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Review of *Vicious Circles* by
Jon Barwise and Lawrence Moss
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1 Overview

Since its very first days, modern set theory had "paradox" written all over it. For the past hundred years some of the best mathematitians of late 19th and 20th century (such as Cantor, Russel and Tarski) would devote their attention to the analysis of set theory, paradoxes it creates and attempts to modify it in order to escape these paradoxes.

As a result of the attempts to rid the set theory of the class of paradoxes (usually associated with Bertrand Russel's name), which appeared when one would start to look at sets that can be elements of themselves, Russel proposed a set theory which would iteratively build all sets in such a manner that self-inclusion became impossible.

This would have sufficed if it wasn't for the fact that quite a number of events and objects, which present interest to scientists exibit circular or repetative behaviour, and / or are self-referential. And in many cases it turns out that Russel's set theory is too prohibitive to model these phenomena.

In the book under review, Jon Barwise and Lawrence Moss set a goal of providing a framework for constructing models for such circular events. The book starts with a brief introduction into set theory. After that the reader is provied with a set of examples of circularity as it is prohibited by set theory from various backgrounds. After that the basic theory of circularity, based on set theory is introduced and its applications are discussed. Finally the theory is enhanced in later chapters and at the end of the book the complex applications are being described.

The more detailed summary of the book is provided below.

2 Summary of Contents

The book is broken into parts, which in turm are broken into chapters.

Part I of the book, **Background**, contains two chapters, *Introduction* and *Background on set theory.* The former chapter contains a brief description of the problem domain and explains the contents of the book. The latter chapter is devoted to brief introduction into the set theory.

The authors describe set theory axioms. The author's approach to the discussion of the axioms is motivated by the goal of establishing the framework for modelling circularity in further chapters. It is shown exactly how and to which extent the axioms prohibit circularity. As a core "nemesis" for circularity, Foundation Axiom is introduced and its implications are discussed.

In **Part II** of the book, **Vicious Circles**, the problem of circularity is introduced. The authors provide a number of examples of circularity in Computer Science (*chapter 3*), philosophy, linguistics and mathematics (*chapter 4*). In the heart of most of the examples is an attempt to operate with a structure that has the form $x = \langle a, x \rangle$, or $x = \{\{a\}, \{a, x\}\}$. It is easy to see that this construction has itself a part and therefore if Foundation Axiom is accepted, it is not a set.

The last chapter of this part, *Circularity and paradox*, is devoted to establishing the link between the well-known set theory paradoxes (such as the Liar Paradox and Russel's paradox) and circularity as it is studied in the book.

Part III, Basic Theory is the core of the book. In *Chapter 6, The Solution Lemma* the alternative to Foundation Axiom is introduced. The first formulation of this alternative, called Anti-Foundation Lemma is due to Forti and Honsell (1983). Among the number of ways to present Anti-Foundation Axiom the authors choose one called Solution Lemma, based on the following approach. The notion of flat system of equations is introduced. Equations considered are of a form similar to described above: $x = \{a, x\}$. The Solution Lemma formulation of the Anti-Foundation Axiom is then defined as a postulate that every flat system of equations has a *unique* solution. Now, constructions like the one above or $x = \{x\}$ become the representations of valid sets in the set theory obtained by replacing the Foundation Axiom with Anti-Foundation Axiom (the axiom system obtained by this will be referenced further as ZFA).

Chapter 7, Bisimulation deals with the question of equivalence of newly defined sets. In which cases can we say that two flat systems of equations as defined in *Chapter 6* have the same solution? This question is answered by introduction of a notion of bisimilation. The main theorem says that two flat systems of equations have the same solution sets iff they are bisimilar. It is shown that bisimilarity is a true equvalence relation (i.e. it is reflexive, symmetric and transitive). Later in the chapter the notion of bisimilation is extended from systems of equations onto sets, which in turn solves the purpose of establishing strong extentionality of newly defined sets. The chapter ends with discussion on how to computer the bisimilation relation for two systems of equations effecitively.

In *Chapter 8, Substitution* the notions of system of equations, solution sets and Solution Lemma introduced in *Chapter 6* are refined and an alternative formulations for Anti-Foundation Axiom are provided. The main motivation for seeking refinement of the notion of system of equation is the fact that although it provides for very simple formulation of Anti-Foundation Axiom, the application of this formulation is rather difficult due to the restrictive nature of the notion of flat system of equations.

In this chapter the general systems of equations which have more complex syntax are introduced. The chapter then proceeds to define the solution set for a general system of equations. The General Form of the Solution Lemma is presented at the end of the chapter.

Chapter 9 is a standalone chapter in which the authors show that the new set theory (with Anti-Foundation axiom) is an extention of the set theory with Foundation Axiom. Also, relative consistency of the new set theory is established.

Part IV, Elementary Applications seeks to establish the connection between the theory described in **Part III** and a variety of notions from Graph Theory, Logic, Game Theory and other disciplines.

In *Chapter 10, Graphs*, yet another version of Anti-Foundation Axiom, based on graph theory is constructed. The key notion, defined in the chapter is a decoration of a graph G. Decoration is such a function d from nodes of G to sets, that

$$d(a) = \{d(b) | (a, b) \text{ is an edge in } G\}$$

After the properties of graph decorations are studied, the formulation of Anti-Foundation Axiom as "Every graph G has a unique decoration" is given. The chapter also translates the notion of bisimilarity onto graphs (two graphs are bisimilar iff they have the same decoration).

Chapter 11 establishes the connection of the new set theory with modal logic. Language $\mathcal{L}_\infty(A)$ is defined (as an extention of standard modal logic longuage $\mathcal{L}(A)$ with conjunction over (possibly infinite) set of formulas). Semantics of this language is given in terms of Kripke structures. The connection between the validity of formulas $\mathcal{L}_\infty(A)$ on a Kripke structure G and its decoration (as Kripke structure is a labeled graph, it has a decoration as defined in previous chapter) is then established and bisimilar Kripke structures are studied.

Second part of this chapter is devoted to characterizations of sets defined in ZFA. A formula Θ (of $\mathcal{L}_\infty(A)$ or $\mathcal{L}(A)$) is said to *characterize* a set a iff Θ is valid only on a. The main result of this part of the chapter is that every set (defined in ZFA) is characterizable by some sentence in $\mathcal{L}_\infty(A)$ or $\mathcal{L}(A)$. Proof of this result however is not simple, and most of rest of the chapter is devoted to it. In the last part of the chapter different axiomatizations of modal logic are given[4].

Capter 12, Games starts with a brief introduction to game theory. A definition of a 2-player game is given and some properties of such games are established. Then, the correspondence between games and the validity of first-order formulas is discussed (Ehrenfeucht-Fraïsse or "pebble" games).

With this information as background, the authors proceed to describe a "pebble game" for determining whether two sets are bisimilar[5]. Then the resolution of *HYPERGAME*, one of the paradoxes presented in *Chapter 5*, is given.

Chapter 13 deals with resolutions of the rest of the paradoxes, described in *Chapter 5*. The main target is the Liar paradox. To approach it, first three valued logic and partial models are introduced. Then, the Liar paradox is studied in the established framework.

Part IV ends with *Chapter 14* devoted to streams. Strams were first introduced, among other examples in *Chapter 3*. As stream is a pair, whose first element is an "atom" and second element is a stream ($s = \langle a, s \rangle$). This is very similar to the notion of *lists* with the sole difference being that lists are finite, while streams are defined to be infinite. In this chapter, a closer look at streams is taken. More than in establishing the actual properties of the streams, however, the authors are interested in developing and applying the new methods (coinduction, corecursion) and demonstrating how these methods can be used in proofs.

Part V of the book consists of three chapters which contain some more theory.

Chapter 15 contains an overview of fixed-point theory. The authors define notions of fixed point, least fixed point and greatest fixed point for monotonic operations on sets. Theorems about the existance of both least fixed points and greatest fixed points for monotonic operators are stated and proven and the properties of both are studied. Last but not least, a connection between fixed points and games is shown at the end of the chapter.

Chapter 16 ties together the notions of a flat system of equations and of greatest fixed points. First, it is shown how a flat system of a equtions can be treated as a Γ-coalgebra $\langle X, e \rangle$ where X is a set and $e : X \leftarrow \Gamma(X)$ for some operator Γ. The notion of a Γ-morphism between the two Γ-coalgebras is introduced. Then the notion of solution of a flat system of equations is extended on coalgebras and a structure of a solution set for coalgebras for monotonic operators which map sets onto pure sets (called proper operators) is described.

The authors then introduce uniform operators as monotonic, proper operators that have one additional property: they commute with almost all substitutions. The main property of uniform operators Γ is that their greatest fixed point is exactly the union of solution sets of all *Gamma*-coalgebras. Finally, if Γ is a uniform operator, then any Γ-coalgebra has a *unique* soultion, which is a subset of the greatest fixed point of Γ.

Chapter 17 is devoted to corecursion. While recursive definitions define mappings from sets to their least-fixed points (and are very well understood and widely studied), the defintions of mappings from sets to their greatest-fixed points (i.e. corecursive definitions) are far less well understood. In the chapter, the notion of uniform operator is strengthened (the new class of operators is called smooth operators), and the notion of corecursion relative to these operators is studied.

Last part of the book (**Part VI**) consists of three chapters that provide more applications of the theory devloped in **Part III** and **Pert V** and a chapter that contains authors' conclusions.

In *Chapter 18* the greatest fixed points of some important operators are studied. *Chapter 19* describes modal logics associated with the operators described in *Chapter 18.*

Chapter 20 contains a detailed philosophical discussion of the ZFA system of axioms (and other related systems of axioms). Finally in the last chapter of the book (*Chapter 21*) the past, the present and future of the area are discussed. Together with a short history of the field, the authors present an extensive set of open problems.

3 Style

From a book on foundations of mathematics (and we can say that **Vicious Circles** is about foundations of mathematics) one should expect very precise and formal presentation of material. One the other hand, a book that seeks to extend upon the foundations of mathematics, and provides formal presentation can easily become unreadable.

In this book the authors make a solid attempt to combine both formality of presentation and clarity of explanations. And for most part (except for rather complicated content of **Part V**) this attempt succeeded. While the notation used in the book is rather diverse and sometimes complex, and while the definitions, statements of theorems and proofs are formalized, authors make every attempt to explain what is going on in plain English. The book contains plenty of examples, that cover virtually everything.

Those who want to test their understanding of the material presented in the book, will be able to do so by doing many exercises found in the book. Some of the exercises are similar to the examples given, while other require deep understanding of the material and proof techniques to be used and are quite challenging. The answers to all exercises are given in the end of the book.

It would be fair to say that the authors also succeeded in their desire to make the book mostly self-contained. As it is seen from the descriptions of various chapters of the book, wherever information from a certain area of mathematics were to be used, this information had been included (f.e. set theory, modal logic, three valued logic etc.). So it is indeed possible for a reader with only general mathematical background to read this book without getting lost. However, because of the plentitude of different areas of mathematics, and logic in particular brought up in the book, extra knowledge in these or similar areas is certainly more than helpful.

Last, but not least, the diagram of the content dependencies between the chapters of the book is very useful, esp. if the book is to be used as a reference or a textbook for a graduate course.

4 Opinion

The Anti-Foundation Axiom, which lies in the core of the theory described in the book was first introduced in 1983, i.e., less than 20 years ago. This fact makes *Vicious Circles* a somewhat unique book: on one hand it is devoted to the very foundations of mathematics, while on the other hand, the material presented is *relatively* new.

As such, this book is a valuable collection of information about the theory of circularity and its relation to other well-established areas of mathematics.