# Truthful Allocation Mechanisms Without Payments: Characterization and Implications on Fairness* 

Georgios Amanatidis $^{\dagger} \quad$ Georgios Birmpas ${ }^{\dagger} \quad$ George Christodoulou ${ }^{\ddagger}$ Evangelos Markakis ${ }^{\dagger}$

November 5, 2018


#### Abstract

We study the mechanism design problem of allocating a set of indivisible items without monetary transfers. Despite the vast literature on this very standard model, it still remains unclear how do truthful mechanisms look like. We focus on the case of two players with additive valuation functions and our purpose is twofold. First, our main result provides a complete characterization of truthful mechanisms that allocate all the items to the players. Our characterization reveals an interesting structure underlying all truthful mechanisms, showing that they can be decomposed into two components: a selection part where players pick their best subset among prespecified choices determined by the mechanism, and an exchange part where players are offered the chance to exchange certain subsets if it is favorable to do so. In the remaining paper, we apply our main result and derive several consequences on the design of mechanisms with fairness guarantees. We consider various notions of fairness, (indicatively, maximin share guarantees and envy-freeness up to one item) and provide tight bounds for their approximability. Our work settles some of the open problems in this agenda, and we conclude by discussing possible extensions to more players.


## 1 Introduction

We study a very elementary and fundamental model for allocating indivisible goods from a mechanism design viewpoint. Namely, we consider a set of indivisible items that need to be allocated to a set of players. An outcome of the problem is an allocation of all the items to the players, i.e., a partition into bundles, and each player evaluates an allocation by his own additive valuation function. Our primary motivation originates from the fair division literature, where such models have been considered extensively. However, the same setting also appears in several domains, including job scheduling, load balancing and many other resource allocation problems.

The focus of our work is on understanding the interplay between truthfulness and fairness in this setting. Hence, we want to identify the effects on fairness guarantees, imposed by eliminating any incentives for the players to misreport their valuation functions. This type of questions has been posed already in previous works and for various notions of fairness, such as envyfreeness, or for the concept of maximin shares [see, among others, Lipton et al., 2004, Caragiannis et al.,

[^0]2009, Amanatidis et al., 2016]. However, the results so far have been rather scarce in the sense that a) in most cases, they concern impossibility results which are far from being tight and $b$ ) the proof techniques are based on constructing specific families of instances that do not enhance our understanding on the structure of truthful mechanisms, with the exception of Caragiannis et al. [2009] which, however, is only for two players and two items.

In order to comprehend the trade-offs that are inherent between incentives and fairness, we first take a step back and focus solely on truthfulness itself. As is quite common in fair division models, we will not allow any monetary transfers, so that a mechanism simply outputs an allocation of the items. Hence, the question we want to begin with is: what is the structure of truthful allocation mechanisms?

There has been already a significant volume of works on characterizing truthful allocation mechanisms for indivisible items, yet there are some important differences from our approach. First, a typical line of work studies this question under the additional assumption of Pareto efficiency or related notions [Pápai, 2000, Klaus and Miyagawa, 2002, Ehlers and Klaus, 2003]. The characterization results that have been obtained show that the combination of truthfulness together with Pareto efficiency tends to make the class of available deterministic mechanisms very poor; only some types of dictatorship survive when imposing both criteria. Second, in some cases the analysis is carried out without any restrictions on the class of valuation functions, i.e., not even monotonicity, which again often results in a very limited class of mechanisms [see, e.g., Pápai, 2001]. When moving to a specific class, such as the class of additive functions which is usualy assumed in fair division, it is conceivable that we can have a much richer class of truthful mechanisms. The results above indicate that the known characterizations of truthful mechanisms are also dependent on further assumptions, which may be well justified in various scenarios, but they are not aligned with the goal of fair division.

### 1.1 Contribution

Our main result is a characterization of deterministic truthful mechanisms that allocate all the items to two players with additive valuations. In doing so, we identify some important allocation properties that every truthful mechanism should satisfy. One such crucial property is the notion of controlling items (Definition 3.7); we say that a player controls an item, whenever it is possible to report values that will guarantee him this item, regardless of the other player's valuation function. We show that truthfulness implies that every item is controlled by some player. Exploiting this property further, greatly helps us in understanding how a mechanism operates. Consequently, our analysis and the characterization we eventually obtain reveals an interesting structure underlying all truthful mechanisms; they can all be essentially decomposed into two components: (i) a selection partwhere players pick their best subset among prespecified choices determined by the mechanism, and (ii) an exchange part where players are offered the chance to exchange certain subsets if it is favorable to do so. Hence, we call them picking-exchange mechanisms.

Next, we apply our main result and derive several consequences on the design of mechanisms with (approximate) fairness guarantees. We consider various notions of fairness in Section 4, starting our discussion with the more standard ones such as proportionality and envyfreeness, and explaining why such concepts cannot be attained-even approximately-by truthful mechanisms. We then focus on more recently studied relaxations of either envy-freeness or proportionality where positive algorithmic results have been obtained (e.g., finding allocations
that are envy-free up to one item, or achieve approximate maximin share guarantees). For these notions, we provide tight bounds on the approximation guarantees of truthful mechanisms, settling some of the open problems in this area Caragiannis et al., 2009, Amanatidis et al., 2016]. Interestingly, our results also reveal that the best truthful approximation algorithms for fair division are achieved by ordinal mechanisms, i.e., mechanisms that exploit only the relative ranking of the items and not the cardinal information of the valuation functions.

The heart of our approach for obtaining lower bounds on the approximability of fairness criteria, is a necessary condition for fairness in view of our notion of control, which we call no control of pairs. It states that no player should control more than one item. We show how this condition summarizes minimum requirements for various fairness concepts previously studied in the literature. Although this condition does not offer an alternative fairness criterion, it is a useful tool for showing lower bounds.

Finally, in Section5we provide a general class of truthful mechanisms for the case of multiple players. This class generalizes picking-exchange mechanisms in a non-trivial way. As indicated by our mechanisms, there is a much richer structure in the case of multiple players. In particular, the notion of control does not convey enough information anymore. Instead, there seem to exist several different levels of control.

### 1.2 Related Work

The only work we are aware of, in which a full characterization is given for truthful mechanisms with indivisible items, additive valuations, and no further assumptions is by Caragiannis et al. [2009]. However, this is only a characterization for two players and two items. Apart from characterizations, there have been several works that try to quantify the effects of truthfulness on several concepts of fairness. For the performance of truthful mechanisms with respect to envyfreeness, see Caragiannis et al. [2009] and Lipton et al. [2004], whereas for max-min fairness see Bezakova and Dani |2005]. Coming to more recent results and along the same spirit, Amanatidis et al. [2016] and Markakis and Psomas [2011] study the notion of maximin share allocations, and a related notion of worst-case guarantees respectively. They obtain separation results, showing that the approximation factors achievable by truthful mechanisms are strictly worse than the known algorithmic (nontruthful) results. Obtaining a better understanding for the structure of truthful mechanisms and how they affect fairness has been an open problem underlying all the above works. For a better and more complete elaboration on fairness and the numerous fairness concepts that have been suggested, we refer the reader to the books Brams and Taylor, 1996, Robertson and Webb, 1998, Moulin, 2003] and the recent surveys Bouveret et al. [2016], Procaccia [2016].

There has been a long series of works on characterizing mechanisms with indivisible items beyond the context of fair divison. Many works characterize the allocation mechanisms that arise when we combine truthfulness with Pareto efficiency [see, e.g., Pápai, 2000, Klaus and Miyagawa, 2002, Ehlers and Klaus, 2003]. Typically, such mechanisms tend to be dictatorial, and it is also well known that economic efficiency is mostly incompatible with fairness [see, e.g., Bouveret et al., 2016]. Another assumption that has been used is nonbossiness, which means that one cannot change the outcome without affecting his own bundle. For instance, Svensson 1999] assumes nonbossiness in a setting where each player is interested in acquiring only one item. For general valuations, this also leads to dictatorial algorithms [Pápai, 2001]. In most of these works ties are ignored by considering strict preference orders over all subsets of the items, while in some cases
it is also allowed for the mechanism not to allocate all the items.
There have also been relevant works for the setting of divisible goods [see, among others, Chen et al., 2013, Cole et al., 2013]. We note that for additive valuation functions, a mechanism for divisible items can be interpreted as a randomized mechanism for indivisible items. This connection is already discussed and explored in Guo and Conitzer [2010], Aziz et al. [2016]. In our work, we do not study randomized mechanisms, however it is an interesting question to have characterization results for such settings as well. Along this direction, see Mennle and Seuken [2014] where a relaxed notion of truthfulness is studied.

Related to our work is also the literature on exchange markets. These are models where players are equipped with an initial endowment, e.g., a house or a set of items. For the case where players can have multiple indivisible items as their initial endowment, see Pápai [2003, 2007]. Exchange markets provide an example where the existing characterizations go well beyond dictatorships and are closely related to the exchange component of our mechanisms.

Finally, for settings with payments, the work of Dobzinski and Sundararajan 2008], and independently of Christodoulou et al. [2008], provided a characterization of truthful mechanisms with two players and additive valuations when all items are allocated. However, their characterization does not apply to our setting because they make an additional assumption, namely decisiveness. It roughly requires that each player should be able to receive any possible bundle of items, by making an appropriate bid. Their motivation is the characterization of truthful mechanisms with bounded makespan (maximum finishing time) for the scheduling problem, and in their case decisiveness is necessary in order to achieve bounded guarantees. In our case, our motivation is fairness, and decisiveness is a very strong assumption which has the opposite effects of what we need; e.g., assigning the full-bundle to a player is unacceptable in terms of fairness. Finally, Christodoulou and Kovács [2011] give a global characterization of envy-free and truthful mechanisms for settings with payments, when there are multiple players but only two items.

## 2 Preliminaries and Notation

With the exception of Section [5, we consider a setting with two players and a set of $m$ indivisible items, $M=\{1, \ldots, m\}=[m]$, to be allocated to the players. We assume that each player $i$ has an additive valuation function $v_{i}$ over the items, so that for every $S \subseteq M, v_{i}(S)=\sum_{j \in S} v_{i}(\{j\})$. For $j \in M$, we write $v_{i j}$ instead of $v_{i}(\{j\})$.

We say that ( $S_{1}, S_{2}, \ldots, S_{k}$ ) is a partition of a set $S$, if $\bigcup_{i \in[k]} S_{i}=S$, and $S_{i} \cap S_{j}=\varnothing$ for any $i, j \in[k]$ with $i \neq j$. Note that we do not require that $S_{i} \neq \varnothing$ for all $i \in[n]$. An allocation of $M$ to the players is a partition in the form $S=\left(S_{1}, S_{2}\right)$. By $\mathscr{M}$ we denote the set of all allocations of $M$.

The set $V_{m}$ of all possible profiles is $\mathbb{R}_{+}^{m} \times \mathbb{R}_{+}^{m}$, i.e., we assume that $\nu_{i j}>0$ for every $i \in\{1,2\}$ and $j \in M$. For some statements we need the assumption that the players' valuation functions are such that no two sets have the same value. So, let $\mathcal{I}_{m}^{\neq}$denote the set of such profiles, i.e.,

$$
\mathscr{V}_{m}^{\neq}=\left\{\left(\nu_{1}, v_{2}\right) \in V_{m} \mid \forall S, T \subseteq[m] \text { with } S \neq T, \text { and } \forall i \in\{1,2\}, \sum_{j \in S} v_{i j} \neq \sum_{j \in T} v_{i j}\right\} .
$$

Definition 2.1. A deterministic allocation mechanism with no monetary transfers, or simply a mechanism, for allocating all the items in $M=[m]$, is a mapping $\mathscr{X}$ from $V_{m}$ to $\mathscr{M}$. That is, for any profile $\mathbf{v}$, the outcome of the mechanism is $\mathscr{X}(\mathbf{v})=\left(X_{1}(\mathbf{v}), X_{2}(\mathbf{v})\right) \in \mathscr{M}$, and $X_{i}(\mathbf{v})$ denotes the set of items player $i$ receives.

A mechanism $\mathscr{X}$ is truthful if for any instance $\mathbf{v}=\left(\nu_{1}, \nu_{2}\right)$, any player $i \in\{1,2\}$, and any $v_{i}^{\prime}$ :

$$
v_{i}\left(X_{i}(\mathbf{v})\right) \geq v_{i}\left(X_{i}\left(v_{i}^{\prime}, v_{-i}\right)\right)
$$

Since we will repeatedly argue about intersections of $X_{i}(\mathbf{v})$ with various subsets of $M$, we use $X_{i}^{S}(\mathbf{v})$ as a shortcut for $X_{i}(\mathbf{v}) \cap S$, where $S \subseteq M$.

### 2.1 Fairness concepts ${ }^{1}$

Several notions have emerged throughout the years as to what can be considered a fair allocation. We define below the concepts that we will examine in Section 4 . Although all concepts can be clearly defined for any number of players, we provide the definitions for two players, since this is the focus of the paper.

We start with two of the most dominant solution concepts in fair division, namely proportionality and envy-freeness.

Definition 2.2. An allocation $S=\left(S_{1}, S_{2}\right)$ is

1. proportional, if $v_{i}\left(S_{i}\right) \geq \frac{1}{2} v_{i}(M)$, for $i \in\{1,2\}$.
2. envy-free, if $v_{1}\left(S_{1}\right) \geq v_{1}\left(S_{2}\right)$, and $v_{2}\left(S_{2}\right) \geq v_{2}\left(S_{1}\right)$.

Proportionality was considered in the very first work on fair division by Steinhaus 1948]. Envy-freeness was suggested later by Gamow and Stern 1958], and with a more formal argumentation by Foley [1967] and Varian 1974].

Envy-freeness is a stricter notion than proportionality, but even for the latter existence cannot be guaranteed under indivisible goods. One can also consider approximation versions of these problems as follows: Given an instance $I$, let $E(I)$ be the minimum possible envy that can be achieved at $I$, among all possible allocations. We say that a mechanism achieves a $\rho$ approximation, if for every instance $I$, it produces an allocation where the envy between any pair of players is at most $\rho E(I)$. Similarly for proportionality, suppose that an instance $I$ admits an allocation where every player receives a value of at least $\frac{c(I)}{2} v_{i}(M)$ for some $c(I) \leq 1$. Then a $\rho$-approximation would mean that each player is guaranteed a bundle with value at least $\frac{\rho c(I)}{2} v_{i}(M)$.

Apart from the approximation versions, the fact that we cannot always have proportional or envy-free allocations gives rise to relaxations of these definitions, with the hope of obtaining more positive results. We describe below three such relaxations, all of which admit either exact or constant-factor approximation algorithms (not necessarily truthful) in polynomial time.

The first such relaxation is the concept of envy-freeness up to one item, where each person may envy another player by an amount which does not exceed the value of a single item in the other player's bundle. Formally:

Definition 2.3. An allocation $S=\left(S_{1}, S_{2}\right)$ is envy-free up to one item, if there exists an item $a_{1} \in S_{1}$, and an item $a_{2} \in S_{2}$, such that $v_{i}\left(S_{i}\right) \geq v_{i}\left(S_{j} \backslash\left\{a_{j}\right\}\right)$, for $i, j \in\{1,2\}$.

It is quite easy to achieve envy-freeness up to one item, e.g., a round-robin algorithm that alternates between the players and gives them in each step their best remaining item suffices. Other algorithms are also known to satisfy this criterion [see Lipton et al., 2004].

[^1]A more interesting relaxation from an algorithmic point of view, comes from the notion of maximin share guarantees, recently proposed by Budish [2011]. For two players, the maximin share of a player $i$ is the value that he could achieve by being the cutter in a discretized form of the cut and choose protocol. This is a guarantee for player $i$, if he would partition the items into two bundles so as to maximize the value of the least valued bundle. We define below the approximate version of this notion. Recall that $\mathscr{M}$ is the set of all allocations of $M$.

Definition 2.4. Given a set of items [m], the maximin share of a player $i \in\{1,2\}$, is

$$
\boldsymbol{\mu}_{i}=\max _{S \in \mathscr{M}} \min \left\{v_{i}\left(S_{1}\right), v_{i}\left(S_{2}\right)\right\} .
$$

For $\rho \leq 1$, an allocation $S=\left(S_{1}, S_{2}\right)$ is called a $\rho$-approximate maximin share allocation if $v_{i}\left(S_{i}\right) \geq$ $\rho \cdot \boldsymbol{\mu}_{i}$, for $i \in\{1,2\}$.

For two players maximin share allocations always exist and even though they are NP-hard to compute, we have a PTAS by reducing this to standard job scheduling problems. Hence each player can receive a value of at least $(1-\epsilon) \boldsymbol{\mu}_{i}$. For a higher number of players, constant factor approximation algorithms also exist [see Procaccia and Wang, 2014, Amanatidis et al., 2015].

Finally, a related approach was undertaken by Hill [1987]. This work examined what is the worst case guarantee that a player can have as a function of the total number of players and the maximum value of an item across all players. For two players, the following function was identified precisely as the guarantee that can be given to each player. Note that the total value of the items is normalized to 1 in this case.

Definition 2.5. Let $V_{2}:[0,1] \rightarrow[0,1 / 2]$ be the unique nonincreasing function satisfying $V_{2}(\alpha)=$ $1 / 2$ for $\alpha=0$, whereas for $\alpha>0$ :

$$
V_{2}(\alpha)= \begin{cases}1-k \alpha & \text { if } \alpha \in I(2, k) \\ 1-\frac{(k+1)}{2(k+1)-1} & \text { if } \alpha \in N I(2, k)\end{cases}
$$

where for any integer $k \geq 1, I(2, k)=\left[\frac{k+1}{k(2(k+1)-1)}, \frac{1}{2 k-1}\right]$ and $N I(2, k)=\left(\frac{1}{2(k+1)-1}, \frac{k+1}{k(2(k+1)-1)}\right)$.
Markakis and Psomas [2011] proved that for two players, there always exists an allocation such that each player $i$ receives at least $V_{2}\left(\alpha_{i}\right)$, where $\alpha_{i}=\max _{j \in[m]} \nu_{i j}$. The approximation version of this notion would be to construct allocations where each player receives a value of at least $\rho V_{2}\left(\alpha_{i}\right)$. Recently, a stricter variant of this guarantee has been provided by Gourvès et al. [2015] (also see Remark 4.9).

## 3 Characterization of Truthful Mechanisms

We present our main characterization result in this section. We start in subsection 3.1] with the main definitions and illustrating examples, and then we state our result in subsection 3.2 along with a road map of the proof. To avoid repetition, when referring to a truthful mechanism $\mathscr{X}$, we mean a truthful mechanism for allocating all the items in $M$ to two players with additive valuation functions.

### 3.1 A Non-Dictatorial Class of Mechanisms

The main result of this section is that every truthful mechanism is a picking-exchange mechanism (Theorem 3.6). Before we make a precise statement, we formally define the types of mechanisms involved and provide illustrating examples.

Picking Mechanisms. We start with a family of mechanisms where players make a selection out of choices that the mechanism offers to them. Given a subset $S$ of items, we define a set of offers $\mathscr{O}$ on $S$, as a nonempty collection of proper subsets of $S$ that exactly covers $S$ (i.e., $\cup_{T \in \mathscr{O}} T=S$ ), and in which there is no common element that appears in all subsets (i.e., $\cap_{T \in \mathscr{O}} T=\varnothing$ ).
Definition 3.1. A mechanism $\mathscr{X}$ is a picking mechanism ${ }^{2}$ if there exists a partition $\left(N_{1}, N_{2}\right)$ of $M$, and sets of offers $\mathscr{O}_{1}$ and $\mathscr{O}_{2}$ on $N_{1}$ and $N_{2}$ respectively, such that for every profile $\mathbf{v}$,

$$
X_{i}(\mathbf{v}) \cap N_{i} \in \underset{S \in \mathscr{O}_{i}}{\arg \max } v_{i}(S)
$$

Technical nuances aside, such a mechanism can be implemented by first letting player 1 choose his best offer from $\mathscr{O}_{1}$ and giving what remains from $N_{1}$ to player 2. Then it lets player 2 choose his best offer from $\mathscr{O}_{2}$ and gives what remains from $N_{2}$ to player 1. The following example illustrates a picking mechanism.
Example 1. Consider the following mechanism $\mathscr{X}$ on a set $M=\{1, \ldots 6\}$, which first partitions $M$ into $N_{1}=\{1,2,3,4\}, N_{2}=\{5,6\}$ and then constructs the offer sets $\mathscr{O}_{1}=\{\{1,2\},\{2,3\},\{4\}\}, \mathscr{O}_{2}=$ $\{\{5\},\{6\}\}$. On input $\mathbf{v}, \mathscr{X}$ first gives to player 1 his best set-with respect to $v_{1}$-among $\{1,2\},\{2,3\}$ and $\{4\}$, and then gives what remains from $N_{1}$ to player 2 . Next, $\mathscr{X}$ gives to player 2 his best setaccording to $v_{2}$-among $\{5\}$ and $\{6\}$, and then gives what remains from $N_{2}$ to player 1. $\mathscr{X}$ resolves ties lexicographically, e.g., in case of a tie, $\{1,2\}$ is preferred to $\{4\}$.

It is not hard to see that $\mathscr{X}$ is truthful. For the following input $v$, the circles denote the allocation.

$$
v=\left(\begin{array}{cccccc}
3 & 5 & 5 & 10 & 4 & 2 \\
2 & 3 & 6 & 1 & 5 & 3
\end{array}\right) .
$$

Exchange Mechanisms. We now move to a quite different class of mechanisms. Let $X, Y$ be two disjoint subsets of $M$. We call the ordered pair ( $X, Y$ ) an exchange deal. Moreover, we say that an exchange deal $(X, Y)$ is favorable with respect to $\mathbf{v}$ if $v_{1}(Y)>v_{1}(X)$ and $v_{2}(Y)<v_{2}(X)$, while it is unfavorable with respect to $\mathbf{v}$ if $\nu_{1}(Y)<v_{1}(X)$ or $\nu_{2}(Y)>v_{2}(X)$. Let $S$ and $T$ be two disjoint subsets of items and let $S_{1}, S_{2}, \ldots, S_{k}$ and $T_{1}, \ldots, T_{k}$ be two collections of nonempty and pairwise disjoint subsets of $S$ and $T$ respectively. We say then that the set of exchange deals $D=\left\{\left(S_{1}, T_{1}\right),\left(S_{2}, T_{2}\right), \ldots,\left(S_{k}, T_{k}\right)\right\}$ on $(S, T)$ is valid.
Definition 3.2. A mechanism $\mathscr{X}$ is an exchange mechanism ${ }^{3}$ if there exists a partition $\left(E_{1}, E_{2}\right)$ of $M$, and a valid set of exchange deals $D=\left\{\left(S_{1}, T_{1}\right), \ldots,\left(S_{k}, T_{k}\right)\right\}$ on ( $E_{1}, E_{2}$ ), such that for every profile $\mathbf{v}$, there exists a set of indices $I=I(\mathbf{v}) \subseteq[k]$ for which

$$
X_{1}(\mathbf{v})=\left(E_{1} \backslash \bigcup_{i \in I} S_{i}\right) \cup \bigcup_{i \in I} T_{i}, \quad X_{2}(\mathbf{v})=M \backslash X_{1} .
$$

[^2]Moreover, $I$ contains the indices of every favorable exchange deal with respect to $\mathbf{v}$, but no indices of unfavorable exchange deals.

On a high level, an exchange mechanism initially partitions the items into endowments for the players, and then examines a list of possible exchange deals. Every exchange that improves both players is performed, while every exchange that reduces the value of even one player is avoided. The mechanism may also perform other exchanges where one player is indifferent and the other player can be either indifferent or improved. Whether such exchange deals are materialized or not is up to the tie-breaking rule employed by the mechanism. The following example illustrates an exchange mechanism.
Example 2. Let $M=\{1, \ldots 5\}$, and consider the following mechanism $\mathscr{Y}$, with $E_{1}=\{1,2,3\}, E_{2}=$ $\{4,5\}$, and a valid set of exchange deals $D=\{(\{2,3\},\{4\})\}$ on ( $E_{1}, E_{2}$ ): One can think of such a mechanism as if $\mathscr{Y}$ initially reserves the set $E_{1}$ for player 1 and the set $E_{2}$ for player 2 . Then it examines whether exchanging $\{2,3\}$ with $\{4\}$ strictly improves both players, and performs the exchange only if the answer is yes. Mechanism $\mathscr{Y}$ is an example of an exchange mechanism with only one possible exchange deal. Again, one can see that no player has an incentive to lie.

For the following input $v$, the circles denote the allocation produced.

$$
v=\left(\begin{array}{ccccc}
6 & 2 & 3 & 7 & 1 \\
1 & 6 & 1 & 4 & 7
\end{array}\right) .
$$

Picking-Exchange Mechanisms Finally, we define the class of picking-exchange mechanisms which is a generalization of both picking and exchange mechanisms.

Definition 3.3. A mechanism $\mathscr{X}$ is a picking-exchange mechanism if there exists a partition ( $N_{1}$, $N_{2}, E_{1}, E_{2}$ ) of $M$, sets of offers $\mathscr{O}_{1}$ and $\mathscr{O}_{2}$ on $N_{1}$ and $N_{2}$ respectively, and a valid set of exchange deals $D=\left\{\left(S_{1}, T_{1}\right), \ldots,\left(S_{k}, T_{k}\right)\right\}$ on $\left(E_{1}, E_{2}\right)$, such that for every profile $\mathbf{v}$,

$$
X_{i}(\mathbf{v}) \cap N_{i} \in \underset{S \in \mathscr{G}_{i}}{\operatorname{argmax}} v_{i}(S) \quad \text { and } \quad X_{1}(\mathbf{v}) \cap\left(E_{1} \cup E_{2}\right)=\left(E_{1} \backslash \bigcup_{i \in I} S_{i}\right) \cup \bigcup_{i \in I} T_{i}
$$

where $I=I(\mathbf{v}) \subseteq[k]$ contains the indices of all favorable exchange deals, but no indices of unfavorable exchange deals.

It is helpful to think that a picking-exchange mechanism runs independently a picking mechanism on $N_{1} \cup N_{2}$ and an exchange mechanism on $E_{1} \cup E_{2}$, like in Example3. Although this is true under the assumption that the players' valuation functions are such that no two sets have the same value, it is not true for general additive valuations. The reason is that the tie-breaking for choosing the offers from $\mathscr{O}_{1}$ and $\mathscr{O}_{2}$ may not be independent from the decision of whether to perform each exchange that is neither favorable nor unfavorable.

The following example illustrates a picking exchange mechanism.
Example 3. Let $M=\{1, \ldots, 11\}$, and consider the mechanism $\mathcal{Z}$ that partitions $M$ into $N_{1}=\{1,2$, $3,4\}, N_{2}=\{5,6\}, E_{1}=\{7,8,9\}$ and $E_{2}=\{10,11\}$, and is the combination of $\mathscr{X}$ and $\mathscr{Y}$ from the previous two examples: On input $\mathbf{v}, \mathcal{Z}$ runs $\mathscr{X}$ on $N_{1} \cup N_{2}$ and $\mathscr{Y}$ on $E_{1} \cup E_{2}$ (where $\mathscr{Y}$ should of course be adjusted to run on $\{7, \ldots, 11\}$ instead of $\{1, \ldots, 5\}$ ). It outputs the union of the outputs of $\mathscr{X}$ and $\mathscr{Y}$.

For the following input $v$, the circles denote the final allocation.

$$
v=\left(\begin{array}{ccccccccccc}
3 & 5 & 5 & 10 & 4 & (2) & 6 & 2 & 3 & 7 & 1 \\
(2) & 3 & 6 & 1 & 5 & 3 & 1 & 6 & 1 & 4 & 7
\end{array}\right) .
$$

### 3.2 Truthfulness and Picking-Exchange Mechanisms

Essentially, we show that a mechanism is truthful if and only if it is a picking-exchange mechanism. We begin with the easier part of our characterization, namely that under the assumption that each valuation function induces a strict preference relation over all possible subsets, every picking-exchange mechanism is truthful. Recall that the set of such profiles is denoted by $V_{m}^{\neq}$.

Theorem 3.4. When restricted to $\mathbb{V}_{m}^{\neq}$, every picking-exchange mechanism $\mathscr{X}$ for allocating $m$ items is truthful.

Remark 3.5. For simplicity, Theorem 3.4 is stated for a subclass of additive valuation functions. However, it holds for general additive valuations as long as the mechanism uses a sensible tiebreaking rule (e.g., label-based or welfare-based, see Remark A.1 in Appendix A). The proof is very similar $4^{4}$

We are now ready to state the main result of this work.
Theorem 3.6. Every truthful mechanism $\mathscr{X}$ can be implemented as a picking-exchange mechanism.

The rest of this subsection is a road map to the proof of Theorem 3.6. The proof is long and technical, so for the sake of presentation, it is broken down to several lemmata. In order to illustrate the high-level ideas, the proofs of those lemmata are deferred to the Appendix A.

For the rest of this subsection we assume a truthful mechanism $\mathscr{X}$ for allocating all the items in $M=[m]$ to two players with additive valuation functions. Every statement is going to be with respect to this $\mathscr{X}$.

### 3.2.1 The Crucial Notion of Control

We begin by introducing the notions of strong desire and of control, which are of key importance for our characterization. We say that player $i$ strongly desires a set $S$ if each item in $S$ has more value for him than all the items of $M \backslash S$ combined, i.e., if for every $x \in S$ we have $v_{i x}>\sum_{y \in M S S} v_{i y}$.

Definition 3.7. We say that player $i$ controls a set $S$ with respect to $\mathscr{X}$, if every time he strongly desires $S$ he gets it whole, i.e., for every $\mathbf{v}=\left(v_{1}, v_{2}\right)$ in which player $i$ strongly desires $S$, then we have that $S \subseteq X_{i}(\mathbf{v})$.

Clearly, given $\mathscr{X}$, any set $S$ can be controlled by at most one player.
The following is a key lemma for understanding how truthful mechanisms operate. The lemma together with Corollary 3.9 below show that every item is controlled by some player under any truthful mechanism.

Lemma 3.8 (Control Lemma). Let $S \subseteq M$. If there exists a profile $\mathbf{v}=\left(\nu_{1}, v_{2}\right)$ such that both players strongly desire $S$, and $S \subseteq X_{i}(\mathbf{v})$ for some $i \in\{1,2\}$, then player $i$ controls every $T \subseteq S$ with respect to $\mathscr{X}$.

[^3]Proof. Let $\mathbf{v}=\left(\nu_{1}, v_{2}\right)$ be a profile such that both players strongly desire $S$ and $S \subseteq X_{1}(\mathbf{v})$ (the case where $S \subseteq X_{2}(\mathbf{v})$ is symmetric). We first prove the statement for $T=S$. Let $\mathbf{v}^{\prime}=\left(v_{1}^{\prime}, v_{2}^{\prime}\right)$ be any profile in which player 1 strongly desires $S$, i.e., $v_{1 x}^{\prime}>\sum_{y \in M S} v_{1 y}^{\prime}, \forall x \in S$. Initially, consider the intermediate profile $\mathbf{v}^{*}=\left(\nu_{1}, v_{2}^{\prime}\right)$. If $S \cap X_{2}\left(\mathbf{v}^{*}\right) \neq \varnothing$ then player 2 would deviate from profile $\mathbf{v}$ to $\mathbf{v}^{*}$ in order to strictly improve his total utility. So by truthfulness we derive that $S \subseteq X_{1}\left(\mathbf{v}^{*}\right)$. Similarly, in the profile $\mathbf{v}^{\prime}$, if $S \cap X_{2}\left(\mathbf{v}^{\prime}\right) \neq \varnothing$ then player 1 would deviate from $\mathbf{v}^{\prime}$ to $\mathbf{v}^{*}$ in order to strictly improve. Thus by truthfulness we have $S \subseteq X_{1}\left(\mathbf{v}^{\prime}\right)$. We conclude that player 1 controls $S$.

Now, suppose that $\mathbf{v}^{\prime \prime}=\left(v_{1}^{\prime \prime}, v_{2}^{\prime \prime}\right)$ is any profile in which player 1 strongly desires $T \subsetneq S$. If $T \nsubseteq$ $X_{1}\left(\mathbf{v}^{\prime \prime}\right)$ then player 1 could strictly improve his utility by playing $v_{1}^{\prime}$ from before (i.e., he declares that he strongly desires $S$ ) and getting $S \supsetneq T$. Thus, by truthfulness, $T \subseteq X_{1}\left(\mathbf{v}^{\prime \prime}\right)$, and we conclude that player 1 controls $T$.

Notice here that the existence of sets that are controlled by some player is always guaranteed. Specifically, each singleton $\{x\}$ is always controlled (only) by one of the players. Indeed, when both players strongly desire $\{x\}$, it is always the case that $\{x\} \subseteq X_{i}(\mathbf{v})$ for some $i \in\{1,2\}$. This is summarized in the following corollary.

Corollary 3.9. Let $\mathscr{X}$ be a truthful mechanism for allocating the items in $M$ to two players with additive valuations. For every $x \in M$ there exists $i \in\{1,2\}$ such that only player $i$ controls $\{x\}$ with respect to $\mathscr{X}$.

Aside from its use in the current proof, the corollary has implications on fairness, that will be explored in Section 4 ,

### 3.2.2 Identifying the Components of a Mechanism

Our goal now is to determine the "exchange component" and the "picking component" of mechanism $\mathscr{X}$. Every picking-exchange mechanism is completely determined by the seven sets $N_{1}$, $N_{2}, \mathscr{O}_{1}, \mathscr{O}_{2}, E_{1}, E_{2}$, and $D$ mentioned in Definition 3.3 (plus a deterministic tie-breaking rule). Below we try to identify these sets. Later we show that the mechanism's behavior is identical to that of a picking-exchange mechanism defined by them.

To proceed, we will need to consider the collection of all maximal sets controlled by each player. For $i \in\{1,2\}$, let

$$
\mathscr{A}_{i}=\{S \subseteq M \mid \text { player } i \text { controls } S \text { and for any } T \supsetneq S, i \text { does not control } T\} .
$$

Clearly, every set controlled by player $i$ is a subset of an element of $\mathscr{A}_{i}$. According to Lemma3.8, if we consider the set $C_{i}=\bigcup_{S \in \mathscr{A} A_{i}} S$, i.e., the union of all the sets in $\mathscr{A}_{i}$, this is exactly the set of items that are controlled-as singletons-by player $i$.

Corollary 3.10. The sets $C_{1}$ and $C_{2}$ define a partition of $M$.
Using the $\mathscr{A}_{i} \mathrm{~s}$ and the $C_{i} \mathrm{~s}$, we define the sets of interest that determine the mechanism. We begin with $E_{i}=\bigcap_{S \in \mathscr{A}_{i}} S$ for $i \in\{1,2\}$. As we are going to see eventually in Lemma3.18, the "exchange component" of $\mathscr{X}$ is observed on $E_{1} \cup E_{2}$.

Defining the corresponding valid set of exchange deals $D$ is trickier, and we need some terminology. Recall that $X_{i}^{S}(\mathbf{v})=X_{i}(\mathbf{v}) \cap S$. For $S \subseteq E_{1}$ and $T \subseteq E_{2}$, we say that $(S, T)$ is a feasible exchange, if there exists a profile $\mathbf{v}$, such that $X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})=\left(E_{1} \backslash S\right) \cup T$. In such a case, each of $S$ and $T$ is called exchangeable. An exchangeable set $S$ is called minimally exchangeable if any $S^{\prime} \subsetneq S$
is not exchangeable. Finally, a feasible exchange $(S, T)$ is a minimal feasible exchange, if at least one of $S$ and $T$ is minimally exchangeable. Now let

$$
D=\{(S, T) \mid(S, T) \text { is a minimal feasible exchange with respect to } \mathscr{X}\} .
$$

Of course, at this point it is not clear whether $D$ is well defined as a valid set of exchange deals, and this is probably the most challenging part of the characterization.

Next, we define $N_{i}=C_{i} \backslash E_{i}$ and $\mathscr{O}_{i}=\left\{S \backslash E_{i} \mid S \in \mathscr{A}_{i}\right\}$ for $i \in\{1,2\}$. As shown in Lemmata 3.11 and 3.12 we identify the "picking component" of $\mathscr{X}$ on $N_{1} \cup N_{2}$, and $\mathscr{O}_{i}$ will correspond to the set of offers.

Note that by Corollary 3.10 and the above definitions, $\left(N_{1}, N_{2}, E_{1}, E_{2}\right)$ is a partition of $M$. The intuition behind breaking $C_{i}$ into $N_{i}$ and $E_{i}$ is that player $i$ has different levels of control on those two sets. The fact that $E_{i}$ is contained in every maximal set controlled by player $i$ will turn out to mean that $\mathscr{X}$ gives the ownership of $E_{i}$ to player $i$. On the other hand, the control of player $i$ on $N_{i}$ is much more restricted as shown below.

### 3.2.3 Cracking the Picking Component

The first step is to show that the $\mathscr{O}_{i}$ s defined above, greatly restrict the possible allocations of the items of $N_{1} \cup N_{2}$. In particular, whatever player $i$ receives from $N_{i}$ must be contained in some set of $\mathscr{O}_{i}$.
Lemma 3.11. For every profile $\mathbf{v}$ and every $i \in\{1,2\}$, there exists $S \in \mathscr{O}_{i}$ such that $X_{i}^{N_{i}}(\mathbf{v}) \subseteq S$.
The idea behind the proof of Lemma 3.11 is that by receiving some $X_{i}^{N_{i}}(\mathbf{v})$ not contained in any set of $\mathscr{O}_{i}$, player $i$ is able to extend his control to subsets not contained in $C_{i}$, thus leading to contradiction. The proof, as many of the proofs of the remaining lemmata, includes the careful construction of a series of profiles, where in each step one has to argue about how the allocation does or does not change.

Given the restriction implied by Lemma 3.11, next we can prove that the subset of $N_{i}$ that player $i$ receives must be the best possible from his perspective, hence the mechanism behaves as a picking mechanism on each $N_{i}$. Intuitively, suppose that player 1 receives a subset $S$ of $N_{1}$ which is not an element of $\mathscr{O}_{1}$. By Lemma 3.11, $S$ is contained in an element $S^{\prime}$ of $\mathscr{O}_{1}$. Since player 1 controls $S^{\prime}$, this means that he gave up part of his control to gain something that he was not supposed to. Actually, it can be shown that it is the case where player 2 also gave part of his control (either on $N_{2}$ or $E_{2}$ ). This mutual transfer of control, combined with truthfulness, eventually leads to profiles where some of the items must be given to both players at the same time, hence a contradiction.
Lemma 3.12. For every profile $\mathbf{v}$ and every $i \in\{1,2\}$ we have $X_{i}^{N_{i}}(\mathbf{v}) \in \operatorname{argmax}_{S \in \mathscr{O}_{i}} \nu_{i}(S)$.
Now we know that $\mathscr{X}$ behaves as the "right" picking-exchange mechanism on $N_{1} \cup N_{2}$. For most of the rest of the proof we would like to somehow ignore this part of $\mathscr{X}$ and focus on $E_{1} \cup E_{2}$.

### 3.2.4 Separating the Two Components

As mentioned right after Definition 3.3, there is some kind of independence between the two components of a picking-exchange mechanism, at least when restricted on $\mathcal{I}_{m}^{\neq}$. This independence should be present in $\mathscr{X}$ as well; in fact we are going to exploit it to get rid of $N_{1} \cup N_{2}$ until the last part of the proof.

Lemma 3.13. Let $\mathbf{v}=\left(\nu_{1}, v_{2}\right), \mathbf{v}^{\prime}=\left(v_{1}^{\prime}, v_{2}^{\prime}\right) \in \mathscr{V}_{m}^{\neq}$such that $v_{i j}=v_{i j}^{\prime}$ for all $i \in\{1,2\}$ and $j \in E_{1} \cup E_{2}$. Then $X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})=X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right)$.

The lemma states that assuming strict preferences over all subsets, the allocation of $E_{1} \cup E_{2}$ does not depend on the values of either player for the items in $N_{1} \cup N_{2}$. What allows this separation is the complete lack of ties in the restricted profile space.

Without loss of generality we may assume that $E_{1} \cup E_{2}=[\ell]$. We can define a mechanism $\mathscr{X}_{E}$ for allocating the items of $[\ell]$ to two players with valuation profiles in $V_{\ell}^{\neq}$as

$$
\mathscr{X}_{E}(\mathbf{v})=\left(X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right), X_{2}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right)\right), \text { for every } \mathbf{v} \in \mathcal{V}_{\ell}^{\neq},
$$

where $\mathbf{v}^{\prime}$ is any profile in $V_{m}^{\neq}$with $v_{i j}=v_{i j}^{\prime}$ for all $i \in\{1,2\}$ and $j \in[\ell]$. This new mechanism is just the projection of $\mathscr{X}$ on $E_{1} \cup E_{2}$ restricted on a domain where it is well-defined. The truthfulness of $\mathscr{X}_{E}$ on $\mathscr{V}_{\ell}^{\neq}$follows directly from the truthfulness of $\mathscr{X}$ on $\mathscr{V}_{m}^{\neq}$. Moreover, it is easy to see that player $i$ controls $E_{i}$ with respect to $\mathscr{X}_{E}$, for $i \in\{1,2\}$.

The plan is to study $\mathscr{X}_{E}$ instead of $\mathscr{X}$, show that $\mathscr{X}_{E}$ is an exchange mechanism, and finally sew the two parts of $\mathscr{X}$ back together and show that everything works properly for any profile in $\mathcal{V}_{m}$. One issue here is that maybe the set of feasible exchanges with respect to $\mathscr{X}_{E}$ is greatly reduced, in comparison to the set of feasible exchanges with respect to $\mathscr{X}$, because of the restriction on the domain. In such a case, it will not be possible to argue about exchanges in $D$ that are not feasible anymore. It turns out that this is not the case; the set of possible allocations (of $E_{1} \cup E_{2}$ ) is the same, whether we consider profiles in $V_{m}$ or in $V_{m}^{\neq}$.
Lemma 3.14. For every profile $\mathbf{v} \in \mathcal{I}_{m}$ there exists a profile $\mathbf{v}^{\prime} \in \mathcal{V}_{m}^{\neq}$such that $\mathscr{X}(\mathbf{v})=\mathscr{X}\left(\mathbf{v}^{\prime}\right)$.
In particular, the set of feasible exchanges on $E_{1} \cup E_{2}$ is exactly the same for $\mathscr{X}$ and $\mathscr{X}_{E}$, and thus we will utilize the following set of exchanges.

$$
D=\left\{(S, T) \mid(S, T) \text { is a minimal feasible exchange with respect to } \mathscr{X}_{E}\right\} .
$$

### 3.2.5 Cracking the Exchange Component

In the attempt to show that $\mathscr{X}_{E}$ is an exchange mechanism, the first step is to show that $D$ is indeed a valid set of exchange deals.

Lemma 3.15. $D$ is a valid set of exchange deals on $\left(E_{1}, E_{2}\right)$.
The above lemma involves three main steps. First we show that each minimally exchangeable set is involved in exactly one exchange deal. Then, we guarantee that minimally exchangeable sets can be exchanged only with minimally exchangeable sets, and finally, we show that minimally exchangeable sets are always disjoint. There is a common underlying idea in the proofs of these steps: whenever there exist two feasible exchanges that overlap in any way, we can construct a profile where both of them are favorable but the two players disagree on which of them is best. On a high level, each player can "block" his least favorable of the conflicting exchanges, and this leads to violation of truthfulness.

Lemma3.15implies that every exchangeable set $S \subseteq E_{1}$ can be decomposed as $S=W \cup \cup_{i \in I} S_{i}$, where $W=S \backslash \bigcup_{i \in I} S_{i}$ does not contain any minimally exchangeable sets. Ideally, we would like two things. First, the set $W$ in the above decomposition to always be empty, i.e., every exchangeable set should be a union of minimally exchangeable sets. Second, we want every union of
minimally exchangeable subsets of $E_{1}$ to be exchangeable only with the corresponding union of minimally exchangeable subsets of $E_{2}$, and vice versa. It takes several lemmas and a rather involved induction to prove those. A key ingredient of the inductive step is a carefully constructed argument about the value that each player must gain from any exchange (see also Lemma. 14 in Appendix (A).
Lemma 3.16. For every exchangeable set $S \subseteq E_{1}$, there exists some $I \subseteq[k]$ such that $S=\bigcup_{i \in I} S_{i}$. Moreover, $S$ is exchangeable with $T=\bigcup_{i \in I} T_{i}$ and only with $T$.

Finally, we have all the ingredients to fully describe $\mathscr{X}_{E}$ as an exchange mechanism on $E_{1} \cup E_{2}$ and set of exchange deals $D$.
Lemma 3.17. Given any profile $\mathbf{v} \in \mathcal{V}_{\ell}^{\neq}$, each exchange in $D$ is performed if and only if it is favorable, i.e., $X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})=\left(E_{1} \backslash \bigcup_{i \in I} S_{i}\right) \cup \bigcup_{i \in I} T_{i}$, where $I \subseteq[k]$ contains exactly the indices of all favorable exchange deals in $D$.

### 3.2.6 Putting the Mechanism Back Together

As a result of Lemma 3.17 (combined, of course, with Lemmata 3.12 and 3.13), the characterization is complete for truthful mechanisms defined on $\mathcal{V}_{m}^{\neq}$. For general additive valuation functions, however, we need a little more work. This is to counterbalance the fact that in the presence of ties the allocations of $N_{1} \cup N_{2}$ and $E_{1} \cup E_{2}$ may not be independent.

By Lemmata 3.14 and 3.16, we know that for any $\mathbf{v} \in \mathcal{I}_{m}, X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})$ is the result of some exchanges of $D$ taking place. There are two things that can go wrong: $\mathscr{X}$ performs an unfavorable exchange, or it does not perform a favorable one. In either of these cases it is possible to construct some profile in $V_{m}^{\neq}$that leads to contradiction. Hence we have the following lemma.
Lemma 3.18. Given any profile $\mathbf{v} \in \mathcal{I}_{m}, X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})=\left(E_{1} \backslash \bigcup_{i \in I} S_{i}\right) \cup \bigcup_{i \in I} T_{i}$, where $I \subseteq[k]$ contains the indices of all favorable exchange deals in $D$, but no indices of unfavorable exchange deals.

Clearly, Lemma3.18, together with Lemma3.12 concludes the proof of Theorem 3.6,

### 3.3 Immediate Implications of Theorem 3.6

As mentioned in Section 1.2, there are several works characterizing truthful mechanisms in combination with other notions, such as Pareto efficiency, nonbossiness, and neutrality (these results are usually for unrestricted, not necessarily additive valuations). Pareto efficiency means that there is no other allocation where one player strictly improves and none of the others are worse-off. Non-bossiness means that a player cannot affect the outcome of the mechanism without changing his own bundle of items. Finally, neutrality refers to a mechanism being consistent with a permutation on the items, i.e., permuting the items results in the corresponding permuted allocation.

Although such notions are not our main focus, the purpose of this short discussion is twofold. On one hand, we illustrate how our characterization immediately implies a characterization for mechanisms that satisfy these extra properties under additive valuations, and on the other hand we see how these properties are either incompatible with fairness or irrelevant in our context.

To begin with, nonbossiness comes for free in our case, since we have two players and all the items must be allocated. Neutrality and Pareto efficiency, however, greatly reduce the space of available mechanisms. Note that it makes more sense to study neutral mechanisms when the valuation functions induce a strict preference order over all sets of items.

Corollary 3.19. Every neutral, truthful mechanism $\mathscr{X}$ on $\mathcal{I}_{m}^{\neq}$can be implemented as a pickingexchange mechanism, such that

1. there exists $i \in\{1,2\}$ such that $E_{i}=[m]$, or
2. there exists $i \in\{1,2\}$ such that $N_{i}=[m]$ and $\mathscr{O}_{i}=\{S \subseteq[m]| | S \mid=\kappa\}$ for some $\kappa<m$.

Corollary 3.20. Every Pareto efficient, truthful mechanism $\mathscr{X}$ can be implemented as a pickingexchange mechanism, such that

1. there exists $i \in\{1,2\}$ such that $E_{i}=[m]$, or
2. there exists $j \in[m]$ such that $E_{i_{1}}=\{j\}, E_{i_{2}}=[m] \backslash\{j\}$, where $\left\{i_{1}, i_{2}\right\}=\{1,2\}$, and $D=\left\{\left(E_{1}, E_{2}\right)\right\}$, or
3. there exists $i \in\{1,2\}$ such that $N_{i}=[m]$ and $\mathscr{O}_{i}=\left\{S \subseteq N_{i}| | S \mid=m-1\right\}$.

The proofs are deferred to the full version.
It is somewhat surprising that the resulting mechanisms are a strict superset of dictatorships, even when we impose both properties together. Pareto efficiency, however, allows only mechanisms that are rather close to being dictatorial, and thus cannot guarantee fairness of any type. On the other hand, most of the mechanisms defined and studied in Section 4 are neutral, yet neutrality is not implied by the fairness concepts we consider, nor the other way around.

## 4 A Necessary Fairness Condition and its Implications

In this section, we explore some implications of Theorem 3.6 on fairness properties, i.e., on the design of mechanisms where on top of truthfulness, we would like to achieve fairness guarantees. All the missing proofs from this section are in Appendix B,

In Section4.1]we show that the Control Lemma implies that truthfulness prevents any bounded approximation for envy-freeness and proportionality. Then, we move on describing a necessary fairness condition, in terms of our notion of "control", that summarizes a common feature of several relaxations of fairness and provide a restricted version of our characterization that follows this fairness condition. This will allow us, in Section 4.2, to examine what this new class of mechanisms can achieve in each of these fairness concepts.

### 4.1 Implications of the Control Lemma.

### 4.1.1 Control of singletons.

The basic restriction that truthfulness imposes to every mechanism (leading to poor results for some fairness concepts) comes from Corollary 3.9 , an immediate corollary of the Control Lemma, stating that every single item is controlled by some player.

We begin by studing how the above corollary affects two of the most researched notions in the fair division literature, namely proportionality and envy-freeness. It is well known that even without the requirement for truthfulness, it is impossible to achieve any of these two objectives,
simply because in the presence of indivisible goods, envy-free or proportional allocations may not exist ${ }^{5}$

This leads to the definition of approximation versions of these two concepts for settings with indivisible goods. Namely Lipton et al. [2004] considered the minimum envy problem and tried to construct algorithms such that for every instance, an approximation to the minimum possible envy admitted by the instance is guaranteed. Similarly Markakis and Psomas [2011] considered approximate proportionality, i.e., find allocations that achieve an approximation to the best possible value that an instance can guarantee to all agents. See also the discussion in Section 2 on defining the approximation versions of these problems. Note that if time complexity is not an issue, we can always identify the allocation with the best possible envy or with the best possible proportionality, achieveable by a given instance.

We are now ready to state our first application, showing that truthfulness prohibits us from having any approximation to the minimum envy or to proportionality. This greatly improves the conclusions of Lipton et al. [2004] and Caragiannis et al. [2009] that truthful mechanisms cannot attain the optimal minimum envy allocation.

Application 4.1. For any truthful mechanism that allocates all the items to two players with additive valuations, the approximation achieved for either proportionality or the minimum envy is arbitrarily bad (i.e., not lower bounded by any positive function of $m$ ).

Proof. Consider a setting with $m$ items, and a truthful mechanism $\mathscr{X}$. Suppose now that item 1 is controlled by player 1 with respect to $\mathscr{X}$. This means that in the profile $\mathbf{v}=\left(\left[\begin{array}{ll}m & 1\end{array} \ldots 1\right]\right.$, [ $m^{d} 11 \ldots 1$ ) player 1 must obtain item 1 , and player 2 ends up with a negligible fraction of his total value for large enough $d$. The optimal solution would be to assign the first item to the second player and the last $m$ items to the first player, which provides an envy-free and proportional allocation. We conclude that the approximation guarantee that can be obtained by a truthful mechanism is arbitrarily high.

So far, the conclusion is that even approximate proportionality or envy-freeness are quite stringent and incompatible with truthfulness because of the Control Lemma. The next step would be to relax these notions. There have been already a few approaches on relaxing proportionality and envy-freeness under indivisible goods, leading to solutions such as the maximin share fairness, envy-freeness up to one item [Budish, 2011], as well as the type of worst-case guarantees proposed by Hill [1987] (recall Definitions [2.3, 2.4 and [2.5 in Section[2). The fact that a truthful mechanism $\mathscr{X}$ yields control of singletons does not seem to have such detrimental effects on these notions. However, if even a single pair of items is controlled by a player, the same situation arises.

### 4.1.2 Control of pairs

We propose the following necessary (but not sufficient) condition that captures a common aspect of all these relaxations of fairness. This allows us to treat all the above concepts of fairness in a unified way.

[^4]Definition 4.2. We say that a mechanism $\mathscr{X}$ yields control of pairs if there exists $i \in\{1,2\}$ and $S \subseteq[m]$ with $|S|=2$, such that player $i$ controls $S$ with respect to $\mathscr{X}$.

The following lemma states that in order to obtain impossibility results for the above concepts, it is enough to focus on mechanisms with control of pairs.

Lemma 4.3. In order to achieve (either exactly or within a bounded approximation) the above mentioned relaxed fairness criteria, a truthful mechanism that allocates all the items to two players with additive valuations cannot yield control of pairs.

Proof. Assuming that $\{1,2\}$ is controlled by player 1 , in a setting with $m$ items, we may consider $\mathbf{v}^{\prime}=\left(\left[\begin{array}{lll}m & m & \ldots\end{array}\right],\left[m^{d} m^{d} 1 \ldots 1\right]\right)$, in analogy to profile $\mathbf{v}$ in the proof of Application4.1, Given that player 1 can always get both items 1 and 2 when he strongly desires them, it is easy to see that envy-freeness up to one item cannot be achieved, while by choosing large enough $d$ the approximation of $\boldsymbol{\mu}_{2}$ or $V_{2}\left(\alpha_{2}\right)$ can be arbitrarily bad.

So now we are ready to move to a complete characterization of truthful mechanisms that do not yield control of pairs. Of course such mechanisms are picking-exchange mechanisms, but our fairness condition allows only singleton offers, and the exchange part is completely degenerate.

Definition 4.4. A mechanism $\mathscr{X}$ for allocating all the items in $[m]$ to two players is a singleton picking-exchange mechanism if it is a picking-exchange mechanism where for each $i \in\{1,2\}$ at most one of $N_{i}$ and $E_{i}$ is nonempty, $\left|E_{i}\right| \leq 1$, and

$$
\mathscr{O}_{i}= \begin{cases}\left\{\{x\} \mid x \in N_{i}\right\} & \text { when } N_{i} \neq \varnothing \\ \{\varnothing\} & \text { otherwise }\end{cases}
$$

i.e., the sets of offers contain all possible singletons.

Hence, typically, in a singleton picking-exchange mechanism player $i$ receives from $N_{i} \cup E_{i}$ only his best item. Moreover, for $m \geq 3$, no exchanges are allowed 6

Lemma 4.5. Every truthful mechanism for allocating all the items to two players with additive valuation functions that does not yield control of pairs can be implemented as a singleton pickingexchange mechanism.

It is interesting to note that, in contrast to Application4.1, proving Lemma 4.5 without Theorem 3.6 is not straightforward. In fact, it requires a partial characterization which (on a high level) is similar to characterizing the picking component of general mechanisms.

### 4.2 Applications to Relaxed Notions of Fairness

It is now possible to apply Lemma4.5]on each fairness notion separately, and characterize every truthful mechanism that achieves each criterion.

[^5]Envy-freeness up to one item. We start with a relaxation of envy-freeness. Below we provide a complete description of the mechanisms that satisfy this criterion.

Application 4.6. For $m \leq 3$, every singleton picking-exchange mechanism achieves envy-freeness up to one item. For $m=4$ every singleton picking-exchange mechanism with $\left|N_{1}\right|=\left|N_{2}\right|=2$ achieves envy-freeness up to one item. Finally, for $m \geq 5$ there is no truthful mechanism that allocates all the items to two players and achieves envy-freeness up to one item.

Maximin share fairness and related notions. For maximin share allocations a truthful mechanism was suggested by Amanatidis et al. [2016] for any number of items and any number of players. For two players, their mechanism is the singleton picking-exchange mechanism with $N_{1}=[m]$ and produces an allocation that guarantees to each player a $\frac{1}{[m / 2]}$-approximation of his maximin share. It was left as an open problem whether a better truthful approximation exists. Here we show that this approximation is tight; in fact, almost any other singleton pickingexchange mechanism performs strictly worse. Note that the best previously known lower bound for two players was $1 / 2$.

Application 4.7. For any $m$ there exists a singleton picking-exchange mechanism that guarantees to player $i$ a $\lfloor\max \{2, m\} / 2\rfloor^{-1}$-approximation of $\boldsymbol{\mu}_{i}$, for $i \in\{1,2\}$. There is no truthful mechanism that allocates all the items to two players and achieves a better guarantee with respect to maximin share fairness.

Regarding now allocations that guarantee an approximation of the function $V_{2}\left(\alpha_{i}\right)$ defined by Hill [1987] (recall the definition in Section[2), the singleton picking-exchange mechanism with $N_{1}=[m]$ was also suggested by Markakis and Psomas [2011] as a $\frac{1}{[m / 2]}$-approximation of $\left.V_{2}\left(\alpha_{i}\right)\right]^{7}$ This comes as no surprise, since there exists a strong connection between maximin shares and the function $V_{n}$, especially for two players. This is illustrated in the following corollary, where both the positive and the negative results coincide with the ones for the maximin share fairness.

Application 4.8. For any $m$ there exists a singleton picking-exchange mechanism that guarantees to player i a $\lfloor\max \{2, m\} / 2\rfloor^{-1}$-approximation of $V_{2}\left(\alpha_{i}\right)$, for $i \in\{1,2\}$, where $\alpha_{i}=\max _{j \in[m]} v_{i j}$. There is no truthful mechanism that allocates all the items to two players and achieves a better guarantee with respect to the $V_{2}\left(\alpha_{i}\right)$ s.

Again, the best previously known lower bound for two players was constant, namely $2 / 3$ due to Markakis and Psomas [2011]. In Applications 4.7 and 4.8 , it is stated that there exists a $\frac{1}{[m / 2]}$ approximate singleton picking-exchange mechanism. It is interesting that any singleton pickingexchange mechanism does not perform much worse. Following the corresponding proofs, we have that even the worst singleton picking-exchange mechanism achieves a $\frac{1}{m-1}$-approximation in each case.

Remark 4.9. Gourvès et al. 2015 introduced a variant of $V_{n}$, called $W_{n}$, and showed that there always exists an allocation such that each player $i$ receives $W_{n}\left(\alpha_{i}\right) \geq V_{n}\left(\alpha_{i}\right)$ (where the inequality is often strict). Since the definition of $W_{n}$ is rather involved even for $n=2$, we defer a formal discussion about it to the full version of the paper. However, it is not hard to show that for every valuation function $v_{i}$ we have $V_{2}\left(\alpha_{i}\right) \leq W_{2}\left(\alpha_{i}\right) \leq \boldsymbol{\mu}_{i}$ and thus the analog of Application4.8 holds.

[^6]Remark 4.10. Amanatidis et al. [2016] made the following interesting observation: every single known truthful mechanism achieving a bounded approximation of maximin share fairness is ordinal, in the sense that it only needs a ranking of the items for each player rather than his whole valuation function. Finding truthful mechanisms that explicitly take into account the players' valuation functions in order to achieve better guarantees was posed as a major open problem. Note that, weird tie-breaking aside, all singleton picking-exchange mechanisms are ordinal! Therefore, from the mechanism designer's perspective, it is impossible to exploit the extra cardinal information given as input and at the same time maintain truthfulness and some nontrivial fairness guarantee.

## 5 Truthful Mechanisms for Many Players

We introduce a family of non-dictatorial, truthful mechanisms for any number of players. Our mechanisms are defined recursively; in analogy to serial dictatorships, the choices of a player define the sub-mechanism used to allocate the items to the remaining players. Here, however, this serial behavior is observed "in parallel" in several sets of a partition of $M$.

A generalized deal between $k$ players is a collection of (up to $k(k-1)$ ) exchange deals between pairs of players. A set $D$ of generalized deals is called valid if all the sets involved in all these exchange deals are nonempty and pairwise disjoint. Given a profile $\mathbf{v}=\left(\nu_{1}, v_{2}, \ldots, v_{n}\right)$ we say that a generalized deal is favorable if it strictly improves all the players involved, while it is unfavorable if there exists a player involved whose utility strictly decreases.

Definition 5.1. A mechanism $\mathscr{X}$ for allocating all the items in $[m$ ] to $n$ players is called a serial picking-exchange mechanism if

1. when $n=1, \mathscr{X}$ always allocates the whole $[m$ ] to player 1 .
2. when $n \geq 2$, there exist a partition $\left(N_{1}, \ldots, N_{n}, E_{1}, \ldots, E_{n}\right.$ ) of [ $m$ ], sets of offers $\mathscr{O}_{i}$ on $N_{i}$ for $i \in[n]$, a valid set $D$ of generalized deals, and a mapping $f$ from subsets of $M$ to serial picking-exchange mechanisms for $n-1$ players, such that for every profile $\mathbf{v}=\left(\nu_{1}, \ldots, v_{n}\right)$ we have for all $i \in[n]$ :

- $X_{i}^{N_{i}}(\mathbf{v}) \in \arg \max _{S \in \mathscr{O}_{i}} v_{i}(S)$,
- $X_{i}^{E}(\mathbf{v})$, where $E=\bigcup_{j \in[n]} E_{j}$, is the result of starting with $E_{i}$ and performing some of the deals in $D$, including all the favorable deals but no unfavorable ones,
- the items of $N_{i} \backslash X_{i}^{N_{i}}(\mathbf{v})$ are allocated to players in $[n] \backslash\{i\}$ using the serial pickingexchange mechanism $f\left(N_{i} \backslash X_{i}^{N_{i}}(\mathbf{v})\right)$.
Clearly, serial picking-exchange mechanisms are a generalization of picking-exchange mechanisms studied in Section3, The following example illustrates how such a mechanism looks like for three players.

Example 4. Suppose that we have three players with additive valuations. For simplicity, assume that each player's valuation induces a strict preference over all possible subsets of items. Let $M=[100]$ be the set of items, and consider the following relevant ingredients of our mechanism:

- $N_{1}=\{1,2, \ldots, 20\}, \mathscr{O}_{1}=\left\{\{1,2,3\}, N_{1} \backslash\{1\}\right\}$
- $N_{2}=\{21,22, \ldots, 50\}, \mathscr{O}_{2}=\left\{S \subseteq N_{2}| | S \mid=6\right\}$
- $N_{3}=\{51,52, \ldots, 70\}, \mathscr{O}_{3}=\{\{51, \ldots, 60\},\{61, \ldots, 70\}\}$
- $E_{1}=\{71, \ldots, 80\}, E_{2}=\{81, \ldots, 90\}, E_{3}=\{91, \ldots, 100\}$
- $D=\left\{\left[(\{75,79\},\{83\})^{1,3}\right],\left[(\{71\},\{88\})^{1,2},(\{72,80\},\{95\})^{1,3},(\{85\},\{99,100\})^{2,3}\right]\right\}$
- $f$ is a mapping from subsets of $M$ to picking-exchange mechanisms (for 2 players)

The above sets are the analog of the corresponding sets of a picking-exchange mechanism. The deals, however, are a bit more complex. E.g., by $\left[(\{71\},\{88\})^{1,2},(\{72,80\},\{95\})^{1,3},(\{85\},\{99,100\})^{2,3}\right]$ we denote the deal in which:

- player 1 gives item 71 to player 2 and items 72,80 to player 3
- player 2 gives item 88 to player 1 and item 85 to player 3
- player 3 gives item 95 to player 1 and items 99, 100 to player 2

The mapping $f$ suggests which truthful mechanism should be used every time there are items left to be allocated to only two players.

We are ready to describe our mechanism $\mathscr{X}$ :

1. The mechanism gives endowments $E_{1}, E_{2}, E_{3}$ to the three players and then performs each exchange deal that strictly improves all the players involved.
2. Then, for each $i \in\{1,2,3\}$, the mechanism gives to player $i$ his best set in $\mathscr{O}_{i}$, say $S_{i}$.
3. Finally, for each $i \in\{1,2,3\}, \mathscr{X}$ uses mechanism $f\left(N_{i} \backslash S_{i}\right)$ to allocate the items of $N_{i} \backslash S_{i}$ to players in $\{1,2,3\} \backslash i$.

Like picking-exchange mechanisms, serial picking-exchange mechanisms are truthful, given an appropriate tie-breaking rule (e.g., a label-based tie-breaking rule). To bypass a general discussion about tie-breaking, however, we may assume that each player's valuation induces a strict preference over all subsets of $M$. We denote by $\nabla_{n, m}^{\neq}$the set of profiles that only include such valuation functions. Following almost the same proof, however, we have that for general additive valuations every serial picking-exchange mechanism is truthful when using label-based tie-breaking.

Theorem 5.2. When restricted to $\mathbb{V}_{n, m}^{\neq}$, every serial picking-exchange mechanism $\mathscr{X}$ for allocating $m$ items to $n$ players is truthful.

The proof is similar in spirit with the proof of Theorem 3.4 and is deferred to the full version of the paper.

## 6 Discussion

We obtained a nontrivial characterization for truthful mechanisms, that has immediate implications on fairness. A natural question to ask is whether our characterization can be extended for more than two players. Characterizing the truthful mechanisms without money for any number of additive players is, undoubtedly, a fundamental open problem. However, as indicated by Definition 5.1, there seems to be a much richer structure when one attempts to describe such mechanisms, even though serial picking-exchange mechanisms are only a subset of nonbossy truthful mechanisms. In particular, the notion of control that was crucial for identifying the structure of truthful mechanisms for two players does not convey enough information anymore. Instead, there seem to exist several different levels of control, and understanding this structure still remains a very interesting and intriguing question.

## Aknowledgements

This work has been partly supported by the COST Action IC1205 on Computational Social Choice, and by an internal grant of the Athens University of Economics and Business. George Christodoulou was supported by EPSRC EP/M008118/1 and Royal Society LT140046. We also wish to acknowledge the Simons institute for hosting the program on Economics and Computation, as some ideas and preliminary discussions began there.

## References

G. Amanatidis, E. Markakis, A. Nikzad, and A. Saberi. Approximation algorithms for computing maximin share allocations. In Automata, Languages, and Programming, ICALP 2015, Proceedings, Part I, pages 39-51, 2015.
G. Amanatidis, G. Birmpas, and E. Markakis. On truthful mechanisms for maximin share allocations. In Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI 2016, pages 31-37, 2016.
H. Aziz, A. Filos-Ratsikas, J. Chen, S. Mackenzie, and N. Mattei. Egalitarianism of random assignment mechanisms: (extended abstract). In International Conference on Autonomous Agents \& Multiagent Systems, AAMAS 2016, pages 1267-1268, 2016.
I. Bezakova and V. Dani. Allocating indivisible goods. ACM SIGecom Exchanges, 5:11-18, 2005.
S. Bouveret and J. Lang. Manipulating picking sequences. In ECAI 2014-21st European Conference on Artificial Intelligence, pages 141-146, 2014.
S. Bouveret, Y. Chevaleyre, and N. Maudet. Fair allocation of indivisible goods. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A.D. Procaccia, editors, Handbook of Computational Social Choice, chapter 12. Cambridge University Press, 2016.
S. J. Brams and A. D. Taylor. Fair Division: from Cake Cutting to Dispute Resolution. Cambridge University press, 1996.
E. Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. Journal of Political Economy, 119(6):1061-1103, 2011.
I. Caragiannis, C. Kaklamanis, P. Kanellopoulos, and M. Kyropoulou. On low-envy truthful allocations. In First International Conference on Algorithmic Decision Theory, ADT 2009, pages 111-119, 2009.
Y. Chen, J. Lai, D. Parkes, and A. Procaccia. Truth, justice, and cake cutting. volume 77, pages 284-297, 2013.
G. Christodoulou and A. Kovács. A global characterization of envy-free truthful scheduling of two tasks. In Internet and Network Economics - 7th International Workshop, WINE 2011, Proceedings, pages 84-96, 2011.
G. Christodoulou, E. Koutsoupias, and A. Vidali. A characterization of 2-player mechanisms for scheduling. In Algorithms - ESA, 16th Annual European Symposium, pages 297-307, 2008.
R. Cole, V. Gkatzelis, and G. Goel. Mechanism design for fair division: allocating divisible items without payments. In ACM Conference on Electronic Commerce, EC '13, pages 251-268. ACM, 2013.
S. Dobzinski and M. Sundararajan. On characterizations of truthful mechanisms for combinatorial auctions and scheduling. In ACM Conference on Electronic Commerce (EC '08), 2008.
L. Ehlers and B. Klaus. Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems. Social Choice and Welfare, 21(2):265-280, 2003.
D. Foley. Resource allocation and the public sector. Yale Economics Essays, 7:45-98, 1967.
G. Gamow and M. Stern. Puzzle-Math. Viking press, 1958.
L. Gourvès, J. Monnot, and L. Tlilane. Worst case compromises in matroids with applications to the allocation of indivisible goods. Theoretical Computer Science, 589:121-140, 2015.
M. Guo and V. Conitzer. Strategy-proof allocation of multiple items between two agents without payments or priors. In 9th International Conference on Autonomous Agents \& Multiagent Systems AAMAS 2010, pages 881-888, 2010.
T. Hill. Partitioning general probability measures. The Annals of Probability, 15(2):804-813, 1987.
B. Klaus and E. Miyagawa. Strategy-proofness, solidarity and consistency for multiple assignment problems. International Journal of Game Theory, 30(3):421-435, 2002.
R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On approximately fair allocations of indivisible goods. In ACM Conference on Electronic Commerce (EC), pages 125-131, 2004.
E. Markakis and C.-A. Psomas. On worst-case allocations in the presence of indivisible goods. In 7th Workshop on Internet and Network Economics (WINE 2011), pages 278-289, 2011.
T. Mennle and S. Seuken. An axiomatic approach to characterizing and relaxing strategyproofness of one-sided matching mechanisms. In ACM Conference on Economics and Computation, $E C^{\prime} 14$, pages 37-38, 2014.
H. Moulin. Fair division and collective welfare. MIT Press, 2003. ISBN 978-0-262-63311-6.
S. Pápai. Strategyproof multiple assignment using quotas. Review of Economic Design, 5(1):91105, 2000.
S. Pápai. Strategyproof and nonbossy multiple assignments. Journal of Public Economic Theory, 3(3):257-271, 2001.
S. Pápai. Strategyproof exchange of indivisible goods. Journal of Mathematical Economics, 39(8): 931-959, 2003.
S. Pápai. Exchange in a general market with indivisible goods. Journal of Economic Theory, 132: 208-235, 2007.
A. D. Procaccia. Cake cutting algorithms. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A.D. Procaccia, editors, Handbook of Computational Social Choice, chapter 13. Cambridge University Press, 2016.
A. D. Procaccia and J. Wang. Fair enough: guaranteeing approximate maximin shares. In ACM Conference on Economics and Computation, EC '14, pages 675-692, 2014.
J. M. Robertson and W. A. Webb. Cake Cutting Algorithms: be fair if you can. AK Peters, 1998.
H. Steinhaus. The problem of fair division. Econometrica, 16:101-104, 1948.
L. Svensson. Strategy-proof allocation of indivisible goods. Social Choice and Welfare, 16:557567, 1999.
H. Varian. Equity, envy and efficiency. Journal of Economic Theory, 9:63-91, 1974.

## A Missing Material from Section 3

Proof of Theorem 3.4. Assume $\mathscr{X}$ is a picking-exchange mechanism with partition ( $N_{1}, N_{2}, E_{1}$, $\left.E_{2}\right)$, offer sets $\mathscr{O}_{i}$ on $N_{i}$, for $i \in\{1,2\}$, and set of exchange deals $D$. Let $\mathbf{v}=\left(\nu_{1}, v_{2}\right) \in \mathcal{V}_{m}^{\neq}$be a profile, and fix $v_{2}$. We are going to show that there is no $\mathbf{v}^{\prime}=\left(v_{1}^{\prime}, v_{2}\right) \in \mathcal{V}_{m}^{\neq}$such that $v_{1}\left(X_{1}\left(\mathbf{v}^{\prime}\right)\right)>v_{1}\left(X_{1}(\mathbf{v})\right)$.

For any $\mathbf{v}^{\prime}=\left(\nu_{1}^{\prime}, v_{2}\right) \in \mathscr{V}_{m}^{\neq}$there exist the following possibilities:
(a) $X_{1}\left(\mathbf{v}^{\prime}\right)=X_{1}(\mathbf{v})$. Then clearly $\nu_{1}\left(X_{1}\left(\mathbf{v}^{\prime}\right)\right)=v_{1}\left(X_{1}(\mathbf{v})\right)$.
(b) $X_{1}^{N_{1} \cup N_{2}}\left(\mathbf{v}^{\prime}\right) \neq X_{1}^{N_{1} \cup N_{2}}(\mathbf{v})$, but $X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right)=X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})$. Then it must be the case where $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \neq$ $X_{1}^{N_{1}}(\mathbf{v})$. Indeed, player 1 has no power over $N_{2}$ where the items that he is allocated depend only on the unique best offer to player 2, i.e., $X_{1}^{N_{2}}\left(\mathbf{v}^{\prime}\right)=X_{1}^{N_{2}}(\mathbf{v})$. But this can only mean $\nu_{1}\left(X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right)\right)<$ $\nu_{1}\left(X_{1}^{N_{1}}(\mathbf{v})\right)$ by the definition of a picking-exchange mechanism and the fact that there are no subsets of equal value. So in total, $\nu_{1}\left(X_{1}\left(\mathbf{v}^{\prime}\right)\right)<\nu_{1}\left(X_{1}(\mathbf{v})\right)$.
(c) $X_{1}^{N_{1} \cup N_{2}}\left(\mathbf{v}^{\prime}\right)=X_{1}^{N_{1} \cup N_{2}}(\mathbf{v})$, but $X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right) \neq X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})$. By the definition of picking-exchange mechanisms, player 1 can never force an exchange that is good for him but not for player 2. That is, by deviating he will lose one or more exchanges that were good for him, and/or force one or more exchanges that were bad for him. We conclude it is the case where $\nu_{1}\left(X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right)\right)<$ $\nu_{1}\left(X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})\right)$, and therefore, $\nu_{1}\left(X_{1}\left(\mathbf{v}^{\prime}\right)\right)<\nu_{1}\left(X_{1}(\mathbf{v})\right)$.
(d) $X_{1}^{N_{1} \cup N_{2}}\left(\mathbf{v}^{\prime}\right) \neq X_{1}^{N_{1} \cup N_{2}}(\mathbf{v})$, and $X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right) \neq X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})$. By the fact that we are restricted to $\mathbb{V}_{m}^{\neq}$, we can derive that the "picking part" on $N_{1} \cup N_{2}$ and the "exchange part" on $E_{1} \cup E_{2}$ are independent. So, by cases (b) and (c) above we have $\nu_{1}\left(X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right)\right)<\nu_{1}\left(X_{1}^{N_{1}}(\mathbf{v})\right)$ and $\nu_{1}\left(X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right)\right)<$ $v_{1}\left(X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})\right)$. Therefore, $v_{1}\left(X_{1}\left(\mathbf{v}^{\prime}\right)\right)<\nu_{1}\left(X_{1}(\mathbf{v})\right)$.

We conclude that every picking-exchange mechanism on $\mathcal{V}_{m}^{\neq}$is truthful.
Remark A.1. With only slight modifications of the above proof, we have that for general additive valuations every picking-exchange mechanism is truthful when using the following two interesting families of tie-breaking rules:
Tie-breaking with labels. Every set in $\mathscr{O}_{1} \cup \mathscr{O}_{2}$ has a distinct label, and whenever $\arg \max _{S \in \mathscr{O}_{i}} \nu_{i}(S)$ is not a singleton, player $i$ receives the set with the smallest label. Further, every deal in $D$ has a label with five possible values, each indicating one of the following: (i) the exchange takes place every time it is not unfavorable, (ii) it only takes place every time it is not unfavorable and at least one player is strictly improved, (iii) it only takes place every time it is not unfavorable and player 1 is strictly improved, (iv) it only takes place every time it is not unfavorable and player 2 is strictly improved, and (v) it only takes place every time it is favorable.
Welfare maximizing tie-breaking. When $\arg \max _{S \in \mathscr{O}_{i}} \nu_{i}(S)$ is not a singleton, player $i$ receives the set that leaves in $N_{i}$ as much value as possible for the other player. If there are still ties, labels are used to resolve those. Further, for every deal in $D$ the exchange takes place every time it is not unfavorable and at least one player is strictly improved.

Proof of Corollary 3.10, From the definition of the $C_{i}$ s and Corollary 3.9, $C_{1} \cup C_{2}=M$ follows. On the other hand, if $z \in C_{1} \cap C_{2}$, then there exist a set $A \in \mathscr{A}_{1}$, such that $z \in A$, and a set $B \in \mathscr{A}_{2}$ such that $z \in B$. By Lemma 3.8 , this implies that the singleton $\{z\}$ is controlled by both players, which is a contradiction. Thus, we have $C_{1} \cap C_{2}=\varnothing$.

Proof of Lemma 3.11. Due to symmetry, it suffices to prove the statement for $i=1$. If $N_{1}=\varnothing$ then the statement is trivially true. So assume $N_{1} \neq \varnothing$ and suppose that the statement does not hold. That is, there exists a profile $\mathbf{v}=\left(\nu_{1}, v_{2}\right)$ such that for any $S \in \mathscr{O}_{1}$ we have $X_{1}^{N_{1}}(\mathbf{v}) \nsubseteq S$. This means $X_{1}^{N_{1}}(\mathbf{v}) \neq \varnothing$. Since the sets in $\mathscr{O}_{1}$ cover $N_{1}$, there exists $S^{\prime}$ such that $S^{\prime} \cap X_{1}^{N_{1}}(\mathbf{v}) \neq \varnothing$. Let $Z$ be a maximum cardinality such intersection between some $S^{\prime} \in \mathscr{O}_{1}$ and $X_{1}^{N_{1}}(\mathbf{v})$, and $x$ be any element of $X_{1}^{N_{1}}(\mathbf{v}) \backslash Z$. Note that $x$ is guaranteed to exist since $X_{1}^{N_{1}}(\mathbf{v})$ is not contained in any set of $\mathscr{O}_{1}$. Also, there is no $S^{\prime \prime} \in \mathscr{O}_{1}$ such that $Z \cup\{x\} \subseteq S^{\prime \prime}$ due to the maximality of $Z$.

The generic values that may appear in $\mathbf{v}$ restrict our ability to argue about the allocation, so our first goal is to reach a profile $\mathbf{u}$ that contradicts the lemma's statement, like $\mathbf{v}$, but has appropriately selected values. Then, having $\mathbf{u}$ as a starting point we can create profiles in which the allocations contradict truthfulness.

Now, recall that in profile $\mathbf{v}$, player 1 gets $Z \cup\{x\}$ (notice that he may get more items as well), and consider profiles $\mathbf{v}^{\prime}=\left(\nu_{1}^{\mathrm{I}}, \nu_{2}\right)$ and $\mathbf{v}^{\prime \prime}=\left(\nu_{1}^{\mathrm{II}}, \nu_{2}\right)$, where

|  | $Z$ | $x$ | $M \backslash(Z \cup\{x\})$ |
| :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{I}}$ | $-m^{2}-$ | $m$ | $-1-$ |

and

|  | $Z$ | $x$ | $M \backslash(Z \cup\{x\})$ |
| :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{II}}$ | $-m-$ | $m^{2}$ | $-1-$ |

By truthfulness, player 1 continues to get $Z \cup\{x\}$ in both cases, i.e., $Z \cup\{x\} \subseteq X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right)$ and $Z \cup\{x\} \subseteq$ $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime \prime}\right)$.

We proceed by changing the values of player 2 this time. Assuming that $M \backslash(Z \cup\{x\})=\left\{i_{1}, i_{2}, \ldots\right.$, $\left.i_{\ell}\right\}$ let $f_{i_{j}}=m$ if $i_{j} \in X_{2}\left(\mathbf{v}^{\prime \prime}\right)$ and $f_{i_{j}}=1$ otherwise. Consider the next profile $\mathbf{u}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{I}}\right)$ :

|  | $Z$ | $x$ | $M \backslash(Z \cup\{x\})$ |
| :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{II}}$ | $-m-$ | $m^{2}$ | $-1-$ |
| $v_{2}^{\mathrm{I}}$ | $-1-$ | $m^{2}$ | $f_{i_{1}, \ldots, f_{i_{\ell}}}$ |

Now notice that player 1 must get item $x$, since $x \in N_{1}$ and thus he controls $\{x\}$. On the other hand, since player 2 can not get $x$ he must continue to get at least the items in $X_{2}\left(\mathbf{v}^{\prime \prime}\right)$ by truthfulness (otherwise he would play $\nu_{2}$ instead). Since this the case, he can not get a strict superset of $X_{2}\left(\mathbf{v}^{\prime \prime}\right)$ either. Indeed, if this was not the case he would deviate from $\mathbf{v}^{\prime \prime}$ to $\mathbf{u}$. So we can conclude that $X_{2}(\mathbf{u})=X_{2}\left(\mathbf{v}^{\prime \prime}\right)$.

Now we move to a profile $\mathbf{u}^{\prime}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\mathrm{II}}\right)$ where eventually player 2 gets item $x$ :

|  | $Z$ | $x$ | $M \backslash(Z \cup\{x\})$ |
| :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{I}}$ | $-m^{2}-$ | $m$ | $-1-$ |
| $v_{2}^{\mathrm{II}}$ | $-m^{2}-$ | $m^{2}$ | $f_{i_{1}}, \ldots, f_{i_{\ell}}$ |

In $\mathbf{u}^{\prime}$, both players strongly desire $Z \cup\{x\}$. But player 1 cannot get both set $Z$ and item $x$, or by Lemma 3.8 he controls $Z \cup\{x\}$ and thus $Z \cup\{x\} \subseteq S$ for some $S \in \mathscr{O}_{1}$. However, he controls $Z$, since there exists some $S^{\prime} \in \mathscr{O}_{1}$ such that $Z=S^{\prime} \cap X_{1}^{N_{1}}(\mathbf{v}) \subseteq S^{\prime}$. So, player 1 has to get $Z$ since he strongly desires it, and item $x$ is given to player 2 (probably with other items in $M \backslash(Z \cup\{x\})$.

Finally, consider our final profile $\mathbf{u}^{\prime \prime}=\left(v_{1}^{\mathrm{I}}, \nu_{2}^{\mathrm{I}}\right)$

|  | $Z$ | $x$ | $M \backslash(Z \cup\{x\})$ |
| :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{II}}$ | $-m^{2}-$ | $m$ | $-1-$ |
| $v_{2}^{\mathrm{II}}$ | $-1-$ | $m^{2}$ | $f_{i_{1}}, \ldots, f_{i_{\ell}}$ |

By truthfulness, player 2 must get item $x$, or he would deviate from $\mathbf{u}^{\prime \prime}$ to $\mathbf{u}^{\prime}$. However, now player 1 can strictly improve his utility by deviating from profile $\mathbf{u}^{\prime \prime}$ to $\mathbf{u}$, something that contradicts truthfulness.

Proof of Lemma3.12. Due to symmetry, it suffices to prove the statement for $i=1$. If $N_{1}=\varnothing$ then the statement is trivially true. So assume $N_{1} \neq \varnothing$ and suppose, towards a contradiction, that the statement does not hold. That is, there exists a profile $\mathbf{v}=\left(\nu_{1}, \nu_{2}\right)$ such that $X_{1}^{N_{1}}(\mathbf{v}) \notin$ $\arg \max _{S \in \mathscr{O}_{1}} \nu_{1}(S)$. We consider two cases, depending on whether $X_{2}^{N_{2}}(\mathbf{v})$ is in $\mathscr{O}_{2}$ or not. In both cases, we create a series of deviations that eventually contradict truthfulness. Like in the proof of Lemma 3.11, our first goal is to reach a profile $\mathbf{u}$ that contradicts the statement, like $\mathbf{v}$, but has appropriately selected values. Using $\mathbf{u}$ as a starting point we create profiles in which the allocations dictated by truthfulness are in conflict.
Case 1. Assume $X_{2}^{N_{2}}(\mathbf{v}) \in \mathscr{O}_{2}$ (note that this includes the case where $\mathscr{O}_{2}=\{\varnothing\}$ ). Intuitively this is the case where the two players trade value between $N_{1}$ and $E_{2}$.

Consider the profile $\mathbf{v}^{\prime}=\left(\nu_{1}, \nu_{2}^{\mathrm{I}}\right)$, where

$$
\begin{array}{c|c|c||c|c||c||c|c} 
& X_{1}^{N_{1}}(\mathbf{v}) & X_{2}^{N_{1}}(\mathbf{v}) & X_{1}^{N_{2}}(\mathbf{v}) & X_{2}^{N_{2}}(\mathbf{v}) & E_{1} & X_{1}^{E_{2}}(\mathbf{v}) & X_{2}^{E_{2}}(\mathbf{v}) \\
\hline v_{2}^{I} & -m- & -m^{3}- & -m- & -m^{4}- & -1- & -m^{2}- & -m^{4}-
\end{array}
$$

By truthfulness, $X_{2}\left(\mathbf{v}^{\prime}\right) \supseteq X_{2}^{N_{1}}(\mathbf{v}) \cup X_{2}^{N_{2}}(\mathbf{v}) \cup X_{2}^{E_{2}}(\mathbf{v})$. This implies $X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right)=X_{2}^{N_{2}}(\mathbf{v})$ due to the maximality of $X_{2}^{N_{2}}(\mathbf{v})$ and Lemma 3.11 , as well as $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \subseteq X_{1}^{N_{1}}(\mathbf{v})$. The latter implies that $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \notin$ $\arg \max _{S \in \mathscr{O}_{1}} \nu_{1}(S)$.
Claim A.2. $X_{1}^{E_{2}}\left(\mathbf{v}^{\prime}\right) \neq \varnothing$.
Proof of Claim A.2. Suppose $X_{1}^{E_{2}}\left(\mathbf{v}^{\prime}\right)=\varnothing$ and let $S^{\prime} \in \arg \max _{S \in \mathscr{O}_{1}} \nu_{1}(S)$. Then player 1, whose total received value in $\mathbf{v}^{\prime}$ would be strictly less than $\nu_{1}\left(S^{\prime} \cup\left(X_{1}^{N_{2}}(\mathbf{v})\right) \cup E_{1}\right)$, could force the mechanism to give him at least that by playing


By the definition of $N_{1}, N_{2}, E_{1}$, and Lemma3.11, player 1 gets $S^{\prime}, N_{2} \backslash X_{2}^{N_{2}}(\mathbf{v})$, and $E_{1}$ (and possibly something from $E_{2}$ ). Since this contradicts truthfulness, it must be the case that $X_{1}^{E_{2}}\left(\mathbf{v}^{\prime}\right) \neq \varnothing$. (In fact, this settles Case 1 when $E_{2}=\varnothing$.)

Next, let $S_{1} \in \mathscr{O}_{1}$ be such that $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \subseteq S_{1}$ (they could possibly be equal). Consider the profile $\mathbf{u}=\left(v_{1}^{\mathrm{II}}, \nu_{2}^{\mathrm{I}}\right)$, where

$$
\begin{array}{c|c|c|c||c||c|c||c|c}
{ }^{E_{1}}\left(\mathbf{N}_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right)\right. & S_{1} \backslash X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) & N_{1} \backslash S_{1} & N_{2} & X_{1}^{E_{1}}\left(\mathbf{v}^{\prime}\right) & X_{2}^{E_{1}}\left(\mathbf{v}^{\prime}\right) & X_{1}^{E_{2}}\left(\mathbf{v}^{\prime}\right) & X_{2}^{E_{2}}\left(\mathbf{v}^{\prime}\right) \\
\hline v_{1}^{\mathrm{II}} & -m- & -1+m^{-1}- & -1- & -1- & -m^{3}- & -1- & -m^{2}- & -1-
\end{array}
$$

Notice that $S_{1}$ is the unique set in $\arg \max _{S \in \mathscr{O}_{1}} v_{1}^{\mathrm{II}}(S)$. By truthfulness, $X_{1}(\mathbf{u}) \supseteq X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \cup X_{1}^{E_{1}}\left(\mathbf{v}^{\prime}\right) \cup$ $X_{1}^{E_{2}}\left(\mathbf{v}^{\prime}\right)$.
Claim A.3. $S_{1} \nsubseteq X_{1}(\mathbf{u})$, and therefore $X_{1}^{N_{1}}(\mathbf{u}) \notin \arg _{\max _{S \in \mathscr{O}_{1}}} v_{1}^{\mathrm{II}}(S)$.
Proof of Claim A.3. Suppose $S_{1} \subseteq X_{1}(\mathbf{u})$. By Lemma 3.11 this means $S_{1}=X_{1}^{N_{1}}(\mathbf{u})$. Then player 2, whose total received value in $\mathbf{u}$ would be strictly less than $v_{2}^{\mathrm{II}}\left(\left(N_{1} \backslash S_{1}\right) \cup X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \cup X_{2}^{E_{2}}\left(\mathbf{v}^{\prime}\right)\right)+m$, could force the mechanism to give him more than that by playing

|  | $N_{1}$ | $X_{1}^{N_{2}}\left(\mathbf{v}^{\prime}\right)$ | $X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right)$ | $E_{1}$ | $E_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\mathrm{II}}$ | $-1-$ | $-1-$ | $-m-$ | $-1-$ | $-m-$ |

By the definition of $N_{2}, E_{2}$, in $\mathbf{v}^{\prime \prime}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{II}}\right)$ player 2 gets $X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right)$ and $E_{2}$ (and possibly something from $N_{1}$ and $E_{1}$ ). Given that, the maximum value that player 1 could achieve in $\mathbf{v}^{\prime \prime}$ is $v_{1}^{\mathrm{II}}\left(S_{1} \cup\right.$ $\left.X_{1}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \cup E_{1}\right)$ and there is no subset of $M \backslash\left(X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \cup E_{2}\right)$ giving this value other than $S_{1} \cup X_{1}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \cup$ $E_{1}$. In fact, player 1 can achieve exactly this by increasing his reported value for each item in $S_{1} \cup E_{1}$ to $m^{3}$. Thus $X_{1}\left(\mathbf{v}^{\prime \prime}\right)=S_{1} \cup X_{1}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \cup E_{1}$ and $\nu_{2}^{\mathrm{II}}\left(X_{2}\left(\mathbf{v}^{\prime \prime}\right)\right)=\nu_{2}^{\mathrm{II}}\left(\left(N_{1} \backslash S_{1}\right) \cup X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \cup E_{2}\right) \geq$ $v_{2}^{\mathrm{II}}\left(\left(N_{1} \backslash S_{1}\right) \cup X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \cup X_{2}^{E_{2}}\left(\mathbf{v}^{\prime}\right)\right)+m^{2}$. Since this contradicts truthfulness, it must be the case that $S_{1} \nsubseteq X_{1}(\mathbf{u})$ (and thus $X_{1}^{N_{1}}(\mathbf{u}) \notin{\arg \max _{S \in \mathscr{O}_{1}}}^{v_{1}^{\mathrm{II}}(S) \text { ). }}$

ClaimA.3 implies that $S_{1} \backslash X_{1}^{N_{1}}(\mathbf{u}) \neq \varnothing$. Since the sets in $\mathscr{O}_{1}$ have empty intersection, there must exist some $T \in \mathscr{O}_{1}$ such that $S_{1} \backslash X_{1}^{N_{1}}(\mathbf{u}) \nsubseteq T$. We are going to concentrate most of player 2's value from $N_{1}$ on $W=\left(S_{1} \backslash X_{1}^{N_{1}}(\mathbf{u})\right) \backslash T \subseteq X_{2}^{N_{1}}(\mathbf{u})$. Notice that $W \neq \varnothing$.

So consider the profile $\mathbf{u}^{\prime}=\left(v_{1}^{\text {II }}, v_{2}^{\text {III }}\right)$, where

$$
\begin{array}{c|c|c||c|c||c||c|c} 
& N_{1} \backslash W & W & X_{1}^{N_{2}}(\mathbf{v}) & X_{2}^{N_{2}}(\mathbf{v}) & E_{1} & X_{1}^{E_{2}}\left(\mathbf{v}^{\prime}\right) & X_{2}^{E_{2}}\left(\mathbf{v}^{\prime}\right) \\
\hline v_{2}^{\mathrm{III}} & -m- & -m^{3}- & -m- & -m^{4}- & -1- & -m^{2}- & -m^{4}-
\end{array}
$$

By the definition of $N_{2}, E_{2}$ and truthfulness, $X_{2}\left(\mathbf{u}^{\prime}\right) \supseteq W \cup X_{2}^{N_{2}}(\mathbf{v}) \cup X_{2}^{E_{2}}\left(\mathbf{v}^{\prime}\right)$.
Claim A.4. $X_{1}^{E_{2}}\left(\mathbf{u}^{\prime}\right) \neq \varnothing$.
Proof of Claim A.4. This is very similar to the proof of Claim A.2. Suppose $X_{1}^{E_{2}}\left(\mathbf{u}^{\prime}\right)=\varnothing$. Then player 1 , whose total received value in $\mathbf{u}^{\prime}$ would be strictly less than $v_{1}^{\mathrm{II}}\left(S_{1} \cup X_{1}^{N_{2}}(\mathbf{v}) \cup E_{1}\right)$, could force the mechanism to give him at least that by playing

|  | $S_{1}$ | $N_{1} \backslash S_{1}$ | $N_{2}$ | $E_{1}$ | $E_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{III}}$ | $-m-$ | $-1-$ | $-1-$ | $-m-$ | $-1-$ |

Since this contradicts truthfulness, it must be the case where $X_{1}^{E_{2}}\left(\mathbf{u}^{\prime}\right) \neq \varnothing$.
Before we examine the final profile of the proof, let us consider the following simple profile $\mathbf{u}^{\prime \prime}=\left(v_{1}^{\mathrm{IV}}, v_{2}^{\mathrm{IV}}\right)$ :

|  | $T$ | $N_{1} \backslash T$ | $X_{1}^{N_{2}}(\mathbf{v})$ | $X_{2}^{N_{2}}(\mathbf{v})$ | $E_{1}$ | $X_{1}^{E_{2}}\left(\mathbf{u}^{\prime}\right)$ | $X_{2}^{E_{2}}\left(\mathbf{u}^{\prime}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{IV}}$ | $-m-$ | $-1-$ | $-1-$ | $-1-$ | $-1-$ | $-m^{2}-$ | $-1-$ |
| $v_{2}^{\mathrm{IV}}$ | $-1-$ | $-1-$ | $-1-$ | $-m-$ | $-1-$ | $-m-$ | $-m-$ |

By the definition of $N_{2}, E_{2}$, in $\mathbf{u}^{\prime \prime}$ player 2 gets $X_{2}^{N_{2}}(\mathbf{v})$ and $E_{2}$ (and possibly something from $N_{1}$ and $E_{1}$ ). Given that, the maximum value that player 1 could achieve in $\mathbf{u}^{\prime \prime}$ is $|T| \cdot m+\left|X_{1}^{N_{2}}(\mathbf{v}) \cup E_{1}\right|$, and there is no subset of $M \backslash\left(X_{2}^{N_{2}}(\mathbf{v}) \cup E_{2}\right)$ giving this value other than $T \cup X_{1}^{N_{2}}(\mathbf{v}) \cup E_{1}$. In fact, player 1 can achieve exactly this by increasing his reported value for each item in $T \cup E_{1}$ to $m^{3}$. Thus $X_{1}\left(\mathbf{u}^{\prime \prime}\right)=T \cup X_{1}^{N_{2}}(\mathbf{v}) \cup E_{1}$ and $X_{2}\left(\mathbf{u}^{\prime \prime}\right)=\left(N_{1} \backslash T\right) \cup X_{2}^{N_{2}}(\mathbf{v}) \cup E_{2}$.

The final profile we need is $\mathbf{u}^{\prime \prime \prime}=\left(v_{1}^{\mathrm{IV}}, v_{2}^{\mathrm{II}}\right)$, and the contradiction follows from the allocation of the items in $X_{1}^{E_{2}}\left(\mathbf{u}^{\prime}\right)$. If $X_{1}^{E_{2}}\left(\mathbf{u}^{\prime}\right) \nsubseteq X_{1}\left(\mathbf{u}^{\prime \prime \prime}\right)$ then player 1 has incentive to deviate to profile $\mathbf{u}^{\prime}=$ $\left(v_{1}^{\text {II }}, v_{2}^{\text {III }}\right)$. So, it must be the case where $X_{1}^{E_{2}}\left(\mathbf{u}^{\prime}\right) \subseteq X_{1}\left(\mathbf{u}^{\prime \prime \prime}\right)$, and therefore $v_{2}^{\text {III }}\left(X_{2}\left(\mathbf{u}^{\prime \prime \prime}\right)\right) \leq v_{2}^{\text {III }}(M \backslash$ $\left.X_{1}^{N_{2}}\left(\mathbf{u}^{\prime}\right)\right)<\nu_{2}^{\text {III }}\left(W \cup X_{2}^{N_{2}}(\mathbf{v}) \cup X_{2}^{E_{2}}\left(\mathbf{u}^{\prime}\right)\right)+m^{2}$. On the other hand, notice that $W \subseteq N_{1} \backslash T$. Using the allocation for $\mathbf{u}^{\prime \prime}$ we derived above, by truthfulness we have that $v_{2}^{\text {III }}\left(X_{2}\left(\mathbf{u}^{\prime \prime \prime}\right)\right) \geq v_{2}^{\text {III }}\left(W \cup X_{2}^{N_{2}}(\mathbf{v}) \cup\right.$ $\left.E_{2}\right) \geq v_{2}^{\text {III }}\left(W \cup X_{2}^{N_{2}}(\mathbf{v}) \cup X_{2}^{E_{2}}\left(\mathbf{u}^{\prime}\right)\right)+m^{2}$, which is a contradiction.
Case 2. Assume $X_{2}^{N_{2}}(\mathbf{v}) \notin \mathscr{O}_{2}$. Case 1 implies that not only $X_{1}^{N_{1}}(\mathbf{v}) \notin \operatorname{argmax}_{S \in \mathscr{O}_{1}} v_{1}(S)$ but $X_{1}^{N_{1}}(\mathbf{v}) \notin$ $\mathscr{O}_{1}$. Intuitively this is the case where the two players trade value between $N_{1}$ and $N_{2}$. The proof uses a sequence of profiles similar to Case 1 .

Consider the profile $\mathbf{v}^{\prime}=\left(\nu_{1}, v_{2}^{\mathrm{I}}\right)$, where

|  | $X_{1}^{N_{1}}(\mathbf{v})$ | $X_{2}^{N_{1}}(\mathbf{v})$ | $X_{1}^{N_{2}}(\mathbf{v})$ | $X_{2}^{N_{2}}(\mathbf{v})$ | $E_{1}$ | $E_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{1}$ | $-1-$ | $-m^{2}-$ | $-m-$ | $-m^{3}-$ | $-1-$ | $-1-$ |

By truthfulness, $X_{2}\left(\mathbf{v}^{\prime}\right) \supseteq X_{2}^{N_{1}}(\mathbf{v}) \cup X_{2}^{N_{2}}(\mathbf{v})$. This implies $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \subseteq X_{1}^{N_{1}}(\mathbf{v})$, and thus $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \notin \mathscr{O}_{1}$. By Case 1 , this means that $X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \notin \mathscr{O}_{2}$.

Next, let $S_{1} \in \mathscr{O}_{1}$ be a minimal set of $\mathscr{O}_{1}$ such that $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \subseteq S_{1}$. Since $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \notin \mathscr{O}_{1}$, we have $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \subsetneq S_{1}$. Consider the profile $\mathbf{u}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$, where

|  | $X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right)$ | $S_{1} \backslash X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right)$ | $N_{1} \backslash S_{1}$ | $X_{1}^{N_{2}}\left(\mathbf{v}^{\prime}\right)$ | $X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right)$ | $E_{1}$ | $E_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{I}}$ | $-m-$ | $-1+m^{-1}-$ | $-1-$ | $-m-$ | $-1-$ | $-1-$ | $-1-$ |

Notice that $S_{1}$ is the unique set in $\arg \max _{S \in \mathscr{O}_{1}} \nu_{1}^{\mathrm{I}}(S)$. By truthfulness, $X_{1}(\mathbf{u}) \supseteq X_{1}^{N_{1}}\left(\mathbf{v}^{\prime}\right) \cup X_{1}^{N_{2}}\left(\mathbf{v}^{\prime}\right)$.
Claim A.5. $S_{1} \nsubseteq X_{1}(\mathbf{u})$, and therefore $X_{1}^{N_{1}}(\mathbf{u}) \notin \arg \max _{S \in \mathscr{O}_{1}} v_{1}^{\mathrm{I}}(S)$.
Proof of Claim A.5. This is similar to the proof of Claim A.3, Suppose $S_{1} \subseteq X_{1}(\mathbf{u})$. By Lemma 3.11 this means $S_{1}=X_{1}^{N_{1}}(\mathbf{u})$. Then player 2, whose total received value in $\mathbf{u}$ would be strictly less than $v_{2}^{\mathrm{I}}\left(X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right)\right)+m$, could force the mechanism to give him at least that by playing

|  | $N_{1}$ | $N_{2} \backslash S_{2}$ | $S_{2}$ | $E_{1}$ | $E_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\text {II }}$ | $-1-$ | $-1-$ | $-m-$ | $-1-$ | $-m-$ |

where $S_{2} \in \mathscr{O}_{2}$ is such that $X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \subseteq S_{2}$. By the definition of $N_{2}, E_{2}$, in $\mathbf{v}^{\prime \prime}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\mathrm{II}}\right)$ player 2 gets $S_{2}$ and $E_{2}$ (and possibly something from $N_{1}$ and $E_{1}$ ). Note, however, that $X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \notin \mathscr{O}_{2}$ and thus $X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right) \subsetneq S_{2}$. Therefore, $v_{2}^{\mathrm{I}}\left(X_{2}\left(\mathbf{v}^{\prime \prime}\right)\right) \geq v_{2}^{\mathrm{I}}\left(S_{2}\right) \geq v_{2}^{\mathrm{I}}\left(X_{2}^{N_{2}}\left(\mathbf{v}^{\prime}\right)\right)+m$. Since this contradicts truthfulness, it must be the case that $S_{1} \nsubseteq X_{1}(\mathbf{u})$ (and thus $X_{1}^{N_{1}}(\mathbf{u}) \notin \arg \max _{S \in \mathscr{O}_{1}} v_{1}^{\mathrm{I}}(S)$ ).

This implies that $S_{1} \backslash X_{1}^{N_{1}}(\mathbf{u}) \neq \varnothing$. Since the sets in $\mathscr{O}_{1}$ have empty intersection, there must exist some $T \in \mathscr{O}_{1}$ such that $S_{1} \backslash X_{1}^{N_{1}}(\mathbf{u}) \nsubseteq T$. We are going to concentrate most of player 2's value from $N_{1}$ on $W=\left(S_{1} \backslash X_{1}^{N_{1}}(\mathbf{u})\right) \backslash T \neq \varnothing$. So consider the profile $\mathbf{u}^{\prime}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\text {III }}\right)$, where

|  | $N_{1} \backslash W$ | $W$ | $X_{1}^{N_{2}}(\mathbf{u})$ | $X_{2}^{N_{2}}(\mathbf{u})$ | $E_{1}$ | $E_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\text {III }}$ | $-1-$ | $-m^{2}-$ | $-m-$ | $-m^{3}-$ | $-1-$ | $-1-$ |

By truthfulness, $X_{2}\left(\mathbf{u}^{\prime}\right) \supseteq W \cup X_{2}^{N_{2}}(\mathbf{u})$. This implies that $S_{1} \nsubseteq X_{1}\left(\mathbf{u}^{\prime}\right)$ and thus $X_{1}^{N_{1}}\left(\mathbf{u}^{\prime}\right) \notin$ $\arg \max _{S \in \mathscr{O}_{1}} v_{1}^{1}(S)$. By Case 1, this means that $X_{2}^{N_{2}}\left(\mathbf{u}^{\prime}\right) \notin \mathscr{O}_{2}$. Therefore, $X_{1}^{N_{2}}\left(\mathbf{u}^{\prime}\right) \neq \varnothing$.

Now let $S_{2}^{\prime} \in \mathscr{O}_{2}$ is such that $X_{2}^{N_{2}}\left(\mathbf{u}^{\prime}\right) \subsetneq S_{2}^{\prime}$. Before we examine the final profile of the proof, let us consider the following profile $\mathbf{u}^{\prime \prime}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{IV}}\right)$ :

|  | $T$ | $N_{1} \backslash T$ | $N_{2} \backslash S_{2}^{\prime}$ | $S_{2}^{\prime} \backslash X_{2}^{N_{2}}\left(\mathbf{u}^{\prime}\right)$ | $X_{2}^{N_{2}}\left(\mathbf{u}^{\prime}\right)$ | $E_{1}$ | $E_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\nu_{1}^{\text {II }}$ | $-m-$ | $-1-$ | $-m^{2}-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-1-$ |
| $v_{2}^{\text {IV }}$ | $-1-$ | $-1-$ | $-1-$ | $-m-$ | $-m-$ | $-1-$ | $-m-$ |

By the definition of $N_{2}, E_{2}$, in $\mathbf{u}^{\prime \prime}$ player 2 gets $S_{2}^{\prime}$ and $E_{2}$ (and possibly something from $N_{1}$ and $\left.E_{1}\right)$. Given that, the maximum value that player 1 could achieve in $\mathbf{u}^{\prime \prime}$ is $|T| \cdot m+\left|N_{2} \backslash S_{2}^{\prime}\right| \cdot m^{2}+\left|E_{1}\right|$. In fact, player 1 can achieve exactly this by increasing his reported value for each item in $T \cup E_{1}$ to $m^{3}$. Thus $X_{1}\left(\mathbf{u}^{\prime \prime}\right)=T \cup\left(N_{2} \backslash S_{2}^{\prime}\right) \cup E_{1}$ and $X_{2}\left(\mathbf{u}^{\prime \prime}\right)=\left(N_{1} \backslash T\right) \cup S_{2}^{\prime} \cup E_{2}$.

The final profile we need is $\mathbf{u}^{\prime \prime \prime}=\left(v_{1}^{\mathrm{II}}, \nu_{2}^{\mathrm{III}}\right)$, and the contradiction follows from the allocation of the items in $X_{1}^{N_{2}}\left(\mathbf{u}^{\prime}\right)$. If $X_{1}^{N_{2}}\left(\mathbf{u}^{\prime}\right) \nsubseteq X_{1}\left(\mathbf{u}^{\prime \prime \prime}\right)$ then player 1 has incentive to deviate to profile
$\mathbf{u}^{\prime}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\text {III }}\right)$. So, it must be the case where $X_{1}^{N_{2}}\left(\mathbf{u}^{\prime}\right) \subseteq X_{1}\left(\mathbf{u}^{\prime \prime \prime}\right)$ and therefore $\nu_{2}^{\text {III }}\left(X_{2}\left(\mathbf{u}^{\prime \prime \prime}\right)\right) \leq v_{2}^{\text {III }}(M \backslash$ $\left.X_{1}^{N_{2}}\left(\mathbf{u}^{\prime}\right)\right)<|W| \cdot m^{2}+\left|X_{2}^{N_{2}}(\mathbf{u})\right| \cdot m^{3}+m$. On the other hand, notice that $W \subseteq N_{1} \backslash T$ and recall that $X_{2}^{N_{2}}(\mathbf{u}) \subseteq X_{2}^{N_{2}}\left(\mathbf{u}^{\prime}\right) \subsetneq S_{2}^{\prime}$. Using the allocation for $\mathbf{u}^{\prime \prime}$ we calculated above, by truthfulness we have that $v_{2}^{\text {III }}\left(X_{2}\left(\mathbf{u}^{\prime \prime \prime}\right)\right) \geq v_{2}^{\text {III }}\left(\left(N_{1} \backslash T\right) \cup S_{2}^{\prime}\right) \geq|W| \cdot m^{2}+\left|X_{2}^{N_{2}}(\mathbf{u})\right| \cdot m^{3}+m$, which is a contradiction.

Proof of Lemma3.13. Suppose that this not true. So there are profiles $\mathbf{v}=\left(\nu_{1}, \nu_{2}\right), \mathbf{v}^{\prime}=\left(\nu_{1}^{\prime}, v_{2}^{\prime}\right) \in$ $V_{m}^{\neq}$such that $v_{i j}=v_{i j}^{\prime}$ for all $i \in\{1,2\}$ and $j \in E_{1} \cup E_{2}$, but $X_{1}^{E_{1} \cup E_{2}}(\mathbf{v}) \neq X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right)$. In such a case, either $\mathbf{v}=\left(\nu_{1}, v_{2}\right), \hat{\mathbf{v}}=\left(v_{1}^{\prime}, v_{2}\right)$, or $\hat{\mathbf{v}}=\left(v_{1}^{\prime}, v_{2}\right), \mathbf{v}^{\prime}=\left(v_{1}^{\prime}, v_{2}^{\prime}\right)$ is also a pair of profiles that violates the statement. Without loss of generality we assume that $\mathbf{v}, \hat{\mathbf{v}}$ is such a pair, and that $\nu_{1}\left(X_{1}^{E_{1}}(\mathbf{v})\right)>$ $\nu_{1}\left(X_{1}^{E_{1}}(\hat{\mathbf{v}})\right)$. Now let $S_{1}, \hat{S}_{1} \in \mathscr{O}_{1}$ be the single best offer in each case. If $S_{1}=\hat{S}_{1}$ then player 1 would deviate from $\hat{\mathbf{v}}$ to $\mathbf{v}$ and strictly improve. So assume that $S_{1} \neq \widehat{S}_{1}$ and multiply the values in $E_{1} \cup E_{2}$ for player 1 with a large enough constant $K$, so that $K\left(\hat{v}_{1}\left(X_{1}^{E_{1}}(\mathbf{v})\right)-\hat{v}_{1}\left(X_{1}^{E_{1}}(\hat{\mathbf{v}})\right)\right)>\hat{v}_{1}\left(N_{1} \cup N_{2}\right)$.

Call $\mathbf{v}^{*}=\left(v_{1}^{*}, v_{2}\right)$ and $\hat{\mathbf{v}}^{*}=\left(\nu_{1}^{\prime *}, v_{2}\right)$ the new profiles and notice that they are still in $\bigvee_{m}^{\neq}$. Also, it is easy to see that truthfulness implies $X_{1}(\mathbf{v})=X_{1}\left(\mathbf{v}^{*}\right)$ and $X_{1}(\hat{\mathbf{v}})=X_{1}\left(\hat{\mathbf{v}}^{*}\right)$. Indeed, by Lemma 3.12, we have $X_{1}^{N_{1} \cup N_{2}}(\mathbf{v})=X_{1}^{N_{1} \cup N_{2}}\left(\mathbf{v}^{*}\right)$, and if it was the case where $X_{1}^{E_{1} \cup E_{2}}(\mathbf{v}) \neq X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{*}\right)$, then player 1 would deviate from profile $\mathbf{v}$ to $\mathbf{v}^{*}$ or vice versa to strictly improve his utility. The same holds for $\hat{\mathbf{v}}$ to $\hat{\mathbf{v}}^{*}$.

Now, however, player 1 would deviate from $\hat{\mathbf{v}}^{*}$ to $\mathbf{v}^{*}$ in order to improve by at least $\hat{v}_{1}^{*}\left(X_{1}^{E_{1}}\left(\mathbf{v}^{*}\right)\right)-$ $\hat{v}_{1}^{*}\left(X_{1}^{E_{1}}\left(\hat{\mathbf{v}}^{*}\right)\right)-\hat{v}_{1}^{*}\left(N_{1} \cup N_{2}\right)=K\left(\hat{v}_{1}\left(X_{1}^{E_{1}}(\mathbf{v})\right)-\hat{v}_{1}\left(X_{1}^{E_{1}}(\hat{\mathbf{v}})\right)\right)-\hat{v}_{1}\left(N_{1} \cup N_{2}\right)>0$, and this contradicts truthfulness.

Remark A.6. Since we are talking about $\mathscr{X}_{E}$ in many of the following proofs, it is correct to write $X_{i}^{E_{1} \cup E_{2}}(\cdot)$, not $X_{i}(\cdot)$. For the sake of readability, though, we drop the superscript wherever it is not necessary. Similarly, in order to avoid the unnecessary use of extra symbols, we prove the statements for $m$ items, although in Subsection $3.2 \mathscr{X}_{E}$ is a mechanism on $\ell \leq m$ items.

Remark A.7. For most of the following proofs we need to construct profiles in $\mathcal{I}_{m}^{\neq}$. To facilitate the presentation, however, the valuation functions we construct only use a few powers of $m$. As a result, the corresponding profiles typically are not in $\mathscr{V}_{m}^{\neq}$. Still, this is without loss of generality; when defining such valuation functions we can add $2^{i} / 2^{\kappa}$ to the value of item $i$, for $i \in[m]$. When $\kappa \in \mathbb{N}$ is large enough (usually $\kappa=m+1$ suffices), our arguments about the allocation are not affected, and a strict preference over all subsets is induced.

Proof of Lemma3.14, Let $\mathbf{v}=\left(\nu_{1}, \nu_{2}\right) \in \tau_{m}$, and consider the intermediate profile $\mathbf{v}^{*}=\left(v_{1}^{\prime}, \nu_{2}\right)$ where $v_{1 x}^{\prime}=m$, if $x \in X_{1}(\mathbf{v})$, and $v_{1 x}^{\prime}=1$ otherwise. By truthfulness, we have that $X_{1}\left(\mathbf{v}^{*}\right)=X_{1}(\mathbf{v})$. By defining $v_{2}^{\prime}$ in a similar way (i.e., $v_{2 x}^{\prime}=m$, if $x \in X_{2}(\mathbf{v})$, and $v_{2 x}^{\prime}=1$ otherwise), we get the profile $\mathbf{v}^{\prime}=\left(v_{1}^{\prime}, v_{2}^{\prime}\right)$. Again by truthfulness, we have $\mathscr{X}\left(\mathbf{v}^{\prime}\right)=\mathscr{X}(\mathbf{v})$. If $\mathbf{v}^{*}$ and $\mathbf{v}^{\prime}$ where defined as described in RemarkA.7, the same arguments would apply, and moreover, $\mathbf{v}^{\prime} \in \mathcal{V}_{m}^{\neq}$.

Proof of Lemma 3.15, To show that $D$ is indeed a valid set of exchange deals, we need to show that for any two distinct deals $(S, T),\left(S^{\prime}, T^{\prime}\right) \in D$ we have $S \cap S^{\prime}=T \cap T^{\prime}=\varnothing$ and $S, T, S^{\prime}, T^{\prime}$ are all nonempty. The latter is straightforward due to truthfulness and the fact that all values are positive. The former is done through the next three lemmata, the first of which states that each minimally exchangeable set is involved in exactly one exchange deal.

Lemma A.8. If $S \subseteq E_{1}$ is a minimally exchangeable set, then there exists a unique $T \subseteq E_{2}$ such that $(S, T)$ is a feasible exchange.

The lemma is stated in terms of minimally exchangeable subsets of $E_{1}$, but due to symmetry it is true for all minimally exchangeable sets. This is done for the following statements as well, for the sake of readability. The three lemmata are proved right after this proof.

It is implied that every minimally exchangeable set appears in exactly one exchange deal in $D$. The second lemma, below, guarantees that minimally exchangeable sets can be exchanged only with minimally exchangeable sets.
Lemma A.9. Let $S \subseteq E_{1}$ be a minimally exchangeable set and $(S, T)$ be the only feasible exchange involving $S$. Then $T$ is a minimally exchangeable set as well.

The result of the two lemmata combined is that $D=\left\{\left(S_{1}, T_{1}\right),\left(S_{2}, T_{2}\right), \ldots,\left(S_{k}, T_{k}\right)\right\}$, where $S_{1}, \ldots$, $S_{k}, T_{1}, \ldots, T_{k}$ are all the minimally exchangeable sets and are all different from each other. What is still needed is that the intersection between any two minimally exchangeable sets is always empty. The third lemma states something stronger that is indeed needed later in the proof of A.12), namely that the intersection between a minimally exchangeable set and any other exchangeable set is always empty, unless the latter contains the former.
Lemma A.10. Let $S \subseteq E_{1}$ be a minimally exchangeable set and $S^{\prime} \subseteq E_{1}$ be an exchangeable set such that $S^{\prime} \cap S \neq \varnothing$. Then $S \subseteq S^{\prime}$.

If the intersection between any two minimally exchangeable sets was nonempty, then by Lemma A.10 one is contained in the other, which contradicts minimality. We can conclude that $D$ is a valid set of exchange deals.

Proof of Lemma A. 8 . Suppose that this does not hold. Without loss of generality, assume that there is some $S_{1} \subseteq E_{1}$ and two profiles $\mathbf{v}^{\mathrm{I}}=\left(\nu_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$ and $\mathbf{v}^{\mathrm{II}}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{II}}\right)$, such that $X_{1}^{E_{1}}\left(\mathbf{v}^{\mathrm{I}}\right)=E_{1} \backslash S_{1}=$ $X_{1}^{E_{1}}\left(\mathbf{v}^{\mathrm{II}}\right)$ and $X_{1}^{E_{2}}\left(\mathbf{v}^{\mathrm{I}}\right)=S_{2} \neq S_{2}^{\prime}=X_{1}^{E_{2}}\left(\mathbf{v}^{\mathrm{II}}\right)$.

For the sake of readability, let $A=S_{2} \backslash S_{2}^{\prime}, B=S_{2} \cap S_{2}^{\prime}, C=S_{2}^{\prime} \backslash S_{2}$, and $D=M \backslash\left(S_{2} \cup S_{2}^{\prime}\right)$. Since $S_{2} \neq S_{2}^{\prime}$, either $A \neq \varnothing$ or $C \neq \varnothing$. Without loss of generality, suppose that $A \neq \varnothing$. Using this notation, $X_{1}\left(\mathbf{v}^{\mathbf{I}^{1}}\right)=\left(E_{1} \backslash S_{1}\right) \cup A \cup B$ and $X_{2}\left(\mathbf{v}^{\mathrm{I}}\right)=S_{1} \cup C \cup D$, while $X_{1}\left(\mathbf{v}^{\mathrm{II}}\right)=\left(E_{1} \backslash S_{1}\right) \cup B \cup C$ and $X_{2}\left(\mathbf{v}^{\mathrm{II}}\right)=$ $S_{1} \cup A \cup D$.

We proceed to profile $\mathbf{v}^{\text {III }}=\left(\nu_{1}^{\mathrm{I}}, v_{2}^{\text {III }}\right)$ by changing the values of player 2 :

|  | $E_{1} \backslash S_{1}$ | $S_{1}$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\nu_{2}^{\text {III }}$ | $-1-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-m^{3}-$ | $-m^{3}-$ |

Since the most valuable items of player 2 are those which he was allocated in profile $\mathbf{v}^{\mathrm{I}}$, by truthfulness, he should still get them, but he should not get any other item. Thus $\mathscr{X}_{E}\left(\mathbf{v}^{\text {III }}\right)=\mathscr{X}_{E}\left(\mathbf{v}^{\mathrm{I}}\right)$.

We move to profile $\mathbf{v}^{\text {IV }}=\left(v_{1}^{\text {III }}, v_{2}^{\text {III }}\right)$ by changing the values of player 1 :

|  | $E_{1} \backslash S_{1}$ | $S_{1}$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{III}}$ | $-m^{3}-$ | $-m-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-1-$ |

By truthfulness we have that player 1 must get $E_{1} \backslash S_{1}$ and $A$ (or else he could deviate to profile $\mathbf{v}^{\text {III }}$ and strictly improve). Since he gets $A$, an exchange takes place. Due to the minimality of $S_{1}$, we can derive that player 2 receives the whole $S_{1}$. In addition, player 2 continues to get $D$, since he
strongly desires it and $D \subseteq E_{2}$. So we can conclude that $\left(E_{1} \backslash S_{1}\right) \cup A \subseteq X_{1}\left(\mathbf{v}^{\text {IV }}\right)$ and $S_{1} \cup D \subseteq X_{2}\left(\mathbf{v}^{\mathrm{IV}}\right)$, while we do not care about the allocation of the remaining items.

Now let us return to profile $\mathbf{v}^{\mathrm{II}}=\left(\nu_{1}^{\mathrm{II}}, \nu_{2}^{\mathrm{II}}\right)$. Starting from here, we change the values of player 2 and to get profile $\mathbf{v}^{\mathrm{V}}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{IV}}\right)$.

|  | $E_{1} \backslash S_{1}$ | $S_{1}$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\text {IV }}$ | $-1-$ | $-m-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-m^{2}-$ |

By truthfulness, like in profile $\mathbf{v}^{\text {III }}$, we have $\mathscr{X}_{E}\left(\mathbf{v}^{\mathrm{V}}\right)=\mathscr{X}_{E}\left(\mathbf{v}^{\mathrm{II}}\right)$.
Next, we proceed to profile $\mathbf{v}^{\mathrm{VI}}=\left(\nu_{1}^{\mathrm{IV}}, v_{2}^{\mathrm{IV}}\right)$, where

|  | $E_{1} \backslash S_{1}$ | $S_{1}$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{IV}}$ | $-m^{4}-$ | $-m-$ | $-m^{3}-$ | $-m^{2}-$ | $-m^{2}-$ | $-1-$ |

Player 2 continues to get $A, D$ since he strongly desires them and $A, D \subseteq E_{2}$. By the same argument, player 1 gets $E_{1} \backslash S_{1}$. Additionally, we know that an exchange happens (otherwise player 1 would deviate to profile $\mathbf{v}^{\mathrm{v}}$ in order to get the items of $B \cup C$ ), so player 2 gets set the whole $S_{1}$ due to its minimality. Thus we can conclude that $X_{1}\left(\mathbf{v}^{\mathrm{VI}}\right)=\left(E_{1} \backslash S_{1}\right) \cup B \cup C$ and $X_{2}\left(\mathbf{v}^{\mathrm{VI}}\right)=S_{1} \cup A \cup D$.

Next, we move to profile $\mathbf{v}^{\mathrm{VII}}=\left(\nu_{1}^{\mathrm{IV}}, v_{2}^{\mathrm{V}}\right)$ by changing player 2 this time:

|  | $E_{1} \backslash S_{1}$ | $S_{1}$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\mathrm{V}}$ | $-1-1-m^{2}-$ | $-m-$ | $-1-$ | $-1-$ | $-m^{3}-$ |  |

By truthfulness, the allocation does not change, i.e., $X_{1}\left(\mathbf{v}^{\mathrm{VII}}\right)=\left(E_{1} \backslash S_{1}\right) \cup B \cup C$ and $X_{2}\left(\mathbf{v}^{\mathrm{VII}}\right)=$ $S_{1} \cup A \cup D$.

Finally, we move to profile $\mathbf{v}^{\mathrm{VIII}}=\left(v_{1}^{\mathrm{III}}, v_{2}^{V}\right)$ by changing the values of player 1 back to the values that he had in profile $\mathbf{v}^{\text {IV }}$. Now recall that $X_{2}\left(\mathbf{v}^{\text {IV }}\right) \supseteq S_{1} \cup D$. Since in this profile $S_{1} \cup D$ contains player 2's most valuable items, he must continue to get them by truthfulness. This means that there is an exchange. Player 1 however must get some items from $A$ in any exchange; if not he can declare that he strongly desires $E_{1}$ and strictly improve. This, however, contradicts the truthfulness of the mechanism, since player 1 can deviate from $\mathbf{v}^{\mathrm{VII}}$ to $\mathbf{v}^{\mathrm{VIII}}$ and become strictly better.

Proof of Lemma A.9. Suppose that this does not hold, i.e., there exists some minimally exchangeable $S_{1} \in E_{1}$, such that ( $S_{1}, S_{2}$ ) is the only feasible exchange involving $S_{1}$, but $S_{2}$ is not minimally exchangeable. So there exists $S_{2}^{\prime} \subseteq S_{2}$ that is minimally exchangeable. So let $S_{1}^{\prime}$ be such that ( $S_{1}^{\prime}, S_{2}^{\prime}$ ) is a feasible exchange (notice that $S_{1} \neq S_{1}^{\prime}$ by lemma A.8).

For the sake of readability, let $A=E_{1} \backslash\left(S_{1} \cup S_{1}^{\prime}\right), B=S_{1}^{\prime} \cap S_{1}, C=S_{1} \backslash S_{1}^{\prime}, D=S_{2} \cap S_{2}^{\prime}, E=S_{2}^{\prime}$, and $F=S_{2} \backslash S_{2}^{\prime}$.

So there is a profile $\mathbf{v}^{\mathrm{I}}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$, where $X_{1}\left(\mathbf{v}^{\mathrm{I}}\right)=\left(E_{1} \backslash S_{1}\right) \cup S_{2}=A \cup B \cup E \cup F$ and $X_{2}\left(\mathbf{v}^{\mathrm{l}}\right)=\left(E_{2} \backslash\right.$ $\left.S_{2}\right) \cup S_{1}=C \cup D \cup\left(E_{2} \backslash S_{2}\right)$. Also there is another profile $\mathbf{v}^{\mathrm{II}}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{II}}\right)$ where $X_{1}\left(\mathbf{v}^{\mathrm{II}}\right)=\left(E_{1} \backslash S_{1}^{\prime}\right) \cup S_{2}^{\prime}=$ $A \cup D \cup E$ and $X_{2}\left(\mathbf{v}^{\mathrm{II}}\right)=\left(E_{2} \backslash S_{2}^{\prime}\right) \cup S_{1}^{\prime}=B \cup C \cup F \cup\left(E_{2} \backslash S_{2}\right)$.

We start from profile $\mathbf{v}^{\mathrm{I}}=\left(\nu_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$ and we proceed to profile $\mathbf{v}^{\mathrm{III}}=\left(\nu_{1}^{\mathrm{III}}, \nu_{2}^{\mathrm{I}}\right)$ by changing the values of player 1 :

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $E_{2} \backslash S_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\text {III }}$ | $-m^{4}-$ | $-m^{4}-$ | $-m-$ | $-m-$ | $-m^{3}-$ | $-m^{2}-$ | $-1-$ |

Since player's 1 most valuable items are those he was allocated in profile $\mathbf{v}^{1}$, due to the truthfulness of the mechanism, he must continue to get them while not getting any other item. Thus the allocation does not change, i.e., $X_{1}\left(\mathbf{v}^{\text {III }}\right)=A \cup B \cup E \cup F$ and $X_{2}\left(\mathbf{v}^{\text {III }}\right)=C \cup D \cup\left(E_{2} \backslash S_{2}\right)$.

Next, move to profile $\mathbf{v}^{\text {IV }}=\left(v_{1}^{\mathrm{III}}, v_{2}^{\mathrm{III}}\right)$ by changing the values of player 2:

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $E_{2} \backslash S_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\text {III }}$ | $-m-$ | $-m^{4}-$ | $-m-$ | $-m^{3}-$ | $-1-$ | $-m^{2}-$ | $-m^{5}-$ |

Player 2 must get $E_{2} \backslash S_{2}$ since he strongly desires them and $E_{2} \backslash S_{2} \subseteq E_{2}$. Similarly, player 1 gets $A \cup B$. Moreover, we know that an exchange should take place (otherwise player 2 would deviate to $\mathbf{v}^{\text {III }}$ and become strictly better). What can be exchanged from $E_{1}$ is a subset of $C \cup D$, and since $C \cup D=S_{1}$ is minimal, it is exchanged with $S_{2}=E \cup F$ (the only set that is exchangeable with $S_{1}$, by Lemma A.8). Thus we conclude that the allocation here is $X_{1}\left(\mathbf{v}^{\mathrm{IV}}\right)=A \cup B \cup E \cup F$ and $X_{2}\left(\mathbf{v}^{\mathrm{IV}}\right)=C \cup D \cup\left(E_{2} \backslash S_{2}\right)$.

Finally we move to profile $\mathbf{v}^{\mathrm{V}}=\left(\nu_{1}^{\mathrm{IV}}, \nu_{2}^{\mathrm{III}}\right)$, by changing the values of player 1 :

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $E_{2} \backslash S_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{IV}}$ | $-m^{4}-$ | $-m^{2}-$ | $-m-$ | $-m-$ | $-m^{3}-$ | $-m^{2}-$ | $-1-$ |

By truthfulness, like above, the allocation does not change, i.e., $X_{1}\left(\mathbf{v}^{\vee}\right)=A \cup B \cup E \cup F$ and $X_{2}\left(\mathbf{v}^{\vee}\right)=$ $C \cup D \cup\left(E_{2} \backslash S_{2}\right)$.

Now let us return to profile $\mathbf{v}^{\mathrm{II}}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{II}}\right)$. Starting from this profile we change the values of player 2 to get profile $\mathbf{v}^{\mathrm{VI}}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{IV}}\right)$.

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $E_{2} \backslash S_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\mathrm{IV}}$ | $-1-$ | $-m^{3}-$ | $-1-$ | $-1-$ | $-m-$ | $-m^{2}-$ | $-m^{4}-$ |

Player 2 must get (at least) $B \cup F \cup\left(E_{2} \backslash S_{2}\right)$ since else he could deviate to profile $\mathbf{v}^{\text {II }}$ and become strictly better. Now since player 1 loses $B$ we know that an exchange takes place with some of the available items in $E$. By the minimality of $E=S_{2}^{\prime}$, player 1 gets the whole $E$ and he loses $B \cup C=S_{1}^{\prime}$. Thus we can conclude that the allocation here is $X_{1}\left(\mathbf{v}^{\mathrm{VI}}\right)=A \cup D \cup E, X_{2}\left(\mathbf{v}^{\mathrm{VI}}\right)=B \cup C \cup F \cup\left(E_{2} \backslash S_{2}\right)$.

In order to conclude, we move to profile $\mathbf{v}^{\mathrm{VII}}=\left(\nu_{1}^{\mathrm{IV}}, \nu_{2}^{\mathrm{IV}}\right)$ by changing the values of player 1 back to what he played in $\mathbf{v}^{\mathrm{V}}$,

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $E_{2} \backslash S_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{IV}}$ | $-m^{4}-$ | $-m^{2}-$ | $-m-$ | $-m-$ | $-m^{3}-$ | $-m^{2}-$ | $-1-$ |
| $v_{2}^{\mathrm{IV}}$ | $-1-$ | $-m^{3}-$ | $-1-$ | $-1-$ | $-m-$ | $-m^{2}-$ | $-m^{4}-$ |

Player 2 gets $E_{2} \backslash S_{2}$ because he strongly desires it. We also know that an exchange should take place, otherwise player 1 would deviate to $\mathbf{v}^{\mathrm{VI}}$ and strictly improve his total value. As a result, player 2 gets at least one item from set $B$, or he could increase to $m^{4}$ his value for any item in $E_{2}$ and improve by getting $E_{2}$. However, now player 2 can deviate from profile $\mathbf{v}^{\mathrm{V}}$ to $\mathbf{v}^{\mathrm{VII}}$ and become strictly better, something that contradicts the truthfulness of the mechanism.

Proof of Lemman.10 Suppose that this does not hold, i.e., there exists a minimally exchangeable set $S_{1} \in E_{1}$ and an exchangeable set $S_{1}^{\prime} \in E_{1}$, such that $S_{1} \cap S_{1}^{\prime} \neq \varnothing$ and $S_{1} \nsubseteq S_{1}^{\prime}$. Choose $S_{1}^{\prime}$ to
be minimal, i.e., if $S_{1}^{\prime \prime} \subsetneq S_{1}^{\prime}$ then either $S_{1} \cap S_{1}^{\prime \prime}=\varnothing$ or $S_{1}^{\prime \prime}$ is not exchangeable. Let $S_{2}, S_{2}^{\prime}$ be such that $\left(S_{1}, S_{2}\right),\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ are feasible exchanges and $S_{2}^{\prime}$ is minimal in the sense that there is no $S_{2}^{\prime \prime} \subsetneq S_{2}^{\prime}$ where ( $S_{1}^{\prime}, S_{2}^{\prime \prime}$ ) being a feasible exchange. From Lemmata A. 8 and A.9 we have that $S_{2}^{\prime} \backslash S_{2} \neq \varnothing$.

For the sake of readability, let $A=E_{1} \backslash\left(S_{1} \cup S_{1}^{\prime}\right), B=S_{1}^{\prime} \backslash S_{1}, C=S_{1}^{\prime} \cap S_{1}, D=S_{1} \backslash S_{1}^{\prime}, E=S_{2} \cap S_{2}^{\prime}$, $F=S_{2} \backslash S_{2}^{\prime}, G=E_{2} \backslash\left(S_{2} \cup S_{2}^{\prime}\right)$, and $H=S_{2}^{\prime} \backslash S_{2}$.

So there is a profile $\mathbf{v}^{\mathrm{I}}=\left(E_{1} \backslash S_{1}\right) \cup S_{2}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$, where $X_{1}\left(\mathbf{v}^{\mathrm{l}}\right)=A \cup B \cup E \cup F, X_{2}\left(\mathbf{v}^{\mathrm{I}}\right)=\left(E_{2} \backslash S_{2}\right) \cup$ $S_{1}=C \cup D \cup G \cup H$. There is also a profile $\mathbf{v}^{\mathrm{II}}=\left(\nu_{1}^{\mathrm{II}}, v_{2}^{\mathrm{II}}\right)$, where $X_{1}\left(\mathbf{v}^{\mathrm{II}}\right)=\left(E_{1} \backslash S_{1}^{\prime}\right) \cup S_{2}^{\prime}=A \cup D \cup E \cup H$, $X_{2}\left(\mathbf{v}^{\mathrm{II}}\right)=\left(E_{2} \backslash S_{2}^{\prime}\right) \cup S_{1}^{\prime}=B \cup C \cup F \cup G$.

We start from profile $\mathbf{v}^{\mathrm{I}}=\left(\nu_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$ and we proceed to profile $\mathbf{v}^{\text {III }}=\left(\nu_{1}^{\mathrm{I}}, \nu_{2}^{\text {III }}\right)$ by changing the values of player 2 :

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\text {III }}$ | $-1-$ | $-1-$ | $-m-$ | $-m-$ | $-1-$ | $-1-$ | $-m^{2}-$ | $-m^{2}-$ |

By truthfulness, we can conclude that the allocation remains the same, i.e., player 1 gets $A \cup B \cup$ $E \cup F$, while player 2 gets $C \cup D \cup G \cup H$.

Next, we move to profile $\mathbf{v}^{\mathrm{IV}}=\left(v_{1}^{\mathrm{III}}, v_{2}^{\mathrm{III}}\right)$ by changing the values of player 1 :

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{III}}$ | $-m^{3}-$ | $-m^{3}-$ | $-1-$ | $-1-$ | $-m^{2}-$ | $-m^{2}-$ | $-1-$ | $-m-$ |

Again, by truthfulness player 1 gets $A \cup B \cup E \cup F$, and player 2 gets $C \cup D \cup G \cup H$.
We continue by moving to profile $\mathbf{v}^{\mathrm{V}}=\left(\nu_{1}^{\mathrm{III}}, v_{2}^{\mathrm{IV}}\right)$ by changing the values of player 2 :

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\mathrm{IV}}$ | $-1-$ | $-m-$ | $-m^{3}-$ | $-1-$ | $-1-$ | $-m^{2}-$ | $-m^{4}-$ | $-1-$ |

Player 2 must get $G$ since he strongly desires it and $H \subseteq E_{2}$. The same goes for player 1 and $A \cup B$. Now we know that an exchange should take place, otherwise player 2 would deviate to $\mathbf{v}^{\text {III }}$ and become strictly better. Since the only available exchangeable set here is $C \cup D=S_{1}$ (because it is minimal), it is exchanged with set $S_{2}=E \cup F$ (the only set exchangeable with $S_{1}$ by lemma A.8). Thus we conclude that the allocation remains the same, player 1 gets $A \cup B \cup E \cup F$, while player 2 gets $C \cup D \cup G \cup H$.

Next proceed to profile $\mathbf{v}^{\mathrm{VI}}=\left(v_{1}^{\mathrm{IV}}, v_{2}^{\mathrm{IV}}\right)$ by changing the values of player 1 :

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\text {IV }}$ | $-m^{4}-$ | $-m-$ | $-1-$ | $-1-$ | $-m^{2}-$ | $-m^{3}-$ | $-1-$ | $-m^{2}-$ |

we can derive by truthfulness that player 1 must get (at least) $A \cup F$, or else he would deviate to profile $\mathbf{v}^{\mathrm{V}}$ and improve. Currently, this is all what we need to know for $\mathbf{v}^{\mathrm{VI}}$.

Now let us return to profile $\mathbf{v}^{\mathrm{II}}=\left(\nu_{1}^{\mathrm{II}}, \nu_{2}^{\mathrm{II}}\right)$. Starting from here we change the values of player 1 to get profile $\mathbf{v}^{\mathrm{VII}}=\left(\nu_{1}^{\mathrm{V}}, \nu_{2}^{\mathrm{II}}\right)$.

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{V}}$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-m^{2}-$ | $-m-$ | $-1-$ | $-1-$ | $-m-$ |

By truthfulness, the allocation remains the same, i.e., player 1 gets $A \cup D \cup E \cup H$, while player 2 gets $B \cup C \cup F \cup G$.

We now move to profile $\mathbf{v}^{\mathrm{VIII}}=\left(\nu_{1}^{\mathrm{V}}, \nu_{2}^{\mathrm{V}}\right)$ and change the values of player 2.

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{v}$ | $-1-$ | $-\alpha m^{3}-$ | $-\alpha m^{4}-$ | $-1-$ | $-m^{4}-$ | $-m^{5}-$ | $-m^{5}-$ | $-m^{4}-$ |

The values in $B \cup C$ are set in such a way so that $v_{2}^{\mathrm{V}}(B \cup C)>v_{2}^{\mathrm{V}}(E \cup H)$, but $v_{2}^{\mathrm{V}}(T)<v_{2}^{\mathrm{V}}(E \cup H)$ for any $T \subsetneq B \cup C C^{8}$

Notice that player 2 must get $G \cup F$ since he strongly desires it. The same goes for player 1 and $A \cup D$. We know that an exchange should take place, otherwise player 2 would deviate to $\mathbf{v}^{\text {VII }}$ and improve. In this exchange, values are such that player 2 should get the whole $S_{1}^{\prime}$. Thus we conclude that the allocation remains the same, i.e., player 1 gets $A \cup D \cup E \cup H$, while player 2 gets $B \cup C \cup F \cup G$.

We now move to profile $\mathbf{v}^{\mathrm{IX}}=\left(v_{1}^{\mathrm{VI}}, v_{2}^{\mathrm{V}}\right)$ and change the values of player 1 .

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{VI}}$ | $-m^{4}-$ | $-m-$ | $-1-$ | $-m^{4}-$ | $-m^{2}-$ | $-m^{3}-$ | $-1-$ | $-m^{2}-$ |

Again player 2 must get $G \cup F$. Given that, player 1 gets at least $A \cup D \cup E \cup H$, and by truthfulness he cannot receive strictly more items. Therefore, the allocation remains the same, i.e., player 1 gets $A \cup D \cup E \cup H$, while player 2 gets $B \cup C \cup F \cup G$.

We now move to profile $\mathbf{v}^{\mathrm{X}}=\left(\nu_{1}^{\mathrm{IV}}, \nu_{2}^{\mathrm{V}}\right)$ by changing the values of player 1 back to what he had at profile $\mathbf{v}^{\mathrm{VI}}$. Recall:

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{IV}}$ | $-m^{4}-$ | $-m-$ | $-1-$ | $-1-$ | $-m^{2}-$ | $-m^{3}-$ | $-1-$ | $-m^{2}-$ |

Like above Player 2 gets $F \cup G$. The same goes for player $1 A$. By truthfulness, an exchange must happen and player 1 gets at least the set $E \cup H$ (else he would deviate to $\mathbf{v}^{\mathrm{IX}}$ and improve). Moreover, since player 2 loses $E \cup H$ he must at least get the set $B \cup C$. We conclude that player 1 gets $A \cup E \cup H$, player 2 gets $B \cup C \cup F \cup G$, while we do not care what happens for items in $D$.

Now notice that player 2 can deviate from profile $\mathbf{v}^{\mathrm{VI}}$ to profile $\mathbf{v}^{\mathrm{X}}$ and become strictly better (recall that at profile $\mathbf{v}^{\mathrm{VI}}$ player 2 loses $G$, while $A, D, E, H$ all have very small value) and this contradicts truthfulness.

Proof of Lemma3.16, We begin with a direct implication of the Lemmata A.8 A.10, Although we are not guaranteed yet that any feasible exchange can be expressed as a union of exchange deals from $D$ as it should, the following corollary is a step towards this direction. Recall that $S_{1}, \ldots, S_{k}$ and $T_{1}, \ldots, T_{k}$ are all the minimally exchangeable subsets of $E_{1}$ and $E_{2}$ respectively, and that ( $S_{i}, T_{i}$ ) is the only feasible exchange involving either one of $S_{i}$ and $T_{i}$, for every $i \in[k]$.

Corollary A.11. For every exchangeable set $S \subseteq E_{1}$, we have that $S=W \cup \bigcup_{i \in I} S_{i}$, where $I \subseteq[k]$ with $|I| \geq 1$, while $W=S \backslash \bigcup_{i \in I} S_{i}$ does not contain any minimally exchangeable sets. Furthermore, this decomposition is unique.

[^7]Ideally, we would like two things. First, the $W$ part in the above decomposition to always be empty, i.e., we want every exchangeable set to be a union of minimally exchangeable sets (umes for short). Second, we want every umes of $E_{1}$ to be exchangeable only with the corresponding umes of $E_{2}$, and vice versa. To be more precise, we say that an umes $S=\bigcup_{i \in I} S_{i}$ is nice if it is exchangeable with $T=\bigcup_{i \in I} T_{i}$ and only with $T$. The definition of a nice umes of $E_{2}$ is symmetric. As it turns out, every umes is nice, but it takes a rather involved induction to prove it. Especially the fact that $\left(\bigcup_{i \in I} S_{i}, \bigcup_{i \in I} T_{i}\right)$ is exchangeable needs a carefully constructed argument about the value that each player must gain from any exchange (see also Lemma A.14).
Lemma A.12. Every umes is nice.
Given the above lemma, we can now show that the set $W$ in the decomposition of Corollary A.11is always empty. In fact the proof idea is the same as the one for Lemmata A.8-A.10.

Lemma A.13. Every exchangeable set is an umes.
The above two lemmata complete the proof. They are proved below, right after LemmaA. 14 ,

For the following lemmas, recall that umes is short for union of minimally exchangeable sets!

Lemma A.14. Let $(S, T)$ be a feasible exchange such that $S$ is a nice umes with the property that if $S^{\prime} \subseteq S$ is exchangeable, then $S^{\prime}$ is a nice umes. In particular, let $S=\bigcup_{i \in[r]} S_{i}$, where $S_{i}$ is minimally exchangeable for all $i \in[r]$. If $\mathbf{v}$ is a profile where $\left(S_{i}, T_{i}\right)$ is favorable for all $i \in[r]$ then $(S, T)$ gives a lower bound on the value gained from exchanges in profile $\mathbf{v}$ for each player.

Proof of Lemma A.14. Due to symmetry, it suffices to prove the lower bound for player 1. Let $\mathbf{v}=\left(v_{1}, v_{2}\right)$ be a profile like in the statement, where the values are $v_{i 1}, v_{i 2}, \ldots, v_{i m}$ for $i=1,2$.. Since $(S, T)$ is a feasible exchange, there exists a profile $\mathbf{v}^{\mathrm{I}}=\left(\nu_{1}^{\mathrm{I}}, \nu_{2}^{\mathrm{I}}\right) \in \mathcal{V}_{m}^{\neq}$such that the exchange $(S, T)$ takes place, i.e., $X_{1}\left(\mathbf{v}^{\mathrm{I}}\right)=\left(E_{1} \backslash S\right) \cup T$ and $X_{2}\left(\mathbf{v}^{\mathrm{l}}\right)=S \cup\left(E_{2} \backslash T\right)$. Starting from this profile we will use a series of intermediate profiles in order to reach $\mathbf{v}=\left(\nu_{1}, \nu_{2}\right)$. Initially consider profile $\mathbf{v}^{\mathrm{II}}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{I}}\right)$ where we change the values of player 1 .

$$
v_{1 j}^{\mathrm{II}}= \begin{cases}\frac{m \cdot \max _{i} v_{1 i}}{\min _{i} v_{1 i}} \cdot v_{1 j} & \text { if } j \in E_{1} \backslash S \\ v_{1 j} & \text { if } j \in S \cup T \\ \frac{\min _{i} v_{1 i}}{m \cdot \max _{i} v_{1 i}} \cdot v_{1 j} & \text { otherwise }\end{cases}
$$

In this profile each item in $E_{1} \backslash S$ has a value which is higher from the sum of the values in all the other sets. On the other hand, items in $E_{2} \backslash T$ have total value less than the value of a single item in the other sets $\sqrt[9]{9}$ Since this is the case, player 1 must get $E_{1} \backslash S$ since he strongly desires it. In addition, an exchange must take place, or player 1 could deviate to profile $\mathbf{v}^{1}$ and become strictly better. Thus an exchange takes place and must involve a subset $S^{\prime}$ of $S$. Now if $S^{\prime}$ was a proper subset of $S$, then it would be a nice umes, i.e., $S^{\prime}=\bigcup_{j \in I} S_{j}, I \subsetneq[r]$, and it is exchanged only with $T^{\prime}=\bigcup_{j \in I} T_{j}$. However, since exchanges $S_{j}, T_{j}, j \in[r] \backslash I$ are also favorable, player 1 would deviate to profile $\mathbf{v}^{\mathrm{I}}$ and become strictly better. Therefore, the exchange involves the whole $S$, and since $S$ is a nice umes it should be exchanged with $T$. So the allocation here is $X_{1}\left(\mathbf{v}^{\mathrm{II}}\right)=\left(E_{1} \backslash S\right) \cup T$, $X_{2}\left(\mathbf{v}^{\text {II }}\right)=S \cup\left(E_{2} \backslash T\right)$.

[^8]By moving to profile $\mathbf{v}^{\text {III }}=\left(v_{1}^{\mathrm{II}}, \nu_{2}\right)$ where we change the values of player 2, we have that, once again, player 1 must get the items in $E_{1} \backslash S$. Moreover, an exchange must take place, or player 2 could deviate to profile $\mathbf{v}^{\text {II }}$ and become strictly better (recall that he prefers $S$ from $T$ ). By following the same arguments as in the previous case, if the exchange involves a proper subset of $S$, player 2 would deviate to profile $\mathbf{v}^{11}$ and become strictly better. Hence player 2 gets the whole $S$, i.e., the allocation here is again $X_{1}\left(\mathbf{v}^{\mathrm{III}}\right)=\left(E_{1} \backslash S\right) \cup T$ and $X_{2}\left(\mathbf{v}^{\mathrm{III}}\right)=S \cup\left(E_{2} \backslash T\right)$.

Finally we move to profile $\mathbf{v}=\left(\nu_{1}, v_{2}\right)$ by changing the values of player 1. It is easy to see that if there is no exchange that improves player 1 by at least $\nu_{1}(T)-\nu_{1}(S)$, then he could deviate to profile $\mathbf{v}^{\text {III }}=\left(\nu_{1}^{\mathrm{II}}, \nu_{2}\right)$ and gain exactly that.

Proof of Lemman.12. We will use induction in the number of minimally exchangeable sets contained in an umes; let us call this number index of the umes. Lemmata A. 8 and A.9 imply that every umes of index 1 is nice. That is the basis of our induction.

Assume that every umes of index lower or equal to $k$ is nice and notice that Lemma A. 10 implies that every exchangeable subset of an umes is also an umes.

Let $S$ be an umes of index $k+1$. In particular, let $S=\bigcup_{i \in[k+1]} S_{i}$, where for any $i \in[k+1]$ we have that $S_{i}$ is minimally exchangeable and $\left(S_{i}, T_{i}\right)$ is a feasible exchange. By the inductive hypothesis we have that both $S_{1}$ and $S^{\prime}=\bigcup_{i=2}^{k+1} S_{i}$ are nice umes and uniquely exchangeable with $S_{1}$ and $T^{\prime}=\cup_{i=2}^{k+1} T_{i}$ respectively.

We first prove that $(S, T)$ is a feasible exchange. Consider the following profile $\mathbf{v}=\left(\nu_{1}, v_{2}\right)$,

|  | $E_{1} \backslash\left(S^{\prime} \cup S_{1}\right)$ | $S^{\prime}$ | $S_{1}$ | $T^{\prime}$ | $T_{1}$ | $E_{2} \backslash\left(T^{\prime} \cup T_{1}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}$ | $-\Delta-$ | $-\delta-$ | $-\epsilon-$ | $-1-$ | $-\zeta-$ | $-\delta-$ |
| $v_{2}$ | $-\delta-$ | $-n_{j}-$ | $-1-$ | $-\theta_{j}-$ | $-\delta-$ | $-\Delta-$ |

where $\Delta \gg 1 \gg \zeta, n_{j}, \theta_{j}, \epsilon \gg \delta \gg \lambda_{i}{ }^{10}$ Regarding the rest values, $\left|T_{1}\right| \cdot \zeta=\left|S_{1}\right| \cdot \epsilon+\lambda_{1}$ and for all $j \in[k+1] \backslash\{1\}$ we have that $\left|S_{j}\right| \cdot n_{j}=\left|T_{j}\right| \cdot \theta_{j}+\lambda_{j}$. Now notice that $S^{\prime}$ is a nice umes such that every exchangeable $S^{\prime \prime} \subseteq S^{\prime}$ is a nice umes and for all $j,\left(S_{j}, T_{j}\right)$ is a favorable exchange with respect to v. Lemma A.14 guarantees that in $\mathbf{v}$, player 1 gains at least $\nu_{1}\left(T^{\prime}\right)-\nu_{1}\left(S^{\prime}\right)=\left|T^{\prime}\right|-\delta\left|S^{\prime}\right|$ from the exchanges. So player 1 gets a superset of $T^{\prime}$, i.e., $T^{\prime} \subseteq X_{1}^{E_{2}}(\mathbf{v})$. By lemma.10, this means that $X_{1}^{E_{2}}(\mathbf{v})$ is either $T^{\prime}$ or $T$.

On the other hand, if we apply lemmanfor ( $S_{1}, T_{1}$ ) we have that in profile $\mathbf{v}$, player 2 should gain at least $v_{2}\left(S_{1}\right)-v_{2}\left(T_{1}\right)=\left|S_{1}\right|-\delta\left|T_{1}\right|$ from the exchanges. So, $X_{2}^{E_{1}}(\mathbf{v}) \supseteq S_{1}$. Since $T^{\prime}$ is nice, however, we have that $X_{1}^{E_{2}}(\mathbf{v})=T^{\prime}$ implies $X_{2}^{E_{1}}(\mathbf{v})=S^{\prime} \nsupseteq S_{1}$. Therefore, it must be the case where $X_{1}^{E_{2}}(\mathbf{v}) \supsetneq T^{\prime}$ or else player 2 does not get enough value.

We conclude that $X_{2}^{E_{1}}(\mathbf{v})=T$. Now we claim that $X_{2}^{E_{1}}(\mathbf{v})=S$ and therefore $(S, T)$ is a feasible exchange. Indeed, every $S^{\prime \prime} \subsetneq S$ that is exchangeable is an umes of index lower or equal to $k$ and therefore is nice. So $S^{\prime \prime}, T$ cannot be a feasible exchange, due to the fact that $S^{\prime \prime}$ has a unique pair $T^{\prime \prime} \subsetneq T$.

Next we show that there is no $\hat{T} \neq T$ such that $(S, \hat{T})$ is a feasible exchange. By the proof so far we have that if such a $\hat{T}$ existed, then it is not a subset of $T$. So suppose that there is a $\hat{T} \neq T$ such that $(S, \hat{T})$ is a feasible exchange and let $T^{*}$ be a minimal such set (that is, if $R \subsetneq T^{*}$ then $(S, R)$ is not a feasible exchange or $R \subseteq T$ ).

[^9]Thus there are two profiles $\mathbf{v}^{\mathrm{I}}=\left(\nu_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$ and $\mathbf{v}^{\mathrm{II}}=\left(\nu_{1}^{\mathrm{II}}, v_{2}^{\mathrm{II}}\right)$ where we have that $X^{E_{1}}\left(\mathbf{v}^{\mathrm{I}}\right)=\left(E_{1} \backslash S\right)=$ $X_{1}^{E_{1}}\left(\mathbf{v}^{\mathrm{II}}\right)$ and $X_{1}^{E_{2}}\left(\mathbf{v}^{\mathrm{I}}\right)=T^{*} \neq T=X_{1}^{E_{2}}\left(\mathbf{v}^{\mathrm{II}}\right)$.

For the sake of readability, let $A=T^{*} \backslash T, B=T^{*} \cap T, C=T \backslash T^{*}, D=E_{2} \backslash\left(T^{*} \cup T\right)$.
We start from profile $\mathbf{v}^{\mathrm{I}}$ where the allocation is $X_{1}\left(\mathbf{v}^{\mathrm{I}}\right)=\left(E_{1} \backslash S\right) \cup A \cup B, X_{2}\left(\mathbf{v}^{\mathrm{I}}\right)=S \cup C \cup D$ and we proceed to profile $\mathbf{v}^{\text {III }}=\left(\nu_{1}^{\mathrm{I}}, \nu_{2}^{\text {III }}\right)$ by changing the values of player 2 :

|  | $E_{1} \backslash S$ | $S$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\text {III }}$ | $-1-$ | $-m-$ | $-1-$ | $-1-$ | $-m^{2}-$ | $-m^{2}-$ |

By truthfulness, the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{\text {III }}\right)=\left(E_{1} \backslash S\right) \cup A \cup B, X_{2}\left(\mathbf{v}^{\text {III }}\right)=S \cup C \cup D$.
Next we move to profile $\mathbf{v}^{\mathrm{IV}}=\left(v_{1}^{\mathrm{III}}, v_{2}^{\mathrm{III}}\right)$ by changing the values of player 1 :

|  | $E_{1} \backslash S$ | $S$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\text {III }}$ | $-m^{3}-$ | $-m-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-1-$ |

Notice that player 1 must receive $E_{1} \backslash S$ since he strongly desires it. The same goes for player 2 and $C \cup D$. Now we know that an exchange should take place and that in this exchange player 1 must get at least set $A=T^{*} \backslash T$ (otherwise he would deviate to $\mathbf{v}^{\text {III }}$ and become strictly better).

We claim that player 1 gets the whole $T^{*}$. If this was not the case then he would get some set $R \supseteq A \neq \varnothing$. Since $R \subsetneq T^{*}$ and $R \nsubseteq T$ we have that the exchange $(S, R)$ is not feasible due to the minimality of $T^{*}$. Thus $R$ is exchanged with some $\hat{S} \subsetneq S$. However $\hat{S}$ is an umes (by Lemma A.8) and by inductive hypothesis it is exchangeable only with strict subsets of $T$ which is a contradiction. Similarly, player 2 must get set the whole $S$, or otherwise he would get some $\hat{S} \subsetneq S$ which is exchangeable only with strict subsets of $T$, something that can not happen. Thus the allocation here is $X_{1}\left(\mathbf{v}^{\text {IV }}\right)=\left(E_{1} \backslash S\right) \cup A \cup B, X_{2}\left(\mathbf{v}^{\text {IV }}\right)=S \cup C \cup D$.

Next we move to profile $\mathbf{v}^{\mathrm{V}}=\left(\nu_{1}^{\mathrm{III}}, \nu_{2}^{\mathrm{IV}}\right)$ by changing the values of player 2 .

|  | $E_{1} \backslash S$ | $S$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\text {IV }}$ | $-1-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-m-$ | $-m^{3}-$ |

By truthfulness, the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{v}\right)=\left(E_{1} \backslash S\right) \cup A \cup B, X_{2}\left(\mathbf{v}^{v}\right)=S \cup C \cup D$.
Now let us return to profile $\mathbf{v}^{\mathrm{II}}=\left(\mathbf{v}_{1}^{\mathrm{II}}, \mathbf{v}_{2}^{\mathrm{II}}\right)$. Starting from this profile we change the values of player 2 and get profile $\mathbf{v}^{\mathrm{VI}}=\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{V}}\right)$.

|  | $E_{1} \backslash S$ | $S$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\mathrm{V}}$ | $-1-$ | $-m-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-m^{2}-$ |

Since player's 2 most valuable items are those which he was allocated in profile $\mathbf{v}^{\text {II }}$, by truthfulness, the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{\mathrm{VI}}\right)=\left(E_{1} \backslash S\right) \cup B \cup C, X_{2}\left(\mathbf{v}^{\mathrm{VI}}\right)=S \cup A \cup D$.

Next we move to profile $\mathbf{v}^{\mathrm{VII}}=\left(v_{1}^{\mathrm{IV}}, v_{2}^{\mathrm{V}}\right)$ by changing the values of player 1 .

|  | $E_{1} \backslash S$ | $S$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{IV}}$ | $-m^{4}-$ | $-m-$ | $-m^{3}-$ | $-m^{2}-$ | $-m^{2}-$ | $-1-$ |

Notice that player 1 must get $E_{1} \backslash S$ since he strongly desires it. The same goes for player 2 and $A \cup D$. Given that, an exchange takes place and in this exchange player 1 must get the whole $B \cup C=T$ (otherwise he would deviate to $\mathbf{v}^{\mathrm{V}}$ and strictly improve). On the other hand, player

2 must get the whole $S$, or player 1 would deviate from $\mathbf{v}^{\mathrm{v}}$ to $\mathbf{v}^{\mathrm{VI}}$ and strictly improve. Thus the allocation here remains the same: $X_{1}\left(\mathbf{v}^{\mathrm{VII}}\right)=\left(E_{1} \backslash S\right) \cup B \cup C, X_{2}\left(\mathbf{v}^{\mathrm{VII}}\right)=S \cup A \cup D$.

Next we move to profile $\mathbf{v}^{\mathrm{VIII}}=\left(\nu_{1}^{\mathrm{IV}}, v_{2}^{\mathrm{VI}}\right)$ by changing the values of player 2 .

|  | $E_{1} \backslash S$ | $S$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\mathrm{VI}}$ | $-1-$ | $-m^{2}-$ | $-m-$ | $-1-$ | $-1-$ | $-m^{3}-$ |

By truthfulness, the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{\mathrm{VIII}}\right)=\left(E_{1} \backslash S\right) \cup B \cup C, X_{2}\left(\mathbf{v}^{\mathrm{VIII}}\right)=S \cup A \cup D$.
Finally we move to profile $\mathbf{v}^{\mathrm{IX}}=\left(\nu_{1}^{\mathrm{III}}, \nu_{2}^{\mathrm{VI}}\right)$ by changing the values of player 1 back to what he had in profile $\mathbf{v}^{\mathrm{V}}$. Recall:

|  | $E_{1} \backslash S$ | $S$ | $A$ | $B$ | $C$ | $D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\text {III }}$ | $-m^{3}-$ | $-m-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-1-$ |

Notice that player 1 must get $E_{1} \backslash S$ since he strongly desires it. The same goes for player 2 and $D$. Now if player 1 gets nothing from set $A$ then there is no exchange at all. However, in this case player 2 would deviate to profile $\mathbf{v}^{v}$ and become strictly better. Thus player 1 should get at least one item from $A$. As a result, however, player 1 would deviate from profile $\mathbf{v}^{\mathrm{VIII}}$ to $\mathbf{v}^{\mathrm{IX}}$ and become strictly better, something that leads to contradiction.

This completes the inductive step.

Proof of Lemma A.13, Let $S$ be an exchangeable subset of $E_{1}$. Then according to corollary A. 11 $S=\bigcup_{i \in I} S_{i} \cup W$ for some $I \subseteq[k]$, with $|I| \geq 1$. We are going to show that $W=\varnothing$. So suppose, towards a contradiction, that $W \neq \varnothing$. In fact, choose $S$ so that it is a minimal exchangeable nonumes subset of $E_{1}$, i.e., for all $S^{\prime} \subsetneq S, S^{\prime}$ is either umes or non-exchangeable. In addition, notice that $W$ does not contain any exchangeable sets.

Let $T$ be such that $(S, T)$ is a feasible exchange. In fact let $T$ be a minimal such set, i.e., for all $T^{\prime} \subsetneq T$, either ( $S, T^{\prime}$ ) is not a feasible exchange or $T^{\prime}$ is not exchangeable at all. Finally, let $S^{*}=\bigcup_{i \in I} S_{i}, T^{*}=\bigcup_{i \in I} T_{i}$ and notice that $T \backslash T^{*} \neq \varnothing$ since otherwise $T$ would be an umes (as an exchangeable subset of an umes, by Lemma.10).

For the sake of readability, let $A=E_{1} \backslash S, B=T \backslash T^{*}, C=T^{*} \cap T, D=T^{*} \backslash T$, and $E=E_{2} \backslash\left(T \cup T^{*}\right)$.
So there are two profiles, $\mathbf{v}^{\mathrm{I}}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$ where $X_{1}\left(\mathbf{v}^{\mathrm{I}}\right)=A \cup B \cup C, X_{2}\left(\mathbf{v}^{\mathrm{I}}\right)=S \cup D \cup E$ and $\mathbf{v}^{\mathrm{II}}=$ $\left(v_{1}^{\mathrm{II}}, v_{2}^{\mathrm{II}}\right)$ where $X_{1}\left(\mathbf{v}^{\mathrm{II}}\right)=A \cup W \cup C \cup D$ and $X_{2}\left(\mathbf{v}^{\mathrm{II}}\right)=S^{*} \cup B \cup E$.

We start from profile $\mathbf{v}^{\mathrm{I}}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$ and we proceed to profile $\mathbf{v}^{\text {III }}=\left(v_{1}^{\mathrm{I}}, \nu_{2}^{\text {III }}\right)$ by changing the values of player 2 :


Since player's 2 most valuable items are those which he was allocated in profile $\mathbf{v}^{1}$, by truthfulness, the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{\text {III }}\right)=A \cup B \cup C, X_{2}\left(\mathbf{v}^{\text {III }}\right)=S \cup D \cup E$.

Next we move to profile $\mathbf{v}^{\mathrm{IV}}=\left(v_{1}^{\mathrm{III}}, v_{2}^{\mathrm{III}}\right)$ by changing the values of player 1 :

|  | $A$ | $S^{*}$ | $W$ | $B$ | $C$ | $D$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\text {III }}$ | $-m^{3}-$ | $-m-$ | $-m-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-1-$ |

Now notice that player 1 must get $A$ since he strongly desires it. The same goes for player 2 and $D \cup E$. Also we know that an exchange should take place and that in this exchange player 1 must get at least $B=T \backslash T^{*}$ (otherwise he would deviate to $\mathbf{v}^{\mathrm{III}}$ ).

We claim that player 1 gets the whole $T$. If this was not the case then he would get some set $R \supseteq B \neq \varnothing$. Since $R \subsetneq T$ and $R \nsubseteq T^{*}$ we have that the exchange $(S, R)$ is not feasible due to the fact that $T$ is minimal. Thus $R$ should be exchanged with some $\hat{S} \subsetneq S$. However, by the minimality of $S, \hat{S}$ is an umes and it is exchangeable only with strict subsets of $T^{*}$, which is a contradiction. On the other hand, player 2 must get the whole $S$, or otherwise he would get some $\hat{S} \subsetneq S$ which is exchangeable only with strict subsets of $T^{*}$, something that can not happen. Thus the allocation is $X_{1}\left(\mathbf{v}^{\mathrm{IV}}\right)=A \cup B \cup C$ and $X_{2}\left(\mathbf{v}^{\mathrm{IV}}\right)=S \cup D \cup E$.

Next we move to profile $\mathbf{v}^{\mathrm{V}}=\left(v_{1}^{\mathrm{III}}, v_{2}^{\mathrm{IV}}\right)$ by changing the values of player 2:

|  | $A$ | $S^{*}$ | $W$ | $B$ | $C$ | $D$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\text {III }}$ | $-1-$ | $-m^{2}-$ | $-m-$ | $-1-$ | $-1-$ | $-m^{2}-$ | $-m^{3}-$ |

Since player's 2 most valuable items are those which he was allocated in profile $\mathbf{v}^{\mathrm{IV}}$, by the truthfulness of the mechanism, he must continue to get them but he can not get any other item. Thus the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{\vee}\right)=A \cup B \cup C, X_{2}\left(\mathbf{v}^{\mathrm{V}}\right)=S \cup D \cup E$.

Now let us return to profile $\mathbf{v}^{\mathrm{II}}=\left(v_{1}^{\mathrm{II}}, \nu_{2}^{\mathrm{II}}\right)$. Starting from this profile we change the values of player 2 and get profile $\mathbf{v}^{\mathrm{VI}}=\left(\nu_{1}^{\mathrm{II}}, \nu_{2}^{\mathrm{IV}}\right)$ :

|  | $A$ | $S^{*}$ | $W$ | $B$ | $C$ | $D$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\text {IV }}$ | $-1-$ | $-m^{2}-$ | $-m-$ | $-m^{4}-$ | $-1-$ | $-1-$ | $-m^{4}-$ |

Again, player's 2 most valuable items are those which he was allocated in profile $\mathbf{v}^{\text {II }}$. So, by truthfulness, the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{\mathrm{VI}}\right)=A \cup W \cup C \cup D$ and $X_{2}\left(\mathbf{v}^{\mathrm{VI}}\right)=S^{*} \cup B \cup E$.

Next we move to profile $\mathbf{v}^{\mathrm{VII}}=\left(\nu_{1}^{\mathrm{IV}}, \nu_{2}^{\mathrm{IV}}\right)$ by changing the values of player 1 :

|  | $A$ | $S^{*}$ | $W$ | $B$ | $C$ | $D$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\nu_{1}^{\mathrm{IV}}$ | $-m^{5}-$ | $-m-$ | $-m^{2}-$ | $-m^{4}-$ | $-m^{3}-$ | $-m^{3}-$ | $-1-$ |

Notice that player 1 must get $A$ and player 2 must get $B \cup E$. Given that, player 1 must get $W \cup$ $C \cup D=$ (otherwise he could deviate to $\mathbf{v}^{\mathrm{V}}$ and strictly improve). Thus the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{\mathrm{VII}}\right)=A \cup W \cup C \cup D$ and $X_{2}\left(\mathbf{v}^{\mathrm{VII}}\right)=S^{*} \cup B \cup E$.

Next we move to profile $\mathbf{v}^{\mathrm{VIII}}=\left(\nu_{1}^{\mathrm{IV}}, \nu_{2}^{\mathrm{V}}\right)$ by changing the values of player 2 :

|  | $A$ | $S^{*}$ | $W$ | $B$ | $C$ | $D$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{v}$ | $-1-$ | $-m^{3}-$ | $-m-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-m^{4}-$ |

Again, by truthfulness, the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{\text {VIII }}\right)=A \cup W \cup C \cup D$ and $X_{2}\left(\mathbf{v}^{\text {VIII }}\right)=$ $S^{*} \cup B \cup E$.

Finally we move to profile $\mathbf{v}^{\mathrm{IX}}=\left(\nu_{1}^{\mathrm{III}}, \nu_{2}^{V}\right)$ by changing the values of player 1 back to what he had in profile $\mathbf{v}^{\mathrm{V}}$.

|  | $A$ | $S^{*}$ | $W$ | $B$ | $C$ | $D$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{1}^{\mathrm{III}}$ | $-m^{3}-$ | $-m-$ | $-m-$ | $-m^{2}-$ | $-1-$ | $-1-$ | $-1-$ |

Player 1 must get $A$ and player 2 must get $E$. Now if player 1 gets nothing from $B$ then there will be no exchange. However, in this case player 2 would deviate to profile $\mathbf{v}^{\mathrm{V}}$ and become strictly better. Thus player 1 should get at least one item from $B$. As a result, player 1 would deviate from profile $\mathbf{v}^{\text {viII }}$ to $\mathbf{v}^{\mathrm{IX}}$ and strictly improve, something that leads to contradiction.

Proof of Lemma 3.17. Without loss of generality, assume that $\left(S_{1}, T_{1}\right), \ldots,\left(S_{r}, T_{r}\right)$ is the set of all favorable exchanges. Then $(S, T)$ where $S=\bigcup_{i \in[r]} S_{i}$ and $T=\bigcup_{i \in[r]} T_{i}$ will give a lower bound on the value of each player. Indeed, $S$ is an umes an using Lemmata A. 14 and A.12 we have that player 1 should gain at least $\nu_{1}(T)-v_{1}(S)$, while player 2 should gain at least $\nu_{2}(S)-v_{2}(T)$ from the exchanges.

Since $\mathbf{v} \in \mathcal{V}_{m}^{\neq}$, it suffices to show that $\nu_{1}\left(X_{1}(\mathbf{v})\right)=\nu_{1}\left(\left(E_{1} \cup T\right) \backslash S\right)=v_{1}\left(E_{1}\right)+v_{1}(T)-v_{1}(S)$. So suppose that $v_{1}\left(X_{1}(\mathbf{v})\right)>\nu_{1}\left(E_{1}\right)+\nu_{1}(T)-\nu_{1}(S)$ and notice that this also implies that $v_{2}\left(X_{2}(\mathbf{v})\right)>$ $v_{2}\left(E_{2}\right)+v_{2}(S)-v_{2}(T)$, since otherwise it would be $\nu_{2}\left(X_{2}(\mathbf{v})\right)=v_{2}\left(E_{2}\right)+v_{2}(S)-v_{2}(T)$ and we have the desired allocation.

As a result, there exists some $S^{*} \subseteq X_{2}^{E_{1}}(\mathbf{v})$, such that $S^{*}$ is an umes but $\left(S^{*}, T^{*}\right)$-where $T^{*}$ is the "pair" of $S^{*}$-is unfavorable. Without loss of generality, we may assume that $\nu_{1}\left(T^{*}\right)<\nu_{1}\left(S^{*}\right)$. Now let $S^{\prime}$ to be the union of all minimally exchangeable sets $S_{j} \subseteq X_{2}^{E_{1}}(\mathbf{v})$ such that $v_{1}\left(T_{j}\right)<$ $\nu_{1}\left(S_{j}\right)$, and notice that $S^{\prime} \subsetneq X_{2}^{E_{1}}(\mathbf{v})$ and $\nu_{1}\left(T^{\prime}\right)<\nu_{1}\left(S^{\prime}\right)$.

Let $S^{*}=X_{2}^{E_{1}}(\mathbf{v})$ and $T^{*}=X_{1}^{E_{2}}(\mathbf{v})$. We begin with profile $\mathbf{v}=\left(\nu_{1}, \nu_{2}\right)$ where the allocation is $X_{1}(\mathbf{v})=\left(E_{1} \backslash S^{*}\right) \cup T^{*}$ and $X_{2}(\mathbf{v})=\left(E_{2} \backslash T^{*}\right) \cup S^{*}$ and we move to profile $\mathbf{v}^{\prime}=\left(\nu_{1}, v_{2}^{\prime}\right)$.

|  | $E_{1} \backslash S^{*}$ | $S^{*}$ | $T^{*}$ | $E_{2} \backslash T^{*}$ |
| :---: | :---: | :---: | :---: | :---: |
| $v_{2}^{\prime}$ | $-1-$ | $-m-$ | $-1-$ | $-m-$ |

By truthfulness, the allocation remains the same, i.e., $X_{1}\left(\mathbf{v}^{\prime}\right)=\left(E_{1} \backslash S^{*}\right) \cup T^{*}$ and $X_{2}\left(\mathbf{v}^{\prime}\right)=\left(E_{2} \backslash T^{*}\right) \cup$ $S^{*}$.

However, now notice that ( $S^{*} \backslash S^{\prime}, T^{*} \backslash T^{\prime}$ ) is a favorable exchange with respect to $\mathbf{v}^{\prime}$. Moreover, for every minimally exchangeable set $S_{i} \subseteq S^{*} \backslash S^{\prime}$ it holds that ( $S_{i}, T_{i}$ ) is favorable. By using lemma A.14] we have that the gain from the exchange in $\mathbf{v}^{\prime}$ for player 1 must be at least $v_{1}\left(T^{*} \backslash T^{\prime}\right)-v_{1}\left(S^{*} \backslash\right.$ $\left.S^{\prime}\right)>\nu_{1}\left(T^{*}\right)-v_{1}\left(S^{*}\right)$ so we arrive at a contradiction.

Proof of Lemma 3.18, Let $\mathbf{v}=\left(v_{1}, v_{2}\right)$ be a profile in $V_{m}$. By Lemmata 3.14 and A.13, we know that $X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})$ is the result of some exchanges of $D$ taking place, i.e., $X_{1}^{E_{1} \cup E_{2}}(\mathbf{v})=\left(E_{1} \backslash \bigcup_{i \in I} S_{i}\right) \cup$ $\bigcup_{i \in I} T_{i}$, where $I \subseteq[k]$. There are two things that can go wrong: either there exists some $x \in I$ such that ( $S_{x}, T_{x}$ ) is unfavorable, or there exists some $z \in[k] \backslash I$ such that $\left(S_{z}, T_{z}\right)$ is favorable. We first examine the former case.

Without loss of generality, we may assume that $\nu_{1}\left(T_{x}\right)<\nu_{1}\left(S_{x}\right)$. Consider the profile $\mathbf{v}^{\prime}=$ ( $\nu_{1}, v_{2}^{\mathrm{I}}$ ) where

$$
v_{2 j}^{\mathrm{I}}= \begin{cases}m+2^{j-m-1} & \text { if } j \in X_{2}(\mathbf{v}) \\ 1+2^{j-m-1} & \text { otherwise }\end{cases}
$$

By truthfulness, $X_{2}\left(\mathbf{v}^{\prime}\right)=X_{2}(\mathbf{v})$. Note also that $v_{2}^{1}$ induces for player 2 a strict preference over all subsets (see also Remark A.7). Moreover, with respect to $v_{2}^{I}$ the set of "good" minimal exchanges is exactly $\left\{\left(S_{i}, T_{i}\right) \mid i \in I\right\}$.

We now claim that player 1 can deviate and strictly improve his utility, thus contradicting truthfulness. In particular, consider the profile $\mathbf{v}^{\prime \prime}=\left(v_{1}^{\mathrm{I}}, v_{2}^{\mathrm{I}}\right)$ where

$$
v_{1 j}^{\mathrm{I}}= \begin{cases}m+2^{j-m-1} & \text { if } j \in\left(X_{1}\left(\mathbf{v}^{\prime}\right) \cup S_{x}\right) \backslash T_{x} \\ 1+2^{j-m-1} & \text { otherwise }\end{cases}
$$

Again, $v_{1}^{\mathrm{I}}$ induces for player 1 a strict preference over all subsets, and thus $\mathbf{v}^{\prime \prime} \in \mathcal{V}_{m}^{\neq}$. As a result, $\arg \max _{S \in \mathscr{O}_{i}} \nu_{i}^{\mathrm{I}}(S)$ only contains $X_{i}^{N_{i}}(\mathbf{v})$, for $i \in\{1,2\}$, and by Lemma 3.12 we have $X_{1}^{N_{1} \cup N_{2}}\left(\mathbf{v}^{\prime \prime}\right)=$ $X_{1}^{N_{1} \cup N_{2}}\left(\mathbf{v}^{\prime}\right)$. Additionally, notice that with respect to $\mathbf{v}^{\prime \prime}$ the set of favorable minimal exchanges is $\left\{\left(S_{i}, T_{i}\right) \mid i \in I \backslash\{x\}\right\}$. So, by Lemma 3.17 we have $X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime \prime}\right)=\left(E_{1} \backslash \bigcup_{i \in I \backslash\{x\}} S_{i}\right) \cup \bigcup_{i \in I\{x\}} T_{i}=$ $\left(X_{1}^{E_{1} \cup E_{2}}\left(\mathbf{v}^{\prime}\right) \cup S_{x}\right) \backslash T_{x}$.

So, by deviating from $\mathbf{v}^{\prime}$ to $\mathbf{v}^{\prime \prime}$, player 1 improves his utility by $\nu_{1}\left(S_{x}\right)-\nu_{1}\left(T_{x}\right)>0$, which contradicts truthfulness. We conclude that there is no $x \in I$ such that ( $S_{x}, T_{x}$ ) is unfavorable with respect to $\mathbf{v}$.

Next, we move on to the second case, i.e., there exists some $z \in[k] \backslash I$ such that $\left(S_{z}, T_{z}\right)$ is favorable with respect to $\mathbf{v}$. Like in the first case, the valuation functions that we define induce strict preferences over all subsets. Consider the profile $Q=\left(\nu_{1}^{\mathrm{II}}, \nu_{2}\right)$ where

$$
v_{1 j}^{\mathrm{II}}= \begin{cases}m^{2}+2^{j-m-1} & \text { if } j \in X_{1}(\mathbf{v}) \backslash S_{z} \\ m+2^{j-m-1} & \text { if } j \in T_{z} \\ 1+2^{j-m-1} & \text { otherwise }\end{cases}
$$

We know, by Lemmata 3.14 and A.13, that $X_{1}^{E_{1} \cup E_{2}}(Q)=\left(E_{1} \backslash \bigcup_{i \in J} S_{i}\right) \cup \bigcup_{i \in J} T_{i}$ for some $J \subseteq[k]$. By truthfulness, $X_{1}^{N_{1} \cup N_{2}}(Q) \supseteq X_{1}^{N_{1} \cup N_{2}}(\mathbf{v})$. In fact, by Lemma 3.12, it must be the case where $X_{1}^{N_{1} \cup N_{2}}(Q)=X_{1}^{N_{1} \cup N_{2}}(\mathbf{v})$. Again by truthfulness, $X_{1}^{E_{1}}(Q) \supseteq X_{1}^{E_{1}}(\mathbf{v}) \backslash S_{z}=E_{1} \backslash \bigcup_{i \in I \cup\{z\}} S_{i}$ and $X_{1}^{E_{2}}(Q) \supseteq$ $X_{1}^{E_{2}}(\mathbf{v})=\bigcup_{i \in I \cup\{z\}} T_{i}$. This implies that $I \subseteq J \subseteq I \cup\{z\}$. If $J=I \cup\{z\}$, then player 1, by deviating from $\mathbf{v}$ to $Q$, improves his utility by $\nu_{1}\left(T_{z}\right)-\nu_{1}\left(S_{z}\right)>0$, which contradicts truthfulness. So, it must be the case where $J=I$.

Now, consider the profile $Q^{\prime}=\left(\nu_{1}^{\mathrm{II}}, \nu_{2}^{\mathrm{II}}\right) \in \mathcal{V}_{m}^{\neq}$where

$$
v_{2 j}^{\mathrm{II}}= \begin{cases}m+2^{j-m-1} & \text { if } j \in X_{2}^{N_{1} \cup N_{2}}(Q) \cup \bigcup_{i \in I \cup\{z\}} S_{i} \cup \bigcup_{i \notin I \cup\{z\}} T_{i} \\ 1+2^{j-m-1} & \text { otherwise }\end{cases}
$$

Since, for $i \in\{1,2\}$, $\arg \max _{S \in \mathscr{O}_{i}} v_{i}^{\mathrm{II}}(S)$ only contains $X_{i}^{N_{i}}(\mathbf{v})$, by Lemma3.12 we have $X_{2}^{N_{1} \cup N_{2}}\left(Q^{\prime}\right)=$ $X_{2}^{N_{1} \cup N_{2}}(Q)$. Further, the set of favorable minimal exchanges with respect to $Q^{\prime}$ is $\left\{\left(S_{i}, T_{i}\right) \mid i \in\right.$ $I \cup\{z\}\}$. So, by Lemma 3.17 we have $X_{2}^{E_{1} \cup E_{2}}\left(Q^{\prime}\right)=\left(E_{2} \backslash \bigcup_{i \in I \cup\{z\}} T_{i}\right) \cup \bigcup_{i \in I \cup\{z\}} S_{i}$.

So, by deviating from $Q$ to $Q^{\prime}$, player 2 improves his utility by $v_{2}\left(S_{z}\right)-v_{2}\left(T_{z}\right)>0$, which contradicts truthfulness. Therefore, there is no $z \in[k] \backslash I$ such that $\left(S_{z}, T_{z}\right)$ is favorable with respect to $\mathbf{v}$, and this concludes the proof.

## B Missing Material from Section 4

Proof of Lemma 4.5, From theorem 3.6 we know that every truthful mechanism can be implemented as a picking-exchange mechanism. So consider such a mechanism and let us examine
the structure of sets $N_{i}, E_{i}$ and $\mathscr{O}_{i}$. Notice that by the definition of picking-exchange mechanisms, each player $i$ controls $N_{i} \cup E_{i}$. If both $N_{i}, E_{i}$ are nonempty, or $\left|E_{i}\right|>1$, or $\mathscr{O}_{i}$ contains a non-singleton set, then the respective player has control over some pair of items. Thus we can conclude that every possible mechanism can be implemented as a singleton picking-exchange mechanism.

Remark B.1. Regarding the remaining proofs, it suffices to focus only on singleton pickingexchange mechanisms. Indeed, by Theorem 3.6 we know that every truthful mechanism can be implemented as a picking-exchange mechanism, and by Lemmata 4.3 and 4.5 only the singleton picking-exchange mechanisms among them may achieve some fairness guarantee.

Proof of Application4.6. Initially it is easy to see that when $m=1$ or $m=2$, the statement holds in a trivial way for every singleton picking-exchange mechanisms. Indeed, in every instance each player gets at most one item and thus the value a player derives in the worst case is greater or equal to the value of the empty set (bundle of the other player minus an item).

In the case of $m=3$, in any instance one player gets one item and the other player two items. The singleton picking-exchange mechanism guarantees that the player who gets one item is allocated with at least his second best in terms of value, so the value he derives is always greater or equal to the value of his least desirable item (bundle of the other player minus an item). On the other hand, the player who is allocated with two items always derives value greater or equal to the value of the empty set.

Finally, in the case of $m=4$ with $\left|N_{1}\right|=\left|N_{2}\right|=2$, every player gets two items at every instance. The singleton picking-exchange mechanism guarantees that each player will receive at least his second best item, the value of which is greater or equal to the value of his third or fourth best item.

On the other hand, in case of $m \geq 5$ consider profile $v_{1}=[1+\epsilon, 1, \ldots, 1] \cup[1, \delta, \ldots, \delta], \quad v_{2}=$ $[1, \delta, \ldots, \delta] \cup[1+\epsilon, 1, \ldots, 1]$ where $1 \gg \epsilon \gg \delta>0$. The first vector of values is for $N_{1}$ (or $E_{1}$ ) and the second is for $N_{2}$ (or $E_{1}$ ); notice that it is possible for one of them to be empty. We only examine singleton picking-exchange mechanisms (see RemarkB.1). It is easy to see that in such a case, by the pigeonhole principle, no singleton picking-exchange mechanism can achieve envy-freeness up to one item for both players.

Proof of Application 4.7. We only need to prove that among all the singleton picking-exchange mechanisms (see Remark B.1) there is no better approximation ratio than $\lfloor m / 2\rfloor^{-1}$ for $m \geq 3$. Consider profile $v_{1}=[1+\epsilon, 1, \ldots, 1] \cup\left[\left|N_{1} \cup E_{1}\right|, \delta, \ldots, \delta\right], \quad v_{2}=\left[\left|N_{2} \cup E_{2}\right|, \delta, \ldots, \delta\right] \cup[1+\epsilon, 1, \ldots, 1]$, where $1 \gg \epsilon \gg \delta>0$. The first vector of values is for $N_{1}$ (or $E_{1}$ ) and the second is for $N_{2}$ (or $E_{1}$ ); notice that it is possible for one of them to be empty.

It is easy to see that when both $N_{1} \cup E_{1}, N_{2} \cup E_{2}$ are nonempty, then $\mu_{i}=\left|N_{i} \cup E_{i}\right|$ while they both receive value that is slightly greater than 1 . Therefore, no singleton picking-exchange mechanism can achieve a better approximation ratio than $\lceil m / 2\rceil^{-1}$ for both players.

On the other hand, if $N_{1} \cup E_{1}=\varnothing$ (the other case is symmetric) then this is the mechanism in Amanatidis et al. [2016] that achieves exactly $\lfloor m / 2\rfloor^{-1}$.


[^0]:    *A conference version to appear in the 18th ACM conference on Economics and Computation (ACM EC '17).
    ${ }^{\dagger}$ Athens University of Economics and Business. Emails: \{gamana, gebirbas, markakis\}@aueb.gr
    ${ }^{\ddagger}$ University of Liverpool. Email: G. Christodoulou@liverpool.ac.uk

[^1]:    ${ }^{1}$ The material of this subsection is needed in the sequel only within Section 4

[^2]:    ${ }^{2}$ Picking mechanisms are a generalization of truthful picking sequences for two players [see Bouveret and Lang, 2014].
    ${ }^{3}$ If we think about $E_{1}, E_{2}$ as fixed a priori, then exchange mechanisms are a generalization of fixed deal exchange rules in general exchange markets for two players [see Pápai, 2007].

[^3]:    ${ }^{4}$ Describing all such tie-breaking rules seems to be an interesting, nontrivial question for future work, but not our main focus here. It is not hard to see, though, that there exist tie-breaking rules that make a picking-exchange mechanism nontruthful, e.g., break ties on offers of player 1 so that the value that player 2 gets from $N_{1}$ is minimized.

[^4]:    ${ }^{5}$ Consider, for instance, a profile where both players desire only the first item and have a negligible value for the other items. Then one of the players will necessarily remain unsatisfied and receive a value close to zero, no matter what the allocation is.

[^5]:    ${ }^{6}$ The only exceptions-and the only such mechanisms where both $E_{1}$ and $E_{2}$ are nonempty-are two mechanisms for the degenerate case of $m=2$, e.g., $N_{1}=N_{2}=\varnothing, \mathscr{O}_{1}=\mathscr{O}_{2}=\{\varnothing\}, E_{1}=\{a\}, E_{2}=\{b\}$ and $D=\{(\{a\},\{b\})\}$, where $\{a, b\}=\{1,2\}$.

[^6]:    ${ }^{7}$ The approximation factor in Markakis and Psomas 2011 is expressed in terms of $V_{2}(1 / m)$, but it simplifies to $\lfloor m / 2\rfloor^{-1}$.

[^7]:    ${ }^{8}$ This is always possible. In particular, if $|B|>0$ then $\alpha=\frac{|E \cup H| m^{4}-m}{(|B|-1) m^{3}+|C| m^{4}}$ works. If $|B|=0$, then $\alpha=\frac{|E \cup H| m^{4}-m}{(|C|-1) m^{4}}$. In order to apply the idea mentioned in RemarkA.7 one can multiply the whole profile with the denominator of $\alpha$.

[^8]:    ${ }^{9}$ Notice that the values are chosen in a way such that if $\mathbf{v} \in \mathcal{V}_{m}^{\neq}$, then $\mathbf{v}^{\mathrm{I}} \in \mathcal{V}_{m}^{\neq}$as well.

[^9]:    ${ }^{10}$ In order to be able to apply the idea mentioned in RemarkA. 7 one can use $m^{7}$ instead of 1 , and $\Delta=m^{8}, \delta=m^{3}$, $\lambda_{i}=\left|T_{i}\right| \cdot\left|S_{i}\right|, n_{i}=\left|T_{j}\right| \cdot m^{4}, \theta_{i}=\left|S_{j}\right| \cdot\left(m^{4}-1\right), \zeta=\left|S_{1}\right| \cdot m^{4}$, and $\epsilon=\left|T_{1}\right| \cdot\left(m^{4}-1\right)$.

