Finding Even Cycles Faster via Capped k-Walks

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Abstract

Finding cycles in graphs is a fundamental problem in algorithmic graph theory. In this paper, we consider the problem of finding and reporting a cycle of length $2k$ in an undirected graph G with n nodes and m edges for constant $k \geqslant 2$. A classic result by Bondy and Simonovits [J. Combinatorial Theory, 1974] implies that if $m \geqslant 100kn^{1+1/k}$, then G contains a $2k$-cycle, further implying that one needs to consider only graphs with $m = O(n^{1+1/k})$.

Previously the best known algorithms were an $O(n^2)$ algorithm due to Yuster and Zwick [J. Discrete Math 1997] as well as a $O(m^{2-(1+\lceil k/2\rceil^{-1})/(k+1)})$ algorithm by Alon et. al. [Algorithmica 1997].

We present an algorithm that uses $O\left(m^{2k/(k+1)}\right)$ time and finds a $2k$-cycle if one exists. This bound is $O(n^2)$ exactly when $m = \Theta(n^{1+1/k})$. When finding 4-cycles our new bound coincides with Alon et. al., while for every $k > 2$ our new bound yields a polynomial improvement in m.

Yuster and Zwick noted that it is "plausible to conjecture that $O(n^2)$ is the best possible bound in terms of n". We show "conditional optimality": if this hypothesis holds then our $O(m^{2k/(k+1)})$ algorithm is tight as well. Furthermore, a folklore reduction implies that no *combinatorial* algorithm can determine if a graph contains a 6-cycle in time $O(m^{3/2-\varepsilon})$ for any $\varepsilon > 0$ unless boolean matrix multiplication can be solved combinatorially in time $O(n^{3-\varepsilon'})$ for some $\varepsilon' > 0$, which is widely believed to be false. Coupled with our main result, this gives tight bounds for finding 6-cycles combinatorially and also separates the complexity of finding 4- and 6-cycles giving evidence that the exponent of m in the running time should indeed increase with k.

The key ingredient in our algorithm is a new notion of *capped k-walks*, which are walks of length k that visit only nodes according to a fixed ordering. Our main technical contribution is an involved analysis proving several properties of such walks which may be of independent interest.

1 Introduction

We study a basic problem in algorithmic graph theory. Namely, given an undirected and unweighted graph $G = (V, E)$ and an integer ℓ, does G contain a cycle of length exactly ℓ (denoted C_ℓ)? If a C_ℓ exists, we would also like the algorithm to return such a cycle. As a special case, when $\ell = n$ is the number of nodes in the graph, we are faced with the well-known problem of

*Research partly supported by Advanced Grant DFF-0602-02499B from the Danish Council for Independent Research

†Research partly supported by Advanced Grant DFF-0602-02499B from the Danish Council for Independent and the FNU project AlgoDisc – Discrete Mathematics, Algorithms, and Data Structures.

‡Research partly supported by Villum Fonden and the DABAI project.

finding a hamiltonian cycle, which was one of Karp's original 21 NP-complete problems [7]. In fact, the problem is NP-complete when $\ell = n^{\Omega(1)}$.

On the other end of the spectrum, when $\ell = O(1)$ is a constant, the problem is in FPT[1]as first shown by Monien in 1985 [9], by giving an $O(f(\ell) \cdot m)$ algorithm to determine if any given node u is contained in a C_ℓ. For $\ell = 3$, this is the classical problem of triangle-finding, which can be done in $O(n^\omega)$ time using matrix multiplication, where $\omega < 2.373$ is the matrix multiplication exponent [8]. This can be extended to finding a C_ℓ for any constant $\ell = O(1)$ in time $O(n^\omega)$ expected and $O(n^\omega \log n)$ deterministically [2]. When ℓ is odd, this is the fastest known algorithm, however for even $\ell = 2k = O(1)$ one can do better. To appreciate the difference, we must first understand the following basic graph theoretic result about even cycles: Bondy and Simonovits [4] showed that if a graph with n nodes has more than $100kn^{1+1/k}$ edges, then the graph contains a C_{2k}. In contrast, a graph on n nodes can have $\Theta(n^2)$ edges without containing any odd cycle, e.g. $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Using this lemma of Bondy and Simonovits, it was shown by Yuster and Zwick [14] how to find a C_{2k} for constant k in time $O(n^2)$. They note that *"it seems plausible to conjecture that $O(n^2)$ is the best possible bound in terms of n"*. Furthermore, when $m \geqslant 100k \cdot n^{1+1/k}$ we can use the algorithm of Yuster Zwick [14] to find a C_{2k} in $O(n)$ expected time. Given this situation, we seek an algorithm with a running time $O(m^{c_k})$, which utilizes the sparseness of the graph, when m is less than $100k \cdot n^{1+1/k}$. By the above discussion, such an algorithm can be turned into a $O(n^{c_k(1+1/k)})$ time algorithm for finding a C_{2k}. Therefore, if we believe that $O(n^2)$ indeed is the correct running time in terms of n, we must also believe that the best possible value for c_k is $2 - 2/(k+1)$. This is further discussed in Section 1.1 below. Our main result is to present an algorithm which obtains exactly this running time in terms of m and k for finding a C_{2k}. We show the following.

Theorem 1. *Let G be an unweighted and undirected graph with n nodes and m edges, and let $k \geqslant 2$ be a positive integer. A C_{2k} in G, if one exists, can be found in $O(k^{O(k)} m^{\frac{2k}{k+1}})$.*

Theorem 1 presents the first improvement in more than 20 years over a result of Alon, et al. [3], who gave an algorithm with $c_k = 2 - (1 + \frac{1}{\lceil k/2 \rceil})/(k+1)$, i.e., a running time of $O(m^{4/3})$ for 4-cycles and $O(m^{13/8})$ for 6-cycles. For 4-cycles we obtain the same bound with Theorem 1, but for any $k > 2$ our new bound presents a polynomial improvement. In fact our algorithm for finding a C_8 is faster than the algorithm of Alon, et al. for finding a C_6. A comparison with known algorithms is shown below in Figure 1.

We present our algorithm as a black box reduction: Let A be any algorithm which can determine *for a given node u* if u is contained in a C_{2k} in $O(f(k) \cdot m)$ time. Then our algorithm can transform A into an algorithm which finds a C_{2k} in $O(g(k) \cdot m^{2k/(k+1)})$ time. Thus, one may pick any such algorithm A such as the original algorithm of Monien [9] or the seminal color-coding algorithm of Alon et al. [2]. Our algorithm is conceptually simple, but the analysis is technically involved and relies on a new understanding of the relationship between the number of k-walks and the existence of a C_{2k}. By introducing the notion of *capped k-walks*, we show that an algorithm enumerating all such capped k-walks starting in nodes with low degree will either find a $2k$-cycle or spend at most $O(m^{2k/(k+1)})$ time. In some sense this is a stronger version of the combinatorial lemma by Bondy and Simonovits, as any graph with many edges must also have many capped k-walks.

[1]Informally, a problem of size n parameterized by k is in FPT if it can be solved in time $f(k) \cdot n^{O(1)}$, where f is a function independent of n.

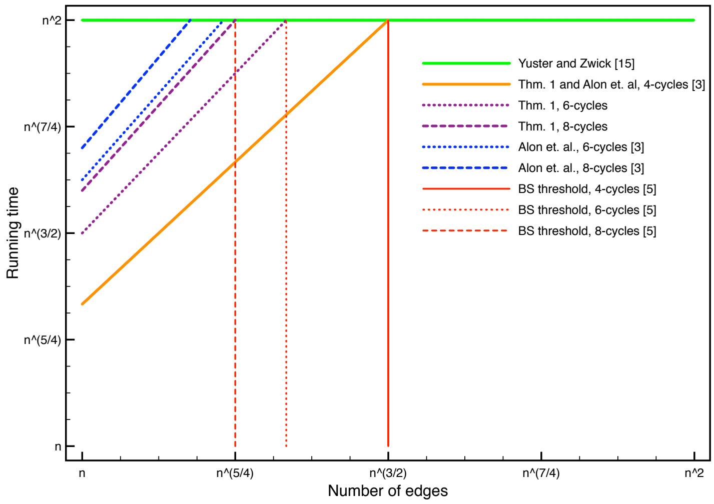

Figure 1: Comparisons of running times in terms of graph density. The illustration shows our algorithm from Theorem 1 compared to [14] and [3], and shows that it uses quadratic time exactly when the threshold from Bondy and Simonovits ensures the existence of a $2k$-cycle.

1.1 Hardness of finding cycles

The literature on finding ℓ-cycles is generally split into two kinds of algorithms: *combinatorial* and *non-combinatorial* algorithms. Where combinatorial algorithms (informally) are algorithms, which do not use the structure of the underlying field and perform Strassen-like cancellation tricks [11]. Interestingly, all known algorithms for finding cycles of even length efficiently are combinatorial. There are several possible explanations for this. One is that the hard instance for even cycles are graphs, which are relatively sparse (i.e. $O(n^{1+1/k})$ edges), and in this case it is difficult to utilize the power of fast matrix-multiplication. Another is that matrix-multiplication based methods allows one to solve the harder problem of directed graphs. Directed graphs are harder because we can no longer make the guarantee that a C_{2k} can always be found if the graph is dense. Furthermore, a simple argument shows that the problem of finding a C_3 can be reduced to the problem of finding a directed C_ℓ for any $\ell > 3$. Especially this problem of finding a C_3 combinatorially has been studied thoroughly in the line of work colloquially referred to as *Hardness in **P***. This line of work is concerned with basing hardness results on widely believed conjectures about problems in **P** such as 3-SUM and APSP. One such popular conjecture (see e.g. [1, 12]) is the combinatorial boolean matrix multiplication (BMM) conjecture stated below.

Conjecture 1. *There exists no* combinatorial *algorithm for multiplying two $n \times n$ boolean matrices in time $O(n^{3-\varepsilon})$ for any $\varepsilon > 0$.*

It is known from [12] that Conjecture 1 above is equivalent to the statement that there exists no truly subcubic[2] *combinatorial* algorithm for finding a C_3 in graphs with n nodes and $\Theta(n^2)$ edges, and a simple reduction shows that this holds for any odd $\ell \geqslant 3$. For even cycles, we show that a simple extension to this folklore reduction gives the following result.

Proposition 1. *Let $k \geqslant 3$ be a fixed integer with $k \neq 4$. Then there exists no* combinatorial *algorithm that can find a $2k$-cycle in graphs with n nodes and m edges in time $O(m^{3/2-\varepsilon})$ unless Conjecture 1 is false.*

[2] An algorithm running polynomially faster than cubic time, i.e. $O(n^{3-\varepsilon})$ for $\varepsilon > 0$.

As noted, the proof of Proposition 1 is a rather simple extension of the reduction for odd cycles, but for completeness, we include the proof in Section 4. In particular, Proposition 1 implies that our $O(m^{3/2})$ time algorithm for finding 6-cycles is optimal among combinatorial algorithms. Interestingly, Proposition 1 also creates a separation between finding 4-cycles and finding larger even cycles, as both Alon, et al. [3] and Theorem 1 provide an algorithm for finding 4-cycles in time $O(m^{4/3})$., which is polynomially smaller than $O(m^{3/2})$. This gives evidence that a trade-off dependent on k like the one obtained in Theorem 1 is indeed necessary.

An important point of Theorem 1, as mentioned earlier, is that it is optimal if we believe that $\Theta(n^2)$ is the correct running time in terms of n. This is formalized in the theorem below. Furthermore, we show that Theorem 1 implies that any hardness result of $n^{2-o(1)}$ would provide a link between the time complexity of an algorithm and the existence of dense graphs without $2k$-cycles. A statement, which is reminiscent of the Erdős Girth Conjecture.

Theorem 2. *Let $k \geqslant 2$ be some fixed integer. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if no algorithm exists which can find a C_{2k}-cycle in graphs with n nodes and m edges in time $O(n^{2-\delta})$, then the following two statements hold.*

1. *There is no algorithm which can detect if a graph contains a C_{2k} in time $O(m^{2k/(k+1)-\varepsilon})$.*
2. *There exists an infinite family of graphs $\mathcal{G}$, such that each $G \in \mathcal{G}$ has $|E(G)| \geqslant |V(G)|^{1+1/k-\varepsilon}$ and contains no C_{2k}.*

1.2 Other results

A problem related to that of finding a given C_ℓ is to determine the girth (length of shortest cycle) of a graph G. In undirected graphs, finding the shortest cycle in general can be done in time $O(n^\omega)$ time due to a seminal paper by Itai and Rodeh [6], and the shortest directed cycle can be found using an extra factor of $O(\log n)$. In undirected graphs they also show that a cycle that exceeds the shortest by at most one can be found in $O(n^2)$ time. It was shown by Vassilevska Williams and Williams [12] that computing the girth exactly is essentially as hard as boolean matrix multiplication, that is, finding a combinatorial, truly subcubic algorithm for computing the girth of a graph would break Conjecture 1. Thus, an interesting question is whether one can approximate the girth faster, and in particular a main open question as noted by Roditty and Vassilevska Williams [10] is whether one can find a $(2-\varepsilon)$-approximation in $O(n^{2-\varepsilon'})$ for any constants $\varepsilon, \varepsilon' > 0$. They answered this question affirmatively for *triangle-free graphs* giving a 8/5-approximation in $O(n^{1.968})$ time [10]. By plugging Theorem 1 into their framework we obtain the following result.

Theorem 3. *There exists an algorithm for computing a 8/5-approximation of the girth in a triangle-free graph G in time $O(n^{1.942})$.*

1.3 Capped k-walks

The main ingredient in our analysis is a notion of *capped k-walks* defined below.

Definition 1. Let $G = (V, E)$ be a graph and let $\preceq$ be a total ordering of V. For a positive integer, k, we say that a $(k+1)$-tuple $(x_0, \ldots, x_k) \in V^{k+1}$ is called a *$\preceq$-capped k-walk* if $(x_0, \ldots, x_k)$ is a walk in G and $x_0 \succeq x_i$ for each $i = 1, 2, \ldots, k$.

When clear from the context we will refer to a $\preceq$-capped k-walk simply by a capped k-walk. Our algorithm for finding $2k$-cycles essentially works by enumerating all $\preceq$-capped k-walks (with some pruning applied), where $\preceq$ is given by ordering nodes according to their degree. We will show that by bounding the number of such $\preceq$-capped k-walks in graphs with a not too large maximum degree, we obtain a bound on the running time of our algorithm. Specifically, we show the following lemma.

Lemma 1. *Let $G = (V, E)$ be a graph, let k be a positive integer, and assume that G has maximum degree at most $m^{2/(k+1)}$. Let $\preceq$ be any ordering of the nodes in G such that $u \preceq v$ for all pairs of nodes u, v such that* $\deg(u) < \deg(v)$*. If G contains no $2k$-cycle, then the number of $\preceq$-capped k-walks is at most $f(k)m^{2k/(k+1)}$, where $f(k) = \left(O(k^2)\right)^{k-1} = k^{O(k)}$.*

We also present a lower bound on the number of $\preceq$-capped k-walk, which implies that graphs with a large number of edges contains a large number of $\preceq$-capped k-walks.

Lemma 2. *Let $G = (V, E)$ be a graph with n nodes and m edges. Let $\preceq$ be any ordering of V. The number of $\preceq$-capped k-walks is at least $n \cdot \left(\frac{m}{2n}\right)^k$*

Lemmas 1 and 2 imply that graphs with more than $Ck^2 n^{1+1/k}$ edges and maximum degree at most $m^{2/(k+1)}$ have a $2k$-cycle, for a sufficiently large constant $C > 0$. Except from the extra factor of k and the bound on the maximum degree, this shows that Lemma 1 is stronger than the lemma of Bondy and Simonovits, which states that graphs with at least than $100kn^{1+1/k}$ edges contain a $2k$-cycle. Indeed, a graph with few edges may still contain many capped k-walks.

1.4 Techniques and overview

Our main technical contribution is the analysis of capped k-walks, outlined in Section 1.3 above. A standard way of reasoning about the number of k-walks in a graph $G = (V, E)$ is to consider the adjacency matrix, X_G, of G, where $X_G[i, j] = 1$ if $(i, j) \in E$ and 0 otherwise. Here we denote the nodes of G by $1, \ldots, n$. Then the number of k-walks in G from i to j is exactly $X_G^k[i, j]$ and the total number of k-walks is $\|X_G^k \mathbf{1}\|_1$, where $\mathbf{1} = (1, 1, \ldots, 1)^n$. Furthermore the number of k-walks starting in a specific node i is $(X_G^k \mathbf{1})_i$. We will be interested in bounding the number of k-walks starting in a specific subset $S \subseteq V$. This number can be calculated as $\langle X_G^k \mathbf{1}, \mathbf{1}_S \rangle$, where $\mathbf{1}_S$ is the vector with 1s in each index, i, such that $i \in S$ and 0s elsewhere. Our goal will be to bound the norm of X_G and use this to bound the number of k-walks. However, bounding the 1-norm leads to a too large bound and cannot be used in proving Lemma 1. We note that the 1-norm of a vector v, can be written as

$$\|v\|_1 = \int_0^\infty |\{i \mid |v_i| \geqslant x\}| \, dx \ .$$

We will instead consider the following related quantity, that we will call the $\|\cdot\|_\phi$-norm.

Definition 2. For a vector $v \in \mathbb{R}^n$ we define the norm $\|v\|_\phi$ by

$$\|v\|_\phi = \int_0^\infty \sqrt{|\{i \mid |v_i| \geqslant x\}|} dx \ .$$

We extend the definition to matrices as

Definition 3. For a real $n \times n$ matrix A we define $\|A\|_\phi$ by:

$$\|A\|_\phi = \sup_{u \neq 0} \left\{ \frac{\|Au\|_\phi}{\|u\|_\phi} \right\} .$$

We analyze this norm in section 3 showing several properties. We use this norm to reason about the number of k-walks starting in a specific set of nodes $S \subseteq V$, by showing that this number is at most $\sqrt{|S|} \left\|X_G^k \mathbf{1}\right\|_\phi$. The main technical lemma of the paper is to show that if G is a graph with no $2k$-cycle and maximum degree at most $m^{2/(k+1)}$, then $\|X_G\|_\phi = O(k^2 m^{1/(k+1)})$.

1.5 Related work

All stated bounds are in the RAM model unless otherwise specified and k is assumed to be fixed. We will review related work of both given even and odd cycles.

Combinatorial upper bounds. We briefly discuss known combinatorial bounds other than the previously mentioned [14, 3, 9]. Alon, et al. [3] also showed several results for directed graphs. In particular, an upper bound of $O(m^{2-1/k})$ to find a C_{2k}, as well as $O(m^{2-\frac{2}{\ell+1}})$ to find C_ℓ for odd ℓ. In the same paper, Alon, et al. [3] also present bounds parameterized on the *degeneracy* of the graph: the degeneracy $d(G)$ of a graph G is the largest minimal degree taken over all the subgraphs of G, and for any G it can be bounded from above by $d(G) \leqslant 1/2m^{1/2}$. They present bounds of the form $O(m^\alpha d(G)^\beta)$. These bounds also apply to directed graphs. We note, that for undirected graphs the result of Theorem 1 is still asymptotically better for $d(G) = \omega(1)$. The problem of combinatorially finding a C_3 has also been studied thoroughly in the literature. The current fastest bound is due to Yu [13] and uses $O(n^3 \operatorname{poly}(\log\log n))/\log^4 n)$ time in the word-RAM model with word-size $\Omega(\log n)$. For sparse graphs a folklore $O(m^{3/2})$ algorithm exists

Non-combinatorial upper bounds. As mentioned, the best algorithm to find general cycles is due to the seminal paper introducing color-coding, Alon et al. [2] who gave an $O(n^\omega)$ expected time upper bound, and an $O(n^\omega \log n)$ worst case upper bound, for finding a C_ℓ in a directed or undirected graph. Other algorithms improve on [2] for finding specific C_ℓ. Alon et al. [3] showed that a C_3 can be found in time $O(m^{\frac{2\omega}{\omega-1}}) = o(m^{1.41})$ in both directed and undirected graphs. Extending this, Eisenbrand and Grandoni [5] showed a $O(n^{1/\omega}m^{2-2/\omega})$ time upper bound for C_4 in directed graphs. Both the former and the latter bounds are asymptotically faster than $O(n^\omega)$ for sufficiently sparse input. Improving asymptotically on Eisenbrand and Grandoni for sparse graphs, Yuster and Zwick [15] showed a $O(m^{(4\omega-1)/(2\omega+1)}) = o(m^{1.48})$ upper bound for directed graphs. For finding a C_6 in graphs with low degeneracy $d(G)$, Alon et al. [3] showed a bound of $O((md(G))^{2\omega/(\omega+1)}) = O((md(G))^{1.41})$.

1.6 Notation

Let $G = (V, E)$ be a graph. For (not necessarily disjoint) sets of nodes $A, B \subseteq V$ we let $E(A, B)$ denote the set of edges between A and B in G, i.e. $E \cap (A \times B)$. We use $E(v, A)$ to denote $E(\{v\}, A)$.

2 Finding even cycles

In this section we describe our algorithm for finding a C_{2k} in an undirected graph $G = (V, E)$ with n nodes and m edges. In our analysis we will assume Lemma 1, but we defer the actual proof of the lemma to Section 3.

Our algorithm works by creating a series of graphs $G^k_{\leqslant 1}, \ldots, G^k_{\leqslant n}$ guaranteed to contain any $2k$-cycle that may exist. Furthermore, the total size of these graphs can (essentially) be bounded by the total number of $\preceq$-capped k-walks which is used to bound the running time.

Proof of Theorem 1. Let A be any algorithm that takes a graph H and a node u in H as input and determines if u is contained in a $2k$-cycle in time $O(g(k) \cdot |E(H)|)$.

Order the nodes of G as $v_1, \ldots, v_n$ non-decreasingly by degree and define $G_{\leqslant i}$ to be the subgraph of G induced by $v_1, \ldots, v_i$. Let $G^k_{\leqslant i}$ denote the subgraph of $G_{\leqslant i}$ containing all edges (and their endpoints) incident to nodes at distance $< k$ from v_i in $G_{\leqslant i}$. Now for each $i \in$

$\{1,\ldots,n\}$ in increasing order we create the graph $G^k_{\leqslant i}$, run algorithm A on $G^k_{\leqslant i}$ and v_i, and return any $2k$-cycle found (stopping the algorithm). If no such cycle is found for any i the algorithm returns that no $2k$-cycle exists in G.

For correctness let C be any $2k$-cycle in G and let v_i be the node in C that is last in the ordering. It then follows from the definition that C is fully contained in $G^k_{\leqslant i}$ and thus either the algorithm returns a $2k$-cycle when A is run on $G^k_{\leqslant i}$ or some other $2k$-cycle when A is run on $G^k_{\leqslant j}$ for $j < i$. For the running time observe first that creating the graphs $G^k_{\leqslant i}$ and running algorithm A on these graphs takes time proportional to the total number of edges in these graphs. Thus what is left is to bound this number of edges. The number of edges in $G^k_{\leqslant i}$ is bounded by the number of capped k-walks starting in v_i in G. Let i be the largest value such that $G^k_{\leqslant i}$ does not contain a $2k$-cycle and $\deg(v_i) \leqslant m^{2/(k+1)}$. It then follows by Lemma 1 that the graphs $G^k_{\leqslant 1},\ldots,G^k_{\leqslant i}$ contain at most a total number of $O(f(k)\cdot m^{2k/(k+1)})$ edges. Furthermore, there are at most $m^{1-2/(k+1)}$ nodes of degree $> m^{2/(k+1)}$, and thus the total number of edges over all the graphs $G^k_{\leqslant 1},\ldots,G^k_{\leqslant n}$ is at most $O(f(k)\cdot m^{2k/(k+1)})$ giving the desired running time. □

As an example, the algorithm A in the above proof could be the algorithm of Monien [9] or Alon et al. [2].

3 Bounding the number of capped k-walks

In this section we will prove Lemma 1. Let $G = (V,E)$ be a given graph. We will denote the nodes of G by $u_1,\ldots,u_n$ or simply $1,\ldots,n$ if it is clear from the context.

Recall the definition of $\|\cdot\|_\phi$ from the introduction. We note that the following basic properties hold.

Lemma 3. *For all vectors $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we have:*

$$\begin{aligned}\|u+v\|_\phi &\leqslant \|u\|_\phi + \|v\|_\phi\,,\\ \|cu\|_\phi &= |c|\cdot\|u\|_\phi\,,\\ \|u\|_\phi = 0 &\iff u = 0\,.\end{aligned}$$

As mentioned in the introduction, we would like to use the $\|\cdot\|_\phi$-norm of X_G to bound the number of k-walks starting in a given subset $S \subseteq V$. We can do this using the following lemma.

Lemma 4. *Let $G = (V,E)$ be a graph with n nodes and adjacency matrix X_G. Let $S \subseteq V$ be a set of nodes. For any integer k the number of k-walks starting in S is bounded by $\sqrt{|S|}\,\left\|X_G^k\mathbf{1}\right\|_\phi$.*

Proof. Let $v = X_G^k\mathbf{1}$ and let w be the vector such that $w_i = v_i$ when $i \in S$ and $w_i = 0$ when $i \notin S$. Then the number of k-walks starting in S is exactly the sum of entries in w, i.e. it is $\|w\|_1$. So the number of k-walks starting in S is bounded by

$$\begin{aligned}\|w\|_1 &= \int_0^\infty |\{i \mid w_i \geqslant x\}|\,dx\\ &\leqslant \sqrt{|S|}\int_0^\infty \sqrt{|\{i \mid w_i \geqslant x\}|}dx\\ &= \sqrt{|S|}\,\|w\|_\phi\\ &\leqslant \sqrt{|S|}\,\|v\|_\phi\ ,\end{aligned}$$

as desired. Here the first inequality follows because w has at most $|S|$ non-zero entries. □

To prove Lemma 1 we want to bound the quantity $\left\|X_G^k\right\|_\phi$ for graphs, G, which do not contain a $2k$-cycle and have maximum degree at most $m^{\frac{2}{k+1}}$. To do this we will need the following lemmas, which are proved in Section 5.

Lemma 5. *Let A be a real $n \times n$ matrix. If, for all vectors $v \in \{0,1\}^n$ we have $\|Av\|_\phi \leqslant C\|v\|_\phi$ for some value C, then $\|A\|_\phi \leqslant 16C$.*

Lemma 6. *Let G be a graph with and let A and B be subsets of nodes in G. Let $k \geqslant 2$ be an integer and assume that G contains no $2k$-cycle. Then*

$$|E(A,B)| \leqslant 100k \cdot \left(\sqrt{|A| \cdot |B|}^{1+1/k} + |A| + |B|\right) . \tag{1}$$

We are now ready to prove the main technical lemma stated below.

Lemma 7. *Let $G = (V,E)$ be a graph with m edges and let k be a positive integer. Assume that G has maximum degree at most $m^{2/(k+1)}$ and does not contain a $2k$-cycle. Let X_G be the adjacency matrix for G, then*

$$\|X_G\|_\phi = O\left(k^2 m^{1/(k+1)}\right) .$$

Proof. We denote the vertices of G by $1,2,\dots,n$ for convenience. By Lemma 5 we only need to show that $\|X_G v\|_\phi = O\left(k^2 m^{1/(k+1)}\|v\|_\phi\right)$ for every vector v where each entry is either 0 or 1. Each such vector, v, can be viewed as a set of nodes $A \subseteq V$, where v_i is 1 whenever $i \in A$ and 0 otherwise. We will adopt this view and denote v by $\mathbf{1}_A$. In this case we have $\|\mathbf{1}_A\|_\phi = \sqrt{|A|}$. Thus it suffices to show that for all $A \subseteq V$ we have

$$\|X_G \mathbf{1}_A\|_\phi = O\left(k^2 m^{1/(k+1)}\sqrt{A}\right) . \tag{2}$$

Now fix an arbitrary $A \subseteq V$. We are going to show that (2) holds. For every non-negative integer i we let B_i denote the set of nodes in G which have more than 2^{i-1} but at most 2^i neighbours in A. That is

$$B_i = \left\{v \in V \mid |E(v,A)| \in \left(2^{i-1}, 2^i\right]\right\} .$$

We note that by the definition of $\|\cdot\|_\phi$ we have that

$$\begin{aligned}
\|X_G \mathbf{1}_A\|_\phi &\leqslant \sum_{i \geqslant 0} 2^i \sqrt{\sum_{j \geqslant i} |B_j|} \\
&\leqslant \sum_{i \geqslant 0} 2^i \sum_{j \geqslant i} \sqrt{|B_j|} \\
&< 2 \cdot \sum_{i \geqslant 0} 2^i \sqrt{|B_i|} .
\end{aligned}$$

So in order to show (2) it suffices to show (3) below

$$\sum_{i \geqslant 0} 2^i \sqrt{|B_i|} = O\left(k^2 m^{1/(k+1)} \sqrt{|A|}\right) . \tag{3}$$

or alternatively to show

$$\sum_{i \geqslant 0} 2^i \frac{\sqrt{|B_i|}}{\sqrt{|A|}} = O\left(k^2 m^{1/(k+1)}\right) . \tag{4}$$

For an integer $i \geqslant 0$ let t_i be defined by

$$t_i = 2^i \cdot \frac{\sqrt{|B_i|}}{\sqrt{|A|}} .$$

We will bound the value t_i by looking at the number of edges between the sets B_i and A. Our plan is to bound the value t_i in several ways, and then taking a geometric mean will yield the result. Observe first, that by the definition of B_i we have at least $2^{i-1}|B_i|$ edges from B_i to A, and hence $2^i |B_i| \leqslant 2|E(B_i, A)| \leqslant 2m$. It follows that t_i is bounded by

$$t_i = \frac{2^i\sqrt{|B_i|}}{\sqrt{|A|}} = \frac{2^{i/2}\sqrt{2^i|B_i|}}{\sqrt{|A|}} \leqslant \frac{2^{i/2}\sqrt{2m}}{\sqrt{|A|}} .$$

Let A_i be the subset of nodes of A that are adjacent to a node in B_i, then $E(B_i, A) = E(B_i, A_i)$. By Lemma 6 it also follows that

$$\begin{aligned} t_i &\leqslant \frac{2|E(B_i, A_i)|}{\sqrt{|B_i| \cdot |A|}} \\ &\leqslant 200k\sqrt{|B_i| \cdot |A_i|}^{1/k} + 200k\sqrt{\frac{|B_i|}{|A|}} + 200k\sqrt{\frac{|A_i|}{|B_i|}} . \end{aligned}$$

We also note that $t_i = 0$ whenever $i > d$ where d is the smallest integer such that $2^{d-1} > m^{2/(k+1)}$, since the maximum degree of the graph is $m^{2/(k+1)}$. It follows that the sum $\sum_{i \geqslant 1} t_i$ can be bounded by:

$$\begin{aligned} &O\left(\sum_{i=1}^{d} \min\left\{\frac{2^i\sqrt{|B_i|}}{\sqrt{|A|}}, k\sqrt{|B_i|\,|A_i|}^{\frac{1}{k}} + k\sqrt{\frac{|B_i|}{|A|}} + k\sqrt{\frac{|A_i|}{|B_i|}}\right\}\right) \\ &= O\left(\Sigma_1 + \sum_{i=1}^{d} \min\left\{\frac{2^i\sqrt{|B_i|}}{\sqrt{|A|}}, k\sqrt{\frac{|B_i|}{|A|}} + k\sqrt{\frac{|A_i|}{|B_i|}}\right\}\right) \\ &= O\left(\Sigma_1 + \sum_{i=1}^{d} \left(k\sqrt{\frac{|B_i|}{|A|}} + k \cdot 2^{i/2}\right)\right) \end{aligned} \tag{5}$$

where

$$\Sigma_1 = \sum_{i=1}^{d} \min\left\{\frac{2^{i/2}\sqrt{2m}}{\sqrt{|A|}}, k\sqrt{|B_i| \cdot |A|}^{1/k}\right\}$$

Here, we have $\sqrt{\frac{|A_i|}{|B_i|}} \leqslant 2^{i/2}$ because each node of B_i has at most 2^i neighbours in A.

Let Σ_1 and Σ_2 denote the two sums of (5) above respectively. We will start by bounding Σ_2. Since, by definition, every node in B_i has at least 2^{i-1} neighbours in A_i and every node in A_i has degree at most $m^{2/(k+1)}$ we see that $|B_i|\, 2^{i-1} \leqslant |A_i|\, m^{2/(k+1)}$. Hence we get that:

$$\Sigma_2 \leqslant \sum_{i=1}^{d} \left(km^{1/(k+1)}2^{(1-i)/2} + k2^{i/2}\right) = O\left(km^{1/(k+1)}\right) .$$

Now we will bound Σ_1. First we note that $|B_i|\, 2^{i-1} \leqslant m$ and therefore $|B_i| \leqslant \frac{2m}{2^i}$. Inserting this gives us:

$$\Sigma_1 \leqslant \sum_{i=1}^{d} \min\left\{\frac{2^{i/2}\sqrt{2m}}{\sqrt{|A|}}, k\sqrt{\frac{2m}{2^i} \cdot |A|}^{1/k}\right\} .$$

Let d_0 be the largest integer such that $2^{d_0} \leqslant \frac{|A|}{(2m)^{(k-1)/(k+1)}}$. Then:

$$\frac{2^{d_0/2}\sqrt{2m}}{\sqrt{|A|}} = \Theta\left(m^{1/(k+1)}\right)$$
$$\sqrt[1/k]{\frac{2m}{2^{d_0}} \cdot |A|} = \Theta\left(m^{1/(k+1)}\right) .$$

Inserting this gives us:

$$\begin{aligned}
\Sigma_1 &\leqslant k \sum_{i=1}^{d} \min\left\{\frac{2^{i/2}\sqrt{2m}}{\sqrt{|A|}}, \sqrt[1/k]{\frac{2m}{2^i} \cdot |A|}\right\} \\
&\leqslant k \sum_{i=-\infty}^{\infty} \min\left\{\frac{2^{i/2}\sqrt{2m}}{\sqrt{|A|}}, \sqrt[1/k]{\frac{2m}{2^i} \cdot |A|}\right\} \qquad (6) \\
&\leqslant k \sum_{i=-\infty}^{d_0} \frac{2^{i/2}\sqrt{2m}}{\sqrt{|A|}} + k \sum_{i=d_0}^{\infty} \sqrt[1/k]{\frac{2m}{2^i} \cdot |A|} \qquad (7) \\
&= O\left(km^{1/(k+1)}\right) \cdot \left(\sum_{i=0}^{\infty} 2^{-i/2} + \sum_{i=0}^{\infty} 2^{-i/k}\right) \qquad (8) \\
&= O\left(k^2 m^{1/(k+1)}\right) . \qquad (9)
\end{aligned}$$

Summarizing, we thus have that

$$\sum_{i \geqslant 0} t_i = O\left(k^2 \cdot m^{\frac{1}{k+1}}\right) ,$$

and combining this with (4), (3) and (2) now gives us the lemma. □

Using Lemma 7 above we are now ready to prove Lemma 1 which we used to bound the number of $\preceq$-capped k-walks in Section 2. The main idea in the proof of Lemma 1 is to split the nodes V into different sets based on their degrees and then use Lemma 7 to bound the $\|\cdot\|_\phi$-norm of the graphs induced by these sets individually.

Proof of Lemma 1. Let V_i be the set of nodes u with $\deg(u) \in \left(2^{i-1}, 2^i\right]$, and let $V_{\leqslant i} = \cup_{j \leqslant i} V_j$ be the set of nodes with $\deg(u) \in (0, 2^i]$. Let $G_{\leqslant i} = (V, E \cap V_{\leqslant i}^2)$ be the subgraph of G induced by $V_{\leqslant i}$. Note that $G_{\leqslant i}$ here is defined slightly differently than we did in Section 2 as we consider entire sets of nodes V_i. Any $\preceq$-capped k-walk starting in from a node $u \in V_i$ is contained in $X_{G_{\leqslant i}}$. It follows by Lemma 4 that the total number of $\preceq$-capped k-walks in G is bounded by

$$\begin{aligned}
\sum_{i \geqslant 0} \sqrt{|V_i|} \left\|X_{G_i}^k \mathbf{1}\right\|_\phi &\leqslant \sum_{i \geqslant 0} \|X_{G_i}\|_\phi^{k-1} \sqrt{|V_i|} \, \|X_{G_i}\mathbf{1}\|_\phi \\
&\leqslant \|X_G\|_\phi^{k-1} \sum_{i \geqslant 0} \sqrt{|V_i|} \, \|X_{G_i}\mathbf{1}\|_\phi . \qquad (10)
\end{aligned}$$

We note that $X_{G_i}\mathbf{1} \leqslant \sum_{j \leqslant i} 2^j \mathbf{1}_{V_j}$, and hence

$$\begin{aligned}
\sum_{i \geqslant 0} \sqrt{|V_i|} \, \|X_{G_i}\mathbf{1}\|_\phi &\leqslant \sum_{i \geqslant 0} \sqrt{|V_i|} \sum_{j \leqslant i} \left\|2^j \mathbf{1}_{V_j}\right\|_\phi \\
&= \sum_{i \geqslant j \geqslant 0} \sqrt{|V_i|} \cdot \sqrt{|V_j|} \cdot 2^j .
\end{aligned}$$

We now note that

$$\sqrt{|V_i|} \cdot \sqrt{|V_j|} \cdot 2^j = \sqrt{2^i\,|V_i|} \cdot \sqrt{2^j\,|V_j|} \cdot 2^{-(i-j)/2} \\ \leqslant \frac{2^i\,|V_i| + 2^j\,|V_j|}{2} \cdot 2^{-(i-j)/2}\,,$$

which implies that

$$\begin{aligned}\sum_{i \geqslant j \geqslant 0} \sqrt{|V_i|} \cdot \sqrt{|V_j|} \cdot 2^j &\leqslant \sum_{i \geqslant j \geqslant 0} \frac{2^i\,|V_i| + 2^j\,|V_j|}{2} \cdot 2^{-(i-j)/2} \\ &= \sum_{i \geqslant 0} 2^i\,|V_i| \sum_{\ell \geqslant 0} 2^{-\ell/2} \\ &= \frac{\sqrt{2}}{\sqrt{2}-1} \sum_{i \geqslant 0} 2^i\,|V_i|\ .\end{aligned}$$

Since $\sum_{i \geqslant 0} 2^i\,|V_i|$ is at most twice as large as the sum of degrees of the nodes in G it is bounded by $4m$, and therefore

$$\sum_{i \geqslant j \geqslant 0} \sqrt{|V_i|} \cdot \sqrt{|V_j|} \cdot 2^j \leqslant 4 \cdot \frac{\sqrt{2}}{\sqrt{2}-1} m < 14m\,. \tag{11}$$

Combining this with (10) and Lemma 7 we get that the number of $\preceq$-capped k-walks is at most

$$14\,\|X_G\|_{\phi}^{k-1}\,m = O\Big((k^2)^{k-1} m^{\frac{2k}{k+1}}\Big)\ ,$$

which is what we wanted to show. □

Below we prove Lemma 2, which gives a lower bound on the number of capped k-walks.

Proof of Lemma 2. Let $\Delta = \frac{m}{2n}$. For a subgraph F of G we let $f(F)$ denote the subgraph F' of F obtained in the following way. Initially we let $F' = F$. As long as there exists a node $v \in F'$ such that $\deg_{F'}(v) < \Delta$ we remove v from F'. We continue this process until no node in F' has fewer than Δ neighbours and let $f(F) = F'$.

We now construct the sequences $(H_i)_{i \geqslant 0}, (H'_i)_{i \geqslant 0}$ of subgraphs of G in the following manner. We let $H'_0 = G$, and $H_0 = f(H'_0)$. If H_i is non-empty, let v_i be the largest element in H_i, i.e. $v_i \succeq v$ for all $v \in H_i$, and define $H'_{i+1} = H_i \backslash \{v_i\}$. If H_i is empty we let $H'_{i+1} = H_i$. In either case we let $H_{i+1} = f(H'_{i+1})$.

For all i such that H_i is non-empty, there exists at least $\deg_{H_i}(v_i)\Delta^{k-1}$ capped k-walks $(x_1, \ldots, x_k)$ with $x_1 = v_i$. By the definition of H'_{i+1} we have that $\deg_{H_i}(v_i) = |E(H_i)| - \left|E(H'_{i+1})\right|$. The total number of capped k-walks in G is therefore at least:

$$\sum_{i \geqslant 0} \left(|E(H_i)| - \left|E(H'_{i+1})\right|\right) \Delta^{k-1}\,. \tag{12}$$

Now note that:

$$\begin{aligned}&\sum_{i \geqslant 0} \left(|E(H_i)| - \left|E(H'_{i+1})\right|\right) \\ &= \left(\sum_{i \geqslant 0} \left|E(H'_i)\right| - \left|E(H'_{i+1})\right|\right) - \left(\sum_{i \geqslant 0} \left|E(H'_i)\right| - |E(H_i)|\right)\,.\end{aligned} \tag{13}$$

The first sum on the right hand side of (13) is a telescoping sum that is equal to m. The second sum on the right hand side of (13) can be bounded by noting that $|E(H_i')| - |E(H_i)|$ is at most $\Delta \cdot |V(H_i' \backslash f(H_i'))|$, since applying f to H_i' removes $|V(H_i' \backslash f(H_i'))|$ nodes, and each node removed had degree at most Δ. Since at most n nodes are removed in total the sum is bounded by $n\Delta$. Hence (13) is at least $m - n\Delta = \frac{m}{2}$. Inserting this into (12) gives that the number of capped k-walks is at least

$$\frac{m}{2} \cdot \Delta^{k-1} = n \cdot \left(\frac{m}{2n}\right)^k ,$$

as desired. □

4 Hardness of finding cycles

Theorem 1 presents an algorithm with a seemingly natural running time in terms of m and k. A natural question to ask is whether the exponent of m has to increase with k and, perhaps more interestingly, what the correct exponent is. In this section we address the possibility of faster algorithms, by proving Theorem 2 and proposition 1 discussed in the introduction.

Proof of Proposition 1. Let $G = (V, E)$ be the graph in which we wish to find a triangle with $|V| = n$ and $|E| = \Theta(n^2)$. By Conjecture 1 it takes $n^{3-o(1)}$ to find a triangle in G. Now create the graph G' consisting of three copies, A, B, and C, of V. Denote each copy of $u \in V$ in A, B, C by u_A, u_B, u_C, respectively. For each edge $(u, v) \in E$ add the edges (u_A, v_B), (u_B, v_C), and (u_C, v_A) to G'. It now follows that G contains a triangle u, v, w if and only if G' contains a triangle u_A, v_B, w_C.

Now Fix $x = \lceil (2k+1)/4 \rceil$ and note that $2k \geqslant 3x$ by the restrictions to k. Create the graph G_k^e by taking a copy of G' and performing the following changes: Replace each edge by a path of length x. If $2k > 3x$ replace each node u_A in G_k^e by a path $u_A^1, \ldots, u_A^{2k-3x+1}$. Otherwise if $2k = 3x$ do nothing. We now claim that G_k^e contains a C_{2k} if and only if G contains a triangle. Observe first, that if G contains a triangle u, v, w, then $u_A^1 \rightsquigarrow v_V \rightsquigarrow w_C \rightsquigarrow u_A^{2k-3x+1} \rightsquigarrow u_A^1$ is a cycle in G_k^e and has length $3x + 2k - 3x = 2k$. Now assume that G_k^e has a cycle of length $2k$. If this cycle contains two nodes u_A^1 and v_A^1 it must have length at least $4x > 2k$ and similar for B and C and $u_A^{2k-3x+1}$ and $v_A^{2k-3x+1}$. Thus, the cycle must exactly be of the form $u_A^1 \rightsquigarrow v_V \rightsquigarrow w_C \rightsquigarrow u_A^{2k-3x+1} \rightsquigarrow u_A^1$ and such a cycle can only have length $2k$ if all edges (u_A, v_B), (v_B, w_C), and (w_C, u_A) are present in G'. Now observe that for constant k the graph G_k^e has $N = \Theta(n^2)$ nodes and $M = \Theta(n^2)$ edges. It now follows from Conjecture 1 that no algorithm can detect a C_{2k} in G_k^e in time $O(M^{3/2-\varepsilon}) = O(n^{3-\varepsilon})$ for any $\varepsilon > 0$. □

The reduction for Proposition 1 is shown in Figure 2 below.

Finally, We show the “conditional optimality” stated in Theorem 2. The theorem states that if $O(n^2)$ time is optimal, then our bound is the best that can be achieved.

Proof of Theorem 2. Let $\varepsilon > 0$ be given and let $\delta = \varepsilon$.

Assume there exists an algorithm which finds a $2k$-cycle in time $O(m^{2k/(k+1)-\varepsilon})$. Now consider the following algorithm: If $m \geqslant 100k \cdot n^{1+1/k}$ answer yes, and otherwise run the given algorithm. This algorithm has running time $O(n^{(1+1/k)\cdot(2k/(k+1)-\varepsilon)}) = o(n^{2-\delta})$. Hence part (1) holds.

Now assume there are finitely many graphs G such that $|E(G)| \geqslant |V(G)|^{1+1/k-\varepsilon}$. Then there must exist some constant n_0 such that no graph with $n \geqslant n_0$ nodes and $m \geqslant n^{1+1/k-\varepsilon}$ edges contains a $2k$-cycle. Now consider the following algorithm: Let $G = (V, E)$ be the graph we wish to detect a C_{2k} in. If $|V| < n_0$ we can answer in constant time. If $|V| \geqslant n_0$ and

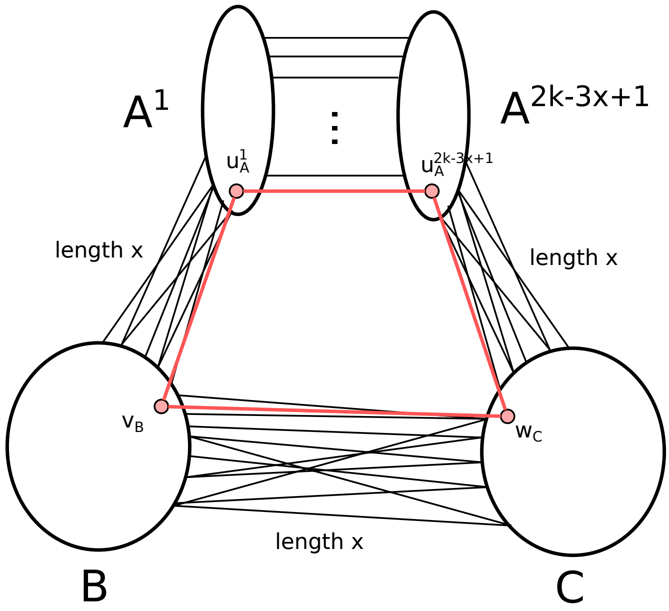

Figure 2: The construction of G_k^e from the proof of Lemma 1 and an example $2k$-cycle highlighted in red.

$|E| \geqslant |V|^{1+1/k-\varepsilon}$ answer no, and otherwise run the algorithm of Theorem 1 to detect a C_{2k} in time $O(|V|^{(1+1/k-\varepsilon)\cdot 2k/(k+1)}) = o(|V|^{2-\delta})$. Hence part (2) holds. $\square$

5 Omitted proofs

This section contains missing proofs from Section 3.

Proof of Lemma 5. Let $v \in \mathbb{R}^n$ be a vector such that each entry either is contained in $\left[2^{-1}, 1\right]$ or is 0. Let $r = |\text{supp}(v)|$ and write v as $v = \sum_{i=1}^{r} \lambda_i e_i$ for vectors e_i such that for each e_i there is a single entry $(e_i)_j = 1$ and all other entries are 0. Let $X_1, \ldots, X_r$ be independent random variables $\in \{0, 1\}$ such that $E(X_i) = \lambda_i$. By the concavity of $\|\cdot\|_\phi$ we then have

$$\begin{aligned} \|Av\|_\phi &= \left\| E\left(A \sum_{i=1}^{r} X_i e_i\right)\right\|_\phi \leqslant E\left(\left\|A \sum_{i=1}^{r} X_i e_i\right\|_\phi\right) \\ &\leqslant E\left(C \left\|\sum_{i=1}^{r} X_i e_i\right\|_\phi\right) \leqslant C\sqrt{r} \\ &\leqslant 2C \|v\|_\phi \ . \end{aligned} \tag{14}$$

Since v was arbitrarily chosen (14) holds for all vector v with entries in $\{0\} \cup [2^{-1}, 1]$.

Let $v \in \mathbb{R}^n$ be a vector where each entry is non-negative. We will show that $\|Av\|_\phi \leqslant 8C \|v\|_\phi$. For each integer k let $v^{(k)} \in \mathbb{R}^n$ be the vector containing the i'th entry of v_i if $v_i \in (2^{k-1}, 2^k]$ and 0 otherwise, i.e.

$$v_i^{(k)} = \left[v_i \in (2^{k-1}, 2^k]\right] v_i \, .$$

Using the triangle inequality and (14) on the vectors $2^{-k}v^{(k)}$ now gives us

$$\begin{aligned}\|Av\|_\phi = \left\|\sum_k Av^{(k)}\right\|_\phi &\leqslant \sum_k 2^k \left\|A2^{-k}v^{(k)}\right\|_\phi \\ &\leqslant \sum_k 2^k \cdot 2C \left\|2^{-k}v^{(k)}\right\|_\phi \\ &= 2C\sum_k \left\|v^{(k)}\right\|_\phi . \end{aligned} \tag{15}$$

Now we have that

$$\begin{aligned}\sum_k \left\|v^{(k)}\right\|_\phi &= \sum_k \int_0^{2^k} \sqrt{\left|\left\{i \mid v_i^{(k)} \geqslant x\right\}\right|}dx \\ &\leqslant \sum_k 2^k \sqrt{\left|\left\{i \mid v_i^{(k)} \geqslant 2^{k-1}\right\}\right|} \\ &= 4\sum_k \int_{2^{k-2}}^{2^{k-1}} \sqrt{\left|\left\{i \mid v_i^{(k)} \geqslant x\right\}\right|}dx \\ &\leqslant 4\sum_k \int_{2^{k-2}}^{2^{k-1}} \sqrt{\left|\{i \mid v_i \geqslant x\}\right|}dx = 4\|v\|_\phi . \end{aligned} \tag{16}$$

Combining (15) and (16) gives that $\|Av\|_\phi \leqslant 8C\|v\|_\phi$ for every non-negative vector $v \in \mathbb{R}^n$ as desired.

Let $v \in \mathbb{R}^n$ be any real vector. Let v^+ and v^- be defined by

$$(v^+)_i = \max\{v_i, 0\}, \quad (v^-)_i = \max\{-v_i, 0\} .$$

Then v^+ and v^- have non-negative coordinates and $v = v^+ - v^-$. It is easy to see that $\|v\|_\phi \geqslant \max\left\{\|v^+\|_\phi, \|v^-\|_\phi\right\}$, and therefore: $\|v^+\|_\phi + \|v^-\|_\phi \leqslant 2\|v\|_\phi$. Now we get the result by the using the triangle inequality:

$$\begin{aligned}\|Av\|_\phi &= \left\|Av^+ - Av^-\right\|_\phi \\ &\leqslant \left\|Av^+\right\|_\phi + \left\|Av^-\right\|_\phi \\ &\leqslant 8C\left(\left\|v^+\right\|_\phi + \left\|v^-\right\|_\phi\right) \\ &\leqslant 16C\|v\|_\phi .\end{aligned}$$

It follows that $\|A\|_\phi \leqslant 16C$. □

Below we show Lemma 6, which can be seen as a modified version of the classic Bondy and Simonovits lemma, as we here argue about edges between any two subsets of the graph, instead of edges in the entire graph as in the original lemma [4].

Proof of Lemma 6. Let $m = |E(A,B)|$ and let $E = E(A,B)$. We will assume that $m \geqslant 100k \cdot (|A| + |B|)$ as the statement is otherwise trivially true. We will assume that the graph contains no $2k$-cycle and show that then $m \leqslant 100k \cdot \sqrt{|A|+|B|}^{1+1/k}$.

Let $2\alpha = \frac{m}{|A|}$ and let $2\beta = \frac{m}{|B|}$ be the average degrees of nodes in A and B respectively when restricted to E. Recursively remove any node from A respectively B which does not have at least α respectively β edges in E. Then we remove strictly less than $\alpha \cdot |A| + \beta \cdot |B| < m$ edges and thus have a non-empty graph left.

Now fix some node $u \in A$ and let $L_0 = \{u\}$. Now define L_{i+1} to be the neighbours of the nodes in L_i using the edges of E for $i = 0, \ldots, k-1$. This gives us the sets $L_0, \ldots, L_k$. Note that if $A \cap B = \emptyset$ we have $L_i \cap L_{i+1} = \emptyset$ for each $i = 0, \ldots, k-1$. We will show by induction that $|L_i| \leqslant |L_{i+1}|$ for each $i = 0, \ldots, k-1$. This is clearly true for $i = 0$ since u has degree at least $\alpha \geqslant 50k$ by assumption. Now fix some $i \geqslant 1$ and assume that the statement is true for all $j < i$. We will assume that i is even (the other case is symmetric). We know from [4, 14] that

$$|E(L_i, L_{i+1})| \leqslant 4k \cdot (|L_i| + |L_{i+1}|) ,$$

as otherwise we can find a $2k$-cycle. By the induction hypothesis this gives us

$$|E(L_{i-1}, L_i)| \leqslant 8k \cdot |L_i| .$$

Since i is even we also know that

$$\alpha \cdot |L_i| \leqslant |E(L_{i-1}, L_i)| + |E(L_i, L_{i+1})| ,$$

and thus

$$(\alpha - 8k) \cdot |L_i| \leqslant |E(L_i, L_{i+1})| \leqslant 4k \cdot (|L_i| + |L_{i+1}|) .$$

This gives us that $(\alpha - 12k) \leqslant 4k \cdot |L_{i+1}|$, and it follows that

$$|L_{i+1}| \geqslant \frac{\alpha - 12k}{4k} \cdot |L_i| .$$

By our assumption on α this proves that $|L_{i+1}| \geqslant L_i$. When i is odd the same argument gives us that $|L_{i+1}| \geqslant \frac{\beta - 12k}{4k} \cdot |L_i|$.

By the above discussion it follows that

$$\begin{aligned} |L_k| &\geqslant \left(\frac{\alpha - 12k}{4k}\right)^{\lceil k/2 \rceil} \cdot \left(\frac{\beta - 12k}{4k}\right)^{\lfloor k/2 \rfloor} \\ &\geqslant \frac{\alpha^{\lceil k/2 \rceil} \beta^{\lfloor k/2 \rfloor}}{(8k)^k} , \end{aligned}$$

where the last inequality follows by our assumption the $\alpha, \beta \geqslant 50k$. Assume now that k is odd (as the even case is handled similar). It then follows that

$$|B| \geqslant |L_k| \geqslant \frac{\alpha^{\lceil k/2 \rceil} \beta^{\lfloor k/2 \rfloor}}{(8k)^k} ,$$

and a symmetric argument gives us

$$|A| \geqslant \frac{\alpha^{\lfloor k/2 \rfloor} \beta^{\lceil k/2 \rceil}}{(8k)^k} ,$$

implying that

$$\sqrt{|A| \cdot |B|} \geqslant \frac{\sqrt{\alpha\beta}^k}{(8k)^k} = \frac{\sqrt{\frac{m^2}{4|A||B|}}^k}{(8k)^k} .$$

Now taking the kth root and isolating m yields exactly the bound we wanted to show

$$m \leqslant 16k \cdot \sqrt{|A||B|}^{1+1/k} .$$

In the above proof we assumed that A and B were disjoint in order to apply the lemma of [4, 14]. Now observe that if this is not the case we can pick subsets $A' \subseteq A$ and $B' \subseteq B$ with $A' \cap B' = \emptyset$ and $E(A', B') \geqslant m/2$ and the argument now follows through. $\square$

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