' \sqsubseteq \Gamma \end{array}
                                                  Given
                                                   Given
             \Gamma \vdash e \Rightarrow A_0
                                                  By the induction hypothesis
           A_0' \sqsubseteq A_0
         A'_{0} \rightsquigarrow A'
B'_{0} \sqsubseteq A'_{0}
B'_{0} \le B'
B' \sqsubseteq A'
                                                  Subderivation
                                                  By inversion on DirConsU
            B_0^{\,\prime}\sqsubseteq A_0
                                                  By Lemma 6 (Transitivity of precision)
           A' \sqsubseteq A
B' \sqsubseteq A
                                                  Given
                                                  By Lemma 6 (Transitivity of precision)
          A_0 \sim A
                                                  By rule DirConsU
               \Gamma \vdash e \Leftarrow A By rule ChkCSub
                  \lambda x. e'_0 \sqsubseteq e
                                                                                       Given
                         e = \lambda x. e_0
                                                                                       From definition of \sqsubseteq
                          e_0' \sqsubseteq e_0
         A'_1 \rightarrow A'_2 \sqsubseteq A
A = A_1 \rightarrow A_2
A'_1 \sqsubseteq A_1
A'_2 \sqsubseteq A_2
                                                                                       Given
                                                                                       By Lemma 4 (Precision inversion)
                         \Gamma' \sqsubset \Gamma
                                                                                       Given
          \Gamma', x : A_1' \sqsubseteq \Gamma, x : A_1

\Gamma', x : A_1' \vdash e_0' \Leftarrow A_2'
                                                                                       By definition of \Box
                                                                                       Subderivation
              \Gamma, x : A_1 \vdash e_0 \Leftarrow A_2
                            A_1 \vdash e_0 \Leftarrow A_2 By the induction hyperimetric A_1 \vdash A_2 \Leftrightarrow A_1 \rightarrow A_2 By rule Chk \rightarrow Intro
                                                                                       By the induction hypothesis
 \begin{tabular}{l} \bullet \ \textbf{Case} \\ \hline \hline \hline $\Gamma' \vdash e_0' \Leftarrow A_i' & \textbf{+}_i^? \leq \delta' \\ \hline \hline $\Gamma' \vdash \underbrace{\inf_i e_0'} \Leftarrow \underbrace{A_1' \, \delta' \, A_2'} \\ \hline \end{tabular}  \ \textbf{ChkSumIntro} 
               \operatorname{inj}_{i} e'_{0} \sqsubseteq e
                                                                                       Given
                         e = \operatorname{inj}_{i} e_{0}
                                                                                      From definition of \sqsubseteq
                        e_0' \sqsubseteq e_0
          A_1' \delta' A_2' \sqsubseteq A
                                                                                       Given
                        A = A_1 \delta A_2
                                                                                       By Lemma 4 (Precision inversion)
                       \begin{array}{c} A_{\mathfrak{i}}' \sqsubseteq A_{\mathfrak{i}} \\ \delta' \sqsubseteq \delta \end{array}
                        \begin{array}{l} \Gamma' \vdash e_0' \Leftarrow A_i' \\ \Gamma' \sqsubseteq \Gamma \end{array}
                                                                                      Subderivation
                                                                                      Given
                          \Gamma \vdash e_0 \Leftarrow A_i
                                                                                       By the induction hypothesis
                       \begin{array}{ll} \textbf{+}_{i}^{?} \leq \delta' & \text{Subderivation} \\ \textbf{+}_{i}^{?} \leq \delta & \text{By Lemma 30 (Relatin} \\ \Gamma \vdash \text{inj}_{i} \ e_{0} \Leftarrow (A_{1} \ \delta \ A_{2}) & \text{By rule ChkSumIntro} \end{array}
                                                                                      By Lemma 30 (Relating +?-subsum and precision)
```

$$e' \subseteq e \\ e = case(e_0, inj_1 x.e_1) \\ e'_0 \subseteq e_0 \\ e'_1 \subseteq e_1 \\ \\ \Gamma' \vdash e'_0 \Rightarrow A'_1 \delta' A'_2 \\ \Gamma' \vdash e \cap \\ \Gamma \vdash e_0 \Rightarrow A_1 \delta A_2 \\ A'_1 \subseteq A_1 \\ \delta' \subseteq A'_1 \\ \delta' \Rightarrow +^{\dagger}_1 \\ By the induction hypothesis \\ A'_1 \subseteq A_1 \\ \delta' \subseteq A \cap \\ \delta' \Rightarrow +^{\dagger}_1 \\ By Lemma 31 (Bidirectional sum precision) \\ \Gamma', x : A'_1 \subseteq \Gamma_1 \\ \Gamma', x : A'_1 = \Gamma_1 \\ A' \subseteq A \cap \\ \Gamma', x : A'_1 = \Gamma_1 \\ A' \subseteq A \cap \\ \Gamma', x : A'_1 \vdash e'_1 \in A' \\ \Gamma', x : A'_1 \vdash e'_1 \in A' \\ \Gamma', x : A'_1 \vdash e'_1 \in A' \\ \Gamma', x : A'_1 \vdash e'_1 \in A' \\ \Gamma', x : A'_1 \vdash e'_1 \in A' \\ \Gamma' \vdash case(e_0, inj_1 x.e_1) \Leftarrow A \cap \\ \Gamma_1 \Rightarrow A \cap \\ \Gamma' \vdash case(e_0, inj_1 x.e_1, inj_2 x2.e_2) \Rightarrow A' \cap \\ ChkSumElim2$$

• Case
$$\Gamma' \vdash e'_0 \Rightarrow A'_1 \delta' A'_2 \quad \Gamma', x_1 : A'_1 \vdash e'_1 \in A' \\ \Gamma' \vdash case(e_0, inj_1 x_1.e_1, inj_2 x_2.e_2) \Rightarrow A' \cap \\ ChkSumElim2 \cap \\ ChkSumElim2 \cap \\ ChkSumElim3 \cap \\ ChkSumElim4 \cap \\ ChkSumElim5 \cap \\ ChkSumElim6 \cap \\ ChkSumElim6 \cap \\ ChkSumElim7 \cap \\ ChkSumElim8 \cap \\ ChkSumElim9 \cap \\ C$$

2. By case analysis on the rule concluding $\Gamma' \vdash e' \Rightarrow A'$.

• Case
$$\frac{\Gamma'(x) = A'}{\Gamma' \vdash \underbrace{x}_{e'} \Rightarrow A'} \operatorname{SynVar}$$
 Let $A = \Gamma(x)$.

$$x \sqsubseteq e \\ e = x \qquad From definition of \sqsubseteq$$

$$\Gamma'(x) = A' \qquad Premise$$

$$\Gamma' \sqsubseteq \Gamma \qquad Given$$

$$\Gamma'(x) \sqsubseteq \Gamma(x) \qquad By definition of \sqsubseteq on contexts$$

$$E \qquad A' \sqsubseteq A \qquad Equivalent$$

$$E \qquad \Gamma' \vdash e'_0 \Leftarrow A' \qquad Equivalent$$

$$E \qquad \Gamma' \vdash e'_0 \Leftarrow A' \qquad Equivalent$$

$$E \qquad Case \qquad \frac{\Gamma' \vdash e'_0 \Leftarrow A'}{\Gamma' \vdash (e'_0 :: A') \Rightarrow A'} \qquad SynAnno$$

$$e'_0 \sqsubseteq e_0 \qquad \qquad "$$

$$A' \sqsubseteq A \qquad \qquad "$$

$$\Gamma' \vdash e'_0 \Leftarrow A' \qquad Subderivation$$

$$\Gamma' \sqsubseteq \Gamma \qquad Given$$

$$\Gamma' \vdash e_0 \Leftarrow A \qquad By the induction hypothesis$$

$$\Gamma \vdash (e_0 :: A) \Rightarrow A \qquad By rule SynAnno$$
• Case
$$\Gamma' \vdash e'_1 \Rightarrow A'_0 \rightarrow A' \qquad \Gamma' \vdash e'_2 \Leftarrow A'_0 \qquad Syn \rightarrow Elim$$

$$E \qquad e'_1 e'_2 \sqsubseteq e \qquad Given$$

$$e'_1 \sqsubseteq e'_1 \sqsubseteq e'_2 \qquad From definition of \sqsubseteq e'_1 \sqsubseteq e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e = e_1 e_2 \qquad From definition of \sqsubseteq e'_1 \sqsubseteq e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad e'_2 \qquad Given$$

$$E \qquad e'_1 \sqsubseteq e'_1 \qquad e'_2 \qquad$$

D.3 Properties of the Static System

Lemma 32 (Static looseness). If $+_{i}^{?} \leq \delta^{S}$ then $+_{i} \leq_{S} \delta^{S}$.

Proof. By case analysis on $+_{i}^{?} \leq \delta^{S}$.

- Case $+_i^? \le +_i$: By definition of static subtyping $+_i \le_S +_i$.
- Case $+_i^? \le +$: By definition of static subtyping $+_i \le_S +$.

Lemma 33 (Static looseness, II).

If
$$\delta^{S} \Rightarrow +_{i}^{*}$$
 then $\delta^{S} = +_{i}$.

Proof. By case analysis on $\delta^{S} \Longrightarrow +_{i}^{*}$.

• Case $+_i \Rightarrow +_i^*$: It is the case that $\delta^S = +_i$.

The following lemma states that static sums are the most precise and incomparable by the precision relation.

Lemma 34 (Precision for static sums).

If
$$\delta_1 \sqsubseteq \delta_2^{\mathsf{S}}$$
 then $\delta_1 = \delta_2^{\mathsf{S}}$.

Proof. Proceed by case analysis on δ_2^{S} .

29 2020/9/22

- Case $\delta_2^S = +_i$: By the definition of imprecision, $\delta_1 = +_i$ only.
- Case $\delta_2^S = +$: By the definition of imprecision, $\delta_1 = +$ only.

Lemma 35 (Precision for static types).

If
$$A_1 \sqsubseteq A_2^{\mathsf{S}}$$
 then $A_1 = A_2^{\mathsf{S}}$.

Proof. By induction on the structure of A_2^{S} .

- Case $A_2^S = \text{Unit:}$ By the definition of imprecision, $A_1^S = \text{Unit only.}$
- Case $A_2^S = A_{12}^S \delta_2^S A_{22}^S$:

$$A_1 \sqsubseteq A_{12}^{\mathsf{S}} \delta_2^{\mathsf{S}} A_{22}^{\mathsf{S}}$$
 Given

$$A_1 \sqsubseteq A_{12}^{\mathsf{S}} \delta_2^{\mathsf{S}} A_{22}^{\mathsf{S}}$$
 Given $A_1 = A_{11} \delta_1 A_{21}$ From the definition of \sqsubseteq

$$A_{11} \sqsubset A_{12}^{\mathsf{S}}$$

$$A_{21} \sqsubseteq A_{22}$$

$$A_{11} \sqsubseteq A_{12}^{\mathsf{S}}$$

$$A_{21} \sqsubseteq A_{22}^{\mathsf{S}}$$

$$\delta_1 \sqsubseteq \delta_2^{\mathsf{S}}$$

$$\begin{array}{ll} A_{11} = A_{12}^{\text{S}} & \text{By the induction hypothesis} \\ A_{21} = A_{22}^{\text{S}} & \text{By the induction hypothesis} \\ \delta_1 = \delta_2^{\text{S}} & \text{By Lemma 34 (Precision for} \\ A_1 = A_2^{\text{S}} & \text{By definition of} = \end{array}$$

$$A_{21} = A_{22}^{S}$$
 By the induction hypothesis

$$\delta_1 = \delta_2^{\mathsf{S}}$$
 By Lemma 34 (Precision for static sums)

$$A_1 = A_2^{S}$$
 By definition of =

• Case
$$A_2^S = A_{12}^S \rightarrow A_{22}^S$$
: Similar to the previous case.

Lemma 36 (Equivalence for static subsum).

1. If
$$\delta_1^{\mathsf{S}} \leq_{\mathsf{S}} \delta_2^{\mathsf{S}}$$
 then $\delta_1^{\mathsf{S}} \leq \delta_2^{\mathsf{S}}$.

2. If
$$\delta_1^{S} < \delta_2^{S}$$
 then $\delta_1^{S} <_{S} \delta_2^{S}$.

Proof.

- 1. By case analysis on $\delta_1^{\mathsf{S}} \leq_{\mathsf{S}} \delta_2^{\mathsf{S}}$.
 - Case $\delta^{S} \leq_{S} \delta^{S}$: By definition of subtyping, $\delta^{S} \leq \delta^{S}$.
 - Case $+_i \le_S +$: By definition of subtyping, $+_i \le +_i^*$ and $+_i^* \le +$. By transitivity of subtyping, $+_i \le +$.
- $\begin{array}{l} \text{2. By case analysis on } \delta_1^{\text{S}} \leq \delta_2^{\text{S}}. \\ \bullet \text{ Case } \delta^{\text{S}} \leq \delta^{\text{S}} \colon \text{ By definition of static subtyping, } \delta^{\text{S}} \leq_{\text{S}} \delta^{\text{S}}. \\ \end{array}$
 - Case $+_i \le +$: By definition of static subtyping, $+_i \le_S +$.

Lemma 37 (Equivalence for static subtyping).

1. If
$$A_1^3 \leq_S A_2^3$$
 then $A_1^3 \leq A_2^3$

1. If
$$A_1^S \leq_S A_2^S$$
 then $A_1^S \leq A_2^S$.
2. If $A_1^S \leq A_2^S$ then $A_1^S \leq_S A_2^S$.

Proof.

- 1. By induction on the structure of the derivation of $A_1^S \leq_S A_2^S$.
 - Case Unit \leq_S Unit: By definition of subtyping, Unit \leq Unit.

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ & \frac{A_{11}^{S} \leq_{S} A_{12}^{S}}{(A_{11}^{S} \delta_{1}^{S} A_{21}^{S}) \leq_{S} A_{22}^{S}} \quad \delta_{1}^{S} \leq_{S} \delta_{2}^{S}}{(A_{11}^{S} \delta_{1}^{S} A_{21}^{S}) \leq_{S} (A_{21}^{S} \delta_{2}^{S} A_{22}^{S})} \\ & A_{11}^{S} \leq_{S} A_{12}^{S} \qquad \text{Subderivation} \\ & A_{21}^{S} \leq_{S} A_{22}^{S} \qquad \text{Subderivation} \\ & \delta_{1}^{S} \leq_{S} \delta_{2}^{S} \qquad \text{Subderivation} \\ & A_{11}^{S} \leq_{S} \delta_{2}^{S} \qquad \text{Subderivation} \\ & A_{21}^{S} \leq_{S} \delta_{2}^{S} \qquad \text{By the induction hypothesis} \\ & A_{21}^{S} \leq_{S} A_{22}^{S} \qquad \text{By the induction hypothesis} \\ & \delta_{1}^{S} \leq_{S} \delta_{2}^{S} \qquad \text{By Lemma 36 (Equivalence} \\ & A_{11}^{S} \delta_{1}^{S} A_{21}^{S} \leq_{S} A_{12}^{S} \delta_{2}^{S} A_{22}^{S} \qquad \text{By definition of } \leq \\ \end{array}$$

$$A_{11}^{S} \le A_{12}^{S}$$
 By the induction hypothesis $A_{21}^{S} < A_{22}^{S}$ By the induction hypothesis

$$\delta_1^{\mathsf{S}} \leq \delta_2^{\mathsf{S}}$$
 By Lemma 36 (Equivalence for static subsum)

$$A_{11}^{S} \delta_{1}^{S} A_{21}^{S} \leq A_{12}^{S} \delta_{2}^{S} A_{22}^{S}$$
 By definition of \leq

$$\begin{tabular}{ll} \bullet \ \textbf{Case} & A_{12}^{\textbf{S}} \leq_{\textbf{S}} A_{11}^{\textbf{S}} & A_{21}^{\textbf{S}} \leq_{\textbf{S}} A_{22}^{\textbf{S}} \\ \hline & (A_{11}^{\textbf{S}} \to A_{21}^{\textbf{S}}) \leq_{\textbf{S}} (A_{12}^{\textbf{S}} \to A_{22}^{\textbf{S}}) \\ \hline \end{array}$$

Similar to the previous case.

- 2. By induction on the structure of the derivation of $A_1^{\mathsf{S}} \leq A_2^{\mathsf{S}}$.
 - Case Unit \leq Unit: By definition of subtyping, Unit \leq_S Unit.

30 2020/9/22

$$\begin{array}{lll} \bullet & \textbf{Case} & \underline{A_{11}^{\textbf{S}} \leq A_{12}^{\textbf{S}}} & \underline{A_{21}^{\textbf{S}} \leq A_{22}^{\textbf{S}}} & \underline{\delta_{1}^{\textbf{S}} \leq \delta_{2}^{\textbf{S}}} \\ & & \underline{(A_{11}^{\textbf{S}} \delta_{1}^{\textbf{S}} A_{21}^{\textbf{S}}) \leq (A_{21}^{\textbf{S}} \delta_{2}^{\textbf{S}} A_{22}^{\textbf{S}})} \\ & & A_{11}^{\textbf{S}} \leq A_{12}^{\textbf{S}} & \text{Subderivation} \\ & A_{21}^{\textbf{S}} \leq A_{22}^{\textbf{S}} & \text{Subderivation} \\ & A_{11}^{\textbf{S}} \leq \delta_{2}^{\textbf{S}} & \text{Subderivation} \\ & A_{11}^{\textbf{S}} \leq A_{12}^{\textbf{S}} & \text{By the induction hypothesis} \\ & A_{21}^{\textbf{S}} \leq A_{22}^{\textbf{S}} & \text{By the induction hypothesis} \\ & A_{21}^{\textbf{S}} \leq A_{22}^{\textbf{S}} & \text{By the induction hypothesis} \\ & \delta_{1}^{\textbf{S}} \leq A_{21}^{\textbf{S}} \leq A_{22}^{\textbf{S}} & \text{By Lemma 36 (Equivalence for static subsum)} \\ & A_{11}^{\textbf{S}} \delta_{1}^{\textbf{S}} A_{21}^{\textbf{S}} \leq A_{12}^{\textbf{S}} \delta_{2}^{\textbf{S}} A_{22}^{\textbf{S}} & \text{By definition of } \leq_{\textbf{S}} \end{array}$$

• Case
$$\frac{A_{12}^{\text{S}} \leq A_{11}^{\text{S}} \qquad A_{21}^{\text{S}} \leq A_{22}^{\text{S}}}{(A_{11}^{\text{S}} \rightarrow A_{21}^{\text{S}}) \leq (A_{12}^{\text{S}} \rightarrow A_{22}^{\text{S}})}$$

Similar to the previous case

Lemma 38 (Directed consistency for static types).

If $A_1^{\mathsf{S}} \rightsquigarrow A_2^{\mathsf{S}}$ then $A_1^{\mathsf{S}} \leq A_2^{\mathsf{S}}$.

Proof. It is given that $A_1^S \leadsto A_2^S$. By inversion on DirConsU, there exist A and B such that $A \sqsubseteq A_1^S$ and $A \le B$ and $B \sqsubseteq A_2^S$. By Lemma 35 (Precision for static types), $A = A_1^S$ and $B = A_2^S$. Therefore, $A \le B$ is equivalent to $A_1^S \le A_2^S$.

Theorem 5 (Static soundness and completeness).

- 1. Soundness:

 - (a) If $\Gamma^{S} \vdash_{S} e^{S} \Leftarrow A^{S}$ then $\Gamma^{S} \vdash e^{S} \Leftarrow A^{S}$ (b) If $\Gamma^{S} \vdash_{S} e^{S} \Rightarrow A^{S}$ then $\Gamma^{S} \vdash e^{S} \Rightarrow A^{S}$.
- 2. Completeness:
 - (a) If $\Gamma^{S} \vdash e^{S} \Leftarrow A^{S}$ then $\Gamma^{S} \vdash_{S} e^{S} \Leftarrow A^{S}$. (b) If $\Gamma^{S} \vdash e^{S} \Rightarrow A^{S}$ then $\Gamma^{S} \vdash_{S} e^{S} \Rightarrow A^{S}$.

- 1. By induction on the structure of the given derivation.
 - Case StVar: Apply rule SynVar.

$$\begin{tabular}{c} \bullet \ \textbf{Case} \\ \hline \hline \hline $\Gamma^{\textbf{S}} \vdash_{\textbf{S}} e^{\textbf{S}} \Rightarrow A_0^{\textbf{S}} \quad A_0^{\textbf{S}} \leq_{\textbf{S}} A^{\textbf{S}} \\ \hline \hline $\Gamma^{\textbf{S}} \vdash_{\textbf{S}} e^{\textbf{S}} \Leftarrow A^{\textbf{S}} \\ \hline \hline \end{tabular} $ \text{StSub} $ \end{tabular}$$

$$\Gamma^{S} \vdash_{S} e^{S} \Rightarrow A_{0}^{S}$$
 Subderivation Subderivation

$$\Gamma^{S} \vdash e^{S} \Rightarrow A_{0}^{S}$$
 By the induction hypothesis

$$A_0^{\mathsf{S}} \leq A^{\mathsf{S}}$$
 By Lemma 37 (Equivalence for static subtyping)

$$\begin{array}{lll} \Gamma^{S} \vdash_{S} e^{S} \Rightarrow A_{0}^{S} & \text{Subderivation} \\ A_{0}^{S} \leq_{S} A^{S} & \text{Subderivation} \\ \Gamma^{S} \vdash e^{S} \Rightarrow A_{0}^{S} & \text{By the induction hypothesis} \\ A_{0}^{S} \leq A^{S} & \text{By Lemma 37 (Equivalence for static subtyping)} \\ A_{0}^{S} \leadsto A^{S} & \text{By Lemma 8 (Subtyping obeys directed consistency)} \\ \Gamma^{S} \vdash e^{S} \Leftarrow A^{S} & \text{By rule ChkCSub} \end{array}$$

- Case StAnno: Use the induction hypothesis and apply rule SynAnno.
- Case StUnitIntro: Apply rule ChkUnitIntro.

$$\begin{tabular}{ll} \bullet \ \, \textbf{Case} \\ & \frac{\Gamma^{\textbf{S}} \vdash_{\textbf{S}} e_{i}^{\textbf{S}} \Leftarrow A_{i}^{\textbf{S}} & \textbf{+}_{i} \leq \delta^{\textbf{S}}}{\Gamma^{\textbf{S}} \vdash_{\textbf{S}} inj_{i} e_{i}^{\textbf{S}} \Leftarrow (A_{1}^{\textbf{S}} \delta^{\textbf{S}} A_{2}^{\textbf{S}})} \ \, \textbf{StSumIntro} \\ \end{tabular}$$

$$\frac{\Gamma^{S} \vdash_{S} \operatorname{inj}_{i} e_{i}^{S} \Leftarrow (A_{1}^{S} \delta^{S} A_{2}^{S})}{\Gamma^{S} \vdash_{S} e_{\underline{i}}^{S} \Leftarrow A_{i}^{S}}$$
 Subderivation

$$\Gamma^{S} \vdash e_{i}^{S} \Leftarrow A_{i}^{S}$$
 By the induction hypothesis

$$+_{i}^{?} \le +_{i}$$
 By definition of \le

$$\begin{array}{lll} \Gamma^{S} \vdash_{S} e_{i}^{S} \Leftarrow A_{i}^{S} & \text{Subderivation} \\ +_{i} \leq_{S} \delta^{S} & \text{Subderivation} \\ \Gamma^{S} \vdash e_{i}^{S} \Leftarrow A_{i}^{S} & \text{By the induction hypothesis} \\ +_{i}^{2} \leq_{i} +_{i} & \text{By definition of } \leq \\ +_{i} \leq_{S} \delta^{S} & \text{By Lemma 36 (Equivalence for static subsum)} \\ +_{i}^{2} \leq_{S} \delta^{S} & \text{By transitivity of } \leq \\ \Gamma^{S} \vdash_{i} \text{inj}_{i} e_{i}^{S} \Leftarrow_{i} (A_{1}^{S} \delta^{S} A_{2}^{S}) & \text{By rule ChkSumIntro} \end{array}$$

$$+_{i}^{?} \le \delta^{S}$$
 By transitivity of \le

- Case StSumElim1: Use the induction hypothesis, the definition of ⇒⇒ and apply rule ChkSumElim1.
- Case StSumElim2: Use the induction hypothesis, the definition of ⇒ and apply rule ChkSumElim2.
- Case St→Intro: Use the induction hypothesis and apply rule Chk→Intro.
- Case St→Elim: Use the induction hypothesis and apply rule Syn→Elim.

- 2. By induction on the structure of the given derivation.
 - Case SynVar: Apply rule StVar.

$$\begin{tabular}{ll} \bullet \ \, \textbf{Case} \\ \hline \frac{\Gamma^{\textbf{S}} \vdash e^{\textbf{S}} \Rightarrow A_0^{\textbf{S}} \qquad A_0^{\textbf{S}} \leadsto A^{\textbf{S}}}{\Gamma^{\textbf{S}} \vdash e^{\textbf{S}} \Leftarrow A^{\textbf{S}}} \ \, \textbf{ChkCSub} \\ \hline \end{tabular}$$

$$\Gamma^{S} \vdash e^{S} \Rightarrow A_{0}^{S}$$
 Subderivation $A_{0}^{S} \rightsquigarrow A^{S}$ Subderivation

$$\begin{array}{lll} \Gamma^{S} \vdash e^{S} \Rightarrow A_{0}^{S} & \text{Subderivation} \\ A_{0}^{S} \leadsto A^{S} & \text{Subderivation} \\ A_{0}^{S} \leq A^{S} & \text{By Lemma 38 (Directed consistency for static types)} \\ \Gamma^{S} \vdash_{S} e^{S} \Rightarrow A_{0}^{S} & \text{By the induction hypothesis} \\ A_{0}^{S} \leq_{S} A^{S} & \text{By Lemma 37 (Equivalence for static subtyping)} \\ \Gamma^{S} \vdash_{S} e^{S} \leftrightarrow A^{S} & \text{By Lemma 37 (Equivalence for static subtyping)} \end{array}$$

$$\Gamma^{S} \vdash_{S} e^{S} \Rightarrow A_{0}^{S}$$
 By the induction hypothesis

$$A_0^{\mathsf{S}} \leq_{\mathsf{S}} A^{\mathsf{S}}$$
 By Lemma 37 (Equivalence for static subtyping)

$$\Gamma^{S} \vdash_{S} e^{S} \Leftarrow A^{S}$$
 By rule StSub

- Case SynAnno: Use the induction hypothesis and apply rule StAnno.
- Case ChkUnitIntro: Apply rule StUnitIntro.

$$\Gamma^{S} \vdash e_{i}^{S} \Leftarrow A_{i}^{S}$$
Subderivation
$$+_{i}^{?} < \delta^{S}$$
Subderivation

$$\begin{array}{ll} \Gamma^{\text{S}} \vdash e_{i}^{\text{S}} \Leftarrow A_{i}^{\text{S}} & \text{Subderivation} \\ \textbf{+}_{i}^{\text{2}} \leq \delta^{\text{S}} & \text{Subderivation} \\ \Gamma^{\text{S}} \vdash_{\text{S}} e_{i}^{\text{S}} \Leftarrow A_{i}^{\text{S}} & \text{By the induction hypothesis} \\ \textbf{+}_{i} \leq_{\text{S}} \delta^{\text{S}} & \text{By Lemma 32 (Static looseness)} \\ \Gamma^{\text{S}} \vdash_{\text{S}} \text{inj}_{i} e_{i}^{\text{S}} \Leftarrow (A_{1}^{\text{S}} \delta^{\text{S}} A_{2}^{\text{S}}) & \text{By rule StSumIntro} \end{array}$$

$$\Gamma^{S} \vdash_{S} \operatorname{inj}_{i} e_{i}^{S} \Leftarrow (A_{1}^{S} \delta^{S} A_{2}^{S})$$
 By rule StSumIntro

• Case
$$\Gamma^{\mathbf{S}} \vdash e_0^{\mathbf{S}} \Rightarrow (A_1^{\mathbf{S}} \delta^{\mathbf{S}} A_2^{\mathbf{S}})$$

$$\begin{array}{c} \bullet \; \mathbf{Case} \\ \Gamma^{\mathbf{S}} \vdash e_0^{\mathbf{S}} \Rightarrow (A_1^{\mathbf{S}} \, \delta^{\mathbf{S}} \, A_2^{\mathbf{S}}) \\ \frac{\delta^{\mathbf{S}} \Rightarrow +_i^*}{\Gamma^{\mathbf{S}} \vdash \mathsf{case}(e_0^{\mathbf{S}}, \mathsf{inj}_i \, x. e_i^{\mathbf{S}}) \Leftarrow A^{\mathbf{S}}} \; \mathsf{ChkSumElim1} \\ \end{array}$$

$$\Gamma^{S} \vdash e_{0}^{S} \Rightarrow (A_{1}^{S} \delta^{S} A_{2}^{S})$$
 Subderivation
$$\Gamma^{S}, x : A_{i}^{S} \vdash e_{i}^{S} \Leftarrow A^{S}$$
 Subderivation
$$\delta^{S} \Rightarrow + \underbrace{}_{i}^{*}$$
 Subderivation Subderivation

$$\Gamma^{S} \vdash e_{0}^{S} \Rightarrow (A_{1}^{S} \delta^{S} A_{2}^{S}) \qquad \text{Subderivation}$$

$$\Gamma^{S}, x : A_{i}^{S} \vdash e_{i}^{S} \Leftarrow A^{S} \qquad \text{Subderivation}$$

$$\delta^{S} \Rightarrow +_{i}^{*} \qquad \text{Subderivation}$$

$$\Gamma^{S} \vdash_{S} e_{0}^{S} \Rightarrow (A_{1}^{S} \delta^{S} A_{2}^{S}) \qquad \text{By the induction hypothesis}$$

$$\Gamma^{S}, x : A_{i}^{S} \vdash_{S} e_{i}^{S} \Leftarrow A^{S} \qquad \text{By the induction hypothesis}$$

$$\delta^{S} = +_{i} \qquad \text{By Lemma 33 (Static looseness, II)}$$

$$\Gamma^{S} \vdash_{S} \text{case}(e_{0}^{S}, \text{inj}_{i} x. e_{i}^{S}) \Leftarrow A^{S} \qquad \text{By rule StSumElim1}$$

- Case ChkSumElim2: Use the induction hypothesis, the definition of \leq_S and apply rule StSumElim2.
- Case Chk→Intro: Use the induction hypothesis and apply rule St→Intro.
- Case Syn→Elim: Use the induction hypothesis and apply rule St→Elim.

D.4 Properties of the Dynamic System

Lemma 39 (Subtyping for dynamic types). If $A_1^D \le A_2^D$ then $A_2^D = A_1^D$.

Proof. By induction on the structure of A_1^D .

- Case $A_1^D = \text{Unit}$: By the definition of subtyping, $A_2^D = \text{Unit}$ only.
- Case $A_1^D = A_{11}^D + A_{21}^D$:

$$A_{11}^{D} + {}^{2}A_{21}^{D} \le A_{2}^{D}$$
 Given
 $A_{2}^{D} = A_{12}^{D} + {}^{2}A_{22}^{D}$ From the definition of $\le A_{11}^{D} \le A_{12}^{D}$ "
 $A_{21}^{D} \le A_{22}^{D}$ "

$$A_{12}^{D} = A_{11}^{D}$$
 By the induction hypothesis $A_{22}^{D} = A_{11}^{D}$ By the induction hypothesis $A_{2}^{D} = A_{1}^{D}$ By definition of $=$

$$A_2^D = A_1^D$$
 By definition of =

• Case $A_1^D = A_{11}^D \rightarrow A_{21}^D$: Similar to the previous case.

Lemma 40 (Precision for dynamic types). If $A_1 \sqsubseteq A_2^D$ then $A_1 = A_2^D$.

Proof. By induction on the structure of A_2^D .

- Case A_2^D = Unit: By the definition of imprecision, A_1 = Unit only.
- Case $A_2^D = A_{12}^D + A_{22}^D$:

$$A_1 \sqsubseteq A_2^D$$
 Given

$$A_1 \sqsubseteq A_2^D$$
 Given
 $A_1 = A_{11} + A_{21}^2$ From the definition of \sqsubseteq
 $A_{11} \sqsubseteq A_{12}^D$ "
 $A_{21} \sqsubseteq A_{22}^D$ "

$$A_{11} \sqsubseteq A_{12}^D$$

$$A_{21} \sqsubset A_{22}^{D}$$
 "

$$A_{11} = A_{12}^D$$
 By the induction hypothesis $A_{21} = A_{22}^D$ By the induction hypothesis $A_1 = A_2^D$ By definition of $=$

$$A_{21} = A_{22}^{D}$$
 By the induction hypothesis

$$A_1 = A_2^D$$
 By definition of =

• Case
$$A_1^D = A_{11}^D \rightarrow A_{21}^D$$
: Similar to the previous case.

Lemma 41 (Directed consistency for dynamic types). If
$$A_1^D \rightsquigarrow A_2^D$$
 then $A_1^D = A_2^D$.

Proof. It is given that
$$A_1^D \rightsquigarrow A_2^D$$
. By inversion on DirConsU, there exist A and B such that $A \sqsubseteq A_1^D$ and $A \le B$ and $B \sqsubseteq A_2^D$. By Lemma 40 (Precision for dynamic types), $A = A_1^D$ and $B = A_2^D$. Therefore, $A \le B$ is equivalent to $A_1^D \le A_2^D$. By Lemma 39 (Subtyping for dynamic types), $A_1^D = A_2^D$.

Theorem 15 (Dynamic soundness and completeness).

1. (a) If
$$\Gamma^D \vdash_D e^D \Leftarrow A^D$$
 then $\Gamma^D \vdash e^D \Leftarrow A^D$.

(b) If
$$\Gamma^{D} \vdash_{D} e^{D} \Rightarrow A^{D}$$
 then $\Gamma^{D} \vdash e^{D} \Rightarrow A^{D}$.

2. (a) If
$$\Gamma^{D} \vdash e^{D} \Leftarrow A^{D}$$
 then $\Gamma^{D} \vdash_{D} e^{D} \Leftarrow A^{D}$.
(b) If $\Gamma^{D} \vdash e^{D} \Rightarrow A^{D}$ then $\Gamma^{D} \vdash_{D} e^{D} \Rightarrow A^{D}$.

(b) If
$$\Gamma^{D} \vdash e^{D} \Rightarrow A^{D}$$
 then $\Gamma^{D} \vdash_{D} e^{D} \Rightarrow A^{D}$

1. By induction on the structure of the given \vdash_{D} -derivation.

- Case DVar: Apply rule SynVar.
- Case DSub: Use the induction hypothesis, reflexivity of directed consistency, and apply rule ChkCSub.
- Case DUnitIntro: Apply rule ChkUnitIntro.

• Case
$$\frac{\Gamma^{\mathsf{D}} \vdash_{\mathsf{D}} e^{\mathsf{D}}_{\mathsf{i}} \Leftarrow \mathsf{A}^{\mathsf{D}}_{\mathsf{i}}}{\Gamma^{\mathsf{D}} \vdash_{\mathsf{D}} \mathrm{inj}_{\mathsf{i}} e^{\mathsf{D}}_{\mathsf{i}} \Leftarrow (\mathsf{A}^{\mathsf{D}}_{\mathsf{1}} + \mathsf{A}^{\mathsf{D}}_{\mathsf{2}})} \mathsf{D} + \mathsf{Intro}$$

$$\Gamma^{D} \vdash_{D} e_{i}^{D} \Leftarrow A_{i}^{D}$$

$$\Gamma^{D} \vdash e^{D} \leftarrow A^{D}$$

$$+\frac{?}{5} < +^{?}$$

$$\begin{array}{ll} \Gamma^{D} \vdash_{D} e^{D}_{i} \Leftarrow A^{D}_{i} & \text{Subderivation} \\ \Gamma^{D} \vdash e^{D}_{i} \Leftarrow A^{D}_{i} & \text{By the induction hypothesis} \\ +^{2}_{i} \leq +^{2} & \text{By definition of} \leq \\ \Gamma^{D} \vdash \operatorname{inj}_{i} e^{D}_{i} \Leftarrow (A^{D}_{1} +^{2} A^{D}_{2}) & \text{By rule ChkSumIntro} \end{array}$$

- Case D+ $^{?}$ Elim1: Use the induction hypothesis, the definition of \Rightarrow and apply rule ChkSumElim1.
- Case D+²Elim2: Use the induction hypothesis, the definition of ⇒⇒ and apply rule ChkSumElim2.
- Case D→Intro: Use the induction hypothesis and apply rule Chk→Intro.
- Case D→Elim: Use the induction hypothesis and apply rule Syn→Elim.
- 2. By induction on the structure of the given ⊢-derivation.
 - Case SynVar: Apply rule DVar.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma^{\textbf{D}} \vdash e^{\textbf{D}} \Rightarrow A^{\textbf{D}}_{0} \qquad A^{\textbf{D}}_{0} \rightsquigarrow A^{\textbf{D}}}{\Gamma^{\textbf{D}} \vdash e^{\textbf{D}} \Leftarrow A^{\textbf{D}}} \ \, \textbf{ChkCSub}$$

$$A_0^D \sim A^D$$

$$A_0^- \sim A^-$$
 Sur

By Lemma 41 (Directed consistency for dynamic types)

$$\Gamma^{D} \vdash e^{D} \rightarrow \Lambda^{D}$$
 Subderivation

 $\begin{array}{lll} A_0^{D} \leadsto A^{D} & \text{Subderivation} \\ A_0^{D} = A^{D} & \text{By Lemma 41} \\ \Gamma^{D} \vdash e^{D} \Rightarrow A_0^{D} & \text{Subderivation} \\ \Gamma^{D} \vdash_{D} e^{D} \Rightarrow A_0^{D} & \text{By the induction} \\ \Gamma^{D} \vdash_{D} e^{D} \Rightarrow A^{D} & \text{Equivalent} \\ \Gamma^{D} \vdash_{D} e^{D} \Leftarrow A^{D} & \text{By rule DSub} \end{array}$ By the induction hypothesis

$$\Gamma^{D} \vdash_{D} e^{D} \Rightarrow A^{D}$$
 Equivalent

$$\Gamma^{D} \vdash_{D} e^{D} \Leftarrow A^{D}$$
 By rule DSub

- Case SynAnno: Use the induction hypothesis and apply rule DAnno.
- Case ChkUnitIntro: Apply rule DUnitIntro.

- Case ChkSumIntro: Use the induction hypothesis, and apply rule D+?Intro.
- Case ChkSumElim1: Use the induction hypothesis, and apply rule D+? Elim1.
- Case ChkSumElim2: Use the induction hypothesis, and apply rule D+? Elim2.
- Case $Chk \rightarrow Intro$: Use the induction hypothesis and apply rule $D \rightarrow Intro$.
- Case Syn→Elim: Use the induction hypothesis and apply rule D→Elim.

D.5 Target System

D.5.1 Subtyping

Lemma 42 (Subtyping inversion).

- 1. If T' < Unit then T' = Unit.
- 2. If Unit < T then T = Unit.
- 3. If $T' \leq T_1 \varphi T_2$ then $T' = T_1' \varphi' T_2'$ where $T_1' \leq T_1$ and $T_2' \leq T_2$ and $\varphi' \leq \varphi$.
- 4. If $T_1' \varphi' T_2' \leq T$ then $T = T_1 \varphi T_2$ where $T_1' \leq T_1$ and $T_2' \leq T_2$ and $\varphi' \leq \varphi$.
- 5. If $T' \leq T_1 \to T_2$ then $T' = T_1' \to T_2'$ where $T_1 \leq T_1'$ and $T_2' \leq T_2$
- 6. If $T_1' \to T_2' \le T$ then $T = T_1 \to T_2$ where $T_1 \le T_1'$ and $T_2' \le T_2$.

Proof.

- 1. By case analysis on T' < Unit.
 - Case Unit \leq Unit: Immediate that T' = Unit.
- 2. Symmetric to part 1.
- 3. By case analysis on $T' \leq T_1 \varphi T_2$.
 - Case $T_1' \varphi' T_2' \le T_1 \varphi T_2$: Immediate as $T' = T_1' \varphi' T_2'$ and subderivations are $T_1' \le T_1$ and $T_2' \le T_2$ and $\varphi' \le \varphi$.
- 4. Symmetric to part 3.
- 5. By case analysis on $T' \leq T_1 \rightarrow T_2$.
 - Case $T_1' \to T_2' \le T_1 \to T_2$: Immediate as $T' = T_1' \to T_2'$ and subderivations are $T_1 \le T_1'$ and $T_2' \le T_2$.
- 6. Symmetric to part 5.

Lemma 43 (Reflexivity of subtyping).

For all types T, it is the case that $T \leq T$.

Proof. By induction on the structure of T.

- Case T = Unit: By the definition of subtyping, $T \le T$.
- Case $T = T_1 \varphi T_2$: By the induction hypothesis, $T_1 \le T_1$ and $T_2 \le T_2$. By the reflexivity of subsum, $\varphi \le \varphi$. Thus, by the definition of subtyping, $T \le T$.
- Case $T = T_1 \rightarrow T_2$: By the induction hypothesis, $T_1 \le T_1$ and $T_2 \le T_2$. Thus, by the definition of subtyping, $T \le T$.

Lemma 44 (Transitivity of subtyping).

If $T_1 \leq T_2$ and $T_2 \leq T_3$ then $T_1 \leq T_2$.

Proof. By induction on the structure of T_2 .

• Case $T_2 = Unit$:

 $T_1 \leq Unit \quad \text{ Given } \quad$

Unit $\leq T_3$ Given

 $T_1 =$ Unit By Lemma 42 (Subtyping inversion)

 $T_3 = Unit$ By Lemma 42 (Subtyping inversion)

Unit ≤ Unit By Lemma 43 (Reflexivity of subtyping)

"

"

 $T_1 \le T_3$ Equivalent

• Case $T_2 = T_{12} \varphi_2 T_{22}$:

 $T_1 \le T_{12} \varphi_2 T_{22}$ Given

 $T_1 = T_{11} \, \varphi_1 \, T_{21}$ By Lemma 42 (Subtyping inversion)

 $\begin{array}{l} T_{11} \leq T_{12} & '' \\ T_{21} \leq T_{22} & '' \end{array}$

 $\phi_1 \leq \phi_2$

 $T_{12} \varphi_2 T_{22} \le T_3$ Given

 $T_3 = T_{13} \, \phi_3 \, T_{23}$ By Lemma 42 (Subtyping inversion)

 $T_{12} \leq T_{13}$

 $T_{22} \le T_{13}$

 $\phi_2 \leq \phi_3$

34 2020/9/22

Corollary 45 (Subtyping inversion).

1. If $T_1' \Leftrightarrow' T_2' \leq T_1 \Leftrightarrow T_2$ then $T_1' \leq T_1$ and $T_2' \leq T_2$ and $\Phi' \leq \Phi$.

2. If
$$T_1' \to T_2' \le T_1 \to T_2$$
 then $T_1 \le T_1'$ and $T_2' \le T_2$.

Proof.

1. Let $T' = T_1' \varphi' T_2'$. We are given $T' \leq T_1 \varphi T_2$. Therefore, by Lemma 42 (Subtyping inversion), $T_1' \leq T_1$ and $T_2' \leq T_2$ and $\varphi' \leq \varphi$.

2. Let
$$T' = T_1' \to T_2'$$
. We are given $T' \le T_1 \to T_2$. Therefore, by Lemma 42 (Subtyping inversion), $T_1 \le T_1'$ and $T_2' \le T_2$.

D.5.2 Values

Lemma 46 (Value inversion).

- 1. If $\cdot \vdash W : T$ and $T \leq (T_1 + T_2)$ then $W = \operatorname{inj}_i W_i$ and $\cdot \vdash W_i : T_i$.
 - Moreover, if $T \leq (T_1 +_k T_2)$ then i = k.
- 2. If $\cdot \vdash W : T$ and $T \leq (T_1 \rightarrow T_2)$ then $W = \lambda x.$ M and $\cdot, x : T_1 \vdash M : T_2.$

Proof.

- 1. By induction on the structure of the derivation of $\cdot \vdash W : \mathsf{T}$.
 - Case TVar: Impossible because context $\Theta = \cdot$ is empty.

• Case
$$\frac{\cdot \vdash W : T' \qquad T' \leq T}{\cdot \vdash W : T}$$
 TSub

$$T' \leq T$$
 Subderivation

$$T \leq T_1 + T_2$$
 Given

 $T' \leq T_1 + T_2$ By Lemma 44 (Transitivity of subtyping).

Immediate from the induction hypothesis.

- Cases TCast, TMatchfail: Impossible because the subject term is not a value.
- Case TUnitIntro: Impossible because T = Unit cannot be a subtype of $T_1 + T_2$.

$$\frac{\cdot \vdash W_i : T_i'}{\cdot \vdash \underbrace{\mathsf{inj}_i \ W_i}_{\mathcal{W}} : \underbrace{(T_1' +_i \ T_2')}_{T}} \ \mathsf{T+_i Intro}$$

By the definition of values W, we know that W_i is a value and $W = inj_i W_i$.

$$\begin{array}{lll} T_1' +_i T_2' \leq T_1 + T_2 & \text{Given} \\ T_i' \leq T_i & \text{By Corollary 45} \\ \cdot \vdash W_i : T_i' & \text{Subderivation} \\ \cdot \vdash W_i : T_i & \text{By rule TSub} \\ T_1' +_i T_2' \leq T_1 +_k T_2 & \text{Suppose} \end{array}$$

$$\begin{array}{ccc} & +_{\mathfrak{i}} \leq +_k & & \text{By Corollary 45} \\ & \mathfrak{i} = k & & \text{From definition of } \leq \end{array}$$

- Cases T+_iElim, T+Elim: Impossible because the subject term is not a value.
- Case $T \rightarrow Intro$: Impossible because $T = T_1' \rightarrow T_2'$ cannot be a subtype of $T_1 + T_2$.
- Case T→Elim: Impossible because the subject term is not a value.
- 2. By induction on the structure of the derivation of $\cdot \vdash W : \mathsf{T}$.
 - Case TVar: Impossible because context $\Theta = \cdot$ is empty.

• Case
$$.\vdash W:T' \qquad T' \leq T \\ \hline .\vdash W:T \qquad TSub$$

$$T' \leq T$$
 Subderivation

$$T \stackrel{-}{\leq} T_1 \rightarrow T_2$$
 Given

 $T' \leq T_1 \rightarrow T_2$ By Lemma 44 (Transitivity of subtyping).

Immediate from the induction hypothesis.

- Cases TCast, TMatchfail: Impossible because the subject term is not a value.
- Case TUnitIntro: Impossible because T = Unit cannot be a subtype of $T_1 \to T_2$.
- Case T_{i} Intro: Impossible because $T = T'_{1} + T'_{2}$ cannot be a subtype of $T_{1} \to T_{2}$.
- Cases T+_iElim, T+Elim: Impossible because the subject term is not a value.

• Case
$$\frac{\cdot,x:T_1'\vdash M:T_2'}{\cdot\vdash\underbrace{\lambda x.\,M}_W:\underbrace{(T_1'\to T_2')}_T} \; T{\to} \mathsf{Intro}$$

By the definition of values W, we know that $W = \lambda x$. M.

$$\begin{array}{ll} T_1' \rightarrow T_2' \leq T_1 \rightarrow T_2 & \text{Given} \\ T_1 \leq T_1' & \text{By Corollary 45} \\ T_2' \leq T_2 & '' \\ \\ \ddots, x: T_1' \vdash M: T_2' & \text{Subderivation} \\ \ddots, x: T_1 \vdash M: T_2' & \text{By Lemma 50 (Context Strengthening)} \\ & \vdots \\ & \ddots, x: T_1 \vdash M: T_2 & \text{By rule TSub} \end{array}$$

• Case T→Elim: Impossible because the subject term is not a value.

Corollary 47 (Target value inversion for $+_{i}$).

If
$$\cdot \vdash W : (T_1 +_i T_2)$$
 then $W = \operatorname{inj}_i W_i$ and $\cdot \vdash W_i : T_i$.

Proof. Let
$$T = T_1 +_i T_2$$
.

Corollary 48 (Target value inversion for +).

If
$$\cdot \vdash W : (T_1 + T_2)$$
 then $W = \operatorname{inj}_i W_i$ and $\cdot \vdash W_i : T_i$.

Proof. By Lemma 46 (Value inversion) with $T = T_1 + T_2$, using Lemma 43 (Reflexivity of subtyping).

Corollary 49.

3

13

If
$$\cdot \vdash W : (T_1 \rightarrow T_2)$$
 then $W = \lambda x$. M_0 and $\cdot, x : T_1 \vdash M_0 : T_2$.

Proof. By Lemma 46 (Value inversion) with $T = T_1 \rightarrow T_2$, using Lemma 43 (Reflexivity of subtyping).

36 2020/9/22

D.5.3 Typing and Evaluation Contexts

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Lemma 50 (Context Strengthening).
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If $\Theta, y : T' \vdash M : T_0$ and $T \leq T'$ then $\Theta, y : T \vdash M : T_0$.

Proof. By induction on the structure of the derivation of $\Theta, y : T' \vdash M : T_0$.

- $\begin{tabular}{ll} \bullet \ \textbf{Case} \\ \hline & \frac{(\Theta, y: T')(M) = T_0}{\Theta, y: T' \vdash M: T_0} \ \textbf{TVar} \\ \hline \end{tabular}$
 - Either M = y, or $M \neq y$.

In the first case:

$$\begin{array}{lll} (\Theta,y:T')(M) = T_0 & \text{Premise} \\ T' = T_0 & \text{By definition} \\ \Theta,y:T \vdash y:T & \text{By rule TVar} \\ T \leq T' & \text{Given} \\ \Theta,y:T \vdash y:T' & \text{By rule TSub} \\ \Theta,y:T \vdash M:T_0 & \text{By above equalities} \end{array}$$

In the second case:

$$\Theta$$
, y : T \vdash M : T₀ By rule TVar

- Case TSub: Use the induction hypothesis and apply rule TSub.
- Case TCast: Use the induction hypothesis and apply rule TCast.
- Case TMatchfail: Immediate from rule TMatchfail.
- Case TUnitIntro: Immediate from rule TUnitIntro.
- Case T+iIntro: Use the induction hypothesis and apply rule T+iIntro.
- Case T+_iElim: Use the induction hypothesis and apply rule T+_iElim.
- Case T+Elim: Use the induction hypothesis and apply rule T+Elim.
- Case $T \rightarrow Intro$: Use the induction hypothesis and apply rule $T \rightarrow Intro$.
- Case $T \rightarrow Elim$: Use the induction hypothesis and apply rule $T \rightarrow Elim$.

Lemma 51 (Substitution).

If $\Theta, x : T' \vdash M : T$ and $\vdash W : T'$ then $\Theta \vdash [W/x]M : T$.

Proof. By induction on the structure of the derivation of Θ , $x : T' \vdash M : T$.

- Case TVar: Use the definition of substitution, well-formedness of Θ , and rule TVar.
- Case TSub: Use the induction hypothesis and apply rule TSub.
- Case TCast: Use the definition of substitution, the induction hypothesis and apply rule TCast.
- Case TMatchfail: Use the definition of substitution and apply rule TMatchfail.
- Case TUnitIntro: Use the definition of substitution and apply rule TUnitIntro.
- Case T+i Intro: Use the definition of substitution, the induction hypothesis and apply rule T+i Intro.
- Case T+_iElim: Use the definition of substitution, the induction hypothesis and apply rule T+_iElim.
- Case T+Elim: Use the definition of substitution, the induction hypothesis and apply rule T+Elim.
- Case T→Intro: Use the definition of substitution, the induction hypothesis and apply rule T→Intro.
- Case T→Elim: Use the definition of substitution, the induction hypothesis and apply rule T→Elim.

Lemma 52 (Evaluation context typing).

If $\Theta \vdash \mathcal{E}[M_0]$: T then there exists T_0 such that $\Theta \vdash M_0 : T_0$.

Proof. By induction on the structure of the derivation of $\Theta \vdash \mathcal{E}[M_0] : T$.

- Case TVar: Immediate as $\mathcal{E}[M_0] = M_0$, so $T_0 = T$.
- Case TSub: Immediate from the induction hypothesis.
- Case TCast: Immediate from the induction hypothesis.
- Case TMatchfail: Immediate as $\mathcal{E}[M_0] = M_0$, so $T_0 = T$.
- Case TUnitIntro: Immediate as $\mathcal{E}[M_0] = M_0$, so $T_0 = T$.
- Case T+iIntro: Immediate from the induction hypothesis.
- Case T+_iElim: Immediate from the induction hypothesis.
- Case T+Elim: Immediate from the induction hypothesis.
- Case $T \rightarrow Intro$: Immediate as $\mathcal{E}[M_0] = M_0$, so $T_0 = T$.

37 2020/9/22

• Case $T \rightarrow \text{Elim}$: Proceed by case analysis on \mathcal{E} . Each case is immediate from the induction hypothesis.

Lemma 53 (Evaluation context replacement).

If
$$\Theta \vdash \mathcal{E}[M_0] : T$$
 and $\Theta \vdash M_0 : T_0$ and $\Theta \vdash M_0' : T_0$ then $\Theta \vdash \mathcal{E}[M_0'] : T$.

Proof. By induction on the structure of the derivation of $\Theta \vdash \mathcal{E}[M_0] : T$.

- Case TVar: Immediate as $\mathcal{E}[M_0]=M_0$, so $T_0=T$ and $\mathcal{E}[M_0']=M_0'$.
- Case TSub: Use the induction hypothesis and apply rule TSub.
- Case TCast: Use the induction hypothesis and apply rule TCast.
- Case TMatchfail: Immediate as $\mathcal{E}[M_0] = M_0$, so $T_0 = T$ and $\mathcal{E}[M'_0] = M'_0$.
- Case TUnitIntro: Immediate as $\mathcal{E}[M_0] = M_0$, so $T_0 = T$ and $\mathcal{E}[M'_0] = M'_0$.
- Case T+iIntro: Use the induction hypothesis and apply rule T+iIntro.
- Case $T+_iElim$: Use the induction hypothesis and apply rule $T+_iElim$.
- Case T+Elim: Use the induction hypothesis and apply rule T+Elim.
- Case $T \rightarrow Intro$: Immediate as $\mathcal{E}[M_0] = M_0$, so $T_0 = T$ and $\mathcal{E}[M'_0] = M'_0$.
- Case $T \rightarrow \text{Elim}$: Proceed by case analysis on \mathcal{E} . For each case, use the induction hypothesis and apply rule $T \rightarrow \text{Elim}$.

D.5.4 Type Safety

Lemma 54 (Type preservation under reduction).

If
$$\cdot \vdash M : T$$
 and $M \mapsto_R M'$ then $\cdot \vdash M' : T$.

Proof. By induction on the structure of the derivation of $\cdot \vdash M : T$.

- Case TVar: Impossible because the context $\Theta = \cdot$ is empty.
- Case TSub: Use the induction hypothesis and apply rule TSub.

• Case

$$\frac{\cdot \vdash M_0 : (T_1 \varphi' T_2)}{\cdot \vdash \underbrace{\langle \varphi \Leftarrow \varphi' \rangle M_0}_{M} : \underbrace{(T_1 \varphi T_2)}_{T}} \mathsf{TCast}$$

Proceed by case analysis on $M \mapsto_R M'$.

■ Case

Case
$$\frac{\varphi' \leq \varphi}{\langle \varphi \Leftarrow \varphi' \rangle} \underbrace{\frac{W}{W_0} \mapsto_R \underbrace{W}_{M'}}_{\text{M}_0} \text{ ReduceUpcast}$$

$$T_1 \leq T_1 \qquad \text{By Lemma 43 (Reflexivity of subtyping)}$$

$$T_2 \leq T_2 \qquad \text{By Lemma 43 (Reflexivity of subtyping)}$$

$$\varphi' \leq \varphi \qquad \qquad \text{Given}$$

$$(T_1 \varphi' T_2) \leq (T_1 \varphi T_2) \qquad \text{Definition of } \leq$$

$$\cdot \vdash W : (T_1 \varphi' T_2) \qquad \text{Subderivation}$$

$$\cdot \vdash W : (T_1 \varphi T_2) \qquad \text{By rule TSub}$$

Case

$$\begin{array}{c} \hline \\ \langle +_i \Leftarrow + \rangle \underbrace{(\operatorname{inj}_i W)}_{M_0} \mapsto_R \underbrace{\operatorname{inj}_i W}_{M'} \end{array} \\ \text{ReduceCastSuccess} \\ \cdot \vdash \operatorname{inj}_i W : (\mathsf{T}_1 + \mathsf{T}_2) \qquad \text{Subderivation} \\ \cdot \vdash W : \mathsf{T}_i \qquad \qquad \text{By Corollary 48} \\ \cdot \vdash \operatorname{inj}_i W : (\mathsf{T}_1 +_i \mathsf{T}_2) \qquad \text{By rule T+}_i \mathsf{Intro} \end{array}$$

Case

$$\frac{\varphi' \in \{\textbf{+}_i, \textbf{+}\}}{\langle \textbf{+}_k \Leftarrow \varphi' \rangle \underbrace{(inj_i W)}_{M_0} \mapsto_{\mathsf{R}} \underbrace{\mathtt{matchfail}}_{M'}} \ \mathsf{ReduceCastFailure}$$

- $\cdot \vdash$ matchfail : T By rule TMatchfail
- Case TMatchfail: Impossible because matchfail $\not\mapsto_R M'$ for any M'.
- Case TUnitIntro: Impossible because () $\not\mapsto_R M'$ for any M'.
- Case T_{i} Intro: Impossible because $M = inj_i M_0 \not\mapsto_R M'$ for any M'.

38 2020/9/22

$$\bullet \ \, \textbf{Case} \ \, \underbrace{ \begin{array}{c} \cdot \vdash M_0 : T_1 +_i T_2 \\ \cdot \vdash \underbrace{\mathsf{case}(M_0, \mathsf{inj}_i \ x.M_i)} : T \end{array}}_{M} : T \vdash_i \mathsf{Elim}$$

Proceed by case analysis on $M \mapsto_R M'$.

Case

- Case T+Elim: Similar to the T+_i Elim case. Apply Corollary 48 instead of Corollary 47 when considering the ReduceCase2 case.
- Case $T \rightarrow Intro$: Impossible because $M = \lambda x$. $M_0 \nleftrightarrow_R M'$ for any M'.
- $\begin{array}{c} \bullet \ \textbf{Case} \\ \\ \frac{\cdot \vdash M_1 : T' \to T \quad \cdot \vdash M_2 : T'}{\cdot \vdash \underbrace{M_1 \, M_2}_{M} : T} \ T {\to} \mathsf{Elim} \\ \end{array}$

Proceed by case analysis on $M \mapsto_R M'$.

Case

$$\begin{array}{cccc} & \underbrace{(\lambda x.\ M_0)}_{M_1} \underbrace{W}_{M_2} \mapsto_R \underbrace{[W/x]M_0}_{M'} & \text{Reduce}\beta \\ & \cdot \vdash W:T' & \text{Subderivation} \\ & \cdot \vdash \lambda x.\ M_0:T' \to T & \text{Subderivation} \\ & \cdot , x:T' \vdash M_0:T & \text{By Corollary 49} \\ & \cdot \vdash [W/x]M_0:T & \text{By Lemma 51 (Substitution)} \end{array}$$

Theorem 6 (Type preservation).

If $\cdot \vdash M : T$ and $M \mapsto M'$ then $\cdot \vdash M' : T$.

Proof. By case analysis on $M \mapsto M'$.

$$\bullet \ \, \textbf{Case} \ \, \frac{M_0 \mapsto_R M_0'}{\mathcal{E}[M_0] \mapsto \mathcal{E}[M_0']} \ \, \textbf{StepContext}$$

 $\cdot \vdash \mathcal{E}[M_0] : T$ Given

 $\cdot \vdash M_0 : T_0$ By Lemma 52 (Evaluation context typing)

 $M_0 \mapsto_R M_0'$ Subderivation

 $\cdot \vdash M'_0 : T_0$ By Lemma 54 (Type preservation under reduction)

 $\cdot \vdash \mathcal{E}[\mathsf{M}_0'] : \mathsf{T}$ By Lemma 53 (Evaluation context replacement)

• Case $\frac{\mathcal{E} \neq []}{\mathcal{E}[\mathsf{matchfail}] \mapsto \mathsf{matchfail}} \, \mathsf{StepMatchfail}$

Immediate by TMatchfail.

Theorem 7 (Progress).

If $\cdot \vdash M$: T then either (a) M is a value, or (b) there exists M' such that $M \mapsto M'$, or (c) M = matchfail.

Proof. By induction on the structure of the derivation of $\cdot \vdash M : T$.

- Case TVar: Impossible, because the context Θ is empty.
- Case TSub: Immediate by the induction hypothesis.
- Case TCast:

We have $M = \langle \varphi \Leftarrow \varphi' \rangle M_0$ and $T = T_1 \varphi T_2$ where $\cdot \vdash M_0 : (T_1 \varphi' T_2)$.

By the induction hypothesis, either M_0 is a value or there exists M'_0 such that $M_0 \mapsto M'_0$.

In the first case, we need to consider all possible assignments to ϕ' and ϕ .

Suppose $\phi' \leq \dot{\phi}$, then $M \mapsto M_0$.

Suppose $\phi' = +_i$ and $\phi = +_k$ where $i \neq k$, then $M_0 = \inf_i W$ by Corollary 47, so $M \mapsto \text{matchfail}$.

39 2020/9/22

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Suppose \varphi' = + and \varphi = +_i, then M_0 = \inf_k W by Corollary 48. Proceed by cases analysis on i, if i = k then M \mapsto M_0, otherwise M \mapsto \mathsf{matchfail}.
```

In the second case, $\langle \varphi \Leftarrow \varphi' \rangle M_0 \mapsto \langle \varphi \Leftarrow \varphi' \rangle M_0'$.

- Case TUnitIntro: We have M = (), a value, which is alternative (a).
- Case TMatchfail: We have M = matchfail, which is alternative (c).
- Case T+iIntro:

We have $M = \operatorname{inj}_i M_0$ and $T = T_1 +_i^2 T_2$ where $\cdot \vdash M_0 : T_i$.

By the induction hypothesis, either M_0 is a value or there exists M'_0 such that $M_0 \mapsto M'_0$.

In the first case, $inj_i M_0 = M$ is a value.

In the second case, $\operatorname{inj}_i M_0 \mapsto \operatorname{inj}_i M'_0$.

• Case T+_iElim:

We have $M = \mathsf{case}(M_0, \mathsf{inj}_i \ x.M_i)$ where $\cdot \vdash M_0 : T_1 +_i T_2$ and $\cdot, x : T_i \vdash M_i : T$. By the induction hypothesis, either M_0 is a value or there exists M'_0 such that $M_0 \mapsto M'_0$. In the first case, $M_0 = \mathsf{inj}_i \ W$ by Corollary 47, so $\mathsf{case}(M_0, \mathsf{inj}_i \ x.M_i) \mapsto [W/x]M_i$. In the second case, $\mathsf{case}(M_0, \mathsf{inj}_i \ x.M_i) \mapsto \mathsf{case}(M'_0, \mathsf{inj}_i \ x.M_i)$.

- Case T+Elim: Similar to the T+_iElim case, using Corollary 48 instead of Corollary 47.
- Case T \rightarrow Intro: We have $M = \lambda x$. M_0 , a value.
- Case T→Elim:

We have $M = M_1 M_2$ where $\cdot \vdash M_1 : T_1 \rightarrow T_2$ and $\cdot \vdash M_2 : T_1$.

By the induction hypothesis, either M_1 is a value or there exists M'_1 such that $M_1 \mapsto M'_1$.

In the first case, $M_1 = \lambda x$. M_0 by Corollary 49.

By the induction hypothesis, either M_2 is a value or there exists M_2' such that $M_2 \mapsto M_2'$.

In the first subcase, M_2 is a value, so $(\lambda x. M_0) M_2 \mapsto [M_2/x]M_0$.

In the second subcase, $(\lambda x. M_0) M_2 \mapsto (\lambda x. M_0) M_2'$.

In the second case, $M_1 M_2 \mapsto M_1' M_2$.

Theorem 8 (matchfail-freeness).

If M is cast-free and matchfail-free and $M \mapsto M'$ then M' is cast-free and matchfail-free.

Proof. By induction on the derivation of $M \mapsto M'$.

By the assumption that M is matchfail-free, rule StepMatchfail is impossible. Therefore, the derivation is by StepContext with subderivation $M_0 \mapsto_R M'_0$, where $M = \mathcal{E}[M_0]$ and $M' = \mathcal{E}[M'_0]$.

- Cases ReduceUpcast, ReduceCastSuccess, ReduceCastFailure: In these cases, M_0 contains a cast, contradicting the assumption that $M = \mathcal{E}[M_0]$ is cast-free. Hence, these cases are impossible.
- Case ReduceCase1:

We have $M_0 = case(inj_i W, inj_i x.M_i)$ and $M'_0 = [W/x]M_i$.

Since M₀ is cast- and matchfail-free, its subterms W and M_i are cast- and matchfail-free.

Therefore, $[W/x]M_i$ is cast- and matchfail-free.

• Cases ReduceCase2, Reduceβ: Similar to the ReduceCase1 case.

D.5.5 Precision

Lemma 55 (Precision on values).

If $W' \leq M$ then M = W for some value M.

Proof. By induction on the structure of the derivation of $W' \leq M$.

- Case () \preccurlyeq M: From definition of \preccurlyeq , it is immediate that M = (), a value.
- Case $x \leq M$: From definition of \leq , it is immediate that M = x, a value.
- Case λx . $M_0' \preccurlyeq M$: From definition of \preccurlyeq , it is immediate that $M = \lambda x$. M_0 , a value.
- Case $\operatorname{inj}_i W_0' \preceq M$: From definition of \preceq , $M = \operatorname{inj}_i M_0$ and $W_0' \preceq M_0$. By the induction hypothesis, $M_0 = W_0$ for some value W_0 . Therefore, $M = \operatorname{inj}_i W_0$, a value.

Lemma 56 (Substitution preserves precision).

If $M' \leq M$ and $W' \leq W$ then $[W'/x]M' \leq [W/x]M$.

Proof. By induction on the structure of the derivation of $M' \leq M$. All cases are immediate by the induction hypothesis, the definition of substitution, and the definition of \leq .

Lemma 57 (Precision inversion on evaluation contexts).

If $\mathcal{E}'[M_0'] \leq M$ then there exists \mathcal{E} and M_0 such that $M = \mathcal{E}[M_0]$ and $M_0' \leq M_0$.

Proof. Proceed by induction on the structure of \mathcal{E}' .

• Case $\mathcal{E}' = []$: Choose $\mathcal{E} = []$ and $M_0 = M$ then $M'_0 \leq M_0$ is given.

40 2020/9/22

- Case $\mathcal{E}' = \mathsf{case}(\mathcal{E}'_0, \mathsf{inj}_i \, x. \mathsf{M}'_i)$: Similar to the $\mathcal{E}' = \mathsf{inj}_i \, \mathcal{E}'_0$ case, hence omitted.
- Case $\mathcal{E}' = \text{case}(\mathcal{E}'_0, \text{inj}_1 \, x_1.M'_1, \text{inj}_2 \, x_2.M'_2)$: Similar to the $\mathcal{E}' = \text{inj}_i \, \mathcal{E}'_0$ case, hence omitted.
- Case $\mathcal{E}' = \langle \varphi_2' \Leftarrow \varphi_1' \rangle \mathcal{E}_0'$:

By inversion on $\langle \varphi_2' \Leftarrow \varphi_1' \rangle \mathcal{E}_0'[M_0'] \leq M$, either $M = \langle \varphi_2 \Leftarrow \varphi_1 \rangle M_1$ or $M \neq \langle \varphi_2 \Leftarrow \varphi_1 \rangle M_1$. In the former case:

$$\begin{array}{ccc} \mathcal{E}_0'[M_0'] \leqslant M_1 & \text{From the definition of} \leqslant \\ M_1 = \mathcal{E}_0[M_0] & \text{By the induction hypothesis} \\ & M_0' \leqslant M_0 & '' \\ & M = \langle \varphi_2 \Leftarrow \varphi_1 \rangle \mathcal{E}_0[M_0] & \text{By above equations} \\ \end{array}$$

In the latter case:

$$\begin{array}{ccc} \mathcal{E}_0'[M_0'] \preccurlyeq M & \text{From the definition of} \preccurlyeq \\ \mathbb{R} & M = \mathcal{E}[M_0] & \text{By the induction hypothesis} \\ \mathbb{R} & M_0' \preccurlyeq M_0 & '' \end{array}$$

• Case $\mathcal{E}' = \mathcal{E}'_0 M'_2$:

$$\begin{array}{lll} \mathcal{E}'[M_0'] \leqslant M & \text{Given} \\ \mathcal{E}'_0[M_0'] \, M_2' \leqslant M & \text{By above equations} \\ M = M_1 \, M_2 & \text{From the definition of} \leqslant \\ \mathcal{E}'_0[M_0'] \leqslant M_1 & \text{"} \\ M_1 = \mathcal{E}_0[M_0] & \text{By the induction hypothesis} \\ M_0' \leqslant M_0 & \text{"} \\ M = \mathcal{E}_0[M_0] \, M_2 & \text{By above equations} \end{array}$$

• Case $\mathcal{E}' = W_1 \mathcal{E}'_0$:

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Lemma 58 (Evaluation contexts preserve precision).

If $\mathcal{E}'[M_0'] \leq \mathcal{E}[M_0]$ and $M_0' \leq M_0$ and $M_1' \leq M_1$ then $\mathcal{E}'[M_1'] \leq \mathcal{E}[M_1]$.

Proof. By induction on the derivation of $\mathcal{E}'[M'_0] \leq \mathcal{E}[M_0]$. All cases are straightforward, using the induction hypothesis and the definition of \preccurlyeq .

Lemma 59 (Reduction preserves precision).

If $\cdot \vdash M_1' : T_1'$ and $\cdot \vdash M_1 : T_1$ and $M_1' \preceq M_1$ and $M_1' \mapsto_R M_2'$ then either (a) M_1 is a value and $M_2' \leq M_1$, or

(b) there exists M_2 such that $M_1 \mapsto_R M_2$ and $M_2' \preccurlyeq M_2$.

Proof. Proceed by case analysis on $M'_1 \mapsto_R M_1$.

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\varphi_1' \leq \varphi_2'}{\langle \varphi_2' \Leftarrow \varphi_1' \rangle W'} \mapsto_{\mathsf{R}} \underbrace{W'}_{M_2'}}_{\mathsf{M}_1'} \ \, \mathsf{ReduceUpcast}$$

Proceed by case analysis on $M'_1 \leq M_1$.

2020/9/22 41

• Case
$$\frac{W' \preccurlyeq M \qquad \langle \varphi_2' \Leftarrow \varphi_1' \rangle \preccurlyeq \langle \varphi_2 \Leftarrow \varphi_1 \rangle }{\langle \varphi_2' \Leftarrow \varphi_1' \rangle W' \preccurlyeq \underbrace{\langle \varphi_2 \Leftarrow \varphi_1 \rangle M}_{M_1} }$$

By Lemma 55 (Precision on values), M=W as $W' \preccurlyeq M$. Since $\varphi_1' \leq \varphi_2'$, it is the case that $\langle \varphi_2' \Leftarrow \varphi_1' \rangle = sc'$. Proceed by cases on the rule deriving $sc' \leq \langle \varphi_2 \Leftarrow \varphi_1 \rangle$.

- Case Cast ≼ Refl: In this case, $\langle \varphi_2 \Leftarrow \varphi_1 \rangle = sc'$. Since $\varphi_1 \leq \varphi_2$ by rule ReduceUpcast it follows that $M_1 \mapsto_R W$, and we already have $M_2' \leq M_2$.
- Cases CastM \preccurlyeq B, CastB \preccurlyeq S, CastM \preccurlyeq S: These rules do not have a safe cast on the left, so these cases are impossible.
- $\begin{array}{l} \textbf{Case} \ \text{Rule deriving} \ \langle \textbf{+}_i \Leftarrow \textbf{+}_i \rangle \ \preccurlyeq \ \langle \textbf{+}_i \Leftarrow \textbf{+} \rangle \\ \text{In this case, sc'} = \langle \textbf{+}_i \Leftarrow \textbf{+}_i \rangle \ \text{and} \ \langle \varphi_2 \Leftarrow \varphi_1 \rangle = \langle \textbf{+}_i \Leftarrow \textbf{+} \rangle. \end{array}$

— Cases Remaining rules:

In the remaining rules, $\langle \varphi_2 \Leftarrow \varphi_1 \rangle = sc.$ Thus, $\varphi_1 \leq \varphi_2$.

By rule ReduceUpcast it follows that $M_1 \mapsto_R \inf_i W_0$ and it was already given that $M_2' \preccurlyeq M_2$.

$$\label{eq:case_equation} \blacksquare \text{ Case } \frac{W' \preccurlyeq M_1}{\langle \varphi_2' \Leftarrow \varphi_1' \rangle W' \preccurlyeq M_1}$$

$$W' \leq M_1$$
 Subderivation

$$M_1 = W$$
 By Lemma 55 (Precision on values)

$$M_2 \leq M_1$$
 By above equations

Case

$$\underbrace{\frac{\langle +_{\mathfrak{i}} \Leftarrow + \rangle \mathrm{inj}_{\mathfrak{i}} \, W'}_{\mathsf{M}'_{1}} \mapsto_{\mathsf{R}} \underbrace{\mathrm{inj}_{\mathfrak{i}} \, W'}_{\mathsf{M}'_{2}}}_{\mathsf{ReduceCastSuccess}}$$

Proceed by case analysis on $M_1' \leq M_1$.

$$\text{ } \overset{\bullet}{\text{ }} \overset{\textbf{Case}}{\underbrace{ \inf_{i} W' \preccurlyeq M}} \overset{\langle \textbf{+}_{i} \Leftarrow \textbf{+} \rangle \preccurlyeq \langle \varphi_{2} \Leftarrow \varphi_{1} \rangle}{\langle \textbf{+}_{i} \Leftarrow \textbf{+} \rangle \text{inj}_{i} W' \preccurlyeq \underbrace{\langle \varphi_{2} \Leftarrow \varphi_{1} \rangle M}_{M_{1}} }$$

Inversion on $\operatorname{inj}_i W' \leq M$ gives $M = \operatorname{inj}_i M_0$ and $W' \leq M_0$.

By Lemma 55 (Precision on values), $M_0 = W$.

Since $\langle +_i \leftarrow + \rangle$ is a backward cast bc', to derive bc' $\preccurlyeq \langle \varphi_2 \leftarrow \varphi_1 \rangle$, we either used Cast \preccurlyeq Refl or CastB \preccurlyeq S.

In the former case, we have $\langle \varphi_2 \Leftarrow \varphi_1 \rangle = bc'$. By rule ReduceCastSuccess we have $M_1 \mapsto_R M$, and we already have $M_2' \preccurlyeq M_2$. In the latter case, we have $\langle \varphi_2 \Leftarrow \varphi_1 \rangle = sc.$ By definition of being a safe cast, $\varphi_1 \leq \varphi_2$. Therefore, by rule ReduceUpcast we have $M_1 \mapsto_R M$, and we already have $M_2' \leq M_2$.

• Case
$$\frac{\inf_i W' \preccurlyeq M_1}{\langle +_i \Leftarrow + \rangle \inf_i W' \preccurlyeq M_1}$$

$$\operatorname{inj}_{i} W' \preceq M_{1}$$
 Subderivation

$$\begin{array}{ll} \operatorname{inj_i} W' \preccurlyeq M_1 & \operatorname{Subderivation} \\ & M_1 = W & \operatorname{By Lemma 55 (Precision on values)} \\ & M_2' \preccurlyeq M_1 & \operatorname{By above equations} \end{array}$$

$$M_2 \leq M_1$$
 By above equations

Case

$$\underbrace{\frac{\varphi' \in \{+_i, +\}}{\langle +_k \Leftarrow \varphi' \rangle \text{inj}_i \ W'}}_{M'_1} \mapsto_{\mathsf{R}} \underbrace{\frac{\mathsf{matchfail}}{M'_2}}_{\mathsf{R}'_2} \ \mathsf{ReduceCastFailure}$$

Proceed by case analysis on $M_1' \leq M_1$

42 2020/9/22

$$\text{ } \overset{\text{Lase}}{\underbrace{\inf_{i} W' \preccurlyeq M}} \underbrace{\langle \textbf{+}_{k} \Leftarrow \varphi' \rangle \preccurlyeq \langle \varphi_{2} \Leftarrow \varphi_{1} \rangle}_{ \langle \textbf{+}_{k} \Leftarrow \varphi' \rangle \text{inj}_{i} W' \preccurlyeq \underbrace{\langle \varphi_{2} \Leftarrow \varphi_{1} \rangle M}_{M_{1}}_{1} }$$

Since $\cdot \vdash M_1 : T_1$ and M_1 is not a value nor is it matchfail, by Theorem 7 there exists M_2 such that $M_1 \mapsto M_2$. By definition, $M_2' = \text{matchfail} \leq M_2$.

• Case
$$\frac{\inf_{i} W' \leq M_{1}}{\langle +_{k} \Leftarrow \varphi' \rangle \inf_{i} W' \leq M_{1}}$$

 $\operatorname{inj}_i W' \leq M_1$ Subderivation

 $M_1 = W$ By Lemma 55 (Precision on values) **13**

 $M_2' \leq M_1$ By definition of \leq

• Case

$$\underbrace{\frac{\mathsf{case}(\mathsf{inj}_{\mathsf{i}} \, W', \mathsf{inj}_{\mathsf{i}} \, x. M'_{\mathsf{i}})}_{M'_{\mathsf{i}}} \mapsto_{\mathsf{R}} \underbrace{[W'/x] M'_{\mathsf{i}}}_{M'_{\mathsf{2}}}}^{\mathsf{ReduceCase1}}$$

Proceed by inversion on case($\inf_i W', \inf_i x.M_i'$) $\leq M_1$. In the first case, $M_1 = case(M, inj_i x.M_i)$:

$$\begin{array}{ll} \operatorname{inj}_i W' \preccurlyeq M & \operatorname{From \ definition \ of} \preccurlyeq \\ M_i' \preccurlyeq M_i & " \\ M = \operatorname{inj}_i M_0 & \operatorname{From \ definition \ of} \preccurlyeq \end{array}$$

 $M = inj_i M_0$ $W' \preccurlyeq M_0$

 $M_0 = W$ By Lemma 55 (Precision on values) $W' \preccurlyeq W$ By above equations

 $M_1 = case(inj_i W, inj_i x.M_i)$ By above equations By rule ReduceCase1

 $M_1 \mapsto_R \underbrace{[W/x]M_i}_{M_2}$ $[W'/x]M_i' \preccurlyeq [W/x]M_i$ By Lemma 56 (Substitution preserves precision)

In the second case, $M_1 = case(M, inj_1 x_1.M_{11}, inj_2 x_2.M_{21})$:

$$\begin{array}{lll} \operatorname{inj}_i W' \preccurlyeq M & \operatorname{From \ definition \ of} \preccurlyeq \\ M'_i \preccurlyeq M_{i1} & " \\ M = \operatorname{inj}_i M_0 & \operatorname{From \ definition \ of} \preccurlyeq \\ W' \prec M & " \end{array}$$

 $W' \preccurlyeq M_0$

 $M_0 = W$ By Lemma 55 (Precision on values) $W' \preccurlyeq W$ By above equations

$$M_1 = \mathsf{case}(\mathsf{inj}_i \ W, \mathsf{inj}_1 \ x_1.M_{11}, \mathsf{inj}_2 \ x_2.M_{21})$$
 By above equations $M_1 \mapsto_R [W/x_i] M_{i1}$ By rule ReduceCase2

 $[W'/x]M_i' \preccurlyeq [W/x_i]M_{i1}$ By Lemma 56 (Substitution preserves precision)

• Case

$$\underbrace{\underbrace{\mathsf{case}(\mathsf{inj}_{\mathsf{i}} \, \mathsf{W}', \mathsf{inj}_{\mathsf{1}} \, \mathsf{x}_{\mathsf{1}}.\mathsf{M}'_{\mathsf{11}}, \mathsf{inj}_{\mathsf{2}} \, \mathsf{x}_{\mathsf{2}}.\mathsf{M}'_{\mathsf{21}})}_{\mathsf{M}'_{\mathsf{2}}} \mapsto_{\mathsf{R}} \underbrace{[\mathsf{W}'/\mathsf{x}_{\mathsf{i}}] \mathsf{M}'_{\mathsf{i}\mathsf{1}}}_{\mathsf{M}'_{\mathsf{2}}}^{\mathsf{ReduceCase2}}$$

$$\begin{array}{ll} M_1' \preccurlyeq M_1 & \text{Given} \\ M_1 = \mathsf{case}(M, \mathsf{inj}_1 \, x_1.M_{11}, \mathsf{inj}_2 \, x_2.M_{21}) & \text{From definition of} \preccurlyeq \\ \mathsf{inj}_i \, W' \preccurlyeq M & " \\ M_{11}' \preccurlyeq M_{11} & " \end{array}$$

$$M_{11} \leq M_{11}$$

 $M'_{21} \leq M_{21}$ "
 $M = inj_i M_0$ From definition of \leq

$$W' \leq M_0$$
 $W' = M_0$
 $W' = M_0$

 $M_0 = W$ By Lemma 55 (Precision on values) $W' \leq W$ By above equations

> 2020/9/22 43

• Case

Theorem 12 (Stepping preserves precision).

If $\cdot \vdash M_1' : T_1'$ and $\cdot \vdash M_1 : T_1$ and $M_1' \preccurlyeq M_1$ and $M_1' \mapsto M_2'$ then either

- (a) M_1 is a value and $M_2' \leq M_1$, or
- (b) there exists M_2 such that $M_1\mapsto M_2$ and $M_2'\preccurlyeq M_2$, or
- (c) $M_1 = \text{matchfail} \text{ and } M'_2 \leq M_1$.

Proof. Proceed by case analysis on $M'_1 \mapsto M'_2$.

• Case

ase
$$\frac{M'_{01} \mapsto_R M'_{02}}{\mathcal{E}'[M'_{01}] \mapsto \mathcal{E}'[M'_{02}]} \text{ StepContext}$$

$$\mathcal{E}'[M'_{01}] \Leftrightarrow M_1 \qquad \text{ Given }$$

$$M_1 = \mathcal{E}[M_{01}] \qquad \text{ By Lemma 57 (Precision inversion on evaluation contexts)}$$

$$M'_{01} \Leftrightarrow M_{01} \qquad "$$

$$\cdot \vdash \mathcal{E}'[M'_{01}] : T'_1 \qquad \text{ Given }$$

$$\cdot \vdash \mathcal{E}[M_{01}] : T_1 \qquad \text{ Given }$$

$$\cdot \vdash \mathcal{E}[M_{01}] : T_0 \qquad \text{ By Lemma 52 (Evaluation context typing)}$$

$$\cdot \vdash M_{01} : T_{01} \qquad \text{ By Lemma 52 (Evaluation context typing)}$$

$$\cdot \vdash M_{01} : T_{01} \qquad \text{ By Lemma 52 (Evaluation context typing)}$$

By Lemma 52 (Evaluation context typing)

Proceed by case analysis on the result of applying Lemma 59 (Reduction preserves precision).

In the first case, M_{01} is a value and $M'_{02} \leq M_{01}$. Since $M'_{01} \leq M_{01}$ and $\mathcal{E}'[M'_{01}] \leq \mathcal{E}[M_{01}]$, by Lemma 58 (Evaluation contexts

preserve precision) it follows that $\mathcal{E}'[M_{02}'] \preccurlyeq \mathcal{E}[M_{01}]$. This is alternative (a). In the second case, $M_{01} \mapsto_R M_{02}$ and $M_{02}' \preccurlyeq M_{02}$. Therefore, by rule StepContext it follows $M_1 \mapsto_R \mathcal{E}[M_{02}]$. Since $M_{01}' \preccurlyeq M_{01}$ and $\mathcal{E}'[M'_{01}] \preccurlyeq \mathcal{E}[M_{01}]$, by Lemma 58 (Evaluation contexts preserve precision) it follows that $\mathcal{E}'[M'_{02}] \preccurlyeq \mathcal{E}[M_{02}]$. This is alternative (b).

Case

$$\underbrace{\frac{\mathcal{E}' \neq []}{\mathcal{E}'[\mathsf{matchfail}]} \mapsto \underbrace{\mathsf{matchfail}}_{\mathsf{M}'_2}}_{\mathsf{StepMatchfail}} \mathsf{StepMatchfail}$$

Since $\cdot \vdash M_1 : T_1$, by Theorem 7 it follows that either M_1 is a value, or there exists M_2 such that $M_1 \mapsto M_2$, or $M_1 = \mathtt{matchfail}$. In the first case, $M'_2 = \text{matchfail} \leq M_1$ by definition of \leq , which is alternative (a).

In the second case, $M'_2 = \text{matchfail} \leq M_2$ by definition of \leq , which is alternative (b).

In the first case, $M'_2 = \text{matchfail} \leq \text{matchfail} = M_1$ by definition of \leq , which is alternative (c).

Theorem 13 (\leq respects convergence).

If $M' \leq M$ where $\cdot \vdash M' : T'$ and $\cdot \vdash M : T$ and M' converges then M also converges.

Proof. It is given that M' converges. By Definition 1, there exists a value W' such that $M' \mapsto^* W'$. Proceed by induction on the number of steps in $M' \mapsto^* W'$.

If M' = W' then $W' \leq M$. By Lemma 55 (Precision on values), M = W for some value W. Therefore, M converges as well.

Otherwise, M' takes at least one step, that is, $M' \mapsto M'_0 \mapsto^* W'$. Then M'_0 must also converge, with $M'_0 \mapsto^* W'$ in fewer steps than $M' \mapsto^* W'$. Since $M' \mapsto M'_0$, proceed by case analysis on the result of applying Theorem 12.

- In the first case (a), M is a value, so M converges.
- In the second case (b), there exists M_0 such that $M \mapsto M_0$ and $M'_0 \leq M_0$. By Theorem 6, $\cdot \vdash M'_0 : T'$ and similarly $\cdot \vdash M_0 : T$.
- By the induction hypothesis, M_0 converges. Since M_0 converges, M must also converge to the same value.
- In the third case (c), M = matchfail and $M'_0 \leq M$. By inversion on $M'_0 \leq \text{matchfail}$ it follows that $M'_0 = \text{matchfail}$. But we know that M'_0 converges, a contradiction. Hence, this case is impossible.

D.6 Translation

D.6.1 Soundness

Theorem 16 (Sum Translation soundness).

Given δ' and δ , there exists \mathcal{C} such that $\delta' \Rightarrow \delta \hookrightarrow \mathcal{C}$.

Moreover, if $\Theta \vdash M : (T_1 \mid \delta' \mid T_2)$ then $\Theta \vdash C[M] : (T_1 \mid \delta \mid T_2)$.

Proof. Proceed by case analysis on whether $|\delta'| \leq |\delta|$.

• Case $|\delta'| \leq |\delta|$:

```
By Lemma 43 (Reflexivity of subtyping)
             T_1 \leq T_1 \\
            T_2 \stackrel{-}{\leq} T_2
                                                     By Lemma 43 (Reflexivity of subtyping)
           |\delta'| \leq |\delta|
                                                     Given
(T_1 | \delta' | T_2) \le (T_1 | \delta | T_2)
                                                     By definition of \leq
            \delta' \Rightarrow \delta \hookrightarrow []
                                                     By rule CoeSub
             \Theta \vdash M : (\mathsf{T}_1 \mid \delta' \mid \mathsf{T}_2)
                                                     Suppose
              \Theta \vdash M : (T_1 \mid \delta \mid T_2)
                                                     By rule TSub
        C[M] = M
                                                     By definition
              \Theta \vdash C[M] : (T_1 \mid \delta \mid T_2)
                                                     By above equations
```

• Case $|\delta'| \not\leq |\delta|$:

$$\begin{array}{ll} |\delta'| \not \leq |\delta| & \text{Given} \\ |\delta'| \Rightarrow |\delta| \hookrightarrow \langle |\delta| \Leftarrow |\delta'| \rangle [] & \text{By rule CoeCast} \\ \Theta \vdash M : (T_1 \, |\delta'| T_2) & \text{Suppose} \\ \Theta \vdash \langle |\delta| \Leftarrow |\delta'| \rangle M : (T_1 \, |\delta| T_2) & \text{By rule TCast} \\ \mathcal{C}[M] = \langle |\delta| \Leftarrow |\delta'| \rangle M & \text{By definition} \\ \Theta \vdash \mathcal{C}[M] : (T_1 \, |\delta| T_2) & \text{By above equations} \end{array}$$

Theorem 17 (Type translation soundness).

If $A' \simeq A$ then there exists C such that $A' \Rightarrow A \hookrightarrow C$.

Moreover, if $\Theta \vdash M : |A'|$ then $\Theta \vdash C[M] : |A|$.

Proof. By induction on the structure of the derivation of $A' \simeq A$.

• Case Unit ≃ Unit:

$$\begin{array}{ll} \mathsf{Unit} \Rightarrow \mathsf{Unit} \hookrightarrow [] & \mathsf{By} \ \mathsf{rule} \ \mathsf{CoeUnit} \\ \Theta \vdash \mathsf{M} : |\mathsf{Unit}| & \mathsf{Suppose} \\ \Theta \vdash \mathcal{C}[\mathsf{M}] : |\mathsf{Unit}| & \mathsf{By} \ \mathsf{definition} \ \mathsf{of} \ \mathcal{C} \end{array}$$

• Case
$$\underbrace{A_1' \simeq A_1}_{A_1'} \underbrace{A_2' \simeq A_2}_{A_2'} \simeq \underbrace{(A_1 \delta A_2)}_{A}$$

Proceed by case analysis on the definition of δ' . In the first case, suppose $\delta' \in \{+\frac{2}{1}, +1\}$.

```
|A_1'| \leq |A_1'|
                                                                                    By Lemma 43 (Reflexivity of subtyping)
                 |A_2'| \leq |A_2'|
                                                                                    By Lemma 43 (Reflexivity of subtyping)
                   |\delta'| \leq +_1
                                                                                   By definition of \leq
   |A_1'|\,|\delta'|\,|A_2'| \leq |A_1'| +_1 |A_2'|
                                                                                   By definition of \leq
                      \Theta \vdash M : |(A_1' \delta' A_2')|
                                                                                   Suppose
                      \Theta \vdash M : (|A_1'| |\delta'| |A_2'|)
                                                                                   By definition of type translation
                      \Theta \vdash M : (|A'_1| +_1 |A'_2|)
                                                                                   By rule TSub
                 |A_1| \leq |A_1|
                                                                                    By Lemma 43 (Reflexivity of subtyping)
                  |A_2| \le |A_2|
                                                                                    By Lemma 43 (Reflexivity of subtyping)
                    +1 \leq |\delta'|
                                                                                    By definition of <
    |A_1| + |A_2| \le |A_1| |\delta'| |A_2|
                                                                                   By definition of \leq
     \Theta, x_1 : |A_1'| \vdash x_1 : |A_1'|
                                                                                   By rule TVar
                   A_1' \simeq A_1
                                                                                   Subderivation
                                                                                    By the induction hypothesis
                   A_1' \Rightarrow A_1 \hookrightarrow C_1
     \Theta, x_1: |A_1'| \vdash \mathcal{C}_1[x_1]: |A_1|
     \Theta, x_1 : |A_1'| \vdash \operatorname{inj}_1 C_1[x_1] : (|A_1| +_1 |A_2|)
                                                                                    By rule T+iIntro
     \Theta, x_1 : |A_1'| \vdash \operatorname{inj}_1 C_1[x_1] : (|A_1| |\delta'| |A_2|)
                                                                                   By rule TSub
                                                                                                        By rule T+<sub>i</sub>Elim
      \Theta \vdash \mathsf{case}(M, \mathsf{inj}_1 \, x_1.\mathsf{inj}_1 \, \mathcal{C}_1[x_1]) : (|A_1| \, |\delta'| \, |A_2|)
    \delta' \Rightarrow \delta \hookrightarrow C_3
                                                                                                        By Theorem 16
      \Theta \vdash \mathcal{C}_3[\mathsf{case}(M, \mathsf{inj}_1 \ x_1.\mathsf{inj}_1 \ \mathcal{C}_1[x_1])] : (|A_1| \ |\delta| \ |A_2|)
       \Theta \vdash \mathcal{C}_3[\mathsf{case}(\mathsf{M},\mathsf{inj}_1 \, \mathsf{x}_1.\mathsf{inj}_1 \, \mathcal{C}_1[\mathsf{x}_1])] : |(\mathsf{A}_1 \, \delta \, \mathsf{A}_2)|
                                                                                                        By definition of type translation
   (A_1'\,\delta'\,A_2') \Rightarrow (A_1\,\delta\,A_2) \hookrightarrow \underbrace{\mathcal{C}_3[\mathsf{case}([\,],\mathsf{inj}_1\,x_1.\mathsf{inj}_1\,\mathcal{C}_1[x_1])]}_{}
                                                                                                                    By rule CoeCase1L
In the second case, suppose \delta' \in \{+\frac{2}{2}, +2\}. Symmetric to the previous case, hence omitted.
In the last case, suppose \delta' \in \{+^2, +_1^*, +_2^*, +\}.
                 |A_1'| \leq |A_1'|
                                                      By Lemma 43 (Reflexivity of subtyping)
                 |A_2'| < |A_2'|
                                                      By Lemma 43 (Reflexivity of subtyping)
                  |\delta'| < +
                                                      By definition of <
  |A_1'| |\delta'| |A_2'| \le |A_1'| + |A_2'|
                                                      By definition of \leq
                     \Theta \vdash M : |(A_1' \delta' A_2')|
                                                                                         Suppose
                     \Theta \vdash M : (|A_1'| |\delta'| |A_2'|)
                                                                                         By definition of type translation
                     \Theta \vdash M : (|A_1'| + |A_2'|)
                                                                                         By rule TSub
    \Theta, x_1 : |A_1'| \vdash x_1 : |A_1'|
                                                                                         By rule TVar
                  A_1' \simeq A_1
                                                                                         Subderivation
                 A_1' \Rightarrow A_1 \hookrightarrow C_1
                                                                                         By the induction hypothesis
    \Theta, x_1 : |A_1'| \vdash C_1[x_1] : |A_1|
    \Theta, x_1: |A_1'| \vdash \text{inj}_1 \ \mathcal{C}_1[x_1]: (|A_1| +_1 |A_2|)
                                                                                         By rule T+iIntro
                  +_1^? \Rightarrow \delta' \hookrightarrow C_1'
                                                                                         By Theorem 16
    \Theta, x_1 : |A_1'| \vdash C_1'[inj_1 C_1[x_1]] : (|A_1| |\delta'| |A_2|)
    \Theta, x_2 : |A_2'| \vdash x_2 : |A_2'|
                                                                                         By rule TVar
                  A_2' \simeq A_2
                                                                                         Subderivation
                 A_2' \Rightarrow A_2 \hookrightarrow C_2
                                                                                         By the induction hypothesis
    \Theta, \mathbf{x}_2 : |\mathbf{A}_2'| \vdash \mathcal{C}_2[\mathbf{x}_2] : |\mathbf{A}_2|
    \Theta, x_2 : |A_2'| \vdash \operatorname{inj}_2 C_2[x_2] : (|A_1| +_2 |A_2|)
                                                                                         By rule T+iIntro
                  +^{?}_{2} \Rightarrow \delta' \hookrightarrow C'_{2}
                                                                                         By Theorem 16
    \Theta, x_2 : |A_2'| \vdash C_2'[inj_2 C_2[x_2]] : (|A_1| |\delta'| |A_2|)
      \Theta \vdash \mathsf{case}(M, \mathsf{inj}_1 \ x_1. \mathcal{C}_1'[\mathsf{inj}_1 \ \mathcal{C}_1[x_1]], \mathsf{inj}_2 \ x_2. \mathcal{C}_2'[\mathsf{inj}_2 \ \mathcal{C}_2[x_2]])] : (|A_1| \, |\delta'| \, |A_2|)
                                                                                                                                                         By rule T+Elim
    \delta' \Rightarrow \delta \hookrightarrow C_3
                                                                                                                                                         By Theorem 16
      \Theta \vdash \mathcal{C}_{3}[\mathsf{case}(M, \mathsf{inj}_{1} \ x_{1}.\mathcal{C}_{1}'[\mathsf{inj}_{1} \ \mathcal{C}_{1}[x_{1}]], \mathsf{inj}_{2} \ x_{2}.\mathcal{C}_{2}'[\mathsf{inj}_{2} \ \mathcal{C}_{2}[x_{2}]])]] : (|A_{1}| \ |\delta| \ |A_{2}|)
       \Theta \vdash \mathcal{C}_{3}[\mathsf{case}(\mathsf{M},\mathsf{inj}_{1} \, x_{1}.\mathcal{C}'_{1}[\mathsf{inj}_{1} \, \mathcal{C}_{1}[x_{1}]],\mathsf{inj}_{2} \, x_{2}.\mathcal{C}'_{2}[\mathsf{inj}_{2} \, \mathcal{C}_{2}[x_{2}]])]]: |\mathsf{A}_{1} \, \delta \, \mathsf{A}_{2})|
                                                                                                                                                         By definition
```

$$(A_1' \, \delta' \, A_2') \Rightarrow (A_1 \, \delta \, A_2) \hookrightarrow \underbrace{\mathcal{C}_3[\mathsf{case}([], \mathsf{inj}_1 \, x_1.\mathcal{C}_1'[\mathsf{inj}_1 \, \mathcal{C}_1[x_1]], \mathsf{inj}_2 \, x_2.\mathcal{C}_2'[\mathsf{inj}_2 \, \mathcal{C}_2[x_2]])]]}_{\mathcal{C}} \quad \text{By rule CoeCase2}$$

$$\begin{array}{lll} \textbf{Case} & \underbrace{A_1' \simeq A_1 & A_2' \simeq A_2}_{(A_1' \to A_2')} & \underbrace{(A_1 \to A_2)}_{A'} \\ \\ \Theta, x: |A_1| \vdash x: |A_1| & \text{By rule TVar} \\ \\ A_1' \simeq A_1 & \text{Subderivation} \\ A_1 \simeq A_1' & \text{By Lemma 12 (Symmetry of Structural Equivalence)} \\ A_1 \Rightarrow A_1' \hookrightarrow C_1 & \text{By the induction hypothesis} \\ \\ \Theta, x: |A_1| \vdash C_1[x]: |A_1'| & " & \\ \\ \Theta \vdash M: |(A_1' \to A_2')| & \text{Suppose} \\ \Theta \vdash M: |(A_1' \to A_2')| & \text{By definition of type translation} \\ \Theta, x: |A_1| \vdash M C_1[x_1]: |A_2'| & \text{By rule T} \to \text{Elim} \\ \\ A_2' \simeq A_2 & \text{Subderivation} \\ A_2' \simeq A_2 & \text{Subderivation} \\ A_2' \simeq A_2 & \text{Subderivation} \\ \Theta, x: |A_1| \vdash C_2[M C_1[x_1]]: |A_2| & \text{By the induction hypothesis} \\ \\ \Theta, x: |A_1| \vdash C_2[M C_1[x_1]]: |A_2| & \text{By rule T} \to \text{Intro} \\ \\ \Theta \vdash \lambda x. C_2[M C_1[x_1]]: |(A_1 \to A_2)| & \text{By rule T} \to \text{Intro} \\ \\ By definition of type translation} \\ \\ (A_1' \to A_2') \Rightarrow (A_1 \to A_2) \hookrightarrow \lambda x. C_2[[|C_1[x]]] & \text{By rule Coe} \\ \\ \end{tabular}$$

Theorem 9 (Translation soundness).

If $\Gamma \vdash e : A$ then there exists M such that $\Gamma \vdash e : A \hookrightarrow M$ and $|\Gamma| \vdash M : |A|$.

Proof. By induction on the structure of the derivation of $\Gamma \vdash e : A$.

• Case SVar: Apply rules STVar and TVar.

• Case
$$\frac{\Gamma \vdash e : A' \qquad A' \leadsto A}{\Gamma \vdash e : A} \text{ SCSub}$$

$$\Gamma \vdash e : A' \qquad \text{Subderivation}$$

$$\Gamma \vdash e : A' \hookrightarrow M' \qquad \text{By the induction hypothesis}$$

$$|\Gamma| \vdash M' : |A'| \qquad "$$

$$A' \leadsto A \qquad \qquad \text{Given}$$

$$A' \simeq A \qquad \qquad \text{By Lemma 17 (Directed consistency obeys Structural Equivalence)}$$

$$A' \Rightarrow A \hookrightarrow \mathcal{C} \qquad \qquad \text{By Theorem 17}$$

$$|\Gamma| \vdash \mathcal{C}[M'] : |A| \qquad "$$

$$\Gamma \vdash e : A \hookrightarrow \mathcal{C}[M'] \qquad \text{By rule STCSub}$$

- Case SAnno: Use the induction hypothesis and apply rule STAnno.
- Case SUnitIntro: Apply rules STUnitIntro and TUnitIntro.
- Case SSumIntro: Use the induction hypothesis and apply rules STSumIntro and T+iIntro.
- Case SSumElim1: Use the induction hypothesis and apply rules STSumElim1 and T+i Elim.
- Case SSumElim2: Use the induction hypothesis and apply rules STSumElim2 and T+Elim.
- Case $S \rightarrow Intro$: Use the induction hypothesis and apply rules $ST \rightarrow Intro$ and $T \rightarrow Intro$.
- Case $S \rightarrow Elim$: Use the induction hypothesis and apply rules $ST \rightarrow Elim$ and $T \rightarrow Elim$.

D.6.2 Precision

Theorem 11 depends on Lemma 60 (Cast insertion preserves precision), which uses a modified version of the translation that always inserts casts, even safe ones. In effect, the modified translation does not have rule CoeSub and always uses rule CoeCast (Figure 12). It also inserts safe casts C'_1 and C'_2 , similar to CoeCase2, in rules *CoeCase1L and *CoeCase1R. See Figure 21.

Lemma 60 (Cast insertion preserves precision).

If
$$\delta_1' \Rightarrow \delta_2' \hookrightarrow C'$$
 and $\delta_1 \Rightarrow \delta_2 \hookrightarrow C$ and $\delta_1' \sqsubseteq \delta_1$ and $\delta_2' \sqsubseteq \delta_2$ and $M' \preccurlyeq M$ then $C'[M'] \preccurlyeq C[M]$.

47 2020/9/22

$$\delta' \Rightarrow \delta \hookrightarrow \mathcal{C}$$
 Coercion \mathcal{C} coerces sum $|\delta'|$ to sum $|\delta|$

$$\frac{\frac{|\delta'| \not\leq |\delta|}{\delta' \Rightarrow \delta \hookrightarrow \langle |\delta| \Leftarrow |\delta'| \rangle[]}}{\delta' \Rightarrow \delta \hookrightarrow \langle |\delta| \Leftarrow |\delta'| \rangle[]} *CoeCast$$

$$A' \Rightarrow A \hookrightarrow \mathcal{C}$$
 Coercion \mathcal{C} coerces target type $|A'|$ to $|A|$

$$\frac{A_1 \Rightarrow A_1' \hookrightarrow \mathcal{C}_1 \quad A_2' \Rightarrow A_2 \hookrightarrow \mathcal{C}_2}{\text{Unit} \Rightarrow \text{Unit} \hookrightarrow []} \text{ CoeUnit}$$

$$\frac{A_1 \Rightarrow A_1' \hookrightarrow \mathcal{C}_1 \quad A_2' \Rightarrow A_2 \hookrightarrow \mathcal{C}_2}{(A_1' \to A_2) \hookrightarrow \lambda x. \, \mathcal{C}_2[[] \, \mathcal{C}_1[x]]} \text{ Coe} \rightarrow$$

$$\frac{\delta' \in \{+^?, +_1\}}{A_1' \Rightarrow A_1 \hookrightarrow \mathcal{C}_1 \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3} \quad *\text{CoeCase1L} \quad \frac{\delta' \in \{+^?, +_2\}}{A_2' \Rightarrow A_2 \hookrightarrow \mathcal{C}_2 \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_2'} \\ \frac{A_2' \Rightarrow A_2 \hookrightarrow \mathcal{C}_2 \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3}{(A_1' \, \delta' \, A_2') \Rightarrow (A_1 \, \delta \, A_2)} \quad *\text{CoeCase1R}$$

$$\hookrightarrow \mathcal{C}_3[\text{case}([], \text{inj}_1 \, x_1. \, \mathcal{C}_1'[\text{inj}_1 \, \mathcal{C}_1[x_1])] \quad \hookrightarrow \mathcal{C}_3[\text{case}([], \text{inj}_2 \, x_2. \, \mathcal{C}_2'[\text{inj}_2 \, \mathcal{C}_2[x_2])]$$

$$+^?_1 \Rightarrow \delta' \hookrightarrow \mathcal{C}_1' \quad +^?_2 \Rightarrow \delta' \hookrightarrow \mathcal{C}_2' \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3$$

$$\frac{\delta' \in \{+^?, +_1^*, +_2^*, +\}}{(A_1' \, \delta' \, A_2') \Rightarrow (A_1 \, \delta \, A_2) \hookrightarrow \mathcal{C}_2} \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3$$

$$\frac{\delta' \in \{+^?, +_1^*, +_2^*, +\}}{(A_1' \, \delta' \, A_2') \Rightarrow (A_1 \, \delta \, A_2) \hookrightarrow \mathcal{C}_2} \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3$$

$$\frac{\delta' \in \{+^?, +_1^*, +_2^*, +\}}{(A_1' \, \delta' \, A_2') \Rightarrow (A_1 \, \delta \, A_2) \hookrightarrow \mathcal{C}_2} \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3$$

$$\frac{\delta' \in \{+^?, +_1^*, +_2^*, +\}}{(A_1' \, \delta' \, A_2') \Rightarrow (A_1 \, \delta \, A_2) \hookrightarrow \mathcal{C}_2} \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3$$

$$\frac{\delta' \in \{+^?, +_1^*, +_2^*, +\}}{(A_1' \, \delta' \, A_2') \Rightarrow (A_1 \, \delta \, A_2) \hookrightarrow \mathcal{C}_2} \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3$$

$$\frac{\delta' \in \{+^?, +_1^*, +_2^*, +\}}{(A_1' \, \delta' \, A_2') \Rightarrow (A_1 \, \delta \, A_2) \hookrightarrow \mathcal{C}_2} \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3$$

$$\frac{\delta' \in \{+^?, +_1^*, +_2^*, +\}}{(A_1' \, \delta' \, A_2') \Rightarrow (A_1 \, \delta \, A_2) \hookrightarrow \mathcal{C}_2} \quad \delta' \Rightarrow \delta \hookrightarrow \mathcal{C}_3$$

Figure 21. Part of the type-directed translation, modified to insert safe casts; differences highlighted

Proof. Note the following reasons for arriving at the result.

- (a) If the translated sums are equal, that is, $|\delta_1'| = |\delta_1|$ and $|\delta_2'| = |\delta_2|$, we have C' = C. (Casts are unique; in this context, this is immediate because we are using a translation that generates casts even if they are safe, so there is only one rule, *CoeCast, that derives the judgment.) Then the result follows from $M' \leq M$ and the definition of \leq .
- (b) If $\mathcal{C}' = \langle \delta_2' \Leftarrow |\delta_1'| \rangle$ [] and $\mathcal{C} = \langle |\delta_2| \Leftarrow |\delta_1| \rangle$ [] and $\langle \delta_2' \Leftarrow \delta_1' \rangle \preccurlyeq \langle \delta_2 \Leftarrow \delta_1 \rangle$ then $\mathcal{C}'[M'] \preccurlyeq \mathcal{C}[M]$ by definition of \preccurlyeq as $M' \preccurlyeq M$.

Proceed by case analysis on $\delta'_1 \sqsubseteq \delta_1$ based on the reflexive, transitive closure of precision on sums.

- Cases $+_i \sqsubseteq +_i$, $+_i \sqsubseteq +_i^?$, $+_i \sqsubseteq +_i^*$, $+_i^? \sqsubseteq +_i^?$, $+_i^* \sqsubseteq +_i^*$: In these cases, $|\delta_1'| = |\delta_1| = +_i$. Proceed by case analysis on $\delta_2' \sqsubseteq \delta_2$.
 - Cases $+_i \sqsubseteq +_i$, $+_i \sqsubseteq +_i^?$, $+_i \sqsubseteq +_i^*$, $+_i^? \sqsubseteq +_i^?$, $+_i^* \sqsubseteq +_i^*$: Here, $|\delta_2'| = |\delta_2| = +_i$.

The translated sums are equal: go to (a) above.

- Cases $+_i \sqsubseteq +^2$, $+_i^2 \sqsubseteq +^2$, $+_i^* \sqsubseteq +^2$: Here, $|\delta_2'| = +_i$ and $|\delta_2| = +$. We have $\langle +_i \Leftarrow +_i \rangle \preccurlyeq \langle + \Leftarrow +_i \rangle$. Go to (b).
- Cases + \sqsubseteq +, + \sqsubseteq +[?], +[?] \sqsubseteq +[?]: Here, $|\delta_2'| = |\delta_2| =$ +. Go to (a) above.
- Cases $+_k \sqsubseteq +_k$, $+_k \sqsubseteq +_k^2$, $+_k \sqsubseteq +_k^*$, $+_k^2 \sqsubseteq +_k^2$, $+_k^* \sqsubseteq +_k^*$: Here, $|\delta_2'| = |\delta_2| = +_k$. Go to (a).
- Cases $+_k \sqsubseteq +^2$, $+_k^2 \sqsubseteq +^2$, $+_k^* \sqsubseteq +^2$: Here, $|\delta_2'| = +_k$ and $|\delta_2| = +$. We have $\langle +_k \Leftarrow +_i \rangle \preccurlyeq \langle + \Leftarrow +_i \rangle$. Go to (b).

- Cases $+_i \sqsubseteq +^2$, $+_i^2 \sqsubseteq +^2$, $+_i^* \sqsubseteq +^2$: In these cases, $|\delta_1'| = +_i$ and $|\delta_1| = +$. Proceed by case analysis on $\delta_2' \sqsubseteq \delta_2$.
 - Cases $+_i \sqsubseteq +_i$, $+_i \sqsubseteq +_i^2$, $+_i \sqsubseteq +_i^*$, $+_i^2 \sqsubseteq +_i^2$, $+_i^* \sqsubseteq +_i^*$: We have $\langle +_i \Leftarrow +_i \rangle \preccurlyeq \langle +_i \Leftarrow + \rangle$. Go to (b).
 - Cases $+_i \sqsubseteq +^2$, $+_i^2 \sqsubseteq +^2$, $+_i^* \sqsubseteq +^2$: We have $\langle +_i \Leftarrow +_i \rangle \preccurlyeq \langle + \Leftarrow + \rangle$. Go to (b).
 - Cases + \sqsubseteq +, + \sqsubseteq +[?], +[?] \sqsubseteq +[?]: We have \langle + \Leftarrow +_i \rangle \leq \langle + \Leftarrow + \rangle . Go to (b).
 - Cases $+_k \sqsubseteq +_k$, $+_k \sqsubseteq +_k^2$, $+_k \sqsubseteq +_k^*$, $+_k^2 \sqsubseteq +_k^2$, $+_k^* \sqsubseteq +_k^*$: We have $\langle +_k \Leftarrow +_i \rangle \preccurlyeq \langle +_k \Leftarrow + \rangle$. Go to (b).

48

- Cases $+_k \sqsubseteq +^2$, $+_k^2 \sqsubseteq +^2$, $+_k^* \sqsubseteq +^2$: We have $\langle +_k \Leftarrow +_i \rangle \preccurlyeq \langle + \Leftarrow + \rangle$. Go to (b).
- Cases $+ \sqsubseteq +, + \sqsubseteq +^2, +^2 \sqsubseteq +^2$: In these cases, $|\delta_1'| = |\delta_1| = +$. Proceed by case analysis on $\delta_2' \sqsubseteq \delta_2$.
 - Cases $+_i \sqsubseteq +_i$, $+_i \sqsubseteq +_i^2$, $+_i \sqsubseteq +_i^*$, $+_i^2 \sqsubseteq +_i^2$, $+_i^* \sqsubseteq +_i^*$: Here, $|\delta_2'| = |\delta_2| = +_i$. Go to (a).
 - Cases $+_i \sqsubseteq +^2$, $+_i^2 \sqsubseteq +^2$, $+_i^* \sqsubseteq +^2$: We have $\langle +_i \Leftarrow + \rangle \preccurlyeq \langle + \Leftarrow + \rangle$. Go to (b).
 - Cases + \Box +, + \Box +, + \Box +, + \Box +. Here, $|\delta_2'| = |\delta_2| = +$. Go to (a).
 - Cases $+_k \sqsubseteq +_k$, $+_k \sqsubseteq +_k^*$, $+_k \sqsubseteq +_k^*$, $+_k^* \sqsubseteq +_k^*$. Here, $|\delta_2'| = |\delta_2| = +_k$. Go to (a).
 - Cases $+_k \sqsubseteq +^?$, $+_k^? \sqsubseteq +^?$, $+_k^* \sqsubseteq +^?$: We have $\langle +_k \Leftarrow + \rangle \preccurlyeq \langle + \Leftarrow + \rangle$. Go to (b).

2020/9/22

Lemma 61 (Coercion preserves precision). If $A_1' \Rightarrow A_2' \hookrightarrow \mathcal{C}'$ and $A_1 \Rightarrow A_2 \hookrightarrow \mathcal{C}$ and $A_1' \sqsubseteq \overline{A}_1$ and $A_2' \sqsubseteq A_2$ and $\overline{M}' \preccurlyeq M$ then $C'[M'] \preceq C[M]$.

Proof. By induction on the structure of the derivation of $A_1' \Rightarrow A_2' \hookrightarrow \mathcal{C}'$.

• Case

$$\begin{array}{lll} \hline \text{Unit} \Rightarrow \text{Unit} \hookrightarrow [] & \text{CoeUnit} \\ \hline \text{Unit} \sqsubseteq A_1 & \text{Given} \\ \text{Unit} \sqsubseteq A_2 & \text{Given} \\ A_1 = \text{Unit} & \text{By Lemma 4 (Precision inversion)} \\ A_2 = \text{Unit} & \text{By Lemma 4 (Precision inversion)} \\ \hline \text{Unit} \Rightarrow \text{Unit} \hookrightarrow \mathcal{C} & \text{Given} \\ \mathcal{C} = [] & \text{By inversion on CoeUnit} \\ \hline M' \preccurlyeq M & \text{Given} \\ \mathcal{C}'[M'] \preccurlyeq \mathcal{C}[M] & \text{By definition of } \mathcal{C}' \text{ and } \mathcal{C} \\ \hline \end{array}$$

• Case
$$\frac{A_{12}' \Rightarrow A_{11}' \hookrightarrow \mathcal{C}_{1}' \qquad A_{21}' \Rightarrow A_{22}' \hookrightarrow \mathcal{C}_{2}'}{(A_{11}' \rightarrow A_{21}') \Rightarrow (A_{12}' \rightarrow A_{22}') \hookrightarrow \lambda x. \, \mathcal{C}_{2}'[[] \, \mathcal{C}_{1}'[x]]} \text{ Coe} \rightarrow$$

$$\begin{array}{ccc} A_{11}' \to A_{21}' \sqsubseteq A_1 & \text{Given} \\ A_1 = A_{11} \to A_{21} & \text{By Lemma 4 (Precision inversion)} \\ A_{11}' \sqsubseteq A_{11} & '' \\ A_{21}' \sqsubseteq A_{21} & '' \end{array}$$

$$\begin{array}{c} A_{12}^{\prime} \rightarrow A_{22}^{\prime} \sqsubseteq A_2 & \text{Given} \\ A_2 = A_{12} \rightarrow A_{22} & \text{By Lemma 4 (Precision inversion)} \\ A_{12}^{\prime} \sqsubseteq A_{12} & '' \\ A_{22}^{\prime} \sqsubseteq A_{22} & '' \end{array}$$

$$\begin{array}{lll} (A_{11} \rightarrow A_{21}) \Rightarrow (A_{12} \rightarrow A_{22}) \hookrightarrow \mathcal{C} & \text{Given} \\ A_{12} \Rightarrow A_{11} \hookrightarrow \mathcal{C}_1 & \text{By inversion on Coe} \rightarrow \\ A_{21} \Rightarrow A_{22} \hookrightarrow \mathcal{C}_2 & '' \\ \mathcal{C} = \lambda x. \, \mathcal{C}_2 \big[[] \, \mathcal{C}_1[x] \big] & '' \\ & x \preccurlyeq x & \text{By definition of } \preccurlyeq \\ A_{12}' \Rightarrow A_{11}' \hookrightarrow \mathcal{C}_1' & \text{Subderivation} \\ \mathcal{C}_1'[x] \preccurlyeq \mathcal{C}_1[x] & \text{By the induction hypothesis} \\ M' \preccurlyeq M & \text{Given} \\ \end{array}$$

$$\begin{array}{ccc} M' \leqslant M & \text{Given} \\ M' \, \mathcal{C}_1'[x] \leqslant M \, \mathcal{C}_1[x] & \text{By definition of } \leqslant \\ A_{21}' \Rightarrow A_{22}' \hookrightarrow \mathcal{C}_2' & \text{Subderivation} \\ \mathcal{C}_2'\left[M' \, \mathcal{C}_1'[x]\right] \leqslant \mathcal{C}_2\left[M \, \mathcal{C}_1[x]\right] & \text{By the induction hypothesis} \\ \lambda x. \, \mathcal{C}_2'\left[M' \, \mathcal{C}_1'[x]\right] \leqslant \lambda x. \, \mathcal{C}_2\left[M \, \mathcal{C}_1[x]\right] & \text{By definition of } \leqslant \end{array}$$

By definition of \leq

• Case

$$\frac{\delta_1' \in \{\textbf{+}_1^2, \textbf{+}_1\}}{A_{11}' \Rightarrow A_{12}' \hookrightarrow \mathcal{C}_1'} \quad \begin{array}{c} \textbf{+}_1^2 \Rightarrow \delta_1' \hookrightarrow \mathcal{C}_{11}' \\ \delta_1' \Rightarrow A_{12}' \hookrightarrow \mathcal{C}_1' & \delta_1' \Rightarrow \delta_2' \hookrightarrow \mathcal{C}_3' \\ \hline (A_{11}' \delta_1' A_{21}') \Rightarrow (A_{12}' \delta_2' A_{22}') \\ \hookrightarrow \mathcal{C}_3' \big[\text{case}([], \text{inj}_1 \ x_1. \mathcal{C}_{11}' [\text{inj}_1 \ \mathcal{C}_1' [x_1]]) \big] \end{array} \\ \text{CoeCase1L}$$

2020/9/22 49

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A_{12}^{\,\prime}\,\delta_2^{\,\prime}\,A_{22}^{\,\prime}\sqsubseteq A_2
                                                                     Given
                       A_2 = A_{12} \, \delta_2 \, A_{22} By Lemma 4 (Precision inversion)
                    A'_{12} \sqsubseteq A_{12}
                                                                     "
                   A_{22}^{12} \sqsubseteq A_{22} \\ \delta_2' \sqsubseteq \delta_2
                                                                     "
    A'_{11} \delta'_1 A'_{21} \sqsubseteq A_1
                                                                    Given
                     A_1 = A_{11} \delta_1 A_{21}
                                                                    By Lemma 4 (Precision inversion)
                   A'_{11} \sqsubseteq A_{11}
A'_{21} \sqsubseteq A_{21}
\delta'_{1} \sqsubseteq \delta_{1}
Since \delta_1' \in \{+^2_1, +_1\} and \delta_1' \sqsubseteq \delta_1, by definition of \sqsubseteq it follows that \delta_1 \in \{+^2_1, +_1, +^*_1, +^*_1\} as well.
Consider the case when \delta_1 \in \{+^2_1, +_1\}.
                        (A_{11}\,\delta_1\,A_{21}) \Rightarrow (A_{12}\,\delta_2\,A_{22}) \hookrightarrow \mathcal{C}
                                            A_{11} \Rightarrow A_{12} \hookrightarrow C_1
                                                                                                                                                       By inversion on CoeCase1L
                                              \begin{aligned}
& +_1^2 \Rightarrow \delta_1 \hookrightarrow \mathcal{C}_{11} \\
& \delta_1 \Rightarrow \delta_2 \hookrightarrow \mathcal{C}_3 \\
& \mathcal{C} = \mathcal{C}_3 \left[ \operatorname{case}([], \operatorname{inj}_1 x_1.\mathcal{C}_{11}[\operatorname{inj}_1 \mathcal{C}_1[x_1]]) \right]
\end{aligned}
                                                                                                                                                       "
                                                                                                                                                       By definition of \leq
                                           A_{11}' \Rightarrow A_{12}' \hookrightarrow \mathcal{C}_1'
                                                                                                                                                       Subderivation
                                        \mathcal{C}_1'[x_1] \preccurlyeq \mathcal{C}_1[x_1]
                                                                                                                                                       By the induction hypothesis
                              \operatorname{inj}_1 \mathcal{C}_1'[x_{\underline{1}}] \preccurlyeq \operatorname{inj}_1 \mathcal{C}_1[x_1]
                                                                                                                                                       By definition of \leq
                                               +_1^? \Rightarrow \delta_1' \hookrightarrow C_{11}'
                                                                                                                                                       Subderivation
                  \underbrace{\mathcal{C}_{11}'[\operatorname{inj}_1\mathcal{C}_1'[x_1]]}_{M_1'} \preccurlyeq \underbrace{\mathcal{C}_{11}[\operatorname{inj}_1\mathcal{C}_1[x_1]]}_{M_1}
                                                                                                                                                       By definition of □
                                                                                                                                                       By Lemma 60 (Cast insertion preserves precision)
                                                                                                                                                       Given
   \underbrace{\frac{\mathsf{case}(\mathsf{M}', \mathsf{inj}_1 \ x_1.\mathsf{M}_1')}_{\mathsf{M}_0'}}_{\delta_1'} \preccurlyeq \underbrace{\underbrace{\mathsf{case}(\mathsf{M}, \mathsf{inj}_1 \ x_1.\mathsf{M}_1)}_{\mathsf{M}_0}}_{\delta_0'}
                                                                                                                                                       By definition of \leq
                                                                                                                                                       Subderivation
                                                                                                                                                       By Lemma 60 (Cast insertion preserves precision)
Consider the case when \delta_1 \in \{+_1^*, +^2\}.
   (A_{11}\,\delta_1\,A_{21}) \Rightarrow (A_{12}\,\delta_2\,A_{22}) \hookrightarrow \mathcal{C} \quad \text{ Given }
                        A_{11} \Rightarrow A_{12} \hookrightarrow C_1
                                                                                          By inversion on CoeCase2
                       C = C_3 \left[ \text{case}([], \text{inj}_1 \ x_1.C_{11}[\text{inj}_1 \ C_1[x_1]], \text{inj}_2 \ x_2.C_{21}[\text{inj}_2 \ C_2[x_2]]) \right]
                                                                                                                                                                                               By definition of \leq
                                  x_1 \preccurlyeq x_1
                             A'_{11} \Rightarrow A'_{12} \hookrightarrow C'_1
                                                                                                                                                                                               Subderivation
                          C_1'[x_1] \preccurlyeq C_1[x_1]
                                                                                                                                                                                               By the induction hypothesis
               \operatorname{inj}_1 \mathcal{C}_1'[x_1] \preccurlyeq \operatorname{inj}_1 \mathcal{C}_1[x_1]
                                                                                                                                                                                               By definition of \leq
                                \begin{array}{c} +_{1}^{?} \Rightarrow \delta_{1}^{\prime} \hookrightarrow \mathcal{C}_{11}^{\prime} \\ +_{1}^{?} \sqsubseteq +_{1}^{?} \end{array}
                                                                                                                                                                                               Subderivation
    \underbrace{\mathcal{C}_{11}'[\operatorname{inj}_1\mathcal{C}_1'[x_1]]}^{+1} \overset{=}{\preccurlyeq} \underbrace{\mathcal{C}_{11}[\operatorname{inj}_1\mathcal{C}_1[x_1]]}_{M_1}
                                                                                                                                                                                               By definition of □
                                                                                                                                                                                               By Lemma 60 (Cast insertion preserves precision)
                                                                                                                                                                           Given
    \underbrace{\operatorname{case}(M', \operatorname{inj}_1 x_1.M'_1)}_{M'_0} \preccurlyeq \underbrace{\operatorname{case}(M, \operatorname{inj}_1 x_1.M_1, \operatorname{inj}_2 x_2.\mathcal{C}_{21}[\operatorname{inj}_2 \mathcal{C}_2[x_2]])}_{M_0}
                                                                                                                                                                           By definition of ≼
                                                                                                                                                                           Subderivation
                                     C_3'[M_0'] \preceq \overline{C_3}[M_0]
                                                                                                                                                                           By Lemma 60 (Cast insertion preserves precision)
```

 $\bullet \ Case \ {\sf CoeCase1R:} \ \ Symmetric \ to \ the \ {\sf CoeCase1L} \ case.$

```
• Case
                           \frac{\textbf{+}_{1}^{?}\Rightarrow\delta_{1}^{'}\hookrightarrow\mathcal{C}_{11}^{'}}{\delta_{1}^{'}\in\{\textbf{+}^{?},\textbf{+}_{1}^{*},\textbf{+}_{2}^{*},\textbf{+}\}} \frac{\textbf{+}_{1}^{?}\Rightarrow\delta_{1}^{'}\hookrightarrow\mathcal{C}_{11}^{'}}{A_{11}^{'}\Rightarrow A_{12}^{'}\hookrightarrow\mathcal{C}_{1}^{'}} \frac{\textbf{+}_{2}^{?}\Rightarrow\delta_{1}^{'}\hookrightarrow\mathcal{C}_{21}^{'}}{A_{21}^{'}\Rightarrow A_{22}^{'}\hookrightarrow\mathcal{C}_{2}^{'}} \frac{\delta_{1}^{'}\Rightarrow\delta_{2}^{'}\hookrightarrow\mathcal{C}_{3}^{'}}{\delta_{1}^{'}\Rightarrow\delta_{2}^{'}\hookrightarrow\mathcal{C}_{3}^{'}} \frac{\delta_{1}^{'}\Rightarrow\delta_{2}^{'}\hookrightarrow\mathcal{C}_{3}^{'}}{(A_{11}^{'}\delta_{1}^{'}A_{21}^{'})\Rightarrow(A_{12}^{'}\delta_{2}^{'}A_{22}^{'})\hookrightarrow\mathcal{C}_{3}^{'}[\mathsf{case}([],\mathsf{inj}_{1}\,x_{1}.\mathcal{C}_{11}^{'}[\mathsf{inj}_{1}\,\mathcal{C}_{1}^{'}[x_{1}]],\mathsf{inj}_{2}\,x_{2}.\mathcal{C}_{21}^{'}[\mathsf{inj}_{2}\,\mathcal{C}_{2}^{'}[x_{2}]])]}
             A'_{12} \delta'_2 A'_{22} \sqsubseteq A_2
                                                                                                               Given
                                       A_2 = A_{12} \delta_2 A_{22} By Lemma 4 (Precision inversion)
                                    A'_{12} \sqsubseteq A_{12}
A'_{22} \sqsubseteq A_{22}
\delta'_{2} \sqsubseteq \delta_{2}
             A'_{11} \delta'_1 A'_{21} \sqsubseteq A_1
                                                                                                              Given
                                        A_1 = A_{11} \delta_1 A_{21} By Lemma 4 (Precision inversion)
                                    A'_{11} \sqsubseteq A_{11}
A'_{21} \sqsubseteq A_{21}
\delta'_{1} \sqsubseteq \delta_{1}
      Since \delta_1' \in \{+^2, +_1^*, +_2^*, +\} and \delta_1' \subseteq \delta_1, by definition of \subseteq it follows that \delta_1 \in \{+^2, +_1^*, +_2^*, +\} as well.
            (A_{11}\,\delta_1\,A_{21}) \Rightarrow (A_{12}\,\delta_2\,A_{22}) \hookrightarrow \mathcal{C} \quad \text{ Given }
                                           A_{11} \Rightarrow A_{12} \hookrightarrow C_1
                                                                                                                                               By inversion on CoeCase2
                                         A_{21} \Rightarrow A_{22} \hookrightarrow C_2 \qquad "
\delta_1 \Rightarrow \delta_2 \hookrightarrow C_3 \qquad "
+_1^2 \Rightarrow \delta_1 \hookrightarrow C_{11} \qquad "
+_2^2 \Rightarrow \delta_1 \hookrightarrow C_{21} \qquad "
             C = C_3 \left[ case([], inj_1 x_1.C_{11}[inj_1 C_1[x_1]], inj_2 x_2.C_{21}[inj_2 C_2[x_2]]) \right]
            \begin{array}{c} x_1 \leqslant x_1 \\ A'_{11} \Rightarrow A'_{12} \hookrightarrow \mathcal{C}'_1 \\ \mathcal{C}'_1[x_1] \leqslant \mathcal{C}_1[x_1] \\ \operatorname{inj}_1 \mathcal{C}'_1[x_1] \leqslant \operatorname{inj}_1 \mathcal{C}_1[x_1] \\ +_1^2 \Rightarrow \delta'_1 \hookrightarrow \mathcal{C}'_{11} \\ +_1^2 \sqsubseteq +_1^2 \\ \underline{\mathcal{C}'_{11}[\operatorname{inj}_1 \mathcal{C}'_1[x_1]]} \leqslant \underbrace{\mathcal{C}_{11}[\operatorname{inj}_1 \mathcal{C}_1[x_1]]}_{M_1} \end{array}
                                                                                                                                            By definition of \leq
                                                                                                                                           Subderivation
                                                                                                                                            By the induction hypothesis
                                                                                                                                          By definition of ≼
                                                                                                                                            Subderivation
                                                                                                                                            By definition of \Box
                                                                                                                                            By Lemma 60 (Cast insertion preserves precision)
                                                                                                                                            By definition of \leq
                                                 A'_{21} \Rightarrow A'_{22} \hookrightarrow \mathcal{C}'_2
                                                                                                                                            Subderivation
                                             C_2'[x_2] \preccurlyeq C_2[x_2]
                                                                                                                                            By the induction hypothesis
            \inf_{\mathbf{j}_{2}} \mathcal{C}'_{2}[\mathbf{x}_{2}] \preceq \inf_{\mathbf{j}_{2}} \mathcal{C}_{2}[\mathbf{x}_{2}]
\inf_{\mathbf{j}_{2}} \mathcal{C}'_{2}[\mathbf{x}_{2}] \preceq \inf_{\mathbf{j}_{2}} \mathcal{C}_{2}[\mathbf{x}_{2}]
+^{2}_{2} \Rightarrow \delta'_{1} \hookrightarrow \mathcal{C}'_{21}
+^{2}_{2} \sqsubseteq +^{2}_{2}
\underbrace{\mathcal{C}'_{21}[\inf_{\mathbf{j}_{2}} \mathcal{C}'_{2}[\mathbf{x}_{2}]]}_{M'_{2}} \preceq \underbrace{\mathcal{C}_{21}[\inf_{\mathbf{j}_{2}} \mathcal{C}_{2}[\mathbf{x}_{2}]]}_{M_{2}}
                                                                                                                                           By definition of \leq
                                                                                                                                            Subderivation
                                                                                                                                             By definition of \Box
                                                                                                                                            By Lemma 60 (Cast insertion preserves precision)
                                                                                                                                                                                                                                                                           Given
             \underbrace{\mathsf{case}(\mathsf{M}',\mathsf{inj}_1\ x_1.\mathsf{M}'_1,\mathsf{inj}_2\ x_2.\mathsf{M}'_2)}_{\mathsf{M}'_0} \preceq \underbrace{\mathsf{case}(\mathsf{M},\mathsf{inj}_1\ x_1.\mathsf{M}_1,\mathsf{inj}_2\ x_2.\mathsf{M}_2)}_{\mathsf{M}_0}
                                                                                                                                                                                                                                                                           By definition of \leq
                              \delta_1' \Rightarrow \delta_2' \hookrightarrow \mathcal{C}_3' Subderivation
```

Theorem 11 (Translation preserves precision).

Suppose $\Gamma' \sqsubseteq \Gamma$ and $e' \sqsubseteq e$.

 $C_3'[M_0'] \leq C_3[M_0]$

1. If
$$\Gamma' \vdash e' \Leftarrow A'$$
 and $\Gamma \vdash e \Leftarrow A$ and $A' \sqsubseteq A$ then $\Gamma' \vdash e' : A' \hookrightarrow M'$ and $\Gamma \vdash e : A \hookrightarrow M$ where $M' \preccurlyeq M$.

2. If $\Gamma' \vdash e' \Rightarrow A'$ and $\Gamma \vdash e \Rightarrow A$ then $\Gamma' \vdash e' : A' \hookrightarrow M'$ and $\Gamma \vdash e : A \hookrightarrow M$ where $A' \sqsubseteq A$ and $M' \preccurlyeq M$.

Proof. By induction on the structure of the derivation of $\Gamma' \vdash e' \Leftarrow A'$ (part 1) or $\Gamma' \vdash e' \Rightarrow A'$ (part 2).

By Lemma 60 (Cast insertion preserves precision)

51 2020/9/22

$$\begin{array}{c} \bullet \text{ Case } \\ \hline \Gamma'(x) = A' \\ \hline \Gamma' \vdash x \Rightarrow A' \end{array} \text{SynVar} \\ \\ x \sqsubseteq e \\ e = x \\ \hline \Gamma \text{From definition of } \sqsubseteq \\ \hline \Gamma \vdash x \Leftarrow A \\ \hline \Gamma(x) = A \\ \hline Given \\ \hline \Gamma(x) = A \\ \hline Given \\ \hline By inversion on SynVar \\ \hline \Gamma'(x) = A' \\ \hline \Gamma' \vdash x : A' \hookrightarrow x \\ \hline By rule STVar \\ \hline \Pi \Rightarrow \Gamma \vdash x : A \hookrightarrow x \\ \hline By rule STVar \\ \hline \end{tabular}$$

$$\begin{tabular}{ll} \bullet \ \textbf{Case} \\ \hline \hline \hline $\Gamma' \vdash e' \Rightarrow A_0' & A_0' \leadsto A' \\ \hline $\Gamma' \vdash e' \Leftarrow A'$ \\ \hline \end{tabular} \ \textbf{ChkCSub}$$

By inversion on $\Gamma \vdash e \Leftarrow A$, rule ChkCSub was applied.

By definition of \leq

Given

• Case
$$\frac{\Gamma' \vdash e_0' \Leftarrow A'}{\Gamma' \vdash (e_0' :: A') \Rightarrow A'} \mathsf{SynAnno}$$

 $(e_0' :: A') \sqsubseteq e$

• Case

$$\Gamma' \vdash () \Leftarrow \mathsf{Unit}$$
 ChkUnitIntro

```
() <u></u> e
                                                                 Given
                                                                From definition of \Box
                   e = ()
                   \Gamma \vdash () \Leftarrow A
                                                                 Given
                  A = Unit
                                                                 By inversion on ChkUnitIntro
                 \Gamma' \vdash () : \mathsf{Unit} \hookrightarrow ()
                                                                By rule STUnitIntro
                  \Gamma \vdash () : Unit \hookrightarrow ()
                                                                By rule STUnitIntro
                                                                By definition of \leq
 \begin{tabular}{ll} \bullet \  \, \textbf{Case} \\ & \frac{\Gamma' \vdash e_0' \Leftarrow A_i' \qquad \textbf{+}_i^? \leq \delta'}{\Gamma' \vdash (inj_i \ e_0') \Leftarrow (A_1' \ \delta' A_2')} \ \text{ChkSumIntro} \\ \end{tabular} 
             \operatorname{inj}_{i} e'_{0} \sqsubseteq e
                                                                                                            Given
                      e = \inf_{i} e_0
e'_0 \sqsubseteq e_0
                                                                                                            From definition of □
                        \Gamma \vdash (\mathtt{inj}_i \ e_0) \Leftarrow A
                                                                                                            Given
                        \Gamma \vdash e_0 \Leftarrow A_i
                                                                                                            By inversion on ChkSumIntro
                       A = A_1 \delta A_2
                      +_{i}^{?} \leq \delta
         A_1' \, \delta' \, A_2' \sqsubseteq A_1 \, \delta \, A_2
                                                                                                            Given
                     A_1' \sqsubseteq A_1
                                                                                                            From definition of □
        A'_{2} \sqsubseteq A_{2}
+^{?}_{i} \sqsubseteq +^{?}_{i}
A'_{1} +^{?}_{i} A'_{2} \sqsubseteq A_{1} +^{?}_{i} A_{2}
                                                                                                            By definition of □
                                                                                                            By definition of □
                      \begin{array}{l} \Gamma' \vdash e_0' \Leftarrow A_i' \\ \Gamma' \vdash e_0' : A_i' \hookrightarrow M_0' \\ \Gamma \vdash e_0 : A_i \hookrightarrow M_0 \end{array}
                                                                                                            Subderivation
                                                                                                            By the induction hypothesis
                    M_0' \leq M_0
                      \Gamma' \vdash (\operatorname{inj}_{\mathfrak{i}} e'_{0}) : (A'_{1} +^{?}_{\mathfrak{i}} A'_{2}) \hookrightarrow (\operatorname{inj}_{\mathfrak{i}} M'_{0})
                                                                                                            By rule STSumIntro
                        \Gamma \vdash (\operatorname{inj}_i e_0) : (A_1 +_i^? A_2) \hookrightarrow (\operatorname{inj}_i M_0)
                                                                                                            By rule STSumIntro
           inj_i M_0' \leq inj_i M_0
                                                                                                            By definition of \leq
                     A'_1 \le A'_1

A_1 \le A_1

A'_2 \le A'_2

A_2 \le A_2
                                                          By Lemma 2 (Reflexivity of subtyping)
                                                          By Lemma 2 (Reflexivity of subtyping)
                                                           By Lemma 2 (Reflexivity of subtyping)
                                                          By Lemma 2 (Reflexivity of subtyping)
          A_1' +_i^? A_2' \le A_1' \delta' A_2'
                                                          By definition of \leq
          A_1 +_i^? A_2 \le A_1 \delta A_2
                                                           By definition of \leq
         A_1' + A_2' \rightarrow A_1' \delta' A_2'
                                                          By Lemma 8 (Subtyping obeys directed consistency)
         A_1 + \stackrel{?}{}_i A_2 \sim A_1 \delta A_2
                                                           By Lemma 8 (Subtyping obeys directed consistency)
                      A_1' +_i^? A_2' \simeq A_1' \delta' A_2'
                                                                                                                            By Lemma 17 (Directed consistency obeys Structural Equivalence)
                     A_1 +_i^? A_2 \simeq A_1 \delta A_2
                                                                                                                            By Lemma 17 (Directed consistency obeys Structural Equivalence)
                     A'_1 + \stackrel{?}{i} A'_2 \Rightarrow A'_1 \delta' A'_2 \hookrightarrow C'
A_1 + \stackrel{?}{i} A_2 \Rightarrow A_1 \delta A_2 \hookrightarrow C
                                                                                                                            By Theorem 17
                                                                                                                            By Theorem 17
                                     \Gamma' \vdash (\operatorname{inj}_i e_0') : (A_1' \delta' A_2') \hookrightarrow \mathcal{C}'[\operatorname{inj}_i M_0']
                                                                                                                            By rule STCSub
                                      \Gamma \vdash (\text{inj}, e_0) : (A_1 \delta A_2) \hookrightarrow \mathcal{C}[\text{inj}, M_0]
                                                                                                                            By rule STCSub
        \quad \quad \mathcal{C}'[\operatorname{inj}_{i} M_{0}'] \preccurlyeq \mathcal{C}[\operatorname{inj}_{i} M_{0}]
                                                                                                                            By Lemma 61 (Coercion preserves precision)
 \begin{array}{c} \bullet \; \mathbf{Case} \\ \frac{\Gamma' \vdash e_0' \Rightarrow (A_1' \, \delta' \, A_2')}{\delta' \Rightarrow \mathbf{+}_i^*} & \Gamma', x : A_i' \vdash e_i' \Leftarrow A' \\ \hline \Gamma' \vdash \mathsf{case}(e_0', \mathsf{inj}_i \, x.e_i') \Leftarrow A' \end{array} \\ \mathsf{ChkSumElim1}
```

```
case(e'_0, inj_i x.e'_i) \sqsubseteq e
                                                                               Given
                           e = \mathsf{case}(e_0, \mathsf{inj}_i \, x.e_i)
                                                                               From definition of □
                          e_0' \sqsubseteq e_0
                          e_i' \sqsubseteq e_i
                            \Gamma \vdash \mathsf{case}(e_0, \mathsf{inj}_i \ x.e_i) \Leftarrow A
                                                                               Given
                            \Gamma \vdash e_0 \Rightarrow (A_1 \delta A_2)
                                                                               By inversion on ChkSumElim1
                 \Gamma\!, x: A_\mathfrak{i} \vdash e_\mathfrak{i} \Leftarrow A
                         \delta \Longrightarrow +_i^*
                          \Gamma' \sqsubset \Gamma
                                                                               Given
                                                                               Subderivation
                          \Gamma' \vdash e'_0 \Rightarrow (A'_1 \delta' A'_2)
                          \Gamma' \vdash e'_0 : (A'_1 \delta' A'_2) \hookrightarrow M'_0
                                                                               By the induction hypothesis
                           \Gamma \vdash e_0 : (A_1 \, \delta \, A_2) \hookrightarrow M_0
              A_1'\,\delta'\,A_2'\sqsubseteq A_1\,\delta\,A_2
                        M_0' \leq M_0
                        A_1' \sqsubseteq A_1
                                                                               From definition of □
                         A_2' \sqsubseteq A_2
              +_{i}^{*} \sqsubseteq +_{i}^{*}
A'_{1} +_{i}^{*} A'_{2} \sqsubseteq A_{1} +_{i}^{*} A_{2}
                                                                               By definition of \Box
                                                                               By definition of □
           \delta' \Longrightarrow +_i^*
                                                                       Subderivation
            \delta' \leq \textbf{+}_i^*
                                                                       By Lemma 24 (⇒⇒ implies subsum)
             \delta \leq \mathbf{+}_{i}^{*}
                                                                       By Lemma 24 (⇒⇒ implies subsum)
           A_1' \leq A_1'
                                                                       By Lemma 2 (Reflexivity of subtyping)
A_{1} \leq A_{1}
A_{1} \leq A_{1}
A'_{2} \leq A'_{2}
A_{2} \leq A_{2}
A'_{1} \delta' A'_{2} \leq A'_{1} + *_{i} A'_{2}
                                                                       By Lemma 2 (Reflexivity of subtyping)
                                                                       By Lemma 2 (Reflexivity of subtyping)
                                                                       By Lemma 2 (Reflexivity of subtyping)
                                                                       By definition of \leq
  A_1 \, \delta \, A_2 \leq A_1 +_i^* A_2
                                                                       By definition of \leq
A_1'\delta'A_2' \rightsquigarrow A_1' +_i^* A_2'
                                                                       By Lemma 8 (Subtyping obeys directed consistency)
  A_1 \delta A_2 \rightsquigarrow A_1 +_i^* A_2
                                                                       By Lemma 8 (Subtyping obeys directed consistency)
 A_1' \delta' A_2' \simeq A_1' +_i^* A_2'
                                                                       By Lemma 17 (Directed consistency obeys Structural Equivalence)
  A_1\,\delta\,A_2\simeq A_1 +_i^* A_2
                                                                       By Lemma 17 (Directed consistency obeys Structural Equivalence)
 A_1' \delta' A_2' \Rightarrow A_1' + i A_2' \hookrightarrow C'
                                                                       By Theorem 17
  A_1 \delta A_2 \Rightarrow A_1 +_i^* A_2 \hookrightarrow C
                                                                       By Theorem 17
             \Gamma' \vdash e'_0 : (A'_1 +_i^* A'_2) \hookrightarrow C'[M'_0]
                                                                       By rule STCSub
              \Gamma \vdash e_0 : (A_1 +_i^* A_2) \hookrightarrow \mathcal{C}[M_0]
                                                                       By rule STCSub
    \mathcal{C}'[M_0'] \preccurlyeq \mathcal{C}[M_0]
                                                                       By Lemma 61 (Coercion preserves precision)
           A' \sqsubseteq A
                                                                       Given
 \Gamma', x: A_i' \stackrel{-}{\sqsubseteq} \Gamma, x: A_i
                                                                       By definition of \Box
 \Gamma', x : A'_i \vdash e'_i \Leftarrow A'
                                                                       Subderivation
 \Gamma', x: A_i' \vdash e_i': A' \hookrightarrow M_i'
                                                                       By the induction hypothesis
    \Gamma, x : A_i \vdash e_i : A \hookrightarrow M_i
          M_i' \leq M_i
          \Gamma' \vdash e' : A' \hookrightarrow \underbrace{\mathsf{case}(\mathcal{C}'[M_0'], \mathsf{inj}_i \, x. M_i')}_{}
                                                                                  By rule STSumElim1
                                                                                  By rule STSumElim1
           \Gamma \vdash e : A \hookrightarrow \mathsf{case}(\mathcal{C}[M_0], \mathsf{inj}_i \ x.M_i)
M' \preceq M
                                                                                  By definition of \leq
```

• Case ChkSumElim2: Similar to the ChkSumElim1 case, hence omitted.

• Case
$$\frac{\Gamma', x : A_1' \vdash e_0' \Leftarrow A_2'}{\Gamma' \vdash (\lambda x. \, e_0') \Leftarrow (A_1' \to A_2')} \, \mathsf{Chk} \to \mathsf{Intro}$$

```
\lambda x. e'_0 \sqsubseteq e
                                                          Given
                   e = \lambda x. e_0
                                                          From definition of □
                 e_0' \sqsubseteq e_0
                   \Gamma \vdash (\lambda x. e_0) \Leftarrow A Given
       \Gamma, x : A_1 \vdash e_0 \Leftarrow A_2
                                                          By inversion on Chk→Intro
                  A = A_1 \rightarrow A_2
                A_1' \to A_2' \sqsubseteq A_1 \to A_2
                                                                                                               Given
                           A_1' \sqsubseteq A_1

A_2' \sqsubseteq A_2
                                                                                                               From definition of □
                              \Gamma' \sqsubset \Gamma
                                                                                                              Given
                 \Gamma', x : A'_1 \sqsubseteq \Gamma, x : A_1
                                                                                                              By definition of □
                 \Gamma', x : A'_1 \vdash e'_0 \Leftarrow A'_2

\Gamma', x : A'_1 \vdash e'_0 : A'_2 \hookrightarrow M'_0
                                                                                                              Subderivation
                                                                                                              By the induction hypothesis
                    \Gamma, x : A_1 \vdash e_0 : A_2 \hookrightarrow M_0
                           M_0' \leq M_0
                               \Gamma' \vdash (\lambda x. e'_0) : (A'_1 \rightarrow A'_2) \hookrightarrow (\lambda x. M'_0)
                                                                                                              By rule ST→Intro
       13
                                \Gamma \vdash (\lambda x. e_0) : (A_1 \rightarrow A_2) \hookrightarrow (\lambda x. M_0)
                                                                                                              By rule ST→Intro
       B
                     \lambda x. M_0' \leq \lambda x. M_0
                                                                                                              By definition of \leq
 \bullet \  \, \textbf{Case} \  \, \frac{\Gamma' \vdash e_1' \Rightarrow (A_0' \to A') \qquad \Gamma' \vdash e_2' \Leftarrow A_0'}{\Gamma' \vdash (e_1' \, e_2') \Rightarrow A'} \  \, \textbf{Syn} \to \textbf{Elim} 
               e_1' e_2' \sqsubseteq e
                                                                                 Given
                     e = e_1 e_2
                                                                                From definition of \sqsubseteq
                    e_1' \sqsubseteq e_1
e_2' \sqsubseteq e_2
                       \Gamma \vdash (e_1 e_2) \Leftarrow A
                                                                                Given
                       \Gamma \vdash e_1 \Rightarrow (A_0 \rightarrow A)
                                                                                By inversion on \mathsf{Syn}{\to}\mathsf{Elim}
                       \Gamma \vdash e_2 \Leftarrow A_0
                     \Gamma' \sqsubset \Gamma
                                                                                 Given
                     \Gamma' \vdash e'_1 \Rightarrow (A'_0 \rightarrow A')
                                                                                 Subderivation
                     \Gamma' \vdash e_1' : (A_0' \to A') \hookrightarrow M_1'
                                                                                By the induction hypothesis
                      \Gamma \vdash e_1 : (A_0 \to A) \hookrightarrow M_1
       A_0' \to A' \sqsubseteq A_0 \to A
                  M_1' \preccurlyeq M_1
                        A' \sqsubseteq A
                                                                                         From definition of □
                        A'_0 \sqsubseteq A_0
\Gamma' \vdash e'_2 \Leftarrow A'_0
\Gamma' \vdash e'_2 : A'_0 \hookrightarrow M'_2
                                                                                         Subderivation
                                                                                         By the induction hypothesis
                           \Gamma \vdash e_2 : A_0 \hookrightarrow M_2
                       M_2' \leq M_2
                           \Gamma' \vdash (e_1' e_2') : A' \hookrightarrow (M_1' M_2')
                                                                                        By rule ST→Intro
                            \Gamma \vdash (e_1 e_2) : A \hookrightarrow (M_1 M_2)
                                                                                         By rule ST→Intro
                M_1' M_2' \leq M_1 M_2
                                                                                         By definition of \leq
```

D.7 Static programs don't go wrong

We write $\Gamma|_{V}$ for Γ restricted to the set of variables V.

Theorem 18 (Static programs don't go wrong).

If $\Gamma \vdash e \Leftarrow A$ by a static derivation then $\Gamma|_{FV(e)} \vdash e : A \hookrightarrow M$ and, for all M' such that $M \mapsto^* M'$, it is the case that M' free.

Proof. Apply Theorem 19 and Theorem 10 to show M free.

The result follows by induction on the number of steps in $M \mapsto^* M'$, using Theorem 8.

D.7.1 Static derivations

Definition 2. We say that a derivation of $\Gamma \vdash e \Leftarrow A$ or $\Gamma \vdash e \Rightarrow A$ is a static derivation if, for all subderivations deriving checking or synthesis judgments, the types checked or synthesized are static.

55 2020/9/22

Note. If a derivation is static, then all of its subderivations must be static.

Lemma 62 (Context thinning).

If $y \notin FV(e)$ then:

1. If
$$\Gamma, y : A' \vdash e \Leftarrow A$$
 then $\Gamma \vdash e \Leftarrow A$.
2. If $\Gamma, y : A' \vdash e \Rightarrow A$ then $\Gamma \vdash e \Rightarrow A$.

Proof. By induction on the structure of the given derivation.

$$\bullet \ \, \mathbf{Case} \ \, \frac{(\Gamma, y : A^{\, \prime})(x) = A}{\Gamma, y : A^{\, \prime} \vdash x \Rightarrow A} \ \, \mathsf{SynVar}$$

$$y \neq x \qquad \text{Since } y \notin FV(x)$$

$$(\Gamma, y : A')(x) = A \qquad \text{Premise}$$

$$\Gamma(x) = A \qquad \text{By definition}$$

$$\Gamma \vdash x \Rightarrow A \qquad \text{By rule SynVar}$$

- Case ChkCSub: Use the induction hypothesis and apply rule ChkCSub.
- Case SynAnno: Use the induction hypothesis, and apply rule SynAnno.
- Case ChkUnitIntro: Apply rule ChkUnitIntro.
- Case ChkSumIntro: Use the induction hypothesis, the definition of FV(-), and apply rule ChkSumIntro.
- Case ChkSumElim1: Use the induction hypothesis, the definition of FV(-), and apply rule ChkSumElim1.
- Case ChkSumElim2: Use the induction hypothesis, the definition of FV(-), and apply rule ChkSumElim2.
- Case Chk \rightarrow Intro: Use the induction hypothesis, the definition of FV(-), and apply rule Chk \rightarrow Intro.
- Case $Syn \rightarrow Elim$: Use the induction hypothesis, the definition of FV(-), and apply rule $Syn \rightarrow Elim$.

Corollary 63 (Context support).

1. If
$$\Gamma \vdash e \Leftarrow A$$
 then $\Gamma|_{FV(e)} \vdash e \Leftarrow A$.

2. If
$$\Gamma \vdash e \Rightarrow A$$
 then $\Gamma|_{FV(e)} \vdash e \Rightarrow A$.

Proof. By induction on $|dom(\Gamma) \setminus FV(e)|$.

If $dom(\Gamma) = FV(e)$, then $\Gamma = \Gamma|_{FV(e)}$ so we already have the result.

Otherwise, use the induction hypothesis, and apply Lemma 62 (Context thinning).

Theorem 19 (Static subformula).

1. If
$$\Gamma \vdash e \Leftarrow A$$
 by a static derivation then $\Gamma^{\mathsf{S}} \vdash e^{\mathsf{S}} \Leftarrow A^{\mathsf{S}}$ where $\Gamma^{\mathsf{S}} = \Gamma|_{FV(e)}$, $e^{\mathsf{S}} = e$, and $A^{\mathsf{S}} = A$.

2. If
$$\Gamma \vdash e \Rightarrow A$$
 by a static derivation then $\Gamma^{S} \vdash e^{S} \Rightarrow A^{S}$ where $\Gamma^{S} = \Gamma|_{FV(e)}$, $e^{S} = e$, and $A^{S} = A$.

Proof. By induction on the height of the given derivation.

Since $\Gamma \vdash e \Leftarrow A$ and $\Gamma \vdash e \Rightarrow A$ by static derivations, all occurrences of types in checking and synthesizing positions are static, including A. Therefore, $A^{S} = A$ already holds.

Applying Corollary 63 individually to $\Gamma \vdash e \Leftarrow A^S$ and $\Gamma \vdash e \Rightarrow A^S$ produces the derivations $\Gamma|_{FV(e)} \vdash e \Leftarrow A^S$ and $\Gamma|_{FV(e)} \vdash e \Rightarrow A^S$ respectively.

Note that $\Gamma|_{FV(e)} \vdash e \Leftarrow A^{S}$ and $\Gamma|_{FV(e)} \vdash e \Rightarrow A^{S}$ are also static derivations.

All cases are then immediate by the induction hypothesis and applying the relevant rule.

D.7.2 Static translations are free of casts and match failures

Notation. We write M free to denote that the target term M contains no casts or matchfails.

Lemma 64 (Subsums don't need casts).

1. If
$$+_i^? \le \delta^S$$
 and $+_i^? \Rightarrow \delta^S \hookrightarrow C$ then $C = []$.

2. If
$$+_i \le +_i^*$$
 and $+_i \Rightarrow +_i^* \hookrightarrow \mathcal{C}$ then $\mathcal{C} = []$.

Proof.

- 1. From definition of subtyping, it is either the case that $\delta^S = +_i$ or $\delta^S = +$. In both cases, by definition of subtyping, $|+_i^2| = +_i \le |\delta^S|$. By inversion on $+_i^2 \Rightarrow \delta^S \hookrightarrow \mathcal{C}$, either rule CoeSub or CoeCast was applied. If rule CoeCast was applied then $+_i \not\le |\delta^S|$, a contradiction. If rule CoeSub was applied, then indeed $\mathcal{C} = []$.
- 2. By definition of subtyping, $|+_i| = +_i \le +_i = |+_i^*|$. By inversion on $+_i \Rightarrow +_i^* \hookrightarrow \mathcal{C}$, either rule CoeCast was applied. If rule CoeCast was applied then $+_i \not \le +_i$, a contradiction. If rule CoeSub was applied, then indeed $\mathcal{C} = []$.

Lemma 65 (Gradual sums in static don't need casts).

1. If
$$A_{11}^{S} +_{i}^{?} A_{21}^{S} \leq A_{12}^{S} \delta^{S} A_{22}^{S}$$
 and $A_{11}^{S} +_{i}^{?} A_{21}^{S} \Rightarrow A_{12}^{S} \delta^{S} A_{22}^{S} \hookrightarrow \mathcal{C}$ and M free then $\mathcal{C}[M]$ free. 2. If $A_{11}^{S} +_{i} A_{21}^{S} \leq A_{12}^{S} +_{i}^{*} A_{22}^{S}$ and $A_{11}^{S} +_{i} A_{21}^{S} \Rightarrow A_{12}^{S} +_{i}^{*} A_{22}^{S} \hookrightarrow \mathcal{C}$ and M free then $\mathcal{C}[M]$ free.

Proof.

2. Similar to the proof for the previous statement, hence omitted.

Lemma 66 (Static sums don't need casts). If $\delta_0^S \leq \delta^S$ and $\delta_0^S \Rightarrow \delta^S \hookrightarrow \mathcal{C}$ then $\mathcal{C} = []$.

Proof. By definition of sum translation, $|\delta_0^{S}| = \delta_0^{S}$ and $|\delta^{S}| = \delta^{S}$. Therefore, $|\delta_0^{S}| \le |\delta^{S}|$. By inversion on $\delta_0^{S} \Rightarrow \delta^{S} \hookrightarrow \mathcal{C}$, either rule CoeSub or CoeCast was applied. If rule CoeCast was applied then $|\delta_0^{S}| \le |\delta^{S}|$, a contradiction. If rule CoeSub was applied, then indeed $\mathcal{C} = []$. \square

Lemma 67 (Static subtypes don't need casts). If $A_0^S \leq A^S$ and $A_0^S \Rightarrow A^S \hookrightarrow \mathcal{C}$ then $\mathcal{C}[M]$ free for any M free.

Proof. By induction on the structure of the derivation of $A_0^S \Rightarrow A^S \hookrightarrow C$.

• Case CoeUnit: Immediate by the definition of C = [].

57 2020/9/22

$$\begin{array}{lll} A_{11}^{S_{1}} + A_{2}^{S_{1}} \leq A_{12}^{S_{2}} & \text{Given} \\ A_{11}^{S_{1}} \leq A_{12}^{S_{2}} & \text{By Lemma 1 (Subtyping inversion)} \\ A_{21}^{S_{1}} \leq A_{22}^{S_{2}} & " \\ +_{1} \Rightarrow \delta^{S} \hookrightarrow \mathcal{C}_{3} & \text{Subderivation} \\ \mathcal{C}_{3} = || & \text{By Lemma 66 (Static sums don't need casts)} \\ & \begin{array}{lll} x_{1} \text{ free} & \text{By the definition of free} \\ A_{11}^{S_{1}} \Rightarrow A_{12}^{S_{2}} \hookrightarrow \mathcal{C}_{1} & \text{Subderivation} \\ \mathcal{C}_{1}[x_{1}] \text{ free} & \text{By the definition of free} \\ A_{11}^{S_{1}} \Rightarrow A_{12}^{S_{2}} \hookrightarrow \mathcal{C}_{1} & \text{Subderivation} \\ \mathcal{C}_{3}[\text{case}(M, \text{inj}_{1} x_{1}, \mathcal{C}_{1}[x_{1}]) \text{ free} & \text{By the definition of free} \\ A_{11}^{S_{1}} \Rightarrow A_{12}^{S_{2}} \hookrightarrow \mathcal{C}_{1} & \text{Suppose} \\ \mathcal{C}_{3}[\text{case}(M, \text{inj}_{1} x_{1}, \mathcal{C}_{1}[x_{1}]) \text{ free} & \text{By the definition of free} \\ \mathcal{C}_{3}[\text{case}(M, \text{inj}_{1} x_{1}, \mathcal{C}_{1}[x_{1}]) \text{ free} & \text{By the definition of } \mathcal{C}_{3} \\ \mathcal{C}_{3} & \text{case}(M, \text{inj}_{1} x_{1}, \mathcal{C}_{1}[x_{1}]) \text{ free} & \text{By the definition of } \mathcal{C}_{3} \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{3} & \text{case}(M, \text{constant of } \mathcal{C}_{3} & \text{case}(M, \text{constant of } \mathcal{C}_{3}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{3} & \text{case}(M, \text{constant of } \mathcal{C}_{3}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{3} & \text{case}(M, \text{constant of } \mathcal{C}_{3}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{3}) & \text{case}(M, \text{constant of } \mathcal{C}_{3} & \text{case}(M, \text{constant of } \mathcal{C}_{3}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{3}) & \text{case}(M, \text{constant of } \mathcal{C}_{3}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{3}) & \text{case}(M, \text{constant of } \mathcal{C}_{3}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{3}) & \text{case}(M, \text{constant of } \mathcal{C}_{3}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{3}) & \text{case}(M, \text{constant of } \mathcal{C}_{3}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{4}) & \text{case}(M, \text{constant of } \mathcal{C}_{4}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{4}) & \text{case}(M, \text{constant of } \mathcal{C}_{4}) \\ \mathcal{C}_{4} & \text{case}(M, \text{constant of } \mathcal{C}_{4}) & \text{case}(M, \text{constant of } \mathcal{C}_{4}) \\ \mathcal{C$$

Suppose

By the definition of free

By definition of C_3

Theorem 10 (Static derivations don't have match failures). If $\Gamma^S \vdash e^S \Leftarrow A^S$ or $\Gamma^S \vdash e^S \Rightarrow A^S$ then there exists M such that $\Gamma^S \vdash e^S : A^S \hookrightarrow M$ and M is free of casts and matchfail.

M free

Proof. By induction on the structure of the given derivation.

• Case SynVar: Apply rule STVar. M = x is free of casts and matchfail.

 $case(M, inj_1 x_1.C_1'[inj_1 C_1[x_1]], inj_2 x_2.C_2'[inj_2 C_2[x_2]])$ free

 $C_3[case(M, inj_1 x_1.C_1'[inj_1 C_1[x_1]], inj_2 x_2.C_2'[inj_2 C_2[x_2]])]$ free

$$\begin{tabular}{ll} \bullet \ Case \\ \hline \hline $\Gamma^{\text{S}} \vdash e^{\text{S}} \Rightarrow A_0^{\text{S}} & A_0^{\text{S}} \leadsto A^{\text{S}} \\ \hline $\Gamma^{\text{S}} \vdash e^{\text{S}} \Leftarrow A^{\text{S}}$ } \\ \hline \hline \end{tabular} \ \ \ \ \ \, \text{ChkCSub} \\ \hline \end{tabular}$$

58 2020/9/22

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\begin{array}{l} \Gamma^{\text{S}} \vdash e^{\text{S}} \Rightarrow A_0^{\text{S}} \\ \Gamma^{\text{S}} \vdash e^{\text{S}} : A_0^{\text{S}} \hookrightarrow M' \end{array}
                                                      Subderivation
                                                      By the induction hypothesis
M' free A_0^S \hookrightarrow A^S

A_0^S \le A^S

A_0^S \le A^S

A_0^S \simeq A^S

A_0^S \Rightarrow A^S \hookrightarrow C
                                                       Subderivation
                                                      By Lemma 38 (Directed consistency for static types)
                                                      By Lemma 15 (Subtyping obeys Structural Equivalence)
                                                      By Theorem 17
   \Gamma^{S} \vdash e^{S} : A^{S} \hookrightarrow \mathcal{C}[M']
                                                      By rule STCSub
                                                       By Lemma 67 (Static subtypes don't need casts)
```

- Case SynAnno: Use the induction hypothesis, the definition of free, and apply rule STAnno.
- Case ChkUnitIntro: Apply rule STUnitIntro. M = () is free of casts and matchfail.

• Case
$$\frac{\Gamma^{S} \vdash e_{i}^{S} \Leftarrow A_{i}^{S} \qquad +_{i}^{2} \leq \delta^{S}}{\Gamma^{S} \vdash \operatorname{inj}_{i} e_{i}^{S} \Leftarrow (A_{1}^{S} \delta^{S} A_{2}^{S})} \text{ ChkSumIntro}$$

$$\Gamma^{S} \vdash e_{i}^{S} \Leftarrow A_{i}^{S} \qquad \text{Subderivation}$$

$$\Gamma^{S} \vdash e_{i}^{S} : A_{i}^{S} \hookrightarrow M_{i} \qquad \text{By the induction hypothesis}$$

$$M_{i} \text{ free} \qquad \qquad By \text{ Lemma 2 (Reflexivity of subtyping)}$$

$$A_{2}^{S} \leq A_{2}^{S} \qquad \text{By Lemma 2 (Reflexivity of subtyping)}$$

$$A_{2}^{S} \leq A_{2}^{S} \qquad \text{By Lemma 2 (Reflexivity of subtyping)}$$

$$A_{1}^{S} +_{i}^{2} \leq \delta^{S} \qquad \text{Subderivation}$$

$$A_{1}^{S} +_{i}^{2} A_{2}^{S} \leq A_{1}^{S} \delta^{S} A_{2}^{S} \qquad \text{By definition of } \leq$$

$$A_{1}^{S} +_{i}^{2} A_{2}^{S} \hookrightarrow A_{1}^{S} \delta^{S} A_{2}^{S} \qquad \text{By Lemma 8 (Subtyping obeys directed consistency)}$$

$$A_{1}^{S} +_{i}^{2} A_{2}^{S} \hookrightarrow A_{1}^{S} \delta^{S} A_{2}^{S} \qquad \text{By Lemma 17 (Directed consistency obeys Structural Equivalence)}$$

$$A_{1}^{S} +_{i}^{2} A_{2}^{S} \hookrightarrow A_{1}^{S} \delta^{S} A_{2}^{S} \qquad \text{By Theorem 17}$$

$$\Gamma^{S} \vdash \operatorname{inj}_{i} e_{i}^{S} : (A_{1}^{S} +_{i}^{2} A_{2}^{S}) \hookrightarrow \mathcal{C}[\operatorname{inj}_{i} M_{i}] \qquad \text{By rule STSumIntro}$$

$$\Gamma^{S} \vdash \operatorname{inj}_{i} e_{i}^{S} : (A_{1}^{S} \delta^{S} A_{2}^{S}) \hookrightarrow \mathcal{C}[\operatorname{inj}_{i} M_{i}] \qquad \text{By rule STCSub}$$

$$M_{i} \text{ free} \qquad \qquad \text{By definition of free}$$

$$\Gamma^{S} \vdash \operatorname{inj}_{i} M_{i} \text{ free} \qquad \qquad \text{By Lemma 65 (Gradual sums in static don't need casts)}$$

 $case(C[M_0], inj, x.M_i)$ free

$$\begin{array}{c} \bullet \quad \textbf{Case} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S \Rightarrow (A_1^S \delta^S A_2^S)}{\delta^S \Rightarrow + \epsilon_0^S + \text{case}(\epsilon_0^S, \text{inj}_i \times \epsilon_1^S) \in A^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \text{case}(\epsilon_0^S, \text{inj}_i \times \epsilon_1^S) \in A^S}{\delta^S \Rightarrow + \epsilon_0^S + \text{case}(\epsilon_0^S, \text{inj}_i \times \epsilon_1^S) \in A^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \text{case}(\epsilon_0^S, \text{inj}_i \times \epsilon_1^S) \in A^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S + \epsilon_0^S}{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^S} \\ & \frac{\delta^S \Rightarrow + \epsilon_0^S + \epsilon_0^$$

59 2020/9/22

By definition of free

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 \begin{array}{ll} \bullet \ \textbf{Case} \\ \frac{\Gamma^{\textbf{S}} \vdash e_0^{\textbf{S}} \Rightarrow (A_1^{\textbf{S}} \, \delta^{\textbf{S}} \, A_2^{\textbf{S}})}{\delta^{\textbf{S}} \implies + \qquad \qquad \Gamma^{\textbf{S}}, x_1 : A_1^{\textbf{S}} \vdash e_1^{\textbf{S}} \Leftarrow A^{\textbf{S}} \\ \frac{\Gamma^{\textbf{S}}, x_2 : A_2^{\textbf{S}} \vdash e_2^{\textbf{S}} \Leftarrow A^{\textbf{S}}}{\Gamma^{\textbf{S}} \vdash \mathsf{case}(e_0^{\textbf{S}}, \mathsf{inj}_1 \, x_1.e_1^{\textbf{S}}, \mathsf{inj}_2 \, x_2.e_2^{\textbf{S}}) \Leftarrow A^{\textbf{S}} \end{array} 
             \begin{array}{c} A_{1}^{S} \leq A_{1}^{S} \\ A_{2}^{S} \leq A_{2}^{S} \\ \delta^{S} \leq + \\ A_{1}^{S} \delta^{S} A_{2}^{S} \leq A_{1}^{S} + A_{2}^{S} \\ A_{1}^{S} \delta^{S} A_{2}^{S} \leadsto A_{1}^{S} + A_{2}^{S} \\ A_{1}^{S} \delta^{S} A_{2}^{S} \leadsto A_{1}^{S} + A_{2}^{S} \\ A_{1}^{S} \delta^{S} A_{2}^{S} \Longrightarrow A_{1}^{S} + A_{2}^{S} \\ A_{1}^{S} \delta^{S} A_{2}^{S} \Longrightarrow A_{1}^{S} + A_{2}^{S} \\ \end{array}
                                                                                                                                                              By Lemma 2 (Reflexivity of subtyping)
                                                                                                                                                              By Lemma 2 (Reflexivity of subtyping)
                                                                                                                                                              By Lemma 23 (All sums below +)
                                                                                                                                                              By definition of \leq
                                                                                                                                                              By Lemma 8 (Subtyping obeys directed consistency)
                                                                                                                                                              By Lemma 17 (Directed consistency obeys Structural Equivalence)
                                                                                                                                                              By Theorem 17
                                        \begin{array}{l} \Gamma^{\text{S}} \vdash e_0^{\text{S}} \Rightarrow (A_1^{\text{S}} \, \delta^{\text{S}} \, A_2^{\text{S}}) \\ \Gamma^{\text{S}} \vdash e_0^{\text{S}} : (A_1^{\text{S}} \, \delta^{\text{S}} \, A_2^{\text{S}}) \hookrightarrow M_0 \\ \qquad \qquad M_0 \text{ free} \\ \Gamma^{\text{S}} \vdash e_0^{\text{S}} : (A_1^{\text{S}} + A_2^{\text{S}}) \hookrightarrow \mathcal{C}[M_0] \\ \qquad \qquad \mathcal{C}[M_0] \text{ free} \end{array}
                                                                                                                                                              Subderivation
                                                                                                                                                              By the induction hypothesis
                                                                                                                                                            By rule STCSub
                                                                                                                                                              By Lemma 67 (Static subtypes don't need casts)
                \Gamma^{\text{S}}, x_1 : A_1^{\text{S}} \vdash e_1^{\text{S}} \Leftarrow A^{\text{S}}
\Gamma^{\text{S}}, x_1 : A_1^{\text{S}} \vdash e_1^{\text{S}} : A^{\text{S}} \hookrightarrow M_1
                                                                                                                              Subderivation
                                                                                                                             By the induction hypothesis
                \begin{array}{ll} \Gamma^{\text{S}}, x_2: A_2^{\text{S}} \vdash e_2^{\text{S}} \Leftarrow A^{\text{S}} & \text{Subderivation} \\ \Gamma^{\text{S}}, x_2: A_2^{\text{S}} \vdash e_2^{\text{S}}: A^{\text{S}} \hookrightarrow M_2 & \text{By the induction hypothesis} \\ M_2 \text{ free} & '' \end{array}
               \Gamma^{\mathsf{S}} \vdash \mathsf{case}(e_0^{\mathsf{S}}, \mathsf{inj}_1 \ \chi_1.e_1^{\mathsf{S}}, \mathsf{inj}_2 \ \chi_2.e_2^{\mathsf{S}}) : A^{\mathsf{S}} \hookrightarrow \mathsf{case}(\mathcal{C}[\mathsf{M}_0], \mathsf{inj}_1 \ \chi_1.\mathsf{M}_1, \mathsf{inj}_2 \ \chi_2.\mathsf{M}_2) \quad \text{ By rule STSumElim2}
                                                  case(C[M_0], inj_1 x_1.M_1, inj_2 x_2.M_2) free
                                                                                                                                                                                                                                                                                                                                       By definition of free
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- Case Chk→Intro: Use the induction hypothesis, the definition of free, and apply rule ST→Intro.
- Case Syn→Elim: Use the induction hypothesis, the definition of free, and apply rule ST→Elim.

60 2020/9/22