

Convex Program Duality, Fisher Markets, and Nash Social Welfare*

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Abstract

We study Fisher markets and the problem of maximizing the Nash social welfare (NSW), and show several closely related new results. In particular, we obtain:

- A new integer program for the NSW maximization problem whose fractional relaxation has a bounded integrality gap. In contrast, the natural integer program has an unbounded integrality gap.
- An improved, and tight, factor 2 analysis of the algorithm of [7]; in turn showing that the integrality gap of the above relaxation is at most 2. The approximation factor shown by [7] was $2e^{1/e} \approx 2.89$.
- A lower bound of $e^{1/e} \approx 1.44$ on the integrality gap of this relaxation.
- New convex programs for natural generalizations of linear Fisher markets and proofs that these markets admit rational equilibria.

These results were obtained by establishing connections between previously known disparate results, and they help uncover their mathematical underpinnings. We show a formal connection between the convex programs of Eisenberg and Gale and that of Shmyrev, namely that their duals are equivalent up to a change of variables. Both programs capture equilibria of linear Fisher markets. By adding suitable constraints to Shmyrev’s program, we obtain a convex program that captures equilibria of the spending-restricted market model defined by [7] in the context of the NSW maximization problem. Further, adding certain integral constraints to this program we get the integer program for the NSW mentioned above.

The basic tool we use is convex programming duality. In the special case of convex programs with linear constraints (but convex objectives), we show a particularly simple way of obtaining dual programs, putting it almost at par with linear program duality. This simple way of finding duals has been used subsequently for many other applications.

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1 Introduction

Recently, Cole and Gkatzelis [7] gave the first constant factor approximation algorithm for the problem of maximizing the Nash social welfare (NSW). In this problem, a set of indivisible goods needs to be allocated to agents with additive utilities, and the goal is to compute an allocation that maximizes the geometric mean of the agents' utilities. The natural integer program for this problem is closely related to the Fisher market model: if we relax the integrality constraint of the allocation, i.e., assume that the goods are divisible, this program reduces to the Eisenberg-Gale (EG) convex program [11], whose solutions correspond to market equilibria for the linear Fisher market. Therefore, a canonical approach for designing a NSW approximation algorithm would be to compute a fractional allocation via the EG program, and then “round” it to get an integral one. However, [7] observed that this program's integrality gap is unbounded, and they were forced to follow an unconventional approach in analyzing their algorithm. This algorithm used an alternative fractional allocation, the *spending-restricted* (SR) equilibrium, and they had to come up with an independent upper bound of the optimal NSW in order to prove that the approximation factor is at most $2e^{1/e} \approx 2.89$.

The absence of a conventional analysis for this problem could be, in part, to blame for the lack of progress on important follow-up problems (e.g., see Section 7). For instance, the SR equilibrium introduces constraints that are incompatible with the EG program, so [7] had to use a complicated algorithm for computing this allocation. Generalizing such an algorithm may be non-trivial, and so would proving new upper bounds for the optimal NSW. In this paper we remove this obstacle by uncovering the underlying structure of the NSW problem and shedding new light on the results of [7]. Specifically, we propose a new integer program which, as we show, also computes the optimal NSW allocation. More importantly, we prove that the relaxation of this program computes the SR equilibrium, and, quite surprisingly, we also show that the objective of this program happens to be precisely the upper bound that was used in [7]. As a result, this new integer program yields a convex program for computing the SR equilibrium and, unlike the standard program, it has an integrality gap that is bounded by 2.89. In addition to this, we give a family of instances showing a lower bound of $e^{1/e} \approx 1.44$ on the integrality gap, and we provide a tight analysis of the algorithm of [7] to show that its approximation factor is 2, which also puts an upper bound of 2 on the integrality gap of the new program.

Apart from the results regarding the NSW problem, we also reveal interesting connections between seemingly disparate results, and we provide convex programs for computing market equilibria in interesting generalizations of Fisher's market model. For instance, besides the EG program, there is another very different convex program for the linear Fisher market, due to Shmyrev [22]; however, there were no known connections between these two programs. Using our techniques, we show that one can define a dual program for each of them, and the two duals are the same, up to a change of variables. Furthermore, by adding suitable constraints to Shmyrev's program, we obtain the convex program that captures the SR equilibria.

The spending-restricted market model is a generalization of Fisher's market model and has potential use beyond its NSW application. Under this model, sellers can declare an upper bound on the money they wish to earn in the market (and take back their unsold good). Therefore, the total amount of money that the buyers can spend on this seller's good is bounded. Assume that each seller is selling his services in the market. In the last half century, society has seen the emergence of a multitude of very high end jobs which call for a lot of expertise and in turn pay very large salaries. Indeed, the holders of such jobs do not need to work full time to make a comfortable living and one sees numerous such people preferring to work for shorter hours and having a lot more time for leisure. High end dentists, doctors and investors fall in this category. The spending restricted model allows such agents to specify a limit on their earnings beyond which they do not wish to sell their services anymore.

Another generalization of the linear Fisher model that we study is the *utility restricted* (UR) model. In this model, buyers can declare an upper bound on the amount of utility they wish to derive (and take back the unused part of their money). This model is natural as well: in thrift, it is reasonable to assume that a

buyer would only want to buy goods that are absolutely necessary, i.e., place an upper bound amount on utility, and not spend all of her money right away.

Thus, in the SR model, the supply of a good is a function of the prices and, in the UR model, the amount of money a buyer spends in the market is a function of the prices. In the presence of these additional constraints, do equilibria exist and can they be computed in polynomial time? We give a convex program for the second model as well, this time by generalizing the EG program. Existence of equilibria for both models follows from these convex programs. We further show that both models admit rational equilibria, i.e., prices and allocations are rational numbers if all parameters specified in the instance are rational. As a consequence, the ellipsoid algorithm will find a solution to the convex programs in polynomial time.

For some of the results listed above, the techniques that we use are based on convex program duality. We consider a special class of convex programs, those with convex objective functions and *linear* constraints, and show that the duals can be constructed using a simple set of rules,¹ which are almost as simple as those for linear programs. We note that convex programming duality is usually stated in its most general form, with convex objective functions and convex constraints, e.g., see the excellent references by Boyd and Vandenberghe [2] and Rockafellar [21]. At this level of generality the process of constructing the dual of a convex program is quite tedious. Following an earlier version of this paper², these rules have found several additional applications in deriving convex programs: for Fisher markets under spending constraint utilities [1], Fisher markets with transaction costs [5], Arrow-Debreu market with linear utilities [10], and Fisher markets with reserve prices [8]. They have also been used in the design of algorithms: for simplex-like algorithms for spending constraint utilities and perfect price discrimination markets [13], in analyzing the convergence of the tatonnement process [6], in designing online algorithms for scheduling [4, 9, 15], and online algorithms for welfare maximization with production costs [14]. Finally, they have also been used in bounding the price of anarchy of certain games [18].

2 Preliminaries

Fisher’s market model is the following: let M be a set of m divisible goods and N be a set of n buyers. Each buyer i comes to the market with a budget of B_i and we may assume w.l.o.g. that the market has one unit of each good. Each buyer i has a utility function, $u_i : \mathbf{R}_+^m \rightarrow \mathbf{R}_+$, giving the utility that i derives from each bundle of goods. The utility of buyer i is said to be *linear* if there are parameters $v_{ij} \in \mathbf{R}_+$, specifying the value derived by i from one unit of good j . Her utility for the entire bundle is additive, i.e., $u_i(x) = \sum_{j \in M} v_{ij} x_{ij}$. Utility function u_i is said to be *quasi-linear* if, agents have utility for the money spent as well, i.e., $u_i(x) = \sum_{j \in M} (v_{ij} - p_j) x_{ij}$. Utility function u_i is said to be *Leontief* if, given parameters $a_{ij} \in \mathbf{R}_+ \cup \{0\}$ for each good $j \in M$, $u_i(x) = \min_{j \in M} x_{ij}/a_{ij}$. Finally, u_i is said to be *constant elasticity of substitution (CES) with parameter ρ* if given parameters α_j for each good $j \in M$, $u_i(x) = \left(\sum_{j=1}^m \alpha_j x_j^\rho \right)^{\frac{1}{\rho}}$. Throughout the main body of the paper we assume that the utilities are linear unless we note otherwise.

Market equilibrium: Let $p_j \in \mathbf{R}_+$ be the price of good j and $x_{ij} \in \mathbf{R}_+$ denote the amount of good j allocated to buyer i . (We use p and x to denote the vectors of all prices and allocations, respectively.) These are said to form an *equilibrium* if the following conditions hold.

1. The allocation of each buyer i maximizes his utility, subject to her budget constraint, $\sum_j p_j x_{ij} \leq B_i$.

¹The dual is obtained using the usual Lagrangian relaxation technique. We show a “short-cut” for applying this technique, making it especially easy to derive the dual for the special case we consider.

²The part of the current paper about convex programming duality had been made available online since 2010 as the following unpublished manuscript: N. R. Devanur, Fisher Markets and Convex Programs. The manuscript is now incorporated into this paper.

2. Each good j that has a price $p_j > 0$ is allocated fully, i.e., $\sum_i x_{ij} = 1$. A good is allowed to have price $p_j = 0$ as long as $\sum_i x_{ij} \leq 1$.

Two natural generalizations of Fisher’s model that we consider are the following. In the first model which we call *Spending-Restricted* (SR) model, each seller j has an upper bound c_j on the amount of money j wants to earn in the market. Once he earns c_j , selling the least amount of his good, he wants to take back the unsold portion of his good. In other words, the amount of money spent on the good of seller j is restricted by c_j . In equilibrium, buyers spend all their money and get an optimal bundle of goods. Formally, the second equilibrium condition above is modified to $\forall j \in M, \sum_i x_{ij} \leq 1$, and $\sum_i p_j x_{ij} \leq c_j$, and either

$$\sum_i x_{ij} = 1, \text{ or } \sum_i p_j x_{ij} = c_j, \text{ or } p_j = 0.$$

In the second model which we call *Utility-Restricted* (UR) model, buyers have upper bounds d_i on the utility they want to derive in the market. Once buyer i derives utility d_i , spending the least amount of money at prices p , she wants to keep the left-over money. In other words, the utility of buyer i is restricted by d_i . In equilibrium, each good with a positive price should be fully sold. Formally, the first equilibrium condition is modified to $\forall i \in N, u_i(x) \leq d_i$, and $\sum_j p_j x_{ij} \leq B_i$, and either

$$x \text{ minimizes } \sum_j p_j x_{ij} \text{ s.t. } u_i(x) = d_i, \text{ or maximizes } u_i(x) \text{ s.t. } \sum_j p_j x_{ij} \leq B_i.$$

Given an equilibrium (p, x) , we denote the total money spent on item j by q_j , and the money that agent i spends on item j by b_{ij} . The *spending graph*, $Q(b)$, of a given spending vector b , is a bipartite graph where the set of agents corresponds to vertices of one side of the graph and the set of items corresponds to vertices of the other side. Each agent i is connected to the items that she spends money on, i.e., there is an edge between i and j if and only if $b_{ij} > 0$. Note that each agent only spends money on the set of her maximum “bang per buck” items, i.e., the set of items that maximize v_{ij}/p_j . Therefore, by assuming some unique tie breaking rule among goods we can rearrange the spending to ensure that the spending graph is a forest of trees. Throughout this paper we assume that the spending graph is always a forest of trees.

Nash Social Welfare: Given a set M of m indivisible items and a set N of n agents, an *integral* allocation of items to agents restricts the allocation x_{ij} to lie in the set $\{0, 1\}$. The *Nash social welfare* (NSW) (also known as Bernoulli-Nash social welfare) of an integral allocation x is defined as the geometric mean of the agents utilities, i.e., $(\prod_{i \in N} u_i(x))^{1/n}$ [17, 20]. The NSW maximization problem is to find an integral allocation that maximizes the NSW. (We may assume w.l.o.g. that $n \leq m$ for this problem.) Cole and Gkatzelis [7] considered this problem when agents have linear utilities, and gave a $2e^{1/e} \approx 2.89$ factor approximation for it. We now state the upper bound on the optimum value that is used in their result.

Consider an SR market with the same items and agents and utilities. Suppose the items are divisible and have spending restriction of 1 on all items, i.e., $\forall j \in M, c_j = 1$. Let \bar{x} and \bar{p} be an equilibrium allocation and price vector of the market. Note that multiplying all the v_{ij} values of a given agent i by the same positive number does not change the optimal solution or the approximation factor for the problem. In an equilibrium allocation all goods allocated to an agent must have the same “bang per buck” ratio v_{ij}/\bar{p}_j (as was shown in [7]). We can therefore normalize each agent’s valuations so that $v_{ij} = \bar{p}_j$ if $\bar{x}_{ij} > 0$, without loss of generality. We henceforth assume that the valuations are normalized this way in every NSW problem instance. Given such a scaling, we define the following quantity which was used in [7] as an upper bound on the optimal NSW value.

$$\text{SR-UB} := \left(\prod_{j \in M: \bar{p}_j \geq 1} \bar{p}_j \right)^{1/n}.$$

We now state the following lemma that is proved by [7].

Lemma 1 ([7]). *For linear utilities, $\max_{x_{ij} \in \{0,1\}} (\prod_{i \in N} u_i(x))^{1/n} \leq \text{SR-UB}$.*

3 Convex programming duality

3.1 Fenchel Conjugate

We now define the *Fenchel conjugate* of a function, and note some of its properties; see Rockafellar [21] for a detailed treatment. This will be the key ingredient in extending the simple set of rules for LP duality to convex programs. Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function. The conjugate of f is $f^* : \mathbb{R}^n \rightarrow \mathbb{R}$ and is defined as $f^*(\mu) := \sup_x \{\mu^T x - f(x)\}$. Although the conjugate is defined for any function f , for the rest of the article we will assume that f is *strictly convex and differentiable*, since this is the case that is most interesting to the applications we discuss.

Properties of f^* : We note some useful properties here. See Appendix A for more properties.

- If μ and x are such that $f(x) + f^*(\mu) = \mu^T x$ then $\nabla f(x) = \mu$ and $\nabla f^*(\mu) = x$.
- Vice versa, if $\nabla f(x) = \mu$ then $\nabla f^*(\mu) = x$ and $f(x) + f^*(\mu) = \mu^T x$.

We say that (x, μ) form a complementary pair w.r.t. f if they satisfy either one of these two conditions.

3.2 Convex programs with linear constraints

Suppose that we have a convex program with a convex/concave objective function and linear constraints. We can derive another convex program that is the *dual* of this, using Lagrangian duality. This is usually a long calculation. The goal of this section is to identify a shortcut for the same.

Lemma 2. *The following pairs of convex programs are duals of each other, i.e., the optimum of the primal is at most the optimum of the dual (weak duality). If the primal is infeasible, then the dual is unbounded (and vice versa).*

$$\text{Primal: } \max \sum_i c_i x_i - f(x) \text{ s.t.}$$

$$\forall j, \sum_i a_{ij} x_i \leq b_j,$$

$$\text{Dual: } \min \sum_j b_j \lambda_j + f^*(\mu) \text{ s.t.}$$

$$\forall i, \sum_j a_{ij} \lambda_j = c_i - \mu_i,$$

$$\forall j, \lambda_j \geq 0.$$

If the primal constraints are strictly feasible, i.e., there exists \hat{x} such that for all j $\sum_i a_{ij} \hat{x}_i < b_j$, then the two optima are the same (strong duality) and the following generalized complementary slackness conditions characterize them:

- $x_i > 0 \Rightarrow \sum_j a_{ij} \lambda_j = c_i - \mu_i, \quad \lambda_j > 0 \Rightarrow \sum_i a_{ij} x_i = b_j$ and
- x and μ form a complementary pair wrt f , i.e., $\mu = \nabla f(x), x = \nabla f^*(\mu)$ and $f(x) + f^*(\mu) = \mu^T x$.

The proofs of all lemmas in this section are in Appendix A. Note the similarity to LP duality. When an LP is infeasible the dual becomes unbounded. The same happens with these convex programs as well. The differences are as follows. Suppose the concave part of the primal objective is $-f(x)$. There is an extra variable μ_i for every variable x_i that occurs in f . In the constraint corresponding to x_i , the term $-\mu_i$ appears on the RHS along with the constant term. Finally the dual objective has $f^*(\mu)$ in addition to the linear terms. In other words, we *relax* the constraint corresponding to x_i by allowing a slack of μ_i , and *charge* $f^*(\mu)$ to the objective function.

Similarly, the primal program with non-negativity constraints on variables and the corresponding dual program take the following form.

$$\text{Primal: } \max \sum_i c_i x_i - f(x) \text{ s.t.}$$

$$\forall j, \sum_i a_{ij} x_i \leq b_j,$$

$$\forall i, x_i \geq 0.$$

The dual of a minimization program has the following form.

$$\text{Primal: } \min \sum_i c_i x_i + f(x) \text{ s.t.}$$

$$\forall j, \sum_i a_{ij} x_i \geq b_j,$$

$$\forall i, x_i \geq 0.$$

$$\text{Dual: } \min \sum_j b_j \lambda_j + f^*(\mu) \text{ s.t.}$$

$$\forall i, \sum_j a_{ij} \lambda_j \geq c_i - \mu_i,$$

$$\forall j, \lambda_j \geq 0.$$

$$\text{Dual: } \max \sum_j b_j \lambda_j - f^*(\mu) \text{ s.t.}$$

$$\forall i, \sum_j a_{ij} \lambda_j \leq c_i + \mu_i,$$

$$\forall j, \lambda_j \geq 0.$$

4 Convex programs for Fisher markets

We now use the technology developed in the previous section to show a formal connection between the Eisenberg-Gale and Shmyrev convex programs, both of which are known to capture equilibria of linear Fisher markets as their optima. As a first step we construct the dual of the Eisenberg-Gale convex program.

Lemma 3. *The following pairs of convex programs are duals of each other. The dual variables p_j of an optimal solution are equilibrium prices of the corresponding linear Fisher market.*

$$\text{EG Program: } \max \sum_i B_i \log u_i \text{ s.t.}$$

$$\forall i, u_i \leq \sum_j v_{ij} x_{ij},$$

$$\forall j, \sum_i x_{ij} \leq 1,$$

$$x_{ij} \geq 0.$$

$$\min \sum_j p_j - \sum_i B_i \log(\beta_i) \text{ s.t. (1)}$$

$$\forall i, j, p_j \geq v_{ij} \beta_i.$$

In fact, we can even eliminate the β_i 's by observing that in an optimal solution, $\beta_i = \min_j \{p_j/v_{ij}\}$. This gives a convex (but not strictly convex) function of the p_j 's that is minimized at equilibrium. Note that this is an unconstrained³ minimization. The function is $\sum_j p_j - \sum_i B_i \log(\min_j \{p_j/v_{ij}\})$. An interesting property of this function is that the (sub)gradient of this function at any price vector corresponds to the (set of) excess supply of the market with the given price vector. This implies that a tatonnement style price update, where the price is increased if the excess supply is negative and is decreased if it is positive, is actually equivalent to gradient descent. This fact was used to analyze the convergence of the tatonnement process in [6]. A convex program that is very similar to (1) was also discovered independently by Garg [12]. However it is not clear how they arrived at it, or if they realize that this is the dual of the Eisenberg-Gale convex program. Going back to Convex Program (1), we write an equivalent program by taking the logs in each of the constraints.

$$\min \sum_j p_j - \sum_i B_i \log(\beta_i) \text{ s.t.}$$

$$\forall i, j, \log p_j \geq \log v_{ij} + \log \beta_i.$$

Replacing $q_j = \log p_j$ and $\gamma_i = -\log \beta_i$ as the variables, we get the following convex program (2), and its dual (CP).

³Although with some analysis, one can derive that the optimum solution satisfies that $p_j \geq 0$, and $\sum_j p_j = \sum_i B_i$, the program itself has no constraints.

Lemma 4. *The following convex programs are duals of each other.*

$$\begin{array}{l|l}
 \min \sum_j e^{q_j} + \sum_i B_i \gamma_i \text{ s.t.} & (2) \\
 \forall i, j, \gamma_i + q_j \geq \log v_{ij}. & \\
 \hline
 \max \sum_{i,j} b_{ij} \log v_{ij} - \sum_j (p_j \log p_j - p_j) \text{ s.t.} & (\text{CP}) \\
 \forall j, \sum_i b_{ij} = p_j, & \\
 \forall i, \sum_j b_{ij} = B_i, & \\
 \forall i, j, b_{ij} \geq 0. &
 \end{array}$$

By abuse of notation, we use p_j for the variables in (CP) since it turns out that these once again correspond to equilibrium prices. We can remove the $-p_j$ at the end of the objective in (CP) since the constraints imply that $\sum_j p_j = \sum_i B_i$, which is a constant. On removing these terms, we get the convex program of Shmyrev [22]. Thus (CP) and EG convex programs have the same dual, modulo a change of variables!

Quasi-linear utilities: For some markets, it is not clear how to generalize the Eisenberg-Gale convex program, but the dual generalizes easily, and the optimality conditions can be easily seen to be equivalent to equilibrium conditions. We now show an example of this. Recall that a buyer i has a quasi-linear utility if it is of the form $\sum_j (v_{ij} - p_j)x_{ij}$. In particular, if all the prices are such that $p_j > v_{ij}$, then the buyer prefers to not be allocated any good and go back with his budget unspent. It is easy to see that the following convex program (3) captures equilibrium prices for such utilities. In fact, given this convex program, one could take its dual to get an EG-type convex program as well. Although this is a small modification of the EG program, it is not clear how one would arrive at this directly without going through the dual.

Lemma 5. *The following pairs of convex programs are duals of each other, and capture the equilibria of Fisher markets with quasi-linear utilities as their optima.*

$$\begin{array}{l|l}
 \min \sum_j p_j - \sum_i B_i \log(\beta_i) \text{ s.t.} & (3) \\
 \forall i, j, p_j \geq v_{ij} \beta_i, & \\
 \forall i, \beta_i \leq 1. & \\
 \hline
 \max \sum_i B_i \log u_i - v_i \text{ s.t.} & \\
 \forall i, u_i \leq \sum_j v_{ij} x_{ij} + v_i, & \\
 \forall j, \sum_i x_{ij} \leq 1, & \\
 \forall i, j, x_{ij}, v_i \geq 0. &
 \end{array}$$

Summary and Extensions: In this section we showed two applications of the convex programming duality in Section 3, the relation between the EG and Shmyrev convex programs, and a convex program for Quasi-linear utilities. We mention other applications of this tool in the introduction, some of which are in Appendix A. We give a convex program that captures SR equilibrium, and study existence, uniqueness and rationality of equilibrium in Appendix B. Further, the same analysis can be extended to what are called spending constraint utilities (Appendix C). We do the same (convex programs, existence, uniqueness and rationality) for UR markets with linear, Leontief and CES utilities in Appendix D. The convex program for the SR model is closely related to NSW maximization, as we will discuss in the next section.

5 A new program for the Nash social welfare problem

In this section we focus on the APX-hard problem of maximizing the NSW with indivisible items [7, 19]. When the agents have linear valuations, this problem has a natural representation as a convex program (see program on the left below). In this program, there is a variable x_{ij} for each agent i and item j and its value is either 0 or 1, depending on whether the agent is allocated the item or not. An appealing property of this

program is that, if we relax the constraint that $x_{ij} \in \{0, 1\}$, then the program reduces to the Eisenberg-Gale program⁴, which can be solved in polynomial time. This opens the way for a standard approach for designing an approximation algorithm: compute the fractional allocation using the EG program and then use a rounding algorithm to get a good integral allocation. Unfortunately, as was shown in [7], the integrality gap of this program is unbounded, so this approach is doomed to fail.

Facing the unbounded integrality gap obstacle, [7] take a non-standard approach in designing an approximation algorithm. Motivated by the market equilibrium interpretation of the EG program, they propose the spending-restricted equilibrium, and they then independently prove an upper bound for the optimal NSW value (which we call SR-UB, see Lemma 1). They then “round” the fractional allocation implied by the SR equilibrium, and compare the NSW of the rounded solution to SR-UB. In this section, we propose a new integer program, which we refer to as the *spending-restricted* (SR) program (see program on the right below)⁵, and show the following results.

- The optimal solution of the SR program corresponds to the NSW maximizing integral allocation, and the optimal objective function value of this program is equal to the optimal NSW value.
- The fractional relaxation of this program computes the SR equilibrium.
- The objective value of the fractional relaxation is equal to the upper bound SR-UB.
- This relaxation therefore has an integrality gap of at most $2e^{1/e} \approx 2.89$. We also show a lower bound of $e^{1/e} \approx 1.44$ on this integrality gap.

$$\begin{array}{l|l}
 \max (\prod_i u_i)^{1/n} \text{ s.t.} & \max \left(\frac{\prod_i \prod_j v_{ij}^{b_{ij}}}{\prod_j q_j^{q_j}} \right)^{1/n} \text{ s.t.} \quad (\text{SR}) \\
 \forall i, u_i = \sum_j x_{ij} v_{ij} & \forall j, \sum_i b_{ij} = q_j \\
 \forall j, \sum_i x_{ij} = 1 & \forall i, \sum_j b_{ij} = 1 \\
 \forall i, j, x_{ij} \in \{0, 1\}. & \forall i, j, q_j \leq 1, b_{ij} \in \{0, q_j\}
 \end{array}$$

Unlike the standard program for the NSW problem, the SR program uses variables q_j and $b_{ij} \in \{0, q_j\}$. Any solution to this program, corresponds to an allocation of indivisible items to agents. In particular, an agent i is allocated an item j if and only if $b_{ij} = q_j$.⁶ If we relax the constraint that $b_{ij} \in \{0, q_j\}$ and apply a logarithmic transformation of the objective function, we get a convex program, which we can compute in polynomial time. We call this relaxation the f-SR program. Note that the spending constraint ($q_j \leq 1$) is not binding in the SR program, but this is not true for f-SR.

The following lemma shows that the two programs above do, in fact, compute the same allocation.

Lemma 6. *The optimal solution of the SR program corresponds to the NSW maximizing allocation of indivisible items to agents. The objective function value of this solution is equal to the optimal NSW value.*

Proof. Suppose that we fix the integral choices, i.e., for each i and j we fix whether $b_{ij} = 0$ or $b_{ij} = q_j$. For all j , due to the constraint that $\sum_i b_{ij} = q_j$, there can only be one i such that $b_{ij} = q_j$. Hence determining the integral choices is equivalent to determining an integral allocation. Let S_i denote the set of items allocated to i in this integral allocation. We show that given these integral choices, setting $b_{ij} = \frac{v_{ij}}{\sum_{k \in S_i} v_{ik}}$ makes the

⁴To verify this fact, apply a logarithmic transformation to the objective.

⁵The SR program is not, strictly speaking, presented as an integer program, but we could introduce a new variable a_j for each item j and replace the constraint $b_{ij} \in \{0, q_j\}$ with the constraints $b_{ij} = a_j q_j$ and $a_j \in \{0, 1\}$ to make it an integer program.

⁶Note that we can assume $\forall j, q_j > 0$ in an equilibrium w.l.o.g. because if $q_j = 0$ then the equilibrium conditions imply the value of item j is zero for all agents.

objective function equal to the NSW of the allocation, and this is indeed the optimal (objective maximizing) choice of these variables. The first part follows from this sequence of equalities.

$$\left(\frac{\prod_i \prod_j v_{ij}^{b_{ij}}}{\prod_j q_j} \right)^{1/n} = \left(\prod_i \prod_{j \in S_i} \frac{v_{ij}^{b_{ij}}}{b_{ij}} \right)^{1/n} = \left(\prod_i \prod_{j \in S_i} (\sum_{k \in S_i} v_{ik})^{b_{ij}} \right)^{1/n} = \left(\prod_i \sum_{k \in S_i} v_{ik} \right)^{1/n}$$

For the rest of the proof, we work with the log transformation of the objective. Given the integral choices, the **SR** program decomposes into a sum of separate mathematical programs, one for each buyer i .

$$\begin{aligned} & \max \sum_{j \in S_i} (b_{ij} \log v_{ij} - b_{ij} \log b_{ij}) \text{ s.t.} \\ & \forall i, \sum_{j \in S_i} b_{ij} = 1, \text{ and } \forall i, j \in S_i, \quad b_{ij} \geq 0. \end{aligned}$$

This is the same as minimizing the relative entropy, or KL-divergence, between two probability distributions, where the b_{ij} s form one probability distribution, and the other distribution is given by $\frac{v_{ij}}{\sum_{k \in S_i} v_{ik}}$. By Gibbs' inequality, it is known that this is minimized when the two distributions are the same, i.e., when $b_{ij} = \frac{v_{ij}}{\sum_{k \in S_i} v_{ik}}$. (We give an alternate proof of Gibbs' inequality using convex program duality in Appendix A.) \square

5.1 Relaxation of the SR program

In designing their approximation algorithm for the NSW problem in [7], they used, as an intermediate step, a fractional allocation, which was the equilibrium of a spending-restricted market with $c_j = 1$ for all j . If the price of an item j is p_j , then this constraint could be expressed as $\sum_i x_{ij} p_j \leq 1$. But, they could not introduce this constraint into the EG program, since it combines both the primal variables x_{ij} and the dual variable p_j . In the absence of a program that could compute this fractional solution, they instead had to propose a complicated market equilibrium computation algorithm. Lemma 7 shows that in the **SR** program, once we drop the constraint that $b_{ij} \in \{0, q_j\}$, the relaxed program, f-SR, computes the SR equilibrium. Unlike the EG program, the constraint that the total spending on any given item is at most 1 involves only the primal variables q_j . If we also apply a logarithmic transformation to the objective function, then we get the convex program (**CP**) of Section 4, with the additional constraint that $q_j \leq 1$. As a result, we provide a simple convex program that can compute the SR equilibrium. The proof of the following lemma essentially shows that the complementary slackness conditions are equivalent to market equilibrium conditions.

Lemma 7. *The f-SR program computes the SR equilibrium. The variables b_{ij} capture the amount of money spent by buyer i on good j , and the variables q_j capture the total spending on good j . The prices p_j can be recovered from the optimal dual variables.*

Existence and uniqueness of the SR equilibrium: We study existence and uniqueness of the SR equilibrium in Appendix B. We show an SR equilibrium exists if and only if $\sum_j c_j \geq \sum_i B_i$. On the uniqueness side, we show that the spending vector $q = (q_1, \dots, q_m)$, where q_j is the money spent on good j , is unique. Although in the Fisher model we have the uniqueness of price equilibrium, it is easy to see that this is not true for the SR equilibrium. Consider a market with only one buyer with utility function $u(x) = x_1$ and one seller. Let $B_1 = 1$ and $c_1 = 1$. It is easy to see every price bigger than 1 is an SR equilibrium price.

Relation to SR-UB: Quite surprisingly, we also show that the optimal objective value of the f-SR program is the same, up to scaling of the valuations, as the upper bound used by [7], which we called SR-UB.

Lemma 8. *The optimal value of the f-SR program is equal to SR-UB.*

Proof. Let \bar{b}_{ij} and \bar{q}_j be an optimum solution to the f-SR program, and \bar{x} and \bar{p} be equilibrium allocation and price vectors resp. From Lemma 7, the relation between these is that $\bar{b}_{ij} = \bar{p}_j \bar{x}_{ij}$ and $\bar{q}_j = \min\{1, \bar{p}_j\}$. Recall that, from the definition of SR-UB, we normalize each agent's valuations so that $v_{ij} = \bar{p}_j$ if $\bar{x}_{ij} > 0$. With this scaling of the valuations, the objective function of the f-SR program becomes

$$\left(\frac{\prod_i \prod_j v_{ij}^{\bar{b}_{ij}}}{\prod_j \bar{q}_j^{q_j}} \right)^{1/n} = \left(\frac{\prod_j \bar{p}_j^{\sum_i \bar{b}_{ij}}}{\prod_j \bar{q}_j^{q_j}} \right)^{1/n} = \left(\prod_j (\bar{p}_j / \bar{q}_j)^{\bar{q}_j} \right)^{1/n} = \left(\prod_{j: \bar{p}_j \geq 1} \bar{p}_j \right)^{1/n},$$

where in the last equality, we used the fact that $\bar{p}_j = \bar{q}_j$ if $\bar{p}_j < 1$ and $\bar{q}_j = 1$ otherwise. \square

The SR program integrality gap: Given Lemmas 6, 7, and 8, a lower bound on the integrality gap of the SR program also implies a lower bound on the best approximation factor that one can show by rounding a solution to f-SR, and comparing the objective obtained to SR-UB. The next lemma provides such a lower bound for the integrality gap.

Lemma 9. *The integrality gap of the program above is at least $e^{1/e} \approx 1.44$.*

Proof. Consider an instance with n bidders and $m = (1 + f)n$ items, where $f \in (0, 1)$ is a constant. Each agent i has a value of 0 for the first n items, except item i , for which his value is $(1 - f)$. The value of every agent for items $n + 1$ to m , hence referred to as the “valuable” items, is equal to V , which is much higher than 1. In the SR equilibrium for this instance, the prices will be $(1 - f)$ for the first n items and V for the rest. Each agent i will be spending $(1 - f)$ of his budget on item i and the remaining budget of f on the valuable items.

The objective value for this fractional solution would therefore be equal to V^f . On the other hand, any integral allocation would have to assign each one of the valuable items to a distinct agent, so the optimal NSW would be $(1 - f)^{1-f} \cdot (1 - f + V)^f$. If we let V go to infinity, this leads to an integrality gap of

$$\lim_{V \rightarrow \infty} \left(\frac{V^f}{(1-f)^{1-f} \cdot (1-f+V)^f} \right) = \frac{1}{(1-f)^{1-f}}$$

which, for $f = (e - 1)/e$, yields the desired $e^{1/e}$ integrality gap⁷. \square

6 A Tight Analysis of the Spending-Restricted Rounding Algorithm

Using the SR equilibrium as a starting point, [7] proposed the a rounding algorithm called the *Spending-Restricted Rounding* (SRR) algorithm. Using SR-UB as an upper bound, they showed that the approximation factor of this algorithm is at most $2e^{1/e} \approx 2.89$. The first step of the SRR algorithm is to compute the SR equilibrium which, in light of the previous section's results, we can now do using the f-SR convex program. Then, for each tree of the spending graph $Q(b)$, it chooses an arbitrary agent as the root and assigns all items that are either leaves or have $q_j \leq 1/2$ to their parent-agent. The remaining items are matched to agents using the matching with the optimal NSW value, given the previous assignments. This matching can be computed in polynomial using a maximum weight matching algorithm and $\log v_{ij}$ as weights instead of v_{ij} (see [7] for more details). The (full) proofs of this section are deferred to Appendix E.

Using a careful analysis, we now show that the approximation factor of the SRR algorithm is, in fact, better than 2.89 by proving an upper bound of 2. We conclude this section with a matching lower bound.

Theorem 1. *The approximation factor of the SRR algorithm is at most 2.*

⁷To be precise, to make sure that m is an integer, fn would also have to be an integer. Therefore, we as we let n be arbitrarily large, f can take values arbitrarily close to $(e - 1)/e$ while fn remains an integer.

Algorithm 1: Spending-Restricted Rounding (SRR) [7].

- 1 Compute a spending-restricted equilibrium (b, q) .
 - 2 Choose a root-agent for each tree in the spending graph $Q(b)$.
 - 3 Assign any leaf-item in the trees to its parent-agent.
 - 4 Assign any item j with $q_j \leq 1/2$ to its parent-agent.
 - 5 Compute the optimal matching of the remaining items to adjacent agents.
-

Proof Sketch. For each item j that has more than one child-agent in the spending graph $Q(b)$, remove the edges connecting it to all but the one child-agent that spends the most money on j , i.e., the one with the largest b_{ij} value. This yields a pruned spending graph $P(b)$ that is also a forest of trees. We refer to the trees of the pruned graph $P(b)$ as the *matching-trees*. In every matching-tree T with $k \geq 2$ agents, when the algorithm reaches its last step, every remaining item has exactly one parent-agent and one child-agent, so all but one agent can be matched to one of these items. Our proof shows that there exists a matching of the remaining items such that the agents within T have a “high” NSW.

A naive way to match the agents in the last step of the algorithm would be to match all of them, except the one that has accrued the highest value during the previous steps. It was already observed in [7] that, for any matching-tree T of k agents, there exists an agent who was assigned value at least $1/(2k)$ during Steps 3 and 4 of the algorithm, so we could match every agent in T , except him. But, what is the worst case distribution of value that can arise in this matching? We show that the worst case arises for matching-trees that contain a single agent and no items with $p_j > 1/2$. But, even in this case, such an agent got all the items that he was spending on in the SR equilibrium, except one, and he could not be spending more than half of his budget on the one he lost. To verify this fact, note that he either lost this item because the total money spent on the item was less than half, i.e., $q_j \leq 1/2$, and it was assigned to its parent at Step 4, or because the edge connecting him to this item was pruned in the transition from $Q(b)$ to $P(b)$. But, in both of these cases, he could not be spending more than $1/2$ on that item, so he got at least half of his SR equilibrium value.

The more demanding part of the proof is to show that the worst case arises for matching-trees of size 1. In contrast to the analysis of [7], we use the vital observation that, if the agent of some matching-tree T who does not get matched to an item has value v_α , then every other agent $i \in T$ gets value at most $v_{ij} + v_\alpha$, where j is the item that he was matched to in the last step. Lemma 25 uses this fact to prove that in the worst case distribution of value, at least $\lfloor \frac{k-2v_\alpha}{1+2v_\alpha} \rfloor$ agents get value greater than, or equal to, 1. In other words, this new lemma shows that, if the unmatched agent were to leave a lot of value on the table, then this value would not end up with just a few agents but, rather, it would have to be well distributed among the remaining agents. Building further on this observation, Lemma 26 shows that, for any matching-tree T with k agents, the allocation x' induced by the naive matching algorithm satisfies

$$\prod_{i \in T} v_i(x') \geq \frac{1}{2^k} \prod_{j \in T: p_j \geq 1} p_j.$$

Since the allocation x that the SRR algorithm outputs is at least as good as the one by the naive matching, we can combine this inequality with the SR-UB upper bound to get the desired approximation factor bound:

$$\left(\prod_i v_i(x)\right)^{1/n} = \left(\prod_T \prod_{i \in T} v_i(x)\right)^{1/n} \geq \frac{1}{2} \left(\prod_{j: p_j \geq 1} p_j\right)^{1/n}.$$

□

Lemma 10. *The approximation factor of the SRR algorithm is exactly 2.*

7 Discussion

Regarding additional Fisher market extensions, an obvious open question is to obtain a convex program for the common generalization of the spending-restricted and utility-restricted markets, in which buyers have utility bounds and sellers have earning bounds, for the case of linear utilities.

Regarding the NSW problem, we have addressed the symmetric case of NSW, which assumes that all agents have equal budget (or clout). While introducing the Nash bargaining problem [20], Nash only considered the symmetric case but, soon after that, Kalai proposed the non-symmetric case as well, which is also well-studied. Hence a natural open problem is to obtain a constant factor approximation algorithm for the non-symmetric case of NSW. The objective in this generalization is to maximize $\left(\prod_i u_i^{B_i}\right)^{1/B}$, where B_i is the budget of agent i and $B = \sum_i B_i$. Another important generalization of NSW would be to consider utilities that are subadditive instead of additive. In particular, the case of submodular utilities would definitely deserve more attention.

References

- [1] Benjamin Birnbaum, Nikhil R Devanur, and Lin Xiao. Distributed algorithms via gradient descent for Fisher markets. In *Proceedings of the 12th ACM conference on Electronic commerce*, pages 127–136. ACM, 2011.
- [2] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2009.
- [3] W. C. Brainard and H. E. Scarf. How to compute equilibrium prices in 1891. *Cowles Foundation Discussion Paper*, (1270), 2000.
- [4] Niv Buchbinder, Shahar Chen, Anupam Gupta, Viswanath Nagarajan, et al. Online packing and covering framework with convex objectives. *arXiv preprint arXiv:1412.8347*, 2014.
- [5] Sourav Chakraborty, Nikhil R Devanur, and Chinmay Karande. Market equilibrium with transaction costs. In *Internet and Network Economics*, pages 496–504. Springer, 2010.
- [6] Yun Kuen Cheung, Richard Cole, and Nikhil Devanur. Tatonnement beyond gross substitutes?: gradient descent to the rescue. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pages 191–200. ACM, 2013.
- [7] Richard Cole and Vasilis Gkatzelis. Approximating the Nash social welfare with indivisible items. In *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing*, pages 371–380, 2015.
- [8] Richard Cole and Yixin Tao. When does the price of anarchy tend to 1 in large walrasian auctions and fisher markets? *arXiv preprint arXiv:1508.07370*, 2015.
- [9] Nikhil R Devanur and Zhiyi Huang. Primal dual gives almost optimal energy efficient online algorithms. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1123–1140. SIAM, 2014.
- [10] Nikhil R Devanur, Jugal Garg, and László A Végh. A rational convex program for linear Arrow-Debreu markets. *arXiv preprint arXiv:1307.8037*, 2013.
- [11] E. Eisenberg and D. Gale. Consensus of subjective probabilities: the Pari-Mutuel method. *The Annals of Mathematical Statistics*, 30:165–168, 1959.

- [12] Jugal Garg. *Algorithms for Market Equilibrium*. PhD thesis, Dept. of Computer Science and Engineering, IIT Bombay, 2008. URL <http://www.cse.iitb.ac.in/~jugal/aps1.pdf>.
- [13] Jugal Garg, Ruta Mehta, Milind Sohoni, and Nisheeth K Vishnoi. Towards polynomial simplex-like algorithms for market equilibria. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1226–1242. SIAM, 2013.
- [14] Zhiyi Huang and Anthony Kim. Welfare maximization with production costs: a primal dual approach. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 59–72. SIAM, 2015.
- [15] Sungjin Im, Jitendra Kulkarni, Kamesh Munagala, and Kirk Pruhs. Selfishmigrate: A scalable algorithm for non-clairvoyantly scheduling heterogeneous processors. In *Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on*, pages 531–540. IEEE, 2014.
- [16] Kamal Jain and Vijay V. Vazirani. Eisenberg-Gale markets: Algorithms and game-theoretic properties. *Games and Economic Behavior*, 70(1):84–106, September 2010.
- [17] Mamoru Kaneko and Kenjiro Nakamura. The Nash social welfare function. *Econometrica*, pages 423–435, 1979.
- [18] Janardhan Kulkarni and Vahab Mirrokni. Robust price of anarchy bounds via lp and fenchel duality. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1030–1049. SIAM, 2015.
- [19] Euiwoong Lee. Apx-hardness of maximizing Nash social welfare with indivisible items. *CoRR*, abs/1507.01159, 2015.
- [20] J. F. Nash. The bargaining problem. *Econometrica*, 18:155–162, 1950.
- [21] R Tyrrell Rockafellar. *CONVEX ANALYSIS*. Princeton University Press, 1970.
- [22] VI Shmyrev. An algorithm for finding equilibrium in the linear exchange model with fixed budgets. *Journal of Applied and Industrial Mathematics*, 3(4):505–518, 2009.

A Convex Programming Duality

Properties of f^* : We note some useful properties of f^* here.

- f^* is strictly convex and differentiable. (even if f is not strictly convex and differentiable)
- $f^{**} = f$. (using the assumption that f is strictly convex and differentiable)
- If f is separable, that is $f(x) = \sum_i f_i(x_i)$, then $f^*(\mu) = \sum_i f_i^*(\mu_i)$.
- If $g(x) = cf(x)$ for some constant c , then $g^*(\mu) = cf^*(\mu/c)$.
- If $g(x) = f(cx)$ for some constant c , then $g^*(\mu) = f^*(\mu/c)$.
- If $g(x) = f(x + a)$ for some constant a , then $g^*(\mu) = f^*(\mu) - \mu^T a$.
- If μ and x are such that $f(x) + f^*(\mu) = \mu^T x$ then $\nabla f(x) = \mu$ and $\nabla f^*(\mu) = x$.
- Vice versa, if $\nabla f(x) = \mu$ then $\nabla f^*(\mu) = x$ and $f(x) + f^*(\mu) = \mu^T x$.

Conjugates of some simple strictly convex and differentiable functions

- If $f(x) = \frac{1}{2}x^2$, then $\nabla f(x) = x$. Letting $\mu = x$ in $\mu^T x - f(x)$, leads to $f^*(\mu) = \frac{1}{2}\mu^2$.
- If $f(x) = -\log(x)$, then $\nabla f(x) = \frac{-1}{x}$. Set $\mu = \frac{-1}{x}$ to get $f^*(\mu) = -1 + \log(x) = -1 - \log(-\mu)$.
- If $f(x) = x \log x$, then $\nabla f(x) = \log x + 1 = \mu$. So $x = e^{\mu-1}$. $f^*(\mu) = \mu x - f(x) = x(\log x + 1) - x \log x = x = e^{\mu-1}$. That is, $f^*(\mu) = e^{\mu-1}$.

Lemma 2. *The following pairs of convex programs are duals of each other, i.e., the optimum of the primal is at most the optimum of the dual (weak duality). If the primal is infeasible, then the dual is unbounded (and vice versa).*

$$\begin{array}{l}
 \text{Primal: } \max \sum_i c_i x_i - f(x) \text{ s.t.} \\
 \forall j, \sum_i a_{ij} x_i \leq b_j,
 \end{array}
 \quad \left| \quad
 \begin{array}{l}
 \text{Dual: } \min \sum_j b_j \lambda_j + f^*(\mu) \text{ s.t.} \\
 \forall i, \sum_j a_{ij} \lambda_j = c_i - \mu_i, \\
 \forall j, \lambda_j \geq 0.
 \end{array}$$

If the primal constraints are strictly feasible, i.e., there exists \hat{x} such that for all j $\sum_i a_{ij} \hat{x}_i < b_j$, then the two optima are the same (strong duality) and the following generalized complementary slackness conditions characterize them:

- $x_i > 0 \Rightarrow \sum_j a_{ij} \lambda_j = c_i - \mu_i$, $\lambda_j > 0 \Rightarrow \sum_i a_{ij} x_i = b_j$ and
- x and μ form a complementary pair wrt f , i.e., $\mu = \nabla f(x)$, $x = \nabla f^*(\mu)$ and $f(x) + f^*(\mu) = \mu^T x$.

Proof. Suppose first that the set of linear constraints is itself infeasible, that is, there is no solution to the set of inequalities

$$\forall j, \sum_i a_{ij} x_i \leq b_j. \tag{4}$$

Then by Farkas' lemma, we know that there exists numbers $\lambda_j \geq 0$ for all j such that

$$\forall i, \sum_j a_{ij} \lambda_j = 0, \text{ and } \sum_j \lambda_j b_j < 0.$$

Now consider the dual solution with these λ_j s and $\mu_i = c_i$. This is feasible, and the dual objective is $f^*(c) + \sum_j \lambda_j b_j$. By multiplying all the λ_j s by a large positive number, the dual objective can be made arbitrarily small (goes to $-\infty$).

Now suppose that the feasible region defined by the inequalities (4) and the domain of f defined as $dom(f) = \{x : f(x) < \infty\}$ are disjoint. Further assume for now that $f^*(c) < \infty$ and that there is a strict separation between the two, meaning that for all x feasible and $y \in dom(f)$, $d(x, y) > \epsilon$ for some $\epsilon > 0$. Then once again by Farkas' lemma we have that there exist $\lambda_j \geq 0$ for all j and $\delta > 0$ such that

$$\forall y \in dom(f), \sum_{i,j} a_{ij} \lambda_j y_i > \sum_j \lambda_j b_j (1 + \delta).$$

This implies that the dual objective is $< f^*(c) - \delta \sum_j \lambda_j b_j$, and as before, by multiplying all the λ_j by a large positive number, g can be made arbitrarily small.

Now we may assume that the primal is feasible. Define the Lagrangian function

$$L(x, \lambda) := \sum_i c_i x_i - f(x) + \sum_j \lambda_j (b_j - \sum_i a_{ij} x_i).$$

We say that x is feasible if it satisfies all the constraints of the primal problem. Note that for all $\lambda \geq 0$ and x feasible, $L(x, \lambda) \geq \sum_i c_i x_i - f(x)$. Define the dual function

$$g(\lambda) = \max_x L(x, \lambda).$$

So for all λ, x , $g(\lambda) \geq L(x, \lambda)$. Thus $\min_{\lambda \geq 0} g(\lambda)$ is an upper bound on the optimum value for the primal program. The dual program is essentially $\min_{\lambda \geq 0} g(\lambda)$. We further simplify it as follows. Letting $\mu_i = c_i - \sum_j a_{ij} \lambda_j$, we can rewrite the expression for L as

$$L = \sum_i \mu_i x_i - f(x) + \sum_j b_j \lambda_j.$$

Now note that $g(\lambda) = \max_x L(x, \lambda) = \max_x \{\sum_i \mu_i x_i - f(x)\} + \sum_j b_j \lambda_j = f^*(\mu) + \sum_j b_j \lambda_j$. Thus we get the dual optimization problem:

$$\begin{aligned} \min \sum_j b_j \lambda_j + f^*(\mu) \text{ s.t.} \\ \forall i, \sum_j a_{ij} \lambda_j = c_i - \mu_i, \\ \forall j, \lambda_j \geq 0. \end{aligned}$$

□

Lemma 3. *The following pairs of convex programs are duals of each other. The dual variables p_j of an optimal solution are equilibrium prices of the corresponding linear Fisher market.*

$$\text{EG Program: } \max \sum_i B_i \log u_i \text{ s.t.}$$

$$\forall i, u_i \leq \sum_j v_{ij} x_{ij},$$

$$\forall j, \sum_i x_{ij} \leq 1,$$

$$x_{ij} \geq 0.$$

$$\min \sum_j p_j - \sum_i B_i \log(\beta_i) \text{ s.t. (1)}$$

$$\forall i, j, p_j \geq v_{ij} \beta_i.$$

Proof. We let the dual variable corresponding to the constraint $u_i \leq \sum_j u_{ij} x_{ij}$ be β_i and the dual variable corresponding to the constraint $\sum_i x_{ij} \leq 1$ be p_j . We also need a variable μ_i that corresponds to the variable u_i in the primal program since it appears in the objective in the form of a concave function, $m_i \log u_i$. We now calculate the conjugate of this function. Recall that if $f(x) = -\log x$ then $f^*(\mu) = -1 - \log(-\mu)$, and if $g(x) = cf(x)$ then $g^*(\mu) = cf^*(\mu/c)$. Therefore if $g(x) = -c \log x$ then $g^*(\mu) = -c - c \log(-\mu/c) = c \log c - c - c \log(-\mu)$. In the dual objective, we can ignore the constant terms, $c \log c - c$. We are now ready to write down the dual program which is as follows.

$$\begin{aligned} \min \sum_j p_j - \sum_i m_i \log(-\mu_i) \text{ s.t.} \\ \forall i, j, p_j \geq u_{ij} \beta_i, \\ \forall i, \beta_i = -\mu_i. \end{aligned}$$

We can easily eliminate μ_i from the above to get the program as stated in the lemma. \square

Lemma 4. *The following convex programs are duals of each other.*

$$\begin{array}{l|l} \min \sum_j e^{q_j} + \sum_i B_i \gamma_i \text{ s.t.} & \max \sum_{i,j} b_{ij} \log v_{ij} - \sum_j (p_j \log p_j - p_j) \text{ s.t.} \\ \forall i, j, \gamma_i + q_j \geq \log v_{ij}. & \forall j, \sum_i b_{ij} = p_j, \\ & \forall i, \sum_j b_{ij} = B_i, \\ & \forall i, j, b_{ij} \geq 0. \end{array} \quad (2) \quad \text{(CP)}$$

Proof. We construct the dual of (2) as outlined in the Section 3. Again, we need to calculate the conjugate of the convex function that appears in the objective, namely e^x . We could calculate it from scratch, or derive it from the ones we have already calculated. Recall that if $f(x) = e^{x-1}$, then $f^*(\mu) = \mu \log \mu$, and if $g(x) = f(x+a)$ then $g^*(\mu) = f^*(\mu) - \mu^T a$. Thus if $g(x) = e^x = f(x+1)$ then $g^*(\mu) = f^*(\mu) - \mu = \mu \log \mu - \mu$. The dual variable corresponding to the constraint $\gamma_i + q_j \geq \log v_{ij}$ is b_{ij} and the dual variable corresponding to e^{q_j} is p_j . The structure of the dual program now follows from Lemma 2. \square

A.1 Extensions

The Eisenberg-Gale convex program can be generalized to capture the equilibrium of many other markets, such as markets with Leontief utilities, or network flow markets. In fact, [16] identify a whole class of such markets whose equilibrium is captured by convex programs similar to that of Eisenberg and Gale (called *EG markets*). We can take the dual of all such programs to get corresponding generalizations for the convex program (1). For instance, a Leontief utility is of the form $U_i = \min_j \{x_{ij}/\phi_{ij}\}$ for some given values ϕ_{ij} . The Eisenberg-Gale-type convex program for Fisher markets with Leontief utilities is as follows, along with its dual (after some simplification as before).

$$\begin{array}{l|l} \text{Primal: } \max \sum_i m_i \log u_i \text{ s.t.} & \\ \forall i, j, u_i \leq x_{ij}/\phi_{ij}, & \\ \forall j, \sum_i x_{ij} \leq 1, & \\ x_{ij} \geq 0. & \\ \text{Dual: } \min \sum_j p_j - \sum_i m_i \log(\beta_i) \text{ s.t.} & \\ \forall i, \sum_j \phi_{ij} p_j = \beta_i. & \end{array}$$

In general for an EG-type convex program, the dual has the objective function $\sum_j p_j - \sum_i m_i \log(\beta_i)$ where β_i is the minimum cost buyer i has to pay in order to get one unit of utility. For instance, for the network flow market, where the goods are edge capacities in a network and the buyers are source-sink pairs

looking to maximize the flow routed through the network, then β_i is the cost of the cheapest path between the source and the sink given the prices on the edges.

However, for some markets, it is not clear how to generalize the Eisenberg-Gale convex program, but the dual generalizes easily. In each of the cases, the optimality conditions can be easily seen to be equivalent to equilibrium conditions. We now show some examples of this.

Quasi-linear utilities

Suppose the utility of buyer i is $\sum_j (u_{ij} - p_j)x_{ij}$. In particular, if all the prices are such that $p_j > u_{ij}$, then the buyer prefers to not be allocated any good and go back with his budget unspent. It is easy to see that the following convex program captures the equilibrium prices for such utilities. In fact, given this convex program, one could take its dual to get an EG-type convex program as well.

$$\begin{aligned} \text{Primal: } \min \sum_j p_j - \sum_i m_i \log(\beta_i) \text{ s.t.} \\ (5) \\ \forall i, j, p_j \geq u_{ij} \beta_i, \\ \forall i, \beta_i \leq 1. \end{aligned}$$

$$\begin{aligned} \text{Dual: } \max \sum_i m_i \log u_i - v_i \text{ s.t.} \\ \forall i, u_i \leq \sum_j u_{ij} x_{ij} + v_i, \\ \forall j, \sum_i x_{ij} \leq 1, \\ x_{ij}, v_i \geq 0. \end{aligned}$$

Although this is a small modification of the Eisenberg-Gale convex program, it is not clear how one would arrive at this directly without going through the dual.

Transaction costs

Suppose that we are given, for every pair, buyer i and good j , a transaction cost c_{ij} that the buyer has to pay per unit of the good in addition to the price of the good. Thus the total money spent by buyer i is $\sum_j (p_j + c_{ij})x_{ij}$. Chakraborty et al. [5] show that the following convex program captures the equilibrium prices for such markets.

$$\begin{aligned} \min \sum_j p_j - \sum_i m_i \log(\beta_i) \text{ s.t.} \\ (6) \\ \forall i, j, p_j + c_{ij} \geq v_{ij} \beta_i, \\ \forall i, \beta_i \leq 1. \end{aligned}$$

Alternate proof of Gibbs' inequality

Consider the following convex program.

$$\begin{aligned} \max \sum_{j \in S_i} (b_{ij} \log v_{ij} - b_{ij} \log b_{ij}) \text{ s.t.} \\ \sum_{j \in S_i} b_{ij} = 1, \\ b_{ij} \geq 0 \quad \forall j \in S_i. \end{aligned}$$

Using the duality techniques developed in this paper, we write the following dual of this program.

$$\begin{aligned} \min \alpha_i + \sum_{j \in S_i} e^{\mu_{ij} - 1} \text{ s.t.} \\ \forall j \in S_i, \alpha_i \geq \log v_{ij} - \mu_{ij}. \end{aligned}$$

Suppose that we fix the value of α_i . Given this, we want to set μ_{ij} to be as small as possible s.t. the constraint $\alpha_i \geq \log v_{ij} - \mu_{ij}$ is satisfied, which gives us $\mu_{ij} = \log v_{ij} - \alpha_i$. Then $e^{\mu_{ij}-1} = v_{ij}e^{-1-\alpha_i}$, and the objective can be written as a function of α_i as

$$\alpha_i + \sum_{j \in S_i} v_{ij} e^{-1-\alpha_i}.$$

This can be minimized by setting the derivative to zero, which gives

$$\begin{aligned} 1 - \sum_{j \in S_i} v_{ij} e^{-1-\alpha_i} &= 0 \\ \Leftrightarrow e^{\alpha_i+1} &= \sum_{j \in S_i} v_{ij} \Leftrightarrow \alpha_i + 1 = \log\left(\sum_{j \in S_i} v_{ij}\right). \end{aligned}$$

The minimum value of the objective is then $\alpha_i + 1 = \log(\sum_{j \in S_i} v_{ij})$, which is also obtained in the primal by setting $b_{ij} = \frac{v_{ij}}{\sum_{k \in S_i} v_{ik}}$.

B Convex Program, Existence and Uniqueness for the SR equilibrium

In this section, we give the proof of Lemma 7, that the f-SR program captures the SR equilibrium. We then study the existence and the uniqueness of the SR equilibrium and we show a necessary and sufficient condition for its existence. On the uniqueness side, we show that the spending vector $q = (q_1, \dots, q_m)$, where q_j is the money spent on good j , is unique. Although in the Fisher model we have the uniqueness of price equilibrium, it is easy to see that this is not true for the SR equilibrium. Consider a market with only one buyer with utility function $u(x) = x_1$ and one seller. Let $B_1 = 1$ and $c_1 = 1$. It is easy to see that every price bigger than 1 is an SR equilibrium price.

We first state the f-SR program, with a log transformation of the objective function, and generalized for arbitrary spending limits for each good, as in the definition of the general SR equilibrium model. This convex program is a natural extension of program \mathcal{CP} presented in Section 4, with an additional set of constraints for sellers having earning limits:

$$\max \sum_{i,j} b_{ij} \log v_{ij} - \sum_j (q_j \log q_j - q_j) \text{ s.t.} \quad (\text{f-SR})$$

$$\forall j, \sum_i b_{ij} = q_j, \quad (7)$$

$$\forall i, \sum_j b_{ij} = B_i, \quad (8)$$

$$\forall j, q_j \leq c_j, \quad (9)$$

$$\forall i, j, b_{ij} \geq 0. \quad (10)$$

Here b_{ij} is the amount of money buyer i spends on good j , and q_j is the total amount of spending on good j . Constraint 9 makes sure that the spending on good j does not exceed the earning limit of seller j .

Lemma 7. *The f-SR program computes the SR equilibrium. The variables b_{ij} capture the amount of money spent by buyer i on good j , and the variables q_j capture the total spending on good j . The prices p_j can be recovered from the optimal dual variables.*

Proof. Let λ_j, μ_j, η_i be the dual variables corresponding to the first three constraints of the SR program. By the KKT conditions, optimal solutions must satisfy the following:

1. $\forall i \in B, j \in A : \log v_{ij} - \lambda_j - \eta_i \leq 0$
2. $\forall i \in B, j \in A : b_{ij} > 0 \Rightarrow \log v_{ij} - \lambda_j - \eta_i = 0$
3. $\forall j \in A : -\log q_j + \lambda_j - \mu_j = 0$
4. $\forall j \in A : \mu_j \geq 0$
5. $\forall j \in A : \mu_j > 0 \Rightarrow q_j = c_j$

From the first 3 conditions, we have $\forall i \in B, j \in A : \frac{v_{ij}}{q_j e^{\mu_j}} \leq e^{\eta_i}$ and if $b_{ij} > 0$ then $\frac{v_{ij}}{q_j e^{\mu_j}} = e^{\eta_i}$. Let $p_j = q_j e^{\mu_j}$. We will show that p is an equilibrium price with spending b . From the above observation, it is easy to see that each buyer i only spends money on his maximum bang-per-buck (MBB) goods at price p , i.e., goods that give her maximum utility per unit money spent. We also have to check that an optimal solution given by the convex program satisfies the market clearing conditions. The constraint that $\sum_j b_{ij} = 1$ guarantees that each buyer i must spend all his money. Therefore, we only have to show that the amount seller j earns is the minimum between p_j and c_j . If $q_j = c_j$ and $q_j \leq q_j e^{\mu_j} = p_j$. If $q_j < c_j$ then $\mu_j = 0$ and $p_j = q_j < c_j$. Thus, in both cases, $q_j = \min(p_j, c_j)$ as desired. \square

Lemma 11. *An SR equilibrium price exists if and only if $\sum_j c_j \geq \sum_i B_i$.*

Proof. An equilibrium price exists if and only if the feasible region of the f-SR convex program is not empty. We first prove that for the case of linear utility function, the program is feasible if and only if $\sum_j c_j \geq \sum_i B_i$. If $\sum_j c_j < \sum_i B_i$ then the feasible region is empty because the set of constraints 7, 9 and 8 can not be satisfied together. If $\sum_j c_j \geq \sum_i B_i$ then $y_{ij} = \frac{B_i c_j}{\sum_j c_j}$ gives a feasible solution because $\sum_i y_{ij} = c_j \frac{\sum_i B_i}{\sum_j c_j} \leq c_j$ and $\sum_j y_{ij} = B_i \frac{\sum_j c_j}{\sum_j c_j} = B_i$. \square

Lemma 12. *The spending vector q of the SR equilibrium is unique.*

Proof. Consider two distinct price equilibria p and p' , their corresponding spending vectors q and q' and their corresponding demand vectors x and x' . Note that $p_j \geq p'_j \Rightarrow q_j \geq q'_j$ because $q_j = x_j p_j = \min(1, \frac{c_j}{p_j}) p_j \geq \min(1, \frac{c_j}{p'_j}) p'_j = q'_j$. Consider price vector $r = (r_1, \dots, r_m)$ where $\forall k, r_k = \max(p_k, p'_k)$, its corresponding spending vector q^r and its corresponding demand vectors x^r . Note that by changing prices from p to r we may only increasing the prices. Therefore, it is easy to see under linear utility functions the demand of good j going from prices p to r would not decrease if $p'_j < p_j = r_j$. Therefore, we have $q_j^r = x_j^r r_j = x_j^r p_j \geq x_j p_j = q_j \geq q'_j$. We can do the same for all j and show $\forall j, q_j^r = \max(q_j, q'_j)$. For the sake of a contradiction suppose $\exists j, q_j > q'_j$ then using the later it is easy to show $\sum_j q_j^r > \sum_j q_j = \sum_j q'_j = \sum_i B_i$ which is contradiction because the money spent on goods cannot be more than the total budget. Therefore, $\forall j, q_j = q'_j$ and the lemma follows. \square

B.1 Rationality of the SR equilibrium

In this section, we prove rationality results for the spending restricted outcome. Specifically, we show that for those market models, a rational equilibrium exists if an equilibrium exists and all the parameters are rational numbers.

Lemma 13. *In spending-restricted market model under linear utility functions, a rational equilibrium exists if $\sum_j c_j \geq \sum_i B_i$ and all the parameters specified are rational numbers.*

Proof. Let A_i be the set of goods that buyer i spends money on, \mathcal{A} be the family of A_i 's, and L be the set of sellers reaching their earning limits. An equilibrium price p , the corresponding spending b and inverse MBB value α , if existed, must be a point inside the polyhedron $P(\mathcal{A}, L)$ bounded by the following constraints:

$$\begin{aligned}
\forall i \in N, \forall j \in A_i & \quad v_{ij}\alpha_i = p_j \\
\forall j \in M & \quad v_{ij}\alpha_i \leq p_j \\
\forall i \in N, \forall j \notin A_i & \quad b_{ij} = 0 \\
\forall i \in N, & \quad \sum_j b_{ij} = B_i \\
\forall j \in L & \quad \sum_i b_{ij} = c_j \quad p_j \geq c_j \\
\forall j \notin L & \quad \sum_i b_{ij} = p_j \quad p_j \leq c_j \\
\forall i \in N, j \in M & \quad b_{ij} \geq 0
\end{aligned}$$

If an equilibrium price exists, then \mathcal{A} and L such that $P(\mathcal{A}, L)$ is non-empty must also exist. Every point inside that non-empty polyhedron must also correspond to an equilibrium price. Since v_{ij} 's, B_i 's and c_j 's are rational numbers, a vertex of $P(\mathcal{A}, L)$ gives a rational equilibrium price. It then follows from Lemma 11 that a rational equilibrium exists if and only if $\sum_j c_j \geq \sum_i B_i$. \square

C SR equilibrium with Spending Constraint Utilities

We next define the spending constraint model. As before, let M be a set of divisible goods and N a set of buyers, $|M| = m$, $|N| = n$. Assume that the goods are numbered from 1 to m and the buyers are numbered from 1 to n . Each buyer $i \in N$ comes to the market with a specified amount of money, say $B_i \in \mathbf{Q}^+$, and we are specified the quantity, $b_j \in \mathbf{Q}^+$ of each good $j \in M$. For $i \in N$ and $j \in M$, let $f_j^i : [0, B_i] \rightarrow \mathbf{R}_+$ be the *rate function* of buyer i for good j ; it specifies the rate at which i derives utility per unit of j received, as a function of the amount of her budget spent on j . If the price of j is fixed at p_j per unit amount of j , then the function f_j^i/p_j gives the rate at which i derives utility per dollar spent, as a function of the amount of her budget spent on j . Define $g_j^i : [0, B_i] \rightarrow \mathbf{R}_+$ as follows:

$$g_j^i(x) = \int_0^x \frac{f_j^i(y)}{p_j} dy.$$

This function gives the utility derived by i on spending x dollars on good j at price p_j .

In this paper, we will deal with the case that f_j^i 's are decreasing step functions. If so, g_j^i will be a piecewise-linear and concave function. The linear version of Fisher's problem [3] is the special case in which each f_j^i is the constant function so that g_j^i is a linear function. Given prices $\mathbf{p} = (p_1, \dots, p_m)$ of all goods, each buyer wants a utility maximizing bundle of goods. Prices \mathbf{p} are equilibrium prices if each good with a positive price is fully sold.

The convex program for spending restricted model under spending constraint utility functions is as follows:

$$\max \sum_{i,j,l} b_{ij}^l \log v_{ij}^l - \sum_j (q_j \log q_j - q_j) \text{ s.t.} \quad (\text{P2})$$

$$\forall j, \sum_{i,l} b_{ij}^l = q_j, \quad (11)$$

$$\forall i, \sum_{j,l} b_{ij}^l = B_i, \quad (12)$$

$$\forall i, j, l \in S, b_{ij}^l \leq B_{ij}^l, \quad (13)$$

$$\forall j, q_j \leq c_j, \quad (14)$$

$$\forall i, j, l \in S, b_{ij}^l \geq 0. \quad (15)$$

Here b_{ij}^l is the amount of money buyer i spends on good j under segment l , B_{ij}^l is length of the segment l , and q_j is the total amount of spending on good j .

Lemma 14. *Convex program P2 captures SR equilibrium prices of SR market model under spending constraint utility function.*

Proof. Let $\lambda_j, \mu_j, \eta_i, \gamma_{ijl}$ be the dual variables for constraints 11, 14, 12, 13 respectively. By the KKT conditions, optimal solutions must satisfy the following:

1. $\forall i \in N, j \in M, l \in S : \log v_{ij}^l - \lambda_j - \eta_i - \gamma_{ijl} \leq 0$
2. $\forall i \in N, j \in M, l \in S : b_{ij}^l > 0 \Rightarrow \log v_{ij}^l - \lambda_j - \eta_i - \gamma_{ijl} = 0$
3. $\forall j \in M : -\log q_j + \lambda_j - \mu_j = 0$
4. $\forall j \in M : \mu_j \geq 0$
5. $\forall j \in M : \mu_j > 0 \Rightarrow q_j = c_j$
6. $\forall i \in N, j \in M, l \in S : \gamma_{ijl} \geq 0$
7. $\forall i \in N, j \in M, l \in S : \gamma_{ijl} > 0 \Rightarrow b_{ij}^l = B_{ij}^l$

Let $p_j = q_j e^{\mu_j}$. We will prove that p is an equilibrium price with spending b . The second KKT condition says that for a fixed pair of buyer i and good j , $b_{ij}^l > 0$ implies

$$\frac{v_{ij}^l}{e^{\gamma_{ijl}}} = e^{\lambda_j} e^{\eta_i}$$

Therefore, the ratio $v_{ij}^l / e^{\gamma_{ijl}}$ is the same for every segment l under which i spends money on j . From KKT condition 7, $\gamma_{ijl} > 0$ implies $b_{ij}^l = B_{ij}^l$. It follows that for each good j , i must finish spending money on a segment with higher rate before starting spending money on a segment with lower rate.

From the first 3 KKT conditions, we have:

$$\frac{v_{ij}^l}{q_j e^{\gamma_{ijl}} e^{\mu_j}} \leq e^{\eta_i}$$

and equality happens when $b_{ij}^l > 0$. For every segment that i can still spend money on, b_{ij}^l must be less than B_{ij}^l , and thus $\gamma_{ijl} = 0$. Therefore, for every j and l such that $B_{ij}^l > b_{ij}^l > 0$, we have

$$\frac{v_{ij}^l}{p_j} = \frac{v_{ij}^l}{q_j e^{\mu_j}} = e^{\eta_i}$$

and this ratio $\frac{v_{ij}^l}{p_j}$ is maximized among all segments that i can spend money on, i.e. segments such that $b_{ij}^l < B_{ij}^l$. Therefore, we can conclude that each buyer i is spending according to his best spending strategy.

By complementary slackness condition, if $q_j < c_j$ then $\mu_i = 0$ and $q_j = p_j$. Otherwise, if $p_j = c_j$ then $q_j \leq p_j$. Therefore, in this model, the amount seller j earns is the minimum between c_j and p_j . \square

Existence and Uniqueness We first show that the same condition that works for linear utilities also works for spending constraint utilities.

Lemma 15. *For spending constraint utility functions, an equilibrium price exists if and only if $\sum_j c_j \geq \sum_i B_i$.*

Proof. An equilibrium price exists if and only if the feasible region of the convex program is not empty. Similarly to the proof of Lemma 11, we can prove that the program is feasible if and only if $\sum_j c_j \geq \sum_i B_i$. If $\sum_j c_j < \sum_i B_i$ then the feasible region is empty because the set of constraints 11, 14 and 12 can not be satisfied together. Using a similar argument as in the previous part, we can show that if the amount of money that i spends on j is $B_i c_j / \sum_j c_j$ then constraints 11, 14 and 12 are all satisfied. We only need to guarantee that constraint 13 is satisfied as well. This can be done by choosing appropriate y_{ij}^l 's such that $\sum_l y_{ij}^l = \frac{B_i c_j}{\sum_j c_j}$ and $y_{ij}^l \leq B_{ij}^l$. \square

Then, following the same steps as those in the proof of Lemma 12, we also show that the spending vector for spending constraint utilities is unique as well.

Lemma 16. *For spending constraint utility functions the spending vector q is unique.*

C.1 Rationality of SR equilibria under spending constraint utility

Lemma 17. *In spending restricted market model under spending constraint utility functions, a rational equilibrium exists if $\sum_j c_j \geq \sum_i B_i$ and all the parameters specified are rational numbers.*

Proof. For a buyer i and good j , let S_{ij}^+ be the set of segments l such that $b_{ij}^l = B_{ij}^l$, S_{ij}^0 be the set of segments such that $B_{ij}^l > b_{ij}^l > 0$, and S_{ij}^- be the set of segments such that $b_{ij}^l = 0$. Also, let \mathcal{S} be the family of all $S_{ij}^+, S_{ij}^0, S_{ij}^-$ sets, and L be the set of sellers reaching their earning limits. An equilibrium price p , the corresponding spending b and inverse MBB value α , if existed, must be a point inside the polyhedron $P(\mathcal{S}, L)$ bounded by the following constraints:

$$\begin{aligned}
\forall i \in N, \forall j \in M, \forall l \in S_{ij}^+ & \quad v_{ij}^l \alpha_i \geq p_j & \quad b_{ij}^l = B_{ij}^l \\
\forall i \in N, \forall j \in M, \forall l \in S_{ij}^0 & \quad v_{ij}^l \alpha_i = p_j & \quad 0 \leq b_{ij}^l \leq B_{ij}^l \\
\forall i \in N, \forall j \in M, \forall l \in S_{ij}^- & \quad v_{ij}^l \alpha_i \leq p_j & \quad b_{ij}^l = 0 \\
\forall i \in N & \quad \sum_{j,l} b_{ij}^l = B_i \\
\forall j \in L & \quad \sum_{i,l} b_{ij}^l = c_j & \quad p_j \geq c_j \\
\forall j \notin L & \quad \sum_{i,l} b_{ij}^l = p_j & \quad p_j \leq c_j
\end{aligned}$$

Suppose that all the parameters specified are rational numbers. Again, we can see that a rational equilibrium must also exist if an equilibrium exists. It then follows that a rational equilibrium exists if and only if $\sum_j c_j \geq \sum_i B_i$. \square

D Utility restricted market model

D.1 Linear utilities

The convex program for the linear utility with buyers having utility limits is a natural extension of the Eisenberg-Gale program:

$$\max \sum_i B_i \log u_i \text{ s.t.} \quad (\text{P3})$$

$$\forall i, \sum_j x_{ij} v_{ij} = u_i, \quad (16)$$

$$\forall i, u_i \leq d_i, \quad (17)$$

$$\forall j, \sum_i x_{ij} \leq 1, \quad (18)$$

$$\forall i, j, x_{ij} \geq 0. \quad (19)$$

In this program, x_{ij} is the amount of good j allocated to buyer i , and u_i is the amount of utility that buyer i obtains. Constraint 17 guarantees that the amount of utility buyer i gets does not exceed his utility limit d_i .

Lemma 18. *Convex program P3 captures the equilibrium prices of utility restricted market model under linear utility function.*

Proof. Let λ_i, μ_i, p_j be the dual variables for constraints 16, 17, 18 respectively. By the KKT conditions, optimal solutions must satisfy the following:

1. $\forall i \in N, j \in M : -\lambda_i v_{ij} - p_j \leq 0$
2. $\forall i \in N, j \in M : x_{ij} > 0 \Rightarrow -\lambda_i v_{ij} - p_j = 0$
3. $\forall i \in N : \frac{B_i}{u_i} + \lambda_i - \mu_i = 0$
4. $\forall i \in N : \mu_i \geq 0$
5. $\forall i \in N : \mu_i > 0 \Rightarrow u_i = d_i$
6. $\forall j \in N : p_j \geq 0$
7. $\forall j \in N : p_j > 0 \Rightarrow \sum_i x_{ij} = 1$

From the first 3 conditions, we have $\forall i \in N, j \in M : \frac{v_{ij}}{p_j} \leq \frac{u_i}{B_i - \mu_i u_i}$ and if $x_{ij} > 0$ then $\frac{v_{ij}}{p_j} = \frac{u_i}{B_i - \mu_i u_i}$.

We will show that p is an equilibrium price with allocation x . From the above observation, it is easy to see that each buyer i only spends money on his MBB goods at price p . Moreover, we know that if $p_j > 0$ then good j must be fully sold. Therefore, the only remaining thing to prove is that at price p each buyer either spends all his money or attains his utility limit. If $u_i = d_i$ then buyer i reaches his utility limit and the amount of money he spends is $B_i - \mu_i d_i$, which is at most B_i . If $u_i < d_i$ then $\mu_i = 0$ and the amount of money he spends is B_i . \square

We now extend these results to Leontief and CES utility functions. Utility function f_i is said to be *Leontief* if, given parameters $a_{ij} \in \mathbf{R}_+ \cup \{0\}$ for each good $j \in M$, $f_i(x) = \min_{j \in M} x_{ij}/a_{ij}$. Finally, f_i is said to be *constant elasticity of substitution (CES)* with parameter ρ if given parameters α_j for each good $j \in M$,

$$f_i(x) = \left(\sum_{j=1}^n \alpha_j x_j^\rho \right)^{\frac{1}{\rho}}.$$

D.2 Utility restricted market model under Leontief utilities

The convex program for the Leontief utility model is as follows:

$$\max \sum_i B_i \log u_i \text{ s.t.} \tag{P4}$$

$$\forall i, j, u_i \phi_{ij} = x_{ij}, \tag{20}$$

$$\forall i, u_i \leq d_i, \tag{21}$$

$$\forall j, \sum_i x_{ij} \leq 1 \tag{22}$$

$$\forall i, j, x_{ij} \geq 0. \tag{23}$$

Lemma 19. *Convex program P4 captures the equilibrium prices of utility restricted market model under Leontief utility function.*

Proof. Let λ_{ij}, μ_i, p_j be the dual variables for constraints 20, 21, 22 respectively. By the KKT conditions, optimal solutions must satisfy the following:

1. $\forall i \in N, j \in M : -\lambda_{ij} - p_j \leq 0$
2. $\forall i \in N, j \in M : x_{ij} > 0 \Rightarrow -\lambda_{ij} - p_j = 0$
3. $\forall i \in N : \frac{B_i}{u_i} + \sum_j \lambda_{ij} \phi_{ij} - \mu_i = 0$
4. $\forall i \in N : \mu_i \geq 0$
5. $\forall i \in N : \mu_i > 0 \Rightarrow u_i = d_i$
6. $\forall j \in M : p_j \geq 0$
7. $\forall j \in M : p_j > 0 \Rightarrow \sum_i x_{ij} = 1$

Notice that in this model, we may assume that $u_i > 0$ for all $i \in N$. It follows from constraint 20 that $x_{ij} = 0$ if and only if $\phi_{ij} = 0$. From the second KKT condition, we know that if $\phi_{ij} > 0$, we must have $\lambda_{ij} = -p_j$. Substituting in the third condition we have:

$$\frac{B_i}{u_i} - \mu_i = \sum_j p_j \phi_{ij}$$

Therefore,

$$B_i - \mu_i u_i = \sum_j p_j \phi_{ij} \frac{x_{ij}}{\phi_{ij}} = \sum_j p_j x_{ij}$$

It follows that $B_i - \mu_i u_i$ is actually the amount of money that buyer i spends. By complementary slackness condition, if $u_i < d_i$ then $\mu_i = 0$ and i spends all his budget. Otherwise, if $u_i = d_i$ then $B_i - \mu_i u_i \leq B_i$. Therefore, in this model, a buyer i either spends all his budget or attains his utility limit. Moreover, we know that if $p_j > 0$ then good j is fully sold. Thus, p is an equilibrium price with allocation x . \square

D.3 Utility restricted marked model under CES utilities

The convex program for the CES utility model with parameter ρ is as follows:

$$\max \sum_i B_i \log u_i \text{ s.t.} \tag{P5}$$

$$\forall i, u_i = \left(\sum v_{ij} x_{ij}^\rho \right)^{\frac{1}{\rho}}, \tag{24}$$

$$\forall i, u_i \leq d_i, \tag{25}$$

$$\forall j, \sum_i x_{ij} \leq 1, \tag{26}$$

$$\forall i, j, x_{ij} \geq 0. \tag{27}$$

Notice that in this model, $\partial u_i / \partial x_{ij} = u_i^{1-\rho} v_{ij} x_{ij}^{\rho-1}$ has the same term $u_i^{1-\rho} v_{ij}$ for all x_{ij} 's. Moreover, $\partial u_i / \partial x_{ij}$ decreases when x_{ij} increases. It follows that the best spending strategy for a buyer i is to start with $x_{ij} = 0 \quad \forall j \in M$ and spend money on goods j that maximize the ratio $\frac{\partial u_i / \partial x_{ij}}{p_j}$ at every point. At the end of the procedure, all goods j such that $x_{ij} > 0$ will have the same value for $\frac{\partial u_i / \partial x_{ij}}{p_j}$, and that value is the maximum over all goods.

Lemma 20. *Convex program P5 captures the equilibrium prices of utility restricted market model under CES utility function.*

Proof. Let λ_i, μ_i, p_j be the dual variables for constraints 24, 25, 26 respectively. By the KKT conditions, optimal solutions must satisfy the following:

1. $\forall i \in N, j \in M : \quad -\lambda_i u_i^{1-\rho} v_{ij} x_{ij}^{\rho-1} - p_j \leq 0$
2. $\forall i \in N, j \in M : \quad x_{ij} > 0 \Rightarrow -\lambda_i u_i^{1-\rho} v_{ij} x_{ij}^{\rho-1} - p_j = 0$
3. $\forall i \in N : \quad \frac{B_i}{u_i} + \lambda_i - \mu_i = 0$
4. $\forall i \in N : \quad \mu_i \geq 0$
5. $\forall i \in N : \quad \mu_i > 0 \Rightarrow u_i = d_i$
6. $\forall j \in M : \quad p_j \geq 0$
7. $\forall j \in M : \quad p_j > 0 \Rightarrow \sum_i x_{ij} = 1$

We will prove that p is an equilibrium price with allocation x . From the first there KKT conditions, we have

$$\frac{u_i^{1-\rho} v_{ij} x_{ij}^{\rho-1}}{p_j} \leq \frac{u_i}{B_i - \mu_i u_i}$$

and equality happens when $x_{ij} > 0$. Therefore, x is in agreement with the best spending strategy of the buyers, which says that for each buyer i , if $x_{ij} > 0$ then $\frac{\partial u_i / \partial x_{ij}}{p_j}$ is maximized over all j 's. Moreover, we can see that $B_i - \mu_i u_i$ is the amount of money buyer i spends. By complementary slackness condition, if $u_i < d_i$ then $\mu_i = 0$ and i spends all his budget. Otherwise, if $u_i = d_i$ then $B_i - \mu_i u_i \leq B_i$. Therefore, in this model, a buyer i either spends all his budget or attains his utility limit. Moreover, we know that if $p_j > 0$ then good j is fully sold. Thus, p is an equilibrium price with allocation x . \square

D.4 Rationality of equilibria for UR market model under linear utilities

Lemma 21. *In UR market model under linear utility functions, a rational equilibrium exists if all the parameters specified are rational numbers.*

Proof. Let A_i be the set of goods that buyer i spends money on, \mathcal{A} be the family of A_i 's, and L be the set of buyers reaching their utility limits. An equilibrium price p , the corresponding spending b and inverse MBB value α , if existed, must be a point inside the polyhedron $P(\mathcal{A}, L)$ bounded by the following constraints:

$$\begin{aligned} \forall i \in N, \forall j \in A_i & \quad v_{ij} \alpha_i = p_j \\ \forall j \in M & \quad v_{ij} \alpha_i \leq p_j \\ \forall i \in N, \forall j \notin A_i & \quad b_{ij} = 0 \\ \forall j \in N & \quad \sum_i b_{ij} = p_j \\ \forall i \in L & \quad \sum_j b_{ij} = \alpha_i d_i \quad \sum_j b_{ij} \leq B_i \\ \forall i \notin L & \quad \sum_j q_{ij} \leq \alpha_i d_i \quad \sum_j b_{ij} = B_i \\ \forall i \in N, j \in M & \quad b_{ij} \geq 0 \end{aligned}$$

Suppose that all the parameters specified in this model are rational numbers. By a similar argument to Lemma 13, we can see that an equilibrium exists if and only if a rational equilibrium exists. It follow from Lemma 22 that a rational equilibrium price must always exist if all the parameters specified are rational numbers. \square

D.5 Existence and Uniqueness of UR equilibrium

For UR market model, we show that an equilibrium always exists for all utility functions we mentioned in the previous section. On the uniqueness side, the utility vector is unique. To verify that the price vector is not unique, consider a market with only one buyer with utility function $u(x) = x_1$ and one seller. Let $d_1 = 1$ and $B_1 = 2$. It is easy to see every price in interval $[1, 2]$ is an equilibrium price.

Lemma 22. *In UR market model under linear, Leontief and CES utility functions, an equilibrium price always exists.*

Proof. An equilibrium price exists if and only if the feasible region of the convex program is not empty. In P3, P4 and P5, $x_{ij} = 0$ for all i, j is a feasible solution. Therefore, the feasible region is not empty and an equilibrium exists. \square

Lemma 23. *In UR market model under linear, Leontief and CES utility functions, the utilities of an equilibrium are unique.*

Proof. In section D, we showed every equilibrium correspond to an solution of a convex program with an objective function of the form $\sum_i B_i \log u_i$. It is easy to see that the objective function is strictly concave. Therefore, there is a unique vector u that maximizes the objective function and the lemma follows. \square

E Proofs of Theorem 1 and Lemma 10 (Approximation Factor Bounds)

E.1 Approximation Factor Upper Bound

For each item j that has more than one child-agent in the spending graph $Q(b)$, remove the edges connecting it to all but the one child-agent that spends the most money on j , i.e., the one with the largest b_{ij} value. This yields a pruned spending graph $P(b)$ that is also a forest of trees. We refer to the trees of the pruned graph $P(b)$ as the *matching-trees*. In every matching-tree T with $k \geq 2$ agents, when the algorithm reaches its last step, every remaining item has exactly one parent-agent and one child-agent, so all but one agent can be matched to one of these items. Our proof shows that there exists a matching of the remaining items such that the agents within T have a “high” NSW.

A naive way to match the agents in the last step of the algorithm would be to match all of them, except the one that has accrued the highest value during the previous steps. It was already observed in [7] that, for any matching-tree T of k agents, there exists an agent who was assigned value at least $1/(2k)$ during Steps 3 and 4 of the algorithm, so we could match every agent in T , except him. But, what is the worst case distribution of value that can arise in this matching?

If T is some matching-tree of the pruned spending graph $P(b)$, then let M_T denote the union of items in T with the items that were assigned to agents in T in Steps 3 and 4. Also, let H be the set of items with $p_j > 1$ in the SR equilibrium and H_T the subset of those items that belong to T . In proving this theorem, we use the following lemma from [7].

Lemma 24 ([7]). *For any matching-tree T with k agents, there exists an agent $i \in T$ who, during Steps 3 and 4 received one or more items that she values at least $1/(2k)$. Also, for items in M_T :*

$$\sum_{j \in M_T} q_j \geq k - \frac{1}{2}. \quad (28)$$

Let x' be the integral allocation that would arise if we follow the SRR algorithm up to Step 4, and then use the naive matching described above. For simplicity, we assume that the valuations of the agents are scaled in such a way that $v_{ij} = p_j$ if $b_{ij} > 0$, which allows us to use SR-UB as an upper bound of OPT. We begin by showing that, if every agent receives a value of at least $1/2$ in x' , then the theorem follows. To verify this fact, note that every agent who is matched to an item j with price $p_j > 1$ has a value of at least p_j , and every other agent has a value of at least $1/2$, so

$$\left(\prod_i v_i(x') \right)^{1/n} \geq \left(\frac{1}{2^n} \cdot \prod_{j \in H} p_j \right)^{1/n} \geq \frac{1}{2} \cdot \left(\prod_i v_i(x^*) \right)^{1/n}.$$

For any matching-tree with $k = 1$ agent, Inequality (28) implies that this agent will receive value at least $k - 1/2 = 1/2$. Therefore, we now, assume that there exists some matching-tree T with $k \geq 2$ agents such that some agent α in T gets a value less than $1/2$ in x' . Let $v_\alpha(x')$, or v_α for short, be the value that this agent receives. Since $v_\alpha < 1/2$, this agent is the only one in T that was not matched to an item with $p_j > 1/2$, so every other agent i in T has $v_i(x') \geq 1/2$.

Lemma 25. *Over all the possible allocations x' , the one with the minimum product of the valuations, has at least $\left\lfloor \frac{k-2v_\alpha}{1+2v_\alpha} \right\rfloor$ agents with value $v_i(x') \geq 1$.*

Proof. Let k_1 be the number of agents with value at least 1 in x' , and assume that $k_1 \leq \left\lfloor \frac{k-2v_\alpha}{1+2v_\alpha} \right\rfloor - 1$. Since every agent other than α was matched to an item with $q_j \geq 1/2$, we know that the value of the k_1 agents before the matching was at most v_α . Hence, for each such agent i the sum of the q_j values of the items that were assigned to i in x' is at most $1 + v_\alpha$. As a result, if M' is the union of items that were assigned to agent α and the k_1 agents, we know that $\sum_{j \in M'} q_j \leq k_1(1 + v_\alpha) + v_\alpha$.

Using Inequality (28), we get

$$\sum_{j \in M \setminus M'} q_j \geq k - \frac{1}{2} - (k_1(1 + v_\alpha) + v_\alpha) = \frac{1}{2}(k - k_1) + \frac{1}{2}k - \left(\frac{1}{2} + v_\alpha\right)(k_1 + 1).$$

But, we have assumed that $k_1 + 1 \leq \left\lfloor \frac{k-2v_\alpha}{1+2v_\alpha} \right\rfloor \leq \frac{k-2v_\alpha}{1+2v_\alpha}$, so

$$\sum_{j \in M \setminus M'} q_j \geq \frac{1}{2}(k - k_1) + \frac{1}{2}k - \frac{k - 2v_\alpha}{2} = \frac{1}{2}(k - k_1) + v_\alpha.$$

Therefore, the remaining $k - k_1 - 1$ agents have a total value more than $(k - k_1)/2$, i.e., strictly more than $1/2$ on average. It also implies that at least two of these agents have value strictly more than $1/2$. If $k - k_1 - 2$ agents had value equal to $1/2$ then the remaining agent would have a value more than $\frac{1}{2}(k - k_1) - \frac{1}{2}(k - k_1 - 2) = 1$, which contradicts our assumption that only k_1 agents have value at least 1. Let $v_1, v_2 \in (1/2, 1)$ be the values of two such agents in the worst case outcome. It is then easy to verify that, if we were to instead give value $1/2$ to the one agent and $v_1 + v_2 - 1/2$ to the other, the NSW would drop. This contradicts our assumption that this is a worst case outcome. \square

Lemma 26. *For any matching-tree T with k agents, the allocation x of the SRR algorithm satisfies*

$$\prod_{i \in T} v_i(x') \geq \frac{1}{2^k} \prod_{j \in H_T} p_j.$$

Proof. Let k_1 be the number of agents with $v_i(x') \geq 1$. Given any agent i among these k_1 agents, if j is the item that he was matched to, then has value $v_i(x') \geq \max\{1, p_j\}$. As a result, the product of the values of these k_1 players is at least $\prod_{j \in H_T} p_j$. Therefore, it suffices to show that the product of the remaining $k - k_1$ agents is at least $1/2^k$.

Among the $k_2 = k - k_1 - 1$ agents that get value in $[1/2, 1)$, it is easy to verify that their product is minimized when at most one agent among them gets value higher than $1/2$. If we let v_β be the value of that player, and using Inequality (28), we get

$$v_\beta \geq k - \frac{1}{2} - \left[k_1(1 + v_\alpha) + v_\alpha + (k - k_1 - 2)\frac{1}{2} \right]. \quad (29)$$

If we let $\bar{k}_1 = \frac{k-2v_\alpha}{1+2v_\alpha}$ and $\delta = \bar{k}_1 - k_1 = \frac{k-2v_\alpha}{1+2v_\alpha} - \left\lfloor \frac{k-2v_\alpha}{1+2v_\alpha} \right\rfloor$ be the rounding error, then Inequality (29) yields

$$v_\beta \geq \frac{\delta + 1}{2}.$$

This implies that

$$\prod_{i \in T} v_i(x') \geq \frac{v_\alpha v_\beta}{2^{k-k_1-2}} \prod_{j \in H_T} p_j \geq \frac{v_\alpha}{2^{k-\bar{k}_1-1}} \frac{\delta + 1}{2^\delta} \prod_{j \in H_T} p_j \geq \frac{v_\alpha}{2^{k-\bar{k}_1-1}} \prod_{j \in H_T} p_j,$$

where the last inequality comes from the fact that $\delta + 1 \geq 2^\delta$ for $\delta \in [0, 1]$. To verify this fact note that $(\delta + 1 - 2^\delta)'' = -2^\delta \ln^2 2 < 0$, so this is minimized at either $\delta = 0$ or $\delta = 1$, both of which yield $\delta + 1 - 2^\delta = 0$. To conclude the proof, it suffices to show that for every $v_\alpha \in [1/(2k), 1/2]$ and every $k \geq 2$ we have

$$\frac{v_\alpha}{2^{k-k_1-1}} \geq \frac{1}{2^k} \quad \text{or, equivalently,} \quad v_\alpha 2^{\frac{k+1}{1+2v_\alpha}} \geq 1.$$

For $k \geq 7$, it is easy to verify that this inequality holds. In particular, using the fact that $v_\alpha \in [1/(2k), 1/2]$,

$$v_\alpha 2^{\frac{k+1}{1+2v_\alpha}} \geq \frac{1}{2k} 2^{\frac{k+1}{1+2\frac{1}{2}}} \geq \frac{2^{\frac{k+1}{2}}}{2k} \geq 1, \quad \text{for } k \geq 7.$$

Note that $v_\alpha 2^{\frac{k+1}{1+2v_\alpha}}$ is minimized at the same points as $\log v_\alpha + \frac{k+1}{1+2v_\alpha}$. Taking a derivative w.r.t. v_α gives

$$\left(\log v_\alpha + \frac{k+1}{1+2v_\alpha} \right)' = \frac{1}{v_\alpha \ln 2} - \frac{2(k+1)}{(1+2v_\alpha)^2}.$$

For $k \leq 4$, this derivative is positive for any value of v_α , so $v_\alpha 2^{\frac{k+1}{1+2v_\alpha}}$ is minimized at $v_\alpha = 1/(2k)$, where its value is equal to $\frac{1}{2k} 2^{\frac{k(k+1)}{k+1}} = \frac{2^k}{2k} \geq 1$. Finally, replacing $k = 5$ and $k = 6$ and minimizing $v_\alpha 2^{\frac{k+1}{1+2v_\alpha}}$ over all values of v_α also shows that this function is minimized at $v_\alpha = 1/10$ and $v_\alpha = 1/12$ respectively, and its value is at least 1, which concludes the proof. \square

The inequality of Lemma 26 implies the desired approximation factor if we observe that

$$\left(\prod_i v_i(x) \right)^{1/n} = \left(\prod_T \prod_{i \in T} v_i(x) \right)^{1/n} \geq \frac{1}{2} \left(\prod_{j \in H} p_j \right)^{1/n}.$$

E.2 Approximation Factor Lower Bound

Proof. Consider an instance with $m = 2\kappa$ items and $n = \kappa + 1$ agents. Each agent $i \in [1, \kappa]$ has a value of $1/2$ for item i and a value of $1/2 + 1/\kappa$ for item $2i$. The value of these agents for every other item is 0. Finally, agent $\kappa + 1$ values every item from 1 to κ for a value of 1 and has value 0 for the rest. The item prices in the SR equilibrium for this instance are $1/2$ for the first κ items and $1/2 + 1/\kappa$ for the remaining κ items. Agent $\kappa + 1$ spends $1/\kappa$ on each one of the first κ items, while each agent $i \in [1, \kappa]$ spends $1 - 1/\kappa$ on item i and his remaining budget of $1 + 1/\kappa$ on item $2i$.

Facing this SR equilibrium, assume that the SRR algorithm chooses agent $\kappa + 1$ as the root-agent in Step 2, then it would assign all of the first κ items to this agent. To verify this fact note that for every item j among the first κ ones, $q_j = 1/2$ and agent $\kappa + 1$ is the parent-agent. On the other hand, every other agent $i \in [1, \kappa]$ would get only item $2i$. This leads to a product of valuations equal to $\frac{\kappa}{(2+1/\kappa)^\kappa}$. If, on the other hand, agent $\kappa + 1$ was allocated just one of the first κ items and gave each of the other $\kappa - 1$ items to the agents that value them, the product of the valuations would be more than $\frac{1}{2}$. For large values of κ the ratio between the NSW of these two outcomes converges to 2. Finally, note that, even if the algorithm chose some different agent as the root, the result would not be affected in the limit. \square