# Combinatorial Algorithm for Restricted Max-Min Fair Allocation 

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#### Abstract

We study the basic allocation problem of assigning resources to players so as to maximize fairness. This is one of the few natural problems that enjoys the intriguing status of having a better estimation algorithm than approximation algorithm. Indeed, a certain Configuration-LP can be used to estimate the value of the optimal allocation to within a factor of $4+\varepsilon$. In contrast, however, the best known approximation algorithm for the problem has an unspecified large constant guarantee.

In this paper we significantly narrow this gap by giving a 13 -approximation algorithm for the problem. Our approach develops a local search technique introduced by Haxell [Hax95] for hypergraph matchings, and later used in this context by Asadpour, Feige, and Saberi [AFS12]. For our local search procedure to terminate in polynomial time, we introduce several new ideas such as lazy updates and greedy players. Besides the improved approximation guarantee, the highlight of our approach is that it is purely combinatorial and uses the Configuration-LP only in the analysis.


Keywords: approximation algorithms, fair allocation, efficient local search

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## 1 Introduction

We consider the Max-Min Fair Allocation problem, a basic combinatorial optimization problem, that captures the dilemma of how to allocate resources to players in a fair manner. A problem instance is defined by a set $\mathcal{R}$ of indivisible resources, a set $\mathcal{P}$ of players, and a set of nonnegative values $\left\{v_{i j}\right\}_{i \in \mathcal{P}, j \in \mathcal{R}}$ where each player $i$ has a value $v_{i j}$ for a resource $j$. An allocation is simply a partition $\left\{R_{i}\right\}_{i \in \mathcal{P}}$ of the resource set and the valuation function $v_{i}: 2^{\mathcal{R}} \mapsto \mathbb{R}$ for any player $i$ is additive, i.e., $v_{i}\left(R_{i}\right)=\sum_{j \in R_{i}} v_{i j}$. Perhaps the most natural fairness criterion in this setting is the maxmin objective which tries to find an allocation that maximizes the minimum value of resources received by any player in the allocation. Thus, the goal in this problem is to find an allocation $\left\{R_{i}\right\}_{i \in \mathcal{P}}$ that maximizes

$$
\min _{i \in \mathcal{P}} \sum_{j \in R_{i}} v_{i j} .
$$

This problem has also been given the name Santa Claus problem as interpreting the players as kids and the resources as presents leads to Santa's annual allocation problem of making the least happy kid as happy as possible.

A closely related problem is the classic scheduling problem of Scheduling on Unrelated Parallel Machines to Minimize Makespan. That problem has the same input as above and the only difference is the objective function: instead of maximizing the minimum we wish to minimize the maximum. In the scheduling context, this corresponds to minimizing the time at which all jobs (resources) are completed by the machines (players) they were scheduled on. In a seminal paper, Lenstra, Shmoys, and Tardos [LST90] showed that the scheduling problem admits a 2approximation algorithm by rounding a certain linear programming relaxation often referred to as the Assignment-LP. Their approximation algorithm in fact has the often stronger guarantee that the returned solution has value at most OPT $+v_{\max }$, where $v_{\max }:=\max _{i \in \mathcal{P}, j \in \mathcal{R}} v_{i j}$ is the maximum value of a job (resource).

From the similarity between the two problems, it is natural to expect that the techniques developed for the scheduling problem are also applicable in this context. What is perhaps surprising is that the guarantees have not carried over so far, contrary to expectation. While a rounding of the Assignment-LP has been shown [BD05] to provide an allocation of value at least OPT - $v_{\text {max }}$, this guarantee deteriorates with increasing $v_{\max }$. Since in hard instances of the problem (when $v_{\max } \approx O P T$ ) there can be players who are assigned only one resource in an optimal allocation, this result provides no guarantee in general. The lack of guarantee is in fact intrinsic to the AssignmentLP for Max-Min Fair Allocation as the relaxation is quite weak. It has an unbounded integrality gap i.e., the optimal value of the linear program can be a polynomial factor larger than the optimal value of an integral solution.

To overcome the limitations of the Assignment-LP, Bansal and Sviridenko [BS06] proposed to use a stronger relaxation, called Configuration-LP, for Max-Min Fair Allocation. Their paper contains several results on the strength of the Configuration-LP, one negative and many positive. The negative result says that even the stronger Configuration-LP has an integrality gap that grows as $\Omega(\sqrt{|\mathcal{P}|})$. Their positive results apply for the interesting case when $v_{i j} \in\left\{0, v_{j}\right\}$, called Restricted Max-Min Fair Allocation. For this case they give an $O(\log \log |\mathcal{P}| / \log \log \log |\mathcal{P}|)$-approximation algorithm, a substantial improvement over the integrality gap of the Assignment-LP. Notice that the restricted version has the following natural interpretation: each resource $j$ has a fixed value $v_{j}$ but it is interesting only for some subset of the players.

Bansal and Sviridenko further showed that the solution to a certain combinatorial problem on set systems would imply a constant integrality gap. This was later settled positively by

Feige [Fei08a] using a proof technique that repeatedly used the Lovász Local Lemma. At the time of Feige's result, however, it was not known if his arguments were constructive, i.e., if it led to a polynomial time algorithm for finding a solution with the same guarantee. This was later shown to be the case by Haeupler et al. [HSS11], who constructivized the various applications of the Lovász Local Lemma in the paper by Feige [Fei08a]. This led to the first constant factor approximation algorithm for Restricted Max-Min Fair Allocation, albeit with a large and unspecified constant. This approach also requires the solution of the exponentially large Configuration-LP obtained by using the ellipsoid algorithm.

A different viewpoint and rounding approach for the problem was initiated by Asadpour, Feige, and Saberi [AFS12]. This approach uses the perspective of hypergraph matchings where one can naturally interpret the problem as a bipartite hypergraph matching problem with bipartitions $\mathcal{P}$ and $\mathcal{R}$. Indeed, in a solution of value $\tau$, each player $i$ is matched to a subset $R_{i}$ of resources of total value at least $\tau$ which corresponds to a hyperedge ( $i, R_{i}$ ). Previously, Haxell [Hax95] provided sufficient conditions for bipartite hypergraphs to admit a perfect matching, generalizing the well known graph analog, viz., Hall's theorem. Her proof is algorithmic in the sense that when the sufficient conditions hold, then a perfect matching can be found using a local search procedure that will terminate after at most exponentially many iterations. Haxell's techniques were successfully adapted by Asadpour et al. [AFS12] to the Restricted Max-Min Fair Allocation problem to obtain a beautiful proof showing that the Configuration-LP has an integrality gap of at most 4. As the Configuration-LP can be solved to any desired accuracy in polynomial time, this gives a polynomial time algorithm to estimate the value of an optimal allocation up to a factor of $4+\varepsilon$, for any $\varepsilon>0$. Tantalizingly, however, the techniques of [AFS12] do not yield an efficient algorithm for finding an allocation with the same guarantee.

The above results lend the Restricted Max-Min Fair Allocation problem an intriguing status that few other natural problems enjoy (see [Fei08b] for a comprehensive discussion on the difference between estimation and approximation algorithms). Another problem with a similar status is the restricted version of the aforementioned scheduling problem. The techniques in [AFS12] inspired the last author to show [Sve12] that the Configuration-LP estimates the optimal value within a factor $33 / 17+\varepsilon$ improving on the factor of 2 by Lenstra et al. [LST90]. Again, the algorithm in [Sve12] is not known to terminate in polynomial time. We believe that this situation illustrates the need for new tools that improve our understanding of the Configuration-LP especially in the context of basic allocation problems in combinatorial optimization.

Our results. Our main result improves the approximation guarantee for the Restricted Max-Min Fair Allocation problem. Note that $6+2 \sqrt{10} \approx 12.3$.

Theorem 1.1. For every $\varepsilon>0$, there exists a combinatorial $(6+2 \sqrt{10}+\varepsilon)$-approximation algorithm for the Restricted Max-Min Fair Allocation problem that runs in time $n^{O\left(1 / \varepsilon^{2} \log (1 / \varepsilon)\right)}$ where $n$ is the size of the instance.

Our algorithm has the advantage of being completely combinatorial. It does not solve the exponentially large Configuration-LP. Instead, we use it only in the analysis to compare the value of the allocation returned by our algorithm against the optimum. As our hidden constants are small, we believe that our algorithm is more attractive than solving the Configuration-LP for a moderate $\varepsilon$. Our approach is based on the local search procedure introduced in this context by Asadpour et al. [AFS12], who in turn were inspired by the work of Haxell [Hax95]. Asadpour et al. raised the natural question if local search procedures based on alternating trees can be made to run in polynomial time. Prior to this work, the best running time guarantee was a quasi-polynomial
time alternating tree algorithm by Poláček and Svensson [PS12]. The main idea in that paper was to show that the local search can be restricted to alternating paths of length $O(\log n)$ (according to a carefully chosen length function), where $n$ is the number of players and resources. This restricts the search space of the local search giving the running time of $n^{O(\log n)}$. To further reduce the search space seems highly non-trivial and it is not where our improvement comes from. Rather, in contrast to the previous local search algorithms, we do not update the partial matching as soon as an alternating path is found. Instead, we wait until we are guaranteed a significant number of alternating paths, which then intuitively guarantees large progress. We refer to this concept as lazy updates. At the same time, we ensure that our alternating paths are short by introducing greedy players into our alternating tree: a player may claim more resources than she needs in an approximate solution.

To best illustrate these ideas we have chosen to first present a simpler algorithm in Section 3. The result of that section still gives an improved approximation guarantee and a polynomial time local search algorithm. However, it is not combinatorial as it relies on a preprocessing step which in turn uses the solution of the Configuration-LP. Our combinatorial algorithm is then presented in Section 4. The virtue of explaining the simpler algorithm first is that it allows us to postpone some of the complexities of the combinatorial algorithm until later, while still demonstrating the key ideas mentioned above.

Further related work. As mentioned before, the Configuration-LP has an integrality gap of $\Omega(\sqrt{|\mathcal{P}|})$ for the general Max-Min Fair Allocation problem. Asadpour and Saberi [AS07] almost matched this bound by giving a $O\left(\sqrt{|P|} \log ^{3}(|\mathcal{P}|)\right)$-approximation algorithm; later improved by Saha and Srinivisan [SS10] to $O(\sqrt{|\mathcal{P}| \log |\mathcal{P}|} / \log \log |\mathcal{P}|)$. The current best approximation is $O\left(n^{\varepsilon}\right)$ due to Bateni et al. [BCG09] and Chakraborty et al. [CCK09]; for any $\varepsilon>0$ their algorithms run in time $O\left(n^{1 / \varepsilon}\right)$. This leaves a large gap in the approximation guarantee for the general version of the problem as the only known hardness result says that it is NP-hard to approximate the problem to within a factor less than 2 [BD05]. The same hardness also holds for the restricted version.

## 2 The Configuration-LP

Recall that a solution to the Max-Min Fair Allocation problem of value $\tau$ is a partition $\left\{R_{i}\right\}_{i \in \mathcal{P}}$ of the set of resources so that each player receives a set of value at least $\tau$, i.e., $v_{i}\left(R_{i}\right) \geqslant \tau$ for $i \in \mathcal{P}$. Let $\mathcal{C}(i, \tau)=\left\{C \subseteq \mathcal{R}: v_{i}(C) \geqslant \tau\right\}$ be the set of configurations that player $i$ can be allocated in a solution of value $\tau$. The Configuration-LP has a decision variable $x_{i C}$ for each player $i \in \mathcal{P}$ and each $C \in C(i, \tau)$. The intuition is that the variable $x_{i C}$ takes value 1 if and only if she is assigned the bundle $C$. The Configuration-LP is now a feasibility linear program with two sets of constraints: the first set says that each player should receive (at least) one configuration and the second set says that each item should be assigned to at most one player. The formal definition is given in the left box of Figure 1.

It is easy to see that if $\operatorname{CLP}\left(\tau_{0}\right)$ is feasible, then so is $\operatorname{CLP}(\tau)$ for all $\tau \leqslant \tau_{0}$. We say that the value of the Configuration-LP is $\tau_{O P T}$ if it is the largest value such that the above program is feasible. Since every feasible allocation is a feasible solution of the Configuration-LP, $\tau_{O P T}$ is an upper bound on the value of the optimal allocation and therefore $C L P(\tau)$ constitues a valid relaxation.

We note that the LP has exponentially many variables; however, it is known that one can approximately solve it to any desired accuracy by designing a polynomial time (approximate) separation algorithm for the dual [BS06]. For our combinatorial algorithm, the dual shall play an important role in our analysis. By associating the sets of variables $\left\{y_{i}\right\}_{i \in \mathcal{P}}$ and $\left\{z_{j}\right\}_{j \in \mathcal{R}}$ to the

$$
\begin{array}{rl|l}
\sum_{C \in C(i, \tau)} x_{i C} \geqslant 1, & \forall i \in \mathcal{P}, & \max \sum_{i \in \mathcal{P}} y_{i}-\sum_{j \in \mathcal{R}} z_{j} \\
\sum_{i, C: j \in C, C \in C(i, \tau)} x_{i C} \leqslant 1, & \forall j \in \mathcal{R}, & y_{i} \leqslant \sum_{j \in C} z_{j}, \\
x \geqslant 0 . & & y, z \geqslant 0 .
\end{array} \quad \forall i \in \mathcal{P}, \forall C \in C(i, \tau),
$$

Figure 1: The Configuration-LP for a guessed optimal value $\tau$ on the left and its dual on the right.
constraints in the primal corresponding to players and resources respectively, and letting the primal have the objective function of minimizing the zero function, we obtain the dual of $C L P(\tau)$ shown in the right box of Figure 1.

## 3 Polynomial time algorithm

To illustrate our key ideas we first describe a simpler algorithm that works on clustered instances. This setting, while equivalent to the general problem up to constant factors, allows for a simpler exposition of our key ideas. Specifically, we will prove the following theorem in this section.

Theorem 3.1. There is a polynomial time 36-approximation for restricted max-min fair allocation.
We note, however, that producing such clustered instances requires solving the ConfigurationLP. To avoid solving it, and get a purely combinatorial algorithm, we will show how to bypass the clustering step in Section 4.

Before describing our algorithm formally, we begin by giving an informal overview of how it works, while pointing out the key ideas behind it.

### 3.1 Intuitive Algorithm Description and Main Ideas

Our first step towards recovering an approximate solution to an instance of restricted max-min fair allocation, is guessing the value of the Configuration-LP $\tau_{O P T}$ by performing a binary search over the range of its possible values. For a particular guess $\tau$, assuming that $C L P(\tau)$ is feasible, our goal now is to approximately satisfy each player. That is, we will allocate for each player a disjoint collection of resources, whose value for that player is at least $\tau / 36$. Towards this end, we design a local search procedure, that we will apply iteratively in order to find such a 36 -approximate allocation. The input to this procedure will be a partial allocation that satisfies some (possibly empty) subset of the players, and an unsatisfied player. Then, our local search procedure will extend the allocation in order to satisfy the input player as well; hence, applying this procedure iteratively will satisfy all the players.

We now illustrate some key aspects of this local search procedure through an example that appears in Figure 2; for simplicity, in this example we consider only resources of value less than $\tau / 36$. Given a partial allocation of resources to a subset of the players, we wish to extend this to satisfy an additional player $p$. If there are free resources (i.e., not already appearing in our partial allocation) of total value $\tau / 36$ for $p$, then we just satisfy $p$ by assigning those resources to her.

Otherwise, we find a set of resources whose value for $p$ is at least $2 \tau / 5$; these resources constitute a bundle (or, as we will refer to it later on, an edge) $e_{p}$ we would wish to include in our partial allocation in order to satisfy $p$. However, we cannot include this edge right away because there already exist edges in our partial allocation that share resources with $e_{p}$; in other words, such edges are blocking the inclusion of $e_{p}$ into our partial allocation. In Figure 2(a), $e_{p}$ is the gray edge, and its blocking edges are the white ones.

At this point, we should make note of the fact that the size of $e_{p}$ is considerably larger than our goal of $\tau / 36$; this is by design and due to our greedy strategy. By considering edges whose size exceeds our goal, we are able to increase the rate at which blocking edges are inserted into our local search; indeed, in Figure 2(a), a single greedily-constructed edge ( $e_{p}$ ) introduced 3 blocking edges. Ultimately, this will allow us to bound the running time of our local search.

Now, since our goal is to include $e_{p}$ in our partial allocation, we are required to free up some of $e_{p}$ 's resources by finding an alternative way of satisfying the players included in $e_{p}$ 's blocking edges. The steps we take towards this end appear in Figure 2(b): for each player in $e_{p}$ 's blocking edges, we find a new edge that we would wish to include into our partial allocation. But these new gray edges might also be blocked by existing edges in our partial allocation. Therefore this step introduces a second layer of edges comprising a set of edges we would like to include in our allocation, and their corresponding blocking edges; these layers are separated by dashed lines in the example.

Next, we observe that 2 of the 3 gray edges in the second layer actually have a lot of resources that do not appear in any blocking edge. In this case, as one can see in Figure 2(c), we select a subset of free resources from each edge of size at least $\tau / 36$ (drawn with dashed lines), and swap these edges for the existing white edges in our partial allocation. We call this operation a collapse of the first layer, only to be left with $e_{p}$ and a single blocking edge in the first layer. The way we decide when to collapse a layer, is dictated by our strategy of lazy updates: similar to Figure 2(c), we will only collapse a layer if that would mean that a large fraction of its blocking edges will be removed.

Finally, in Figure 2(d), a significant amount of resources of $e_{p}$ has now been freed up. Then, we choose a subset of these resources (again, drawn with a dashed line), and allocate them to $p$. At this point, we have satisfied $p$, and managed to extend our partial allocation to satisfy one more player.

We proceed by formally defining and analyzing the local search algorithm we sketched above.

### 3.2 Parameters

Let $\tau>0$ be a guess on the value of the Configuration-LP. Our algorithm will use the following setting of parameters:

$$
\begin{align*}
\beta & :=36, \\
\alpha & :=5 / 2,  \tag{3.1}\\
\mu & :=1 / 500 .
\end{align*}
$$

Here, $\beta$ is the approximation guarantee, $\alpha$ determines the "greediness" of the edges introduced into the layers, and $\mu$ determines the "laziness" of the updates of our algorithm. As our goal is to expose the main ideas, we have not optimized the constants in this section.

We shall show that whenever $\operatorname{CLP}(\tau)$ is feasible, our algorithm will terminate with a solution of value at least $\tau / \beta$ for the given instance of restricted max-min fair allocation. Combining this with a standard binary search then yields a $\beta$-approximation algorithm.


Figure 2: An example execution of our local search algorithm. In this figure, boxes correspond to players, and circles correspond to resources.

### 3.3 Thin and fat edges, and matchings

We partition the resource set $\mathcal{R}$ into $\mathcal{R}_{f}:=\left\{i \in \mathcal{R}: v_{i} \geqslant \tau / \beta\right\}$ and $\mathcal{R}_{t}:=\left\{i \in \mathcal{R} \mid v_{i}<\tau / \beta\right\}$, fat and thin resources respectively. Note that in a $\beta$-approximate solution, a player is satisfied if she is assigned a single fat resource whereas she needs several thin resources. We will call a pair $(p, R)$, for any $p \in \mathcal{P}$ and $R \subseteq \mathcal{R}$ such that $v_{p}(R)=v(R)$ where $v(R)=\sum_{j \in R} v_{j}$, an edge. Notice that this definition implies that every resource in $R$ is a resource that player $p$ is interested in. We now define thin and fat edges.

Definition 3.2 (Thin and fat edges). We will call an edge $(p, R)$, where $p \in \mathcal{P}$ and $R \subseteq \mathcal{R}$, fat, if $\{j\}=R \subseteq \mathcal{R}_{f}$ contains a single fat resource that $p$ is interested in; this already implies that $v_{p}(R) \geqslant \tau / \beta$. On the other hand, we will call an edge $(p, R)$, where $p \in \mathcal{P}$ and $R \subseteq \mathcal{R}$, thin, if $R \subseteq \mathcal{R}_{t}$ is a set of thin resources that $p$ is interested in.

Finally, for any $\delta \geqslant 1$, we will call an edge $(p, R)$, where $p \in \mathcal{P}$ and $R \subseteq \mathcal{R}$, a $\delta$-edge, if $R$ is a minimal set (by inclusion) of resources of value at least $\tau / \delta$ for $p$, i.e., $v_{p}(R) \geqslant \tau / \delta$.

Remark 3.3. A thin $\delta$-edge has value at most $\tau / \delta+\tau / \beta$ due to the minimality of the edge.
As we have already mentioned, the goal of our local search algorithm is to iteratively extend a partial matching:

Definition 3.4 (Matchings). A set $M$ of $\beta$-edges is called a matching if each player appears in at most one edge and the set of resources used by the edges in $M$ are pairwise disjoint. We say that $M$ matches a player $p \in \mathcal{P}$ if there exists an edge in $M$ that contains $p$. Moreover, it is called a perfect matching if each player is matched by $M$, and otherwise it is called a partial matching.

Using the above terminology, our goal is to find a perfect matching yielding our desired allocation of value $\tau / \beta$. Our approach will be to show that as long as the matching $M$ does not match all players in $\mathcal{P}$ we can extend it to obtain a matching that matches one more player. This ensures that starting with an empty partial matching and repeating this procedure $|\mathcal{P}|$ times we will obtain an allocation of value at least $\tau / \beta$. Thus, it suffices to develop such an algorithm. This is precisely what our algorithm will do. We first state a preprocessing step in Section 3.4 before describing the algorithm in Section 3.5.

### 3.4 Clustering step

This preprocessing phase produces the clustered instances referred to earlier. The clustering step that we use is the following reduction due to Bansal and Sviridenko.

Theorem 3.5 (Clustering Step [BS06]). Assuming that $\operatorname{CLP}(\tau)$ is feasible, we can partition the set of players $\mathcal{P}$ into $m$ clusters $N_{1}, \ldots, N_{m}$ in polynomial time such that

1. Each cluster $N_{k}$ is associated with a distinct subset of $\left|N_{k}\right|-1$ fat items from $\mathcal{R}_{f}$ such that they can be assigned to any subset of $\left|N_{k}\right|-1$ players in $N_{k}$, and
2. there is a feasible solution $x$ to $\operatorname{CLP}(\tau)$ such that $\sum_{i \in N_{k}} \sum_{C \in C_{t}(i, \tau)} x_{i C}=1 / 2$ for each cluster $N_{k}$, where $C_{t}(i, \tau)$ denotes the set of configurations for player $i$ comprising only thin items.

Note that the player that is not assigned a fat item can be chosen arbitrarily and independently for each cluster in the above theorem. Therefore, after this reduction, it suffices to allocate a thin $\beta$-edge for one player in each cluster to obtain a $\beta$-approximate solution for the original instance. Indeed, Theorem 3.5 guarantees that we can assign fat edges for the remaining players. For the rest of the section we assume that our instance has been grouped into clusters $N_{1}, \ldots, N_{m}$ by an application of Theorem 3.5. The second property of these clusters is that each cluster is fractionally assigned $1 / 2$ LP-value of thin configurations. We will use this to prove the key lemma in this section, Lemma 3.6.

We now focus only on allocating one thin $\beta$-edge per cluster and forget about fat items completely. This makes the algorithm in Section 3.5 simpler than our final combinatorial algorithm, where we also need to handle the assignment of fat items to players.

### 3.5 Description of the algorithm

Notation: Recall that it suffices to match exactly one player from each cluster with a thin $\beta$-edge. With this in mind, we say that a cluster $N_{k}$ is matched by $M$ if there exists some player $p \in N_{k}$ such that $p$ is matched by $M$. For a set $S$ of edges, we let $\mathcal{R}(S)=\bigcup_{(p, R) \in S} R$ denote the union of the resources of these edges and we let $\mathcal{P}(S)=\bigcup_{(p, R) \in S}\{p\}$ denote the union of players of these edges. To ease notation, we abbreviate $\mathcal{P}\left(B_{t}\right)$ by $P_{t}$ in the description of the algorithm and its analysis. Finally, for any family of sets $S_{0}, S_{1}, \ldots, S_{\ell}$ we denote $S_{0} \cup S_{1} \cup \cdots \cup S_{t}$ by $S_{\leqslant t}$.

The input to Algorithm 1 is a partial matching $M$ that matches at most one player from each cluster $N_{1}, \ldots, N_{m}$, and a cluster $N_{0}$ that is not matched by $M$; our goal is to extend our partial matching by matching $N_{0}$.

Input: A partial matching $M$ and an unmatched cluster $N_{0}$.
Output: A matching $M^{\prime}$ that matches all clusters matched by $M$ and also matches $N_{0}$.

1. (Initialization) Select an arbitrary player $p_{0} \in N_{0}$ and let $A_{0}=\emptyset, B_{0}=\left\{\left(p_{0}, \emptyset\right)\right\}, \ell=0, \mathcal{L}=\left(A_{0}, B_{0}\right)$.
(Iterative step) Repeat the following until $N_{0}$ is matched by $M$ :
2. (Build phase) Initialize $A_{\ell+1}=\emptyset$; then for each cluster $N_{k}$ with a player in $P_{\ell}$ do:

- If there is a thin $\alpha$-edge $(p, R)$ with $p \in N_{k}$ and $R \cap \mathcal{R}\left(A_{\leqslant \ell+1} \cup B_{\leqslant \ell}\right)=\emptyset$ then $A_{\ell+1}=A_{\ell+1} \cup\{(p, R)\}$.

At the end of the build phase, let first $B_{\ell+1}$ be the edges of $M$ that are blocking the edges in $A_{\ell+1}$. Then update the state of the algorithm by appending $\left(A_{\ell+1}, B_{\ell+1}\right)$ to $\mathcal{L}$ and by incrementing $\ell$ by one.
3. (Collapse phase) While $\exists t: I_{t+1}=\left\{(p, R) \in A_{t+1}: v_{p}\left(R \backslash \mathcal{R}\left(B_{t+1}\right)\right) \geqslant \tau / \beta\right\}$ has cardinality $\geqslant \mu\left|P_{t}\right|$ :

- Choose the smallest such $t$.
//We refer to the following steps as collapsing layer $t$.
- For each cluster $N_{k}$ with players $q, p$ satisfying $q \in P_{t}$ and $(p, R) \in I_{t+1}$ :
- Let $\left(q, R_{q}\right) \in B_{t} \cap M$ be the edge containing $q$.
- Replace $\left(q, R_{q}\right)$ in $M$ with edge $\left(p, R^{\prime}\right)$, where $R^{\prime}$ is a $\tau / \beta$-minimal subset of $R \backslash \mathcal{R}\left(B_{t+1}\right)$, i.e., update $M \leftarrow M \backslash\left\{\left(q, R_{q}\right)\right\} \cup\left\{\left(p, R^{\prime}\right)\right\}$.
- Finally, remove $\left(q, R_{q}\right)$ from $B_{t}$.
- Discard $\left(A_{i}, B_{i}\right)$ from $\mathcal{L}$ for all $i>t$ and set $\ell=t$.

Output the matching $M$ that also matches $N_{0}$.
Algorithm 1: Polynomial Time Algorithm for Clustered Instances

The state of the algorithm is described by a (dynamic) tuple ( $M, \ell, \mathcal{L}$ ), where $M$ is the current partial matching and $\mathcal{L}=\left(\left(A_{0}, B_{0}\right),\left(A_{1}, B_{1}\right), \cdots,\left(A_{\ell}, B_{\ell}\right)\right)$ is a list of pairs of sets of "added" and "blocking" edges that is of length/depth $\ell$. We shall refer to $\left(A_{i}, B_{i}\right)$ as the $i^{\prime}$ th layer ${ }^{1}$.

Invariants. The description of our algorithm appears as Algorithm 1. The algorithm is designed to (apart from extending the matching) maintain the following invariants at the start of each iterative step: for $i=1, \ldots, \ell$,

1. $A_{i}$ is a set of thin $\alpha$-edges that are pairwise disjoint, i.e., for two different edges $(p, R) \neq$ $\left(p^{\prime}, R^{\prime}\right) \in A_{i}$ we have $p \neq p^{\prime}$ and $R \cap R^{\prime}=\emptyset$. In addition, each $\alpha$-edge $(p, R) \in A_{i}$ has $R \cap \mathcal{R}\left(A_{\leqslant i} \cup B_{\leqslant i-1} \backslash\{(p, R)\}\right)=\emptyset$ (its resources are not shared with edges from earlier iterations or edges in $A_{i}$ ).
2. $B_{i}=\left\{(p, R) \in M:(p, R)\right.$ is blocking an edge in $\left.A_{i}\right\}$ contains those edges of $M$ that blocks edges in $A_{i}$, where we say that an edge $(p, R) \in M$ blocks an edge ( $p^{\prime}, R^{\prime}$ ) if $R \cap R^{\prime} \neq \emptyset$.
3. The players of the edges in $A_{i}$ belong to different clusters and any cluster $N_{k}$ with a player $p \in N_{k}$ that appears in an edge in $A_{i}$ has a player $q \in N_{k}$ (that may equal $p$ ) that appears in an edge in $B_{i-1}$.

[^1]4. $\left|I_{i}\right|<\mu\left|P_{i-1}\right|$ where, as in Step 3 of Algorithm 1, $I_{i}=\left\{(p, R) \in A_{i}: v_{p}\left(R \backslash \mathcal{R}\left(B_{i}\right)\right) \geqslant \tau / \beta\right\}$ is defined to be those edges that have sufficient amount of unblocked resources so as to be added to the matching.

In what follows, we further explain the steps of the algorithm and why the invariants are satisfied. It will then also be clear that the algorithm outputs an extended matching whenever it terminates. We then analyze its running time in the next section.

First the algorithm initializes by selecting an arbitrary player $p_{0}$ in the cluster $N_{0}$ that we wish to match. Then each iteration proceeds in two steps. In the build phase, the algorithm adds thin $\alpha$-edges (at most one for each cluster with a player in $P_{\ell}$ ) to $A_{\ell+1}$. Notice that the resources of these edges are disjoint from $\mathcal{R}\left(A_{\leqslant \ell} \cup B_{\leqslant \ell}\right)$ and from each other. We therefore maintain the first invariant. At the end of the build phase, we define $B_{\ell+1}$ to satisfy the second invariant. The third invariant is also satisfied since we only iterate through the clusters with a player in $P_{\ell}$ and add at most one edge to $A_{\ell+1}$ for each such cluster. So after the build phase, the first three invariants are satisfied.

The collapse phase will ensure the fourth invariant while not introducing any violations of the first three. Indeed, the while-loop runs until the fourth invariant is satisfied so we only need to worry about the first three still being satisfied. The first and third invariants remain satisfied because any set $A_{i}$ that was affected in the collapse phase is discarded from the algorithm and if $B_{i}$ was changed then $A_{i+1}$ was also discarded. For the second invariant, note that after updating the matching $M$, we remove the edge that was removed from the matching from $B_{t}$. Hence, $B_{t}$ still only contains edges of the new matching that blocks edges in $A_{t}$. Moreover, by the first invariant, the newly introduced edge in the matching does not share any resources with edges in $A_{\leqslant t} \cup B_{\leqslant t}$. Hence, the second invariant also remains true. Finally, we note that $M$ remains a matching during the update procedure that matches all clusters that were initially matched. Indeed, when $\left(q, R_{q}\right)$ is removed an edge ( $p, R^{\prime}$ ) is added to the matching with $p$ being from the same cluster as $q$ (or the algorithm terminates by having successfully matched a player in $N_{0}$ ). The added edge is a $\beta$-edge and its resources are disjoint from all edges in $M$ since (1) $R^{\prime}$ is a subset of $R \backslash \mathcal{R}\left(B_{t+1}\right)$, (2) $B_{t+1}$ contains all blocking edges of $A_{t+1}$ with respect to the matching before the collapse phase, and (3) the edges in $A_{t+1}$ are disjoint so ( $p, R^{\prime}$ ) is disjoint from any other edges added to the matching in the same collapse phase. We thus maintain a valid matching, in which all edges are pairwise disjoint, and the output is an extended matching that also matches the cluster $N_{0}$.

### 3.6 Analysis of the algorithm

We now proceed to show that the algorithm in Section 3.5 terminates in polynomial time, which then implies Theorem 3.1. Recall that $\alpha$ is the parameter that regulates the "greediness" of the players while $\beta$ is the approximation guarantee, and $\mu$ dictates when we collapse a layer.

The key lemma that we prove in this section is that in each layer $\left(A_{i+1}, B_{i+1}\right)$, the number of edges in $A_{i+1}$ is large compared to the number of blocking edges (or, similarly, the number of players) of lower layers.

Lemma 3.6. Assuming that $\operatorname{CLP}(\tau)$ is feasible, at the beginning of each iterative step, $\left|A_{i+1}\right| \geqslant\left|P_{\leqslant i}\right| / 5$ for each $i=0, \ldots, \ell-1$.

We defer the proof of this statement for now and explain its consequences. As thin items are of value less than $\tau / \beta=\tau / 36$, and each edge in $A_{\leqslant \ell}$ is a thin $\alpha$-edge of value at least $\tau / \alpha=2 \tau / 5$, this implies that $B_{i}$ must be quite large, using $\left|I_{i}\right|<\mu\left|P_{i-1}\right|$ from the fourth invariant. This means that the number of blocking edges will grow quickly as we prove in the next lemma.

Lemma 3.7 (Exponential growth). Assuming that $\operatorname{CLP}(\tau)$ is feasible, at the beginning of the iterative step $\left|P_{i+1}\right|>13\left|P_{\leqslant i}\right| / 10$ for $i=0, \ldots, \ell-1$.
Proof. Fix an $i$ such that $0 \leqslant i<\ell$. By the fourth invariant, $\left|I_{i+1}\right|<\mu\left|P_{i}\right|$ at the beginning of the iterative step. This means that there are at least $\left|A_{i+1}\right|-\mu\left|P_{i}\right|$ many edges in $A_{i+1}$ which are not in $I_{i+1}$. As each edge in $A_{i+1} \backslash I_{i+1}$ has resources of value at least $\tau / \alpha-\tau / \beta$ that are blocked (i.e., contained in $\mathcal{R}\left(B_{i+1}\right)$ ), we can lower bound the total value of blocked resources appearing in $A_{i+1}$ by

$$
\left(\frac{\tau}{\alpha}-\frac{\tau}{\beta}\right)\left(\left|A_{i+1}\right|-\mu\left|P_{i}\right|\right) .
$$

Further, since each edge in $B_{i+1}$ is of value at most $2 \tau / \beta$ by minimality, the total value of such resources is upper bounded by $\left|P_{i+1}\right| \cdot 2 \tau / \beta$. In total,

$$
\left(\frac{\tau}{\alpha}-\frac{\tau}{\beta}\right)\left(\left|A_{i+1}\right|-\mu\left|P_{i}\right|\right) \leqslant\left|P_{i+1}\right| \frac{2 \tau}{\beta} \Longrightarrow\left|P_{i+1}\right| \geqslant \frac{(\beta-\alpha)(1 / 5-\mu)}{2 \alpha}\left|P_{\leqslant i}\right|>13\left|P_{\leqslant i}\right| / 10,
$$

where we have used Lemma 3.6 to bound $\left|A_{i+1}\right|$ by $\left|P_{\leqslant i}\right| / 5$ from below.
Since the number of blocking edges grows exponentially as a function of the layer index, an immediate consequence of Lemma 3.7 is that the total number of layers in the list $\mathcal{L}$ at any step in the algorithm is at most $O(\log |\mathcal{P}|)$. This means that we have to satisfy the condition in the while-loop of the collapse phase after at most logarithmically many iterative steps. When this happens, Algorithm 1 selects the smallest $t$ satisfying the condition and then proceeds to update $A_{t+1}$ and $B_{t}$. Note that, by the condition of the while-loop, and since each edge in $I_{t+1}$ will be updated in the for-loop (using the third invariant), a constant fraction (at least $\mu$ as defined in (3.1)) of the edges in $B_{t}$ are removed. We refer to these steps of the algorithm as the collapse of layer $t$. Furthermore, due to the algorithm's first invariant, we know that the edges that compose $I_{t+1}$ are pairwise disjoint; therefore, we are able to insert all of them simultaneously into our matching, which means that the size of our matching does not decrease during the collapse operation. On the contrary, if $p_{0}$ is part of the edges that are inserted into $M$, then we have actually achieved to extend our matching $M$. Intuitively we make large progress whenever we update $M$ during the collapse of a layer. We prove this by maintaining a signature vector $s:=\left(s_{0}, \ldots, s_{\ell}, \infty\right)$ during the execution of the algorithm, where

$$
s_{i}:=\left\lfloor\log _{1 /(1-\mu)}\left|P_{i}\right|\right\rfloor .
$$

Lemma 3.8. The signature vector always reduces in lexicographic value across each iterative step, and the coordinates of the signature vector are always non-decreasing, i.e., $s_{0} \leqslant s_{1} \ldots \leqslant s_{\ell}$.
Proof. Let $s$ and $s^{\prime}$ be the signature vectors at the beginning and at the end of some iterative step. We now consider two cases depending on whether a collapse operation occurs in this iterative step.

Case 1. No layer was collapsed. Clearly, $s^{\prime}=\left(s_{0}, \ldots, s_{\ell}, s_{\ell+1^{\prime}}^{\prime}, \infty\right)$ has smaller lexicographic value compared to $s$.

Case 2. At least one layer was collapsed. Let $\ell+1$ denote the index corresponding to the newly created layer in the build phase. Let $0 \leqslant t \leqslant \ell$ be the most recent index chosen in the while-loop during the collapse phase. As a result of the collapse operation suppose the layer $P_{t}$ changed to $P_{t}^{\prime}$. Then we know that $\left|P_{t}^{\prime}\right|<(1-\mu)\left|P_{t}\right|$. Since none of the layers with indices less than $t$ were affected during this procedure, $s^{\prime}=\left(s_{0}, \ldots, s_{t-1}, s_{t}^{\prime}, \infty\right)$ where $s_{t}^{\prime}=\left\lfloor\log _{1 /(1-\mu)}\left|P_{t}^{\prime}\right|\right\rfloor \leqslant\left\lfloor\log _{1 /(1-\mu)}\left|P_{t}\right|\right\rfloor-1=s_{t}-1$. This shows that the lexicographic value of the signature vector decreases.

In both cases, the fact that the coordinates of $s^{\prime}$ are non-decreasing follows from Lemma 3.7 and the definition of the coordinates of the signature vector.

Choosing the " $\infty$ " coordinate of the signature vector to be some value larger than $\log _{1 /(1-\mu)}|\mathcal{P}|$ (so that Lemma 3.8 still holds), we see that each coordinate of the signature vector is at most $U$ and the number of coordinates is also at most $U$ where $U=O(\log |\mathcal{P}|)$. Thus, the sum of the coordinates of the signature vector is always upper bounded by $U^{2}$. We now prove that the number of such signature vectors is polynomial in $|\mathcal{P}|$.

A partition of an integer $N$ is a way of writing $N$ as the sum of positive integers (ignoring the order of the summands). The number of partitions of an integer $N$ can be upper bounded by $e^{O(\sqrt{N})}$ by a result of Hardy and Ramanujan [HR18] ${ }^{2}$. Using that the coordinates of our signature vectors are non-decreasing, each signature vector corresponds to a partition of an integer of value at most $U^{2}$, and vice versa: given a partition of an integer of size $\ell$, the largest number of the partition will correspond to the $\ell$-th coordinate, the second largest to the $\ell$-1-th coordinate, and so on. Therefore, we can upper bound the total number of signature vectors by $\sum_{i \leqslant U^{2}} e^{O(\sqrt{i})}=|\mathcal{P}|^{O(1)}$. Since each iteration of the algorithm takes only polynomial time along with Lemma 3.8 this proves Theorem 3.1.

Before we return to the proof of the key lemma in this section, Lemma 3.6, let us note an important property of the algorithm which follows from that in the build-phase we add an $\alpha$-edge for each cluster as long as it is disjoint from the already added resources.

Fact 3.9. Let $q$ be a player from some cluster $N_{k}$. Notice that if a player $q$ is part of some blocking edge in the $i^{\text {th }}$ layer, i.e., $q \in P_{i}$, and further there is no edge $(p, R) \in A_{i+1}$ with $p \in N_{k}$ then it means that none of the players in $N_{k}$ have a set of resources of value at least $\tau / \alpha$ disjoint from the resources $\mathcal{R}\left(B_{\leqslant i} \cup A_{\leqslant i+1}\right)$.

Proof of Lemma 3.6. Notice that since the set $A_{i}$ is discarded if it is modified or any of the sets $A_{0}, A_{1}, \ldots, A_{i-1}, B_{0}, B_{1} \ldots, B_{i-1}$ is modified, it is sufficient to verify the inequality when we construct the new layer $\left(A_{\ell+1}, B_{\ell+1}\right)$ in the build phase. The proof is now by contradiction. Suppose $\left|A_{\ell+1}\right|<\left|P_{\leqslant l}\right| / 5$ after the build phase. Let $\mathcal{N} \subseteq\left\{N_{1}, \ldots, N_{m}\right\}$ be the clusters that have a player in an edge $B_{\leqslant \ell}$ but no player in an edge in $A_{\leqslant \ell+1}$. We have that, $|\mathcal{N}|=\left|P_{\leqslant \ell}\right|-\left|A_{\leqslant \ell+1}\right|$.

Recall that $\mathcal{C}_{t}(i, \tau)$ denotes the set of configurations for player $i$ comprising only thin items. By Theorem 3.5 there exists an $x$ that is feasible for $\operatorname{CLP}(\tau)$ such that $\sum_{i \in N_{k}} \sum_{C \in \mathcal{C}_{t}(i, \tau)} x_{i C}=1 / 2$ for each cluster $N_{k}$. Now form the bipartite hypergraph $\mathcal{H}=\left(\mathcal{N} \cup \mathcal{R}_{t}, E\right)$ where we have vertices for clusters in $\mathcal{N}$ and thin items in $\mathcal{R}$, and edges $\left(N_{k}, C\right)$ for every cluster $N_{k}$ and thin configuration $C$ such that $x_{p C}>0$ and $p \in N_{k}$. To each edge $\left(N_{k}, C\right)$ in $\mathcal{H}$ assign the weight $\left(\sum_{i \in N_{k}} x_{i C}\right) \sum_{j \in C} v_{j}$. The total weight of edges in $\mathcal{H}$ is at least $|\mathcal{N}| \tau / 2$. Let $Z=\mathcal{R}\left(B_{\leqslant \ell} \cup A_{\leqslant \ell+1}\right)$ denote the thin items appearing in the edges of $\mathcal{L}$ and $A_{\ell+1}$. Let $v(Z)=\sum_{j \in Z} v_{j}$ denote their value. Now remove all these items from the hypergraph to form $\mathcal{H}^{\prime}$ which has edges $\left(N_{k}, C \backslash Z\right)$ for each edge $\left(N_{k}, C\right)$ in $\mathcal{H}$. The weight of $\left(N_{k}, C \backslash Z\right)$ is similarly defined to be $\left(\sum_{i \in N_{k}} x_{i C}\right) \sum_{j \in C \backslash Z} v_{j}$.

Let us upper bound the total value of thin items appearing in $Z$. Consider some layer $\left(A_{j}, B_{j}\right)$. The total value of resources in thin $\alpha$-edges in $A_{j}$ is at most $(\tau / \alpha+\tau / \beta)\left|A_{j}\right|$ by the minimality of the edges. The value of resources in $B_{j}$ not already present in some edge in $A_{j}$ is at most $(\tau / \beta)\left|B_{j}\right|$ also by minimality of the thin $\beta$-edges in $B_{j}$. Therefore, $v(Z)$ is at most

$$
\sum_{j=1}^{\ell}\left(\left(\frac{\tau}{\alpha}+\frac{\tau}{\beta}\right)\left|A_{j}\right|+\left(\frac{\tau}{\beta}\right)\left|B_{j}\right|\right)+\left|A_{\ell+1}\right|\left(\frac{\tau}{\alpha}+\frac{\tau}{\beta}\right)<\left|A_{\leqslant \ell+1}\right|\left(\frac{\tau}{\alpha}+\frac{\tau}{\beta}\right)+\left|P_{\leqslant \ell}\right| \frac{\tau}{\beta} .
$$

[^2]As the sum of the edge weights in $\mathcal{H}$ is at least $(|\mathcal{N}| / 2)(\tau)$, the sum of edge weights in $\mathcal{H}^{\prime}$ is at least $|\mathcal{N}| \tau / 2-v(Z)$. And by Fact 3.9, the sum of edge weights in $\mathcal{H}^{\prime}$ must be strictly smaller than $(\mathcal{N} / 2)(\tau / \alpha)$. Thus,

$$
\begin{equation*}
\frac{\left(\left|P_{\leqslant \ell}\right|-\left|A_{\leqslant \ell+1}\right|\right)}{2} \tau-\left|A_{\leqslant \ell+1}\right|\left(\frac{\tau}{\alpha}+\frac{\tau}{\beta}\right)-\left|P_{\leqslant \ell}\right| \frac{\tau}{\beta}<\frac{\left(\left|P_{\leqslant \ell}\right|-\left|A_{\leqslant \ell+1}\right|\right)}{2} \frac{\tau}{\alpha} . \tag{}
\end{equation*}
$$

Note that $\left|A_{\leqslant \ell+1}\right|$ appears with a larger negative coefficient (in absolute terms) on the left-hand-side than on the right-hand-side. Therefore, if ( ${ }^{*}$ ) holds then it also holds for an upper bound of $\left|A_{\leqslant \ell+1}\right|$. We shall compute such a bound and reach a contradiction.

We start by computing an upper bound on $\left|A_{j+1}\right|$ for $j=0, \ldots, \ell-1$. The fourth invariant says that except for at most $\mu\left|P_{j}\right|$ edges in $A_{j+1}$, the remainder have at least $\tau / \alpha-\tau / \beta$ value of resources blocked by the edges in $B_{j+1}$. Using this,

$$
\left(\frac{\tau}{\alpha}-\frac{\tau}{\beta}\right)\left(\left|A_{j+1}\right|-\mu\left|P_{j}\right|\right) \leqslant\left|P_{j+1}\right| \frac{2 \tau}{\beta} \stackrel{\text { summing over } j}{\Longrightarrow}\left(\frac{\tau}{\alpha}-\frac{\tau}{\beta}\right)\left(\left|A_{\leqslant \ell}\right|-\mu\left|P_{\leqslant \ell-1}\right|\right) \leqslant\left|P_{\leqslant \ell}\right| \frac{2 \tau}{\beta} .
$$

Rearranging terms we have,

$$
\left|A_{\leqslant l}\right| \leqslant\left|P_{\leqslant \ell}\right| \frac{2 \alpha}{\beta-\alpha}+\mu\left|P_{\leqslant \ell-1}\right| \leqslant \left\lvert\, P_{\leqslant l}\left(\frac{2 \alpha}{\beta-\alpha}+\mu\right) .\right.
$$

Substituting this upper bound in (*) along with our assumption $\left|A_{\ell+1}\right|<\left|P_{\leqslant \ell}\right| / 5$ we get (after some algebraic manipulations)

$$
\left|P_{\leqslant l}\right|\left(1-\frac{1}{\alpha}-\frac{2}{\beta}\right)-\left|P_{\leqslant \ell}\right|\left(\frac{2 \alpha}{\beta-\alpha}+\mu+1 / 5\right)\left(1+\frac{1}{\alpha}+\frac{2}{\beta}\right)<0 .
$$

This is a contradiction because if we substitute in the values of $\alpha, \beta$, and $\mu$ from (3.1) the left-handside is positive.

## 4 Combinatorial Algorithm

In the previous section, we described a 36-approximation algorithm for restricted max-min fair allocation; however, this algorithm required us to solve the Configuration-LP. In this section, we will design and analyze a purely combinatorial $(6+2 \sqrt{10}+\varepsilon)$-approximation algorithm, for any $0<\varepsilon \leqslant 1$ (for reference, note that $6+2 \sqrt{10}<13$ ). This will prove our main result, Theorem 1.1.

We start by providing an informal overview of how the combinatorial algorithm works.

### 4.1 Intuitive Algorithm Description

To begin with, the general framework of our combinatorial algorithm is similar to that of the simpler algorithm we described in Section 3: we guess an optimal value $\tau$ for the ConfigurationLP, and we then try to find an allocation of resources which approximately satisfies every player, i.e., assigns to each player a set of resources of total value at least $\tau / 13$ for that player. To do so, we will again design a local search procedure, whose goal will be to extend a given partial allocation of resources, so as to satisfy one more player.

An example execution of our combinatorial algorithm appears in Figure 3: there, given a partial allocation of resources to players, we want to extend this allocation to satisfy player $p$. Naturally,
if there is a set of resources, that do not appear in the given partial allocation, and whose total value for $p$ is at least $\tau / 13$, we will assign these resources to player $p$. Otherwise, we find an edge $e_{p}$ whose total value for $p$ is at least $\tau / 2$ (the bottom gray edge in Figure 3(a)), and consider all the edges in our given partial allocation that share resources with that set (the white edges intersecting $e_{p}$ in Figure 3(a)); these edges constitute the first layer that is shown in Figure 3(a).

At this point, we should make note of the fact that, similar to the simpler algorithm we described in Section 3, we will again be using a greedy strategy with respect to the edges we wish to include in our partial matching. Specifically, even though we wish to only assign resources of total value at least $\tau / 13$ to each player, the gray edges we attempt to include in our matching are significantly more valuable (i.e., of total value at least $\tau / 2$ ). Again, this will imply that every gray edge will intersect with multiple white/blocking edges, which will eventually help us prove that the algorithm's running time is polynomial in the size of the input.

Next, similar to the simpler algorithm we described in Section 3, we want to free up the resources that appear in edge $e_{p}$. We do this by finding disjoint sets of resources that satisfy the players appearing in the white edges of the first layer. However, here we encounter the first major difference compared to our previous algorithm: some of the players that appear in the white edges of the first layer can be satisfied by using fat resources, i.e., resources whose value for their corresponding players is at least $\tau / 13$. Since every fat edge we would like to include in our partial allocation can only be blocked by exactly one edge that already belongs to our allocation, alternating paths of fat edges are created. Such a path, that ends in a gray thin edge, is displayed in Figure 3(b); if we wish to include the gray edge that contains $q_{2}$ into our partial allocation, then we would have to replace the white fat edges with the gray ones.

However, considering such alternating paths of fat edges brings up one issue: since, as is shown in Figure 3(a), the alternating paths that originate at players $p_{1}$ and $p_{2}$ end at two distinct gray thin edges, if we were to include both of these edges into our matching, then we would have to guarantee that we will not use the same fat resource to satisfy two different players. In order to do this, we will include the gray edges that contain players $q_{1}$ and $q_{2}$ into our partial allocation, only if the alternating paths that end in these players are vertex-disjoint, as is the case in Figure 3(c).

Next, since we have solved the problem of deciding if we can update our partial matching by replacing white edges with gray ones, the question that arises is when should we do that. Similar to our simpler algorithm, we will employ the strategy of lazy updates. In other words, we will be replacing the white edges of some layer with gray ones (or, as we will call this operation, collapse a specific layer), only if that would mean that a significant amount of the white edges gets replaced. Replacing a significant amount of white (i.e., blocking) edges then implies that we make significant progress towards matching player $p$.

Finally, after we update our partial allocation, by inserting the gray edges containing players $q_{1}$ and $q_{2}$, inserting the gray fat edges that belong to the corresponding alternating paths, and removing the white fat edges that belong to the corresponding alternating paths, we have managed to free up a significant amount of resources of edge $e_{p}$. Hence, we choose a subset of the resources contained in $e_{p}$, whose total value is at least $\tau / 13$, and include it into our partial allocation. At this point, we have managed to extend our partial allocation to include one more player, namely, player $p$.


Figure 3: An illustration of our combinatorial algorithm. In this figure, boxes correspond to players and circles correspond to resources.

### 4.2 Parameters

Let $\tau>0$ be a guess on the value of the Configuration-LP, and fix some $0<\varepsilon \leqslant 1$. Our algorithm will use the following setting of parameters:

$$
\begin{align*}
& \beta:=2(3+\sqrt{10})+\varepsilon, \\
& \alpha:=2,  \tag{4.1}\\
& \mu:=\varepsilon / 100 .
\end{align*}
$$

Similar to our simpler algorithm, $\beta$ is the approximation guarantee, $\alpha$ determines the "greediness" of the algorithm, and $\mu$ determines the "laziness" of the updates of our algorithm.

We shall show that whenever $\operatorname{CLP}(\tau)$ is feasible, our algorithm will terminate with a solution of value at least $\tau / \beta$ for the given instance of restricted max-min fair allocation. Combining this with a standard binary search then yields a $\beta$-approximation algorithm.

### 4.3 Description of the Algorithm

We begin by noting that we will be re-using the definitions of fat and thin edges, $\delta$-edges, and (partial) matchings that we introduced in Section 3.3. However, we remind the reader that the parameters
we used in the above definitions have now changed, see (4.1).
The goal of our algorithm will be to find a perfect matching. Similar to our simpler algorithm, the way we do this is by designing an augmenting algorithm, that will extend any given partial matching to satisfy one more player. Thus, starting from an empty matching and iteratively applying the augmenting algorithm will yield a perfect matching that corresponds to a $\beta$-approximate allocation. We remark that for the purposes of our algorithm, any partial matching we consider contains the maximum number of fat resources possible. In order to enforce this condition, we find a maximum matching between fat resources and players; this will be our initial partial matching. Starting from this partial matching, we proceed to iteratively extend it, by matching one more player at a time while never decreasing the number of fat items in our allocation.

We proceed to define the concepts of Disjoint Path Networks and Canonical Decompositions, that are necessary to state our combinatorial algorithm. These concepts will be used to implement the idea of updating our partial matching using vertex-disjoint alternating paths, that we mentioned in Section 4.1. We then state our algorithm formally, and we analyze its running time in the subsequent sections.

Disjoint Path Networks. As we discussed in the overview of our combinatorial algorithm, we need a way to ensure that the alternating paths we use to update our partial matching are disjoint. We say that two paths are disjoint if they are vertex-disjoint. To do so, we employ a structure called Disjoint Path Networks.

Given a partial matching $M$, let $H_{M}=\left(\mathcal{P} \cup \mathcal{R}_{f}, E_{M}\right)$ be the directed graph defined as follows: there is a vertex for each player in $\mathcal{P}$ and each fat resource in $\mathcal{R}_{f}$; and, there is an arc from a player in $p \in \mathcal{P}$ to a fat resource $f \in \mathcal{R}_{f}$ if $p$ is interested in $f$ unless the arc ( $p,\{f\}$ ) appears in $M$ in which case there is an $\operatorname{arc}(\{f\}, p)$. Note that the graph $H_{M}$ depends only on the assignment of fat resources to players in $M$.

Now, let $S, T \subseteq \mathcal{P}$ be a set of sources and sinks respectively that are not necessarily disjoint. Let $\mathrm{F}_{M}(S, T)$ denote the flow network we get if we place unit capacities on the vertices of $H_{M}$, and use $S$ and $T$ as sources and sinks respectively. Furthermore, let $\mathrm{DP}_{M}(S, T)$ denote the value of an optimal solution, i.e., the maximum number of disjoint paths from the sources $S$ to the sinks $T$ in the graph $H_{M}$.

In our algorithm, $S$ and $T$ will contain only vertices in $H_{M}$ corresponding to players in $\mathcal{P}$. However, to specify a sink we sometimes abuse notation and specify an edge since the corresponding sink vertex can be deduced from it. For example, if we write $\mathrm{DP}_{M}(X, Y)$, for some set of players $X$ and some set of edges $Y$, then we mean the maximum number of disjoint paths that start at a player in $X$ and end in a player that appears in some edge in $Y$.

For basic concepts related to flows, such as flow networks and augmenting paths, we refer the reader to the textbook by Cormen, Leiserson, Rivest and Stein [CLRS09].

State of the Algorithm. The state of the algorithm is described by a dynamic tuple ( $M, \ell, \mathcal{L}, I$ ), where $M$ is the current partial matching, $\mathcal{L}=\left(\left(A_{0}, B_{0}, d_{0}\right),\left(A_{1}, B_{1}, d_{1}\right), \cdots,\left(A_{\ell}, B_{\ell}, d_{\ell}\right)\right)$ is a list of $\ell$ layers and $I$ is a set of "immediately addable" edges. Each layer $L_{i}=\left(A_{i}, B_{i}, d_{i}\right)$ consists of a set of "added" edges $A_{i}$, a set of "blocking" edges $B_{i}$, and a positive integer $d_{i}$. We note that $d_{i}$ is redundant for the formal statement of our algorithm, but will be handy in our analysis.

Canonical Decompositions. We proceed to define the last concept necessary to describe our combinatorial algorithm. Recall that we denote $\cup_{i \leqslant t} S_{i}$ by $S_{\leqslant t}$, for some sequence of sets $S_{0}, \ldots S_{t}$, and that $P_{i}$ denotes the players that appear in $B_{i}$. Moreover, for a set $S$ of edges we use $\mathcal{P}(S)$ to
denote the set of players that appear in an edge in $S$ and we use $\mathcal{R}(S)$ to denote the set of resources that appear in an edge in $S$.

Definition 4.1 (Canonical Decomposition of $I$ ). Given a state ( $M, \ell, \mathcal{L}, I$ ) of the algorithm, we call a collection of disjoint subsets $\left\{I_{0}, I_{1}, \ldots, I_{\ell}\right\}$ of $I$ a canonical decomposition if it satisfies the following conditions:

1. For $i=0,1, \ldots, \ell,\left|I_{\leqslant i}\right|=\mathrm{DP}_{M}\left(P_{\leqslant i}, I_{\leqslant i}\right)=\mathrm{DP}_{M}\left(P_{\leqslant i} I\right)$.
2. There exists an optimal solution $W$ to $\mathrm{F}_{M}\left(P_{\leqslant \ell, I}\right)$ such that, for $i=0,1 \ldots, \ell,\left|I_{i}\right|$ paths in $W$ go from players $Q_{i} \subseteq P_{i}$ to the sinks in $I_{i}$. We denote these paths by $W_{i}$. We also refer to $W$ as the canonical solution corresponding to the decomposition.

As we will see in Section 4.4, canonical decompositions and their corresponding canonical solutions can be computed in polynomial time.

Algorithm Statement. The combinatorial algorithm behind the proof of Theorem 1.1 is stated as Algorithm 2. We remark that the computation of canonical decompositions and solutions to flow networks that are carried out in Steps 3 and 3.b respectively can be carried out in polynomial time; this fact is proved in Section 4.4.

Similar to Algorithm 1, Algorithm 2 preserves the following invariants:

1. For $i=0, \ldots, \ell, A_{i}$ is a set of thin $\alpha$-edges and each $\alpha$-edge $(p, R) \in A_{i}$ has $R \cap \mathcal{R}\left(A_{\leqslant i} \cup B_{\leqslant i-1} \cup\right.$ $I \backslash\{(p, R)\})=\emptyset$ (its resources are not shared with edges from earlier iterations, edges in $A_{i}$, or edges in $I)$.
2. For any edge $(p, R) \in I$, it holds that $R \cap \mathcal{R}\left(A_{\leqslant \ell} \cup I \backslash\{(p, R)\}\right)=\emptyset$ and $v_{p}(R \backslash \mathcal{R}(M)) \geqslant \tau / \beta .^{3}$
3. Given a canonical decomposition $\left\{I_{0}, \ldots, I_{\ell}\right\}$ of $I$, for $i=0, \ldots, \ell$ it holds that $\left|I_{i}\right|<\mu\left|P_{i}\right|$.

The similarities between these invariants and those of the simpler algorithm follow from the same basic ideas. However, since Algorithm 2 is more involved, its analysis requires more invariants that we present in the subsequent sections.

Before proceeding with analyzing Algorithm 2, we explain its steps in more detail and why the algorithm satisfies the above invariants. The algorithm begins with a partial matching $M$ and a player $p_{0}$ that we wish to include in our partial matching. Furthermore, as pointed out earlier, we make sure that $M$ contains a maximum matching between fat resources and players. Every iteration of our algorithm involves two main phases: the build phase, and the collapse phase.

During the build phase of layer $\ell+1$, the algorithm finds thin $\alpha$-edges for the players in $P_{\ell}$ that we then insert into either $I$ (if the $\alpha$-edge contains sufficient resources that do not appear in $M$ ) or to $A_{\ell+1}$. By the design of Algorithm 2, any edge that is inserted into $A_{\ell+1}$ will be disjoint from edges in $A_{\leqslant \ell+1} \cup B_{\leqslant \ell} \cup I$; the same holds for any edge $(p, R)$ that is inserted into $I$, while in addition we have $v_{p}(R \backslash \mathcal{R}(M)) \geqslant \tau / \beta$. Therefore, the first two invariants are preserved during the build phase.

Furthermore, edges inserted into $A_{\ell+1}$ or $I$ need to either contain a player from $P_{\leqslant \ell}$, or to be the final edge in an alternating path that includes fat edges originating at a player in $P_{\leqslant \ell}$. Even though we will not store such alternating paths explicitly, it is required that after we insert any such thin $\alpha$-edge into $A_{\ell+1}$ and $I$, the value of the flow network $\mathrm{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell+1} \cup I\right)$ increases; this

[^3]Input: A partial matching $M$ and an unmatched player $p_{0}$.
Output: A matching $M^{\prime}$ that matches all players matched by $M$ and also matches $p_{0}$.

1. (Initialization) Set $A_{0}=\emptyset, B_{0}=\left\{\left(p_{0}, \emptyset\right)\right\}, \ell=0, d_{0}=0$ and $\mathcal{L}=\left(A_{0}, B_{0}, d_{0}\right)$.
(Iterative step) Repeat the following until $p_{0}$ is matched by $M$ :
2. (Build phase) Initialize $A_{\ell+1}=\emptyset$. While there exists a thin $\alpha$-edge $(p, R)$ such that $R \cap \mathcal{R}\left(A_{\leqslant \ell+1} \cup\right.$ $\left.B_{\leqslant \ell} \cup I\right)=\emptyset$ and $\mathrm{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell+1} \cup I \cup\{(p, R)\}\right)>\operatorname{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell+1} \cup I\right):$

- If $v_{p}(R \backslash \mathcal{R}(M))<\tau / \beta$, then set $A_{\ell+1}=A_{\ell+1} \cup\{(p, R)\}$, else set $I=I \cup\{(p, R)\}$.

At the end of the build phase, let $B_{\ell+1}$ be the edges of $M$ that are blocking the edges in $A_{\ell+1}$. Set $d_{\ell+1} \leftarrow \mathrm{DP}_{M}\left(P_{\leqslant l}, A_{\leqslant l+1} \cup I\right)$; then update the state of the algorithm by appending $\left(A_{\ell+1}, B_{\ell+1}, d_{\ell+1}\right)$ to $\mathcal{L}$ and by incrementing $\ell$ by one.
3. (Collapse phase) Compute the canonical decomposition $\left\{I_{0}, \ldots, I_{\ell}\right\}$ of $I$, and the corresponding canonical solution $W$.
While $\exists t:\left|I_{t}\right| \geqslant \mu\left|P_{t}\right|:$
(a) Choose the smallest such $t$.
//We refer to the following steps as collapsing layer $t$.
(b) Compute optimal solution $X$ to $\mathrm{F}_{M}\left(P_{\leqslant t-1}, A_{\leqslant t} \cup I_{\leqslant t-1}\right)$ whose paths are disjoint from $W_{t}$. $/ / W e ~ r e f e r ~ t o ~ t h e ~ f o l l o w i n g ~ s t e p ~ a s ~ a l t e r n a t i n g ~ a l o n g ~ t h e ~ p a t h s ~ o f ~ W ~ W ~ . ~ . ~$
(c) For each path $\Pi$ in $W_{t}$ that ends at a player $p_{e}$ with an edge $\left(p_{e}, R\right) \in I_{t}$
i. Set $M \leftarrow M \backslash\{(p,\{f\}) \mid(f, p) \in \Pi\} \cup\{(p,\{f\}) \mid(p, f) \in \Pi\}$.
ii. Remove from $M$ and $B_{t}$ the edge containing the source of the path $\Pi$.
iii. Add to $M$ some $\beta$-edge ( $p_{e}, R^{\prime}$ ), where $R^{\prime} \subseteq R$ and $R^{\prime} \cap \mathcal{R}(M)=\emptyset$.
(d) Set $I=I_{0} \cup \ldots \cup I_{t-1}$. For every edge $(p, R) \in A_{t}$, if $v_{p}(R \backslash \mathcal{R}(M)) \geqslant \tau / \beta$, then:

- Remove $(p, R)$ from $A_{t}$ and remove those edges from $B_{t}$ that only block $(p, R)$ in $A_{t}$.
- If $X$ contains a path that ends in $p$, insert $(p, R)$ in $I$.
(e) Discard $\left(A_{i}, B_{i}, d_{i}\right)$ from $\mathcal{L}$ with $i>t$ and set $\ell=t$.

Output the matching $M$ that also matches $p_{0}$.
Algorithm 2: Combinatorial Augmenting Algorithm
will ensure that there are enough disjoint paths of fat edges to permit the inclusion of all such thin edges into our partial matching $M$.

After the algorithm has finished the build phase, it proceeds to the collapse phase. The condition $\exists t:\left|I_{t}\right| \geqslant \mu\left|P_{t}\right|$ of the while-loop guarantees that the third invariant is satisfied once the collapse phase terminates (since the cardinality of $I_{i}$ always equals $\mathrm{DP}_{M}\left(P_{\leqslant i}, I\right)$ no matter the chosen canonical decomposition). We now describe this phase in more detail and show that it maintains a valid matching and that it does not introduce any violations of the first two invariants.

The first step of the collapse phase is to compute a canonical decomposition of $I$, and a corresponding canonical solution $W$. Now suppose that we have $I_{t} \geqslant \mu\left|P_{t}\right|$ and that the algorithm collapses layer $t$. We refer to the edges of $I_{t}$ as "immediately addable" as they have enough free resources (by Invariant 2) to be added to the matching. Indeed, these are the edges we will insert into our partial matching, using the paths of $W_{t}$. Specifically, for each path $\Pi$ of $W_{t}$, the algorithm proceeds as follows. By definition of the sources and the sinks, $\Pi$ is a path that starts
with a player $p_{s} \in P_{t}$ and ends with a player $p_{e}$ such that $\left(p_{e}, R\right) \in I_{t}$. Between $p_{s}$ and $p_{e}$, the path alternates between fat edges that belong to $M$ and fat edges we want to insert into $M$, i.e., $\Pi=\left(p_{s}=p_{1}, f_{1}, p_{2}, f_{2}, \ldots, p_{k}, f_{k}, p_{k+1}=p_{e}\right)$ where $p_{s}$ is interested in $f_{1}, p_{k+1}$ is currently assigned $f_{k}$, and $p_{i}$ is currently assigned $f_{i-1}$ and interested in $f_{i}$ for $i=2, \ldots, k$. To update the matching, we find a $\beta$-edge ( $p_{e}, R^{\prime}$ ) with $R^{\prime} \subseteq R$ that is disjoint from the resources of matching $M$ (guaranteed to exist by the second invariant) and we let ( $p_{s}, R_{s}$ ) denote the edge in $B_{t} \subseteq M$ incident to player $p_{s}$. Step 3.c now updates the matching by inserting $\left(p_{e}, R^{\prime}\right)$ and ( $\left.p_{s}, f_{1}\right),\left(p_{2}, f_{2}\right), \ldots,\left(p_{k}, f_{k}\right)$ to the matching while removing $\left(p_{s}, R_{s}\right)$ and $\left(p_{2}, f_{1}\right),\left(p_{3}, f_{2}\right), \ldots,\left(p_{t}, f_{k}\right)$. This process is called alternating along path П.

As a result, some of the resources of edges in $A_{t}$ are freed up, and we move those edges of $A_{t}$ that now have $\tau / \beta$ free resources to $I$ (Step 3.d). Finally, we discard all layers above the one we collapsed. Let us now see why our first invariant is upheld after the collapse phase. When we collapse layer $t$, we might remove edges from $A_{t}$, we discard all $A_{t^{\prime}}$ for $t^{\prime}>t$ and we preserve $A_{t^{\prime}}$ for $t^{\prime}<t$. Since the first invariant was upheld before the collapse phase, for any $t^{\prime} \leqslant t$ there were no edges in $A_{t^{\prime}}$ that intersected any edge in $A_{\leqslant t^{\prime}}, B_{\leqslant t^{\prime}-1}$ or $I_{0} \cup \ldots I_{t-1}$. Furthermore, since any edge that was inserted into $I$ during Step 3.d previously belonged to $A_{t}$, no edge inserted into $I$ will intersect any edge in $A_{\leqslant t} \cup B_{\leqslant t-1} \cup I_{0} \cup \ldots I_{t-1}$. Therefore, after the collapse phase, for any $t^{\prime} \leqslant t$ every edge in $A_{t^{\prime}}$ is disjoint from edges in $A_{\leqslant t^{\prime}} \cup B_{\leqslant t^{\prime}-1} \cup I$, and the first invariant holds.

After the collapse phase, $I$ contains the edges that belonged to $I_{0} \cup \ldots I_{t-1}$ (call them old edges), plus the edges that were inserted during Step 3.d (call them new edges). Concerning any old edge $e$, since the second invariant held before the collapse phase, and since during the collapse phase for any $t^{\prime} \leqslant t$ we introduce no new edges into $A_{t^{\prime}}$, the resources of $e$ continue to be disjoint from the resources of $A_{\leqslant t}$ and the old edges. Moreover, $v_{p}(\mathcal{R}(e) \backslash \mathcal{R}(M))$ is still at least $\tau / \beta$ since the resources of the edges added to the matching during the collapse phase are disjoint from $\mathcal{R}(e)$, where we use that the second invariant held before this iteration, i.e., that the resources of edges in $I$ are disjoint. Hence, to verify the second invariant it remains to verify that any new edge $(p, R)$ has $v_{p}(R \backslash \mathcal{R}(M)) \geqslant \tau / \beta$ (follows immediately from Step 3.d) and that its resources are disjoint from the resources of all old and other new edges and edges in $A_{\leqslant t} ;$ but this follows directly from the fact that any new edge belonged to $A_{t}$ before the collapse phase and the fact that the first invariant held before the collapse phase. Hence, the second invariant is satisfied after the collapse phase.

Now, let us see why the output of Algorithm 2 is a partial matching that matches player $p_{0}$. Observe that we only update our partial matching during Step 3.c and, as explained above, we alternate along all paths in $W_{t}$ during this step. As these paths are vertex-disjoint and the edges in I have disjoint resources (by the second invariant), these updates do not interfere with each other. Moreover, note that when we alternate along a path all previously matched players remain matched (albeit to new edges) and, in addition, all fat resources remain matched. This means that our algorithm maintains a matching of the players that were matched by the input and that our matching remains one that maximizes the number of assigned fat resources. By iterating until an edge that contains $p_{0}$ is inserted into $M$, it follows that when Algorithm 2 terminates, the output will be a valid matching that also matches $p_{0}$ in addition to the players that were matched by the original matching that was given as input.

Our running time analysis of Algorithm 2 is carried out in the following sections. Specifically, we begin by analyzing the running time of a single iteration of our augmenting algorithm in Section 4.4. Then, we proceed to state certain additional invariants, and prove that they are upheld by Algorithm 2 in Section 4.5. Finally, using these invariants, we will prove that the total number of iterations executed by Algorithm 2 is polynomial in Section 4.6.

### 4.4 Running Time Analysis of a Single Iteration

In this section, we prove that the running time of a single iteration is polynomial. We begin by studying the build phase. In this phase, in each iteration of the while-loop, we consider those players $p$ and resources $R$ such that $(p, R)$ is a thin $\alpha$-edge satisfying $R \cap \mathcal{R}\left(A_{\leqslant \ell+1} \cup B_{\leqslant \ell} \cup I\right)=\emptyset$. We then check whether adding $p$ as a sink to our flow network strictly increases its value, i.e., the number of disjoint paths from the sources in $P_{\leqslant \ell}$ to the sinks in $A_{\leqslant \ell+1} \cup I \cup\{(p, R)\}$. Both these operations can be done in polynomial time as (1) verifying whether such a set $R$ exists for a player $p$ just amounts to calculating the total value of the resources $p$ is interested in that currently are not in the other relevant edges, and as (2) verifying whether the flow network increases its value reduces to a standard maximum flow problem.

Next, we study the collapse operation. Here, we have two non-trivial operations: computing a canonical decomposition (Step 3 of Algorithm 2) and Step 3.b of Algorithm 2.

Lemma 4.2. Given a state ( $M, \ell, \mathcal{L}, I$ ) of the algorithm, we can find a canonical decomposition of I and the corresponding canonical solution in polynomial time.

Proof. We shall construct an optimal solution $W$ to the flow network $\mathrm{F}_{M}\left(P_{\leqslant \ell} I\right)$ with sources $P_{\leqslant \ell}$ and sinks I iteratively. Compute the maximum flow $X^{(0)}$ in the network $\mathrm{F}_{M}\left(P_{\leqslant 0}, I\right)$. Let $Q_{0} \subseteq P_{0}$ be the set of sources appearing in the flow solution $X^{(0)}$. Now observe that this solution $X^{(0)}$ is also a valid flow in the network $\mathrm{F}_{M}\left(P_{\leqslant 1}, I\right)$. Therefore, by using an augmenting flow algorithm, we can augment the flow $X^{(0)}$ to a maximum flow $X^{(1)}$ in the network $\mathrm{F}_{M}\left(P_{\leqslant 1}, I\right)$. Let $Q_{1} \subseteq P_{1}$ be the set of additional sources appearing in the flow solution $X^{(1)}$. We use here an important property of the flow augmentation process, which states that the set of sources in $X^{(1)}$ is precisely the disjoint union $Q_{0} \cup Q_{1}$ (see, for example, [Sch02]). In other words, a vertex appearing as a source of a flow path in a solution continues to be present as a source of a flow path after an augmentation step. Continuing this process, we end up with a flow solution $X^{(\ell)}$ in the network $\mathrm{F}_{M}\left(P_{\leqslant \ell}, I\right)$. Define $W_{i}$ to be the flow paths in $X^{(\ell)}$ that serve the sources $Q_{i} \subseteq P_{i}$ for each $i=0, \ldots, \ell$. Additionally, let $I_{i} \subseteq I$ denote the sinks of $W_{i}$.

By construction, $\left|I_{\leqslant i}\right|=\mathrm{DP}_{M}\left(P_{\leqslant i}, I_{\leqslant i}\right)$. Further, if $\mathrm{DP}_{M}\left(P_{\leqslant i}, I_{\leqslant i}\right)<\mathrm{DP}_{M}\left(P_{\leqslant i}, I\right)$ then this implies that $X^{(i)}$ is not a maximum flow in $\mathrm{F}_{M}\left(P_{\leqslant i}, I\right)$, and therefore can be augmented by one, contradicting the definition of $X^{(i)}$.

The flow paths $W_{0}, W_{1}, \ldots, W_{\ell}$ collectively form the flow solution $X^{(\ell)}$ which is an optimal solution to $\mathrm{F}_{M}\left(P_{\leqslant \ell}, I\right)$. Thus, $\left\{I_{0}, \ldots, I_{\ell}\right\}$ forms a canonical decomposition (with the corresponding canonical solution $W_{0}, \ldots, W_{\ell}$ ). It is also clear that the process outlined above to realize this decomposition runs in polynomial time as the encountered flow networks have unit capacities.

Next, we prove that Step 3.b can be executed in polynomial time:
Lemma 4.3. Consider a state ( $M, \ell, \mathcal{L}, I$ ) of the algorithm and a canonical decomposition $\left\{I_{0}, I_{1}, \ldots, I_{\ell}\right\}$ of $I$ together with the canonical solution $W$. For $i=0, \ldots, \ell$, let $W_{i}$ be the $\left|I_{i}\right|$ paths that go from the players in $Q_{i} \subseteq P_{i}$ to sinks in $I_{i}$. Then, for $i=0,1, \ldots, \ell-1$, we can find in polynomial time an optimal solution $X$ to $\mathrm{F}_{M}\left(P_{\leqslant i}, A_{\leqslant i+1} \cup I_{\leqslant i}\right)$ that is also an optimal solution to $\mathrm{F}_{M}\left(P_{\leqslant i}, A_{\leqslant i+1} \cup I\right)$ whose paths are disjoint from the paths in $W_{i+1}$ and additionally uses all the sinks in $I_{\leqslant i}$.

Proof. Consider a fixed $i$. We shall form an optimal solution $X$ to $\mathrm{F}_{M}\left(P_{\leqslant i}, A_{\leqslant i+1} \cup I_{\leqslant i}\right)$ that is also an optimal solution to $\mathrm{F}_{M}\left(P_{\leqslant i}, A_{\leqslant i+1} \cup I\right)$ and its paths are disjoint from the paths in $W_{i+1}$ and uses all the sinks in $I_{\leqslant i}$. The initial solution will be the set of unit flow paths $W_{\leqslant i}$ from the canonical solution $W$ which has cardinality $\left|I_{\leqslant i}\right|$. We now augment this solution using augmenting paths to the set of sinks $A_{\leqslant i+1}$. Note that throughout this execution each vertex in $I_{\leqslant i}$ will be used as a
sink by some path and therefore $X$ will use all these sinks. Further, the procedure to calculate $X$ clearly runs in polynomial time. We shall now verify the remaining properties of $X$. First, suppose towards contradiction that some iteration used an augmenting path $P$ intersecting a path in $W_{i+1}$. However, this would imply that there exists an augmenting path that uses a sink in $I_{i+1}$. We could then increase the set of disjoint paths from players in $P_{\leqslant i}$ to sinks in $I$ to be greater than $I_{\leqslant i}$ which contradicts the property $\mathrm{DP}_{M}\left(P_{\leqslant i}, I_{\leqslant i}\right)=\mathrm{DP}_{M}\left(P_{\leqslant i}, I\right)$ of the canonical decomposition. Similarly, suppose $X$ is not an optimal solution to $\mathrm{F}_{M}\left(P_{\leqslant i}, A_{\leqslant i+1} \cup I\right)$. Then there exists an augmenting path to an edge in $I \backslash I_{\leqslant i}$ which again contradicts the property $\mathrm{DP}_{M}\left(P_{\leqslant i}, I_{\leqslant i}\right)=\mathrm{DP}_{M}\left(P_{\leqslant i}, I\right)$ of the canonical decomposition.

Finally, since during a collapse operation we can collapse at most $|\mathcal{P}|$ layers, it follows that any iteration of Algorithm 2 terminates in polynomial time.

### 4.5 Additional Invariants of Combinatorial Algorithm

In Section 4.3, we listed three invariants Algorithm 2 preserves that are similar to the simpler algorithm. We argued why they hold, and how these invariants imply that the output of our algorithm is an extended partial matching. In this section, we list two new invariants that will facilitate our polynomial running time proof.

Lemma 4.4. At the beginning of each iteration:
(a) $\mathrm{DP}_{M}\left(P_{\leqslant l} I\right)=|I|$.
(b) $\mathrm{DP}_{M}\left(P_{\leqslant i-1}, A_{\leqslant i} \cup I\right) \geqslant d_{i}$ for each $i=1, \ldots, \ell$.

Proof. We prove the lemma by induction on the number of times the iterative step has been executed. We observe that both invariants trivially hold before the first execution of the iterative step. Assume that they are true before the $r$-th execution of the iterative step. We now verify them before the $r+1$-th iterative step. We actually prove the stronger statement that they hold after the build phase and after each iteration of the collapse phase.
(a) and (b) hold after the build phase. Let $L_{\ell+1}$ denote the layer that was constructed during the build phase. We start by verifying (a). If no edge is added to $I$ during this phase then $|I| \geqslant \mathrm{DP}_{M}\left(P_{\leqslant \ell+1}, I\right) \geqslant \mathrm{DP}_{M}\left(P_{\leqslant \ell}, I\right)=|I|$. Suppose that $a_{1}, \ldots, a_{k}$ were the edges added to the set $I$ in that order. When edge $a_{i}$ was added to the set $I$, from the definition of Step 2 of Algorithm 2 we have that

$$
\mathrm{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell} \cup I \cup\left\{a_{1}, \ldots, a_{i-1}\right\} \cup\left\{a_{i}\right\}\right)>\operatorname{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell} \cup I \cup\left\{a_{1}, \ldots, a_{i-1}\right\}\right),
$$

which then implies that

$$
\operatorname{DP}_{M}\left(P_{\leqslant \ell} I \cup\left\{a_{1}, \ldots, a_{i-1}\right\} \cup\left\{a_{i}\right\}\right)>\operatorname{DP}_{M}\left(P_{\leqslant \ell} I \cup\left\{a_{1}, \ldots, a_{i-1}\right\}\right)
$$

To see this implication, observe that the first inequality implies that, for any flow in $\mathrm{F}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell} \cup\right.$ $I \cup\left\{a_{1}, \ldots, a_{i-1}\right\}$ ) (and hence, for any flow in $\mathrm{F}_{M}\left(P_{\leqslant \ell} I \cup\left\{a_{1}, \ldots, a_{i-1}\right\}\right)$ ), there exists an augmenting path towards $\operatorname{sink} a_{i}$. Along with the induction hypothesis, these inequalities imply that

$$
\operatorname{DP}_{M}\left(P_{\leqslant \ell+1}, I \cup\left\{a_{1}, \ldots, a_{k}\right\}\right) \geqslant \operatorname{DP}_{M}\left(P_{\leqslant \ell} I \cup\left\{a_{1}, \ldots, a_{k}\right\}\right)=|I|+k=\left|I \cup\left\{a_{1}, \ldots, a_{k}\right\}\right| .
$$

For (b), the inequality for $i=\ell+1$ holds by the definition of $d_{\ell+1}$ during this phase. The remaining inequalities follow from the induction hypothesis since none of $M, P_{\leqslant \ell}$ and $A_{\leqslant \ell}$ were altered during this phase and no elements from I were discarded.
(a) and (b) hold after each iteration of the collapse phase. If no layer is collapsed (i.e., there is no $I_{t}$ satisfying the condition of the while-loop) then there is nothing to prove. Now let $t$ denote the index of the layer that is collapsed. Let $\left(M, \ell,\left\{L_{0}, \ldots, L_{t^{\prime}}\right\}, I\right)$ denote the state of the algorithm before collapsing layer $t$ that satisfy (a) and (b) $\left(t^{\prime} \geqslant t\right.$ and $t^{\prime}=\ell+1$ if this is the first iteration of Step 3). Let $I^{\prime}$ denote $I_{0} \cup \cdots \cup I_{t-1} \cup\left\{a_{1}, \ldots, a_{k}\right\}$ where $a_{1}, \ldots, a_{k}$ are the edges added to $I$ in Step 3.d of the collapse phase and let $M^{\prime}$ denote the partial matching after Step 3.c of the collapse phase. We have that (a), $\mathrm{DP}_{M^{\prime}}\left(P_{\leqslant t} I^{\prime}\right)=\left|I^{\prime}\right|$, now follows from Lemma 4.3. Indeed, the solution $X$ used all the sinks in $I_{0} \cup \ldots I_{t-1} \cup\left\{a_{1}, \ldots, a_{k}\right\}$ which equals $I^{\prime}$; and these paths form a solution to $\mathrm{F}_{M^{\prime}}\left(P_{\leqslant t}, I^{\prime}\right)$ as they are disjoint from the paths in $W_{t}$. Notice that we do not use the induction hypothesis in this case, i.e., that ( $\left.M, \ell,\left\{L_{0}, \ldots, L_{t^{\prime}}\right\}, I\right)$ satisfied (a) and (b).

For (b), we need to verify inequalities for $i=1, \ldots, t$. When $i<t$, none of the sets $A_{i}$ were altered during this iterative step. Further, although $M$ and $I$ changes during the collapse phase, by Lemma 4.3 and the definition of Step 3 this change cannot reduce the number of disjoint paths from $P_{\leqslant i-1}$ to $A_{\leqslant i} \cup I$ and therefore (b) remains true by the induction hypothesis. Indeed, the selection of $X$ in Step 3.b is done so as to make sure that the update of the matching along the alternating paths in $W_{t}$ does not interfere with an optimal solution to the flow network with sources $P_{\leqslant i-1}$ and sinks $A_{\leqslant i} \cup I$. For $i=t$, the claim again follows since the number of disjoint paths from $P_{\leqslant t-1}$ to $A_{\leqslant t} \cup I$ cannot reduce because of Step 3.d in the algorithm that maintains $X$ as a feasible solution by the same arguments as for (a).

### 4.6 Bound on the Total Number of Iterations

In this final section, we will use the above invariants to show that our augmenting algorithm performs a polynomial number of iterations, assuming $\operatorname{CLP}(\tau)$ is feasible. We start with two lemmas that show that $d_{i}$ cannot be too small. The first holds in general and the second holds if $\operatorname{CLP}(\tau)$ is feasible.

Lemma 4.5. At the beginning of each iteration, we have $d_{i} \geqslant\left|A_{\leqslant i}\right|$ for every $i=0, \ldots, \ell$.
Proof. We prove this by induction on the variable $r \geqslant 0$ that counts the number of times the iterative step has been executed. For $r=0$ the statement is trivial. Suppose that it is true for $r \geqslant 0$. We shall show that it holds before the $r+1$-th iterative step. If the iteration collapses a layer, then no new layer was added, and since $d_{i}$ 's remain unchanged and $A_{\leqslant i}$ may only decrease, the statement is true in this case.

Now, suppose that no layer was collapsed in this iteration and let $L_{\ell+1}=\left(A_{\ell+1}, B_{\ell+1}, d_{\ell+1}\right)$ be the newly constructed layer in this phase. Again, we have $d_{i} \geqslant\left|A_{i}\right|$ for $i=0, \ldots, \ell$ since none of these quantities are changed by the build phase. Let us now verify that $d_{\ell+1} \geqslant A_{\ell+1}$. Let $A_{\ell+1}=\left\{a_{1}, \ldots, a_{k}\right\}$ denote the set of edges added to $A_{\ell+1}$ indexed by the order in which they were added. When edge $a_{i}$ was added to the set $A_{\ell+1}$, according to Step 2 of Algorithm 2, we have that

$$
\operatorname{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell} \cup I \cup\left\{a_{1}, \ldots, a_{i-1}\right\} \cup\left\{a_{i}\right\}\right)>\operatorname{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell} \cup I \cup\left\{a_{1}, \ldots, a_{i-1}\right\}\right) .
$$

Using (b) of Lemma 4.4 and the induction hypothesis,

$$
\operatorname{DP}_{M}\left(P_{\leqslant \ell-1}, A_{\leqslant \ell} \cup I\right) \geqslant d_{\ell} \geqslant\left|A_{\leqslant \ell}\right| .
$$

Using the previous inequalities,

$$
d_{\ell+1}=\mathrm{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell+1} \cup I\right) \geqslant\left|A_{\leqslant \ell}\right|+k \geqslant\left|A_{\leqslant \ell+1}\right| .
$$

Lemma 4.6. Assuming $\operatorname{CLP}(\tau)$ is feasible, at the beginning of each iteration

$$
\operatorname{DP}_{M}\left(P_{\leqslant i-1}, A_{\leqslant i} \cup I\right) \geqslant d_{i} \geqslant \gamma\left|P_{\leqslant i-1}\right|, \text { where } \gamma=\frac{1}{3}(\sqrt{10}-2) \text {, }
$$

for every $i=1, \ldots, \ell$.
Remark 4.7. The above condition is the only one that needs to be satisfied for the algorithm to run in polynomial time. Therefore, in a binary search, the algorithm can abort if the above condition is violated at some time, since that violation would imply that the Configuration-LP is infeasible; otherwise it will terminate in polynomial time.

Proof. We will prove that $d_{i} \geqslant \gamma\left|P_{\leqslant i-1}\right|$ for $i=1, \ldots, \ell$ as Lemma 4.4(b) then implies the claim. Notice that $d_{i}$ is defined only at the time when layer $L_{i}$ is created and not altered thereafter. So it suffices to verify that: Assuming $d_{i} \geqslant \gamma\left|P_{\leqslant i-1}\right|$ for $i=1, \ldots, \ell$, then for the newly constructed layer $L_{\ell+1}, d_{\ell+1} \geqslant \gamma\left|P_{\leqslant \ell}\right|$ also.

Suppose towards contradiction that $L_{\ell+1}$ is a newly constructed layer (and that no layer was collapsed), such that

$$
d_{\ell+1}=\operatorname{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell+1} \cup I\right)<\gamma \mid P_{\leqslant \ell} .
$$

Then, since no layer was collapsed at Step 3 of Algorithm 2, we have that $\left|I_{i}\right|<\mu\left|P_{i}\right|$ for $i=0, \ldots, \ell$, where $\left\{I_{0}, \ldots, I_{\ell}\right\}$ is the canonical decomposition of $I$ considered by the algorithm. Together with Lemma 4.4(a), this implies

$$
|I|=\mathrm{DP}_{M}\left(P_{\leqslant \ell} I\right)<\mu\left|P_{\leqslant \ell}\right| .
$$

Moreover, by Lemma 4.5 we have

$$
\left|A_{\leqslant \ell+1}\right| \leqslant d_{\ell+1}=\operatorname{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell+1} \cup I\right)<\gamma\left|P_{\leqslant \ell}\right| .
$$

Hence, we have that $\left|A_{\leqslant \ell+1} \cup I\right|<(\mu+\gamma)\left|P_{\leqslant \ell}\right|$.
The rest of the proof is devoted to showing that this causes the dual of the $\operatorname{CLP}(\tau)$ to become unbounded which leads to the required contradiction by weak duality. That is, we can then conclude that if $\operatorname{CLP}(\tau)$ is feasible then $d_{\ell+1} \geqslant \gamma\left|P_{\leqslant \ell}\right|$.

Consider the flow network $\mathrm{F}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell+1} \cup I \cup Z\right)$ with $P_{\leqslant \ell}$ as the set of sources and $A_{\leqslant \ell+1} \cup I \cup Z$ as the collection of sinks where,

$$
Z:=\left\{p \in \mathcal{P} \mid \exists R \subseteq \mathcal{R}: R \cap \mathcal{R}\left(A_{\leqslant \ell+1} \cup I \cup B_{\leqslant \ell}\right)=\emptyset \text { and } v_{p}(R) \geqslant \tau / \alpha\right\} .
$$

Since, during the construction of layer $\ell+1$ we could not insert any more edges into $A_{\ell+1}$ and $I$, the maximum number of vertex disjoint paths from $P_{\leqslant \ell}$ to the sinks equals $\mathrm{DP}_{M}\left(P_{\leqslant \ell}, A_{\leqslant \ell+1} \cup I\right)$ which, by assumption, is less than $\gamma\left|P_{\leqslant \ell}\right|$. Therefore, by Menger's theorem there exists a set $K \subseteq V$ of vertices of cardinality less than $\gamma\left|P_{\leqslant \ell}\right|$ such that, if we remove $K$ from $H_{M}$, the sources $P_{\leqslant \ell} \backslash K$ and the sinks are disconnected, i.e., no sink is reachable from any source in $P_{\leqslant \ell} \backslash K$. We now claim that we can always choose such a vertex cut so that it is a subset of the players.
Claim 4.8. There exists a vertex cut $K \subseteq \mathcal{P}$ separating $P_{\leqslant \ell} \backslash K$ from the sinks of cardinality less than $\gamma\left|P_{\leqslant l}\right|$.

Proof. Take any minimum cardinality vertex cut $K$ separating $P_{\leqslant \ell} \backslash K$ from the sinks. We already saw that $|K|<\gamma\left|P_{\leqslant \ell}\right|$. Observe that every fat resource that is reachable from $P_{\leqslant \ell} \backslash K$ must have outdegree exactly one in $H_{M}$. It cannot be more than one since $M$ is a collection of disjoint edges,
and it cannot be zero since we could then increase the number of fat edges in $M$ which contradicts that we started with a partial matching that maximized the number of fat edges. Therefore in the vertex cut $K$, if there are vertices corresponding to fat resources, we can replace each fat resource with the unique player to which it has an outgoing arc to, to obtain another vertex cut also of the same cardinality that contains only vertices corresponding to players.

Now call the induced subgraph of $H_{M}-K$ on the vertices that are reachable from $P_{\leqslant \ell} \backslash K$ as $H^{\prime}$. Note that by the definition of $K, H^{\prime}$ will not contain any sinks. Using $H^{\prime}$ we define the assignment of values to the dual variables in the dual of $\operatorname{CLP}(\tau)$ as follows:

$$
\begin{aligned}
& y_{i}:= \begin{cases}(1-1 / \alpha) & \text { if player } i \text { is in } H^{\prime}, \\
0 & \text { otherwise, }\end{cases} \\
& z_{j}:= \begin{cases}v_{j} / \tau & \text { if } j \text { is a thin resource that appears in } A_{\leqslant \ell+1} \cup I \cup B_{\leqslant \ell} \\
(1-1 / \alpha) & \text { if } j \text { is a fat resource in } H^{\prime}, \\
0 & \text { otherwise. }\end{cases}
\end{aligned}
$$

We first verify that the above assignment is feasible. Since all the dual variables are nonnegative we only need to verify that $y_{i} \leqslant \sum_{j \in C} z_{j}$ for every $i \in \mathcal{P}$ and $C \in C(i, \tau)$. Consider a player $i$ that is given a positive $y_{i}$ value by the above assignment. Let $C \in C(i, \tau)$ be a configuration for player $i$ of value at least $\tau$; we will call $C$ thin if it only contains thin resources, and fat otherwise. There are two cases we need to consider.

Case 1. C is a thin configuration. Suppose that $\sum_{j \in C} z_{j}<(1-1 / \alpha)$. Then, by our assignment of $z_{j}$ values, this implies that there exists a set $R \subseteq C$ such that $R$ is disjoint from the resources in $A_{\leqslant \ell+1} \cup I \cup B_{\leqslant \ell}$ and $\sum_{j \in R} v_{j} \geqslant \tau / \alpha$. Together this contradicts the fact that $H^{\prime}$ has no sinks since $i$ is then a $\operatorname{sink}($ it is in $Z$ ).

Case 2. C is a fat configuration. Let $j$ be a fat resource in $C$. Since $i$ was reachable in $H^{\prime}$, all the sources in $H^{\prime}$ are assigned thin edges in $M$ (which implies they have no incoming arcs), and $K$ is a subset of the players, it follows that $j$ is also present in $H^{\prime}$. Thus, by our assignment, $z_{j}=1-1 / \alpha$.

Having proved that our assignment of $y_{i}$ and $z_{j}$ values constitutes a feasible solution to the dual of $C L P(\tau)$, we now compute the objective function value $\sum_{i} y_{i}-\sum_{j} z_{j}$ of the above assignment. To do so we adopt the following charging scheme: for each fat resource $j$ in $H^{\prime}$, charge its $z_{j}$ value against the unique player $i$ such that the outgoing arc $(j, i)$ belongs to $H^{\prime}$. The charging scheme accounts for the $z_{j}$ values of all the fat resources except for the fat resources that are leaves in $H^{\prime}$. There are at most $\left|K_{1}\right|$ such fat resources, where $K_{1} \subseteq K$ is the set of players to which the uncharged fat items have an outgoing arc to. Moreover, note that $K_{1}$ only consists of players that are matched in $M$ by fat edges. Since $P_{\leqslant \ell}$ does not have any players matched by fat edges in $M$, no player in $K_{2}:=P_{\leqslant \ell} \cap K$ is present in $K_{1}$, i.e., $K_{1} \cap K_{2}=\emptyset$. Finally, note that no player in $P_{\leqslant \ell} \backslash K=P_{\leqslant \ell}-K_{2}$ has been charged. Thus, considering all players in $\mathcal{P}$ but only fat configurations, we have

$$
\begin{aligned}
\sum_{i \in \mathcal{P}} y_{i}-\sum_{j \in \mathcal{R}_{f}} z_{j} & \geqslant(1-1 / \alpha)\left(\left|P_{\leqslant \ell}\right|-\left|K_{2}\right|\right)-(1-1 / \alpha)\left|K_{1}\right| \\
& =(1-1 / \alpha)\left(\left|P_{\leqslant \ell}\right|-\left(\left|K_{1}\right|+\left|K_{2}\right|\right)\right) \\
& >(1-1 / \alpha)(1-\gamma)\left|P_{\leqslant \ell}\right| .
\end{aligned}
$$

We now compute the total contribution of thin resources, i.e., $\sum_{j \in \mathcal{R}_{t}} z_{j}$. The total value of thin resources from the edges $A_{\leqslant \ell+1}$ and the edges $I$ is at most $(1 / \alpha+1 / \beta)\left|A_{\leqslant \ell+1} \cup I\right|$, due to the minimality of thin $\alpha$-edges. Besides the resources appearing in $A_{\leqslant \ell+1} \cup I$, the total value of resources appearing only in edges $B_{\leqslant \ell}$ is at most $(1 / \beta)\left(\left|B_{\leqslant \ell}\right|\right)<(1 / \beta)\left(\left|P_{\leqslant \ell}\right|\right)$, by the minimality of $\beta$-edges. Indeed, if an edge in $B_{\ell}$ has more than $\tau / \beta$ resources not appearing in an edge in $A_{\leqslant \ell+1} \cup I$ then those resources would form a thin $\beta$-edge which contradicts its minimality.

Using $\left|A_{\leqslant \ell+1} \cup I\right|<(\mu+\gamma)\left|P_{\leqslant \ell}\right|$ we have

$$
\sum_{i \in \mathcal{P}} y_{i}-\sum_{j \in \mathcal{R}} z_{j}>(1-\gamma)\left(1-\frac{1}{\alpha}\right)\left|P_{\leqslant \ell}\right|-(\mu+\gamma)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\left|P_{\leqslant \ell}\right|-\frac{1}{\beta}\left|P_{\leqslant \ell}\right|
$$

Recall that, given any feasible solution to the dual of $\operatorname{CLP}(\tau)$, we can scale it by any positive number, and it will remain feasible; this will imply that if the optimum of the dual of $\operatorname{CLP}(\tau)$ is positive, then the dual of $\operatorname{CLP}(\tau)$ is unbounded. So, the dual of $\operatorname{CLP}(\tau)$ is unbounded when

$$
(1-\gamma)\left(1-\frac{1}{\alpha}\right)-(\mu+\gamma)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)-\frac{1}{\beta} \geqslant 0 \Leftrightarrow \gamma \leqslant \frac{\alpha \beta-(1+\mu)(\alpha+\beta)}{\alpha \beta+\alpha} .
$$

Recall that $\beta=2(3+\sqrt{10})+\varepsilon, \alpha=2$, and $\mu=\varepsilon / 100$. For $\varepsilon>0$ the last inequality is equivalent to $206 \sqrt{10}+3 \varepsilon \leqslant 676$, which is valid for $\varepsilon \leqslant 1$.

We now use the previous lemma to show that if we create a new layer then the number of players in that layer will increase rapidly. This will allow us to bound the number of layers to be logarithmic and also to bound the running time.
Lemma 4.9 (Exponential growth). At each execution of the iterative step of the algorithm, we have

$$
\left|P_{i}\right| \geqslant \delta\left|P_{\leqslant i-1}\right| \text {, where } \delta:=\varepsilon / 100 \text {, }
$$

for each $i=1, \ldots, \ell$.
Proof. Suppose towards contradiction that the statement is false and let $t$ be the smallest index that violates it, i.e., $\left|P_{t}\right|<\delta\left|P_{\leqslant t-1}\right|$. Due to Invariant $3,\left|I_{i}\right|<\mu\left|P_{i}\right|$ for $0 \leqslant i \leqslant t$. Hence,

$$
\left|I_{\leqslant t}\right|<\mu\left|P_{\leqslant t}\right|<\mu(1+\delta)\left|P_{\leqslant t-1}\right| .
$$

Further,

$$
\left|A_{\leqslant t}\right|+\left|I_{\leqslant t}\right| \geqslant \operatorname{DP}_{M}\left(P_{\leqslant t-1}, A_{\leqslant t} \cup I_{\leqslant t}\right)=\operatorname{DP}_{M}\left(P_{\leqslant t-1}, A_{\leqslant t} \cup I\right) \geqslant \gamma\left|P_{\leqslant t-1}\right|,
$$

where the first inequality is trivial, the equality follows from the definition of canonical decompositions (Definition 4.1), and the last inequality follows from Lemma 4.6. This gives us

$$
\left|A_{\leqslant t}\right|>(\gamma-\mu(1+\delta))\left|P_{\leqslant t-1}\right| .
$$

We now obtain an upper bound on the total number of edges in $A_{\leqslant t}$ by counting the value of resources in each $A_{i}$ and $B_{i}$; observe that any thin $\beta$-edge has resources of total value at most $2 \tau / \beta$ due to minimality, while any thin $\alpha$-edge in $A_{\leqslant t}$ has resources of value at least $\tau / \alpha-\tau / \beta$ that are blocked, i.e., appear in some edge in $B_{\leqslant t}$ (since otherwise this edge would be in $I$ instead of $\left.A_{\leqslant t}\right)^{4}$.

[^4]Hence,

$$
\left|A_{i}\right|(\tau / \alpha-\tau / \beta) \leqslant\left|B_{i}\right|(2 \tau / \beta) \stackrel{\text { summing over } i \text { and rearranging }}{\Longrightarrow}\left|A_{\leqslant t}\right| \leqslant\left|B_{\leqslant t}\right| \frac{2 \alpha}{\beta-\alpha} .
$$

Since $\left|B_{\leqslant t}\right|<\left|P_{\leqslant t}\right|$ and $\left|P_{\leqslant t}\right|<(1+\delta)\left|P_{\leqslant t-1}\right|$ we have the bound

$$
\left|A_{\leqslant t}\right|<\frac{2 \alpha}{\beta-\alpha}(1+\delta)\left|P_{\leqslant t-1}\right| .
$$

Therefore we will have a contradiction when

$$
\frac{2 \alpha}{\beta-\alpha}(1+\delta) \leqslant \gamma-(1+\delta) \mu
$$

It can be verified that for any $\varepsilon>0$ the above inequality is equivalent to

$$
22400+6(52+\sqrt{10}) \varepsilon+3 \varepsilon^{2} \leqslant 9400 \sqrt{10}
$$

which is true for $\varepsilon \in[0,1]$ leading to the required contradiction.
We are now ready to prove that our algorithm executes a polynomial number of iterations. To do this, we define the signature vector $s:=\left(s_{0}, \ldots, s_{\ell}, \infty\right)$, where

$$
s_{i}:=\left\lfloor\log _{1 /(1-\mu)} \frac{\left|P_{i}\right|}{\delta^{i+1}}\right\rfloor
$$

corresponding to the state $(M, \ell, \mathcal{L}, I)$ of the algorithm. The signature vector changes as the algorithm executes; in fact, we prove that its lexicographic value always decreases:

Lemma 4.10. Across each iterative step, the lexicographic value of the signature vector decreases. Furthermore, the coordinates of the signature vector are always non-decreasing.

Proof. We show this by induction as usual on the variable $r$ that counts the number of times the iterative step has been executed. The statement for $r=0$ is immediate. Suppose it is true for $r \geqslant 0$. Let $s=\left(s_{0}, \ldots, s_{\ell}, \infty\right)$ and $s^{\prime}=\left(s_{0}^{\prime}, \ldots, s_{\ell^{\prime}}^{\prime}, \infty\right)$ denote the signature vector at the beginning and at the end of the $(r+1)$-th iterative step. We consider two cases:

No layer was collapsed. Let $L_{\ell+1}$ be the newly constructed layer. In this case, $\ell^{\prime}=\ell+1$. By Lemma 4.9, $\left|P_{\ell+1}\right| \geqslant \delta\left|P_{\leqslant \ell}\right|>\delta\left|P_{\ell}\right|$. Clearly, $s^{\prime}=\left(s_{0}, \ldots, s_{\ell}, s_{\ell+1}^{\prime}, \infty\right)$ where $\infty>s_{\ell+1}^{\prime} \geqslant s_{\ell}^{\prime}=s_{\ell}$. Thus, the signature vector $s^{\prime}$ also has increasing coordinates and smaller lexicographic value compared to $s$.

At least one layer was collapsed. Let $0 \leqslant t \leqslant \ell$ be the index of the last layer that was collapsed during the $r$-th iterative step. As a result of the collapse operation suppose the layer $P_{t}$ changed to $P_{t}^{\prime}$. Then we know that $\left|P_{t}^{\prime}\right|<(1-\mu)\left|P_{t}\right|$. Indeed, during Step 3 of Algorithm 2, at least a $\mu$-fraction of the edges in $B_{t}$ are replaced with edges from $I$. Since none of the layers with indices less than $t$ were affected during this procedure, $s^{\prime}=\left(s_{0}, \ldots, s_{t-1}, s_{t}^{\prime}, \infty\right)$ where $s_{t}^{\prime}=\left\lfloor\log _{1 /(1-\mu)} \frac{\left|P_{t}^{\prime}\right|}{\delta^{t+1}}\right\rfloor \leqslant$ $\left\lfloor\log _{1 /(1-\mu)} \frac{(1-\mu)\left|P_{t}\right|}{\delta^{++1}}\right\rfloor \leqslant\left\lfloor\log _{1 /(1-\mu)} \frac{\left|P_{t}\right|}{\delta^{++1}}\right\rfloor-1=s_{t}-1$. This shows that the lexicographic value of the signature vector decreases. That the coordinates of $s^{\prime}$ are non-decreasing follows from Lemma 4.9.

Finally, due to the above lemma, any upper bound on the number of possible signature vectors is an upper bound on the number of iterations Algorithm 2 will execute; we prove there is such a bound of polynomial size:
Lemma 4.11. The number of signature vectors is at most $|\mathcal{P}|^{O(1 / \mu \cdot 1 / \delta \cdot \log (1 / \delta))}$.
Proof. By Lemma $4.9,|\mathcal{P}| \geqslant P_{\leqslant \ell} \geqslant(1+\delta) P_{\leqslant \ell-1} \geqslant \ldots \geqslant(1+\delta)^{\ell}\left|P_{0}\right|$. This implies that $\ell \leqslant \log _{1+\delta}|\mathcal{P}| \leqslant$ $\frac{1}{\delta} \log |\mathcal{P}|$, where the last inequality is obtained by using Taylor series and that $\delta \in[0,1 / 100]$.

Now consider the $i$-th coordinate of the signature vector $s_{i}$. It can be no larger than $\log _{1 /(1-\mu)} \frac{|\mathcal{P}|}{\delta^{i+1}}$. Using the bound on the index $i$ and after some manipulations, we get

$$
\begin{aligned}
s_{i} & \leqslant\left(\log |\mathcal{P}|+(i+1) \log \frac{1}{\delta}\right) \frac{1}{\log \frac{1}{1-\mu}} \\
& \leqslant\left(\log |\mathcal{P}|+\left(\frac{1}{\delta} \log |\mathcal{P}|+1\right) \log \frac{1}{\delta}\right) \frac{1}{\log \frac{1}{1-\mu}} \\
& =\log |\mathcal{P}| \cdot O\left(\frac{1}{\mu \delta} \log \frac{1}{\delta}\right),
\end{aligned}
$$

where the final bound is obtained by again expanding using Taylor series around 0 . Thus, if we let $U=\log |\mathcal{P}| \cdot O\left(\frac{1}{\mu \delta} \log \frac{1}{\delta}\right)$ be an upper bound on the number of layers and the value of each coordinate of the signature vector, then the sum of coordinates of the signature vector is always upper bounded by $U^{2}$.

Now, as in the simpler algorithm, we apply the bound on the number of partitions of an integer. Recall that the number of partitions of an integer $N$ can be upper bounded by $e^{O(\sqrt{N})}$ [HR18]. Since each signature vector corresponds to some partition of an integer at most $U^{2}$, we can upper bound the total number of signature vectors by $\sum_{i \leqslant U^{2}} e^{O(\sqrt{i})}$.

Now using the bound of $U$, we have that the number of signatures is at most $|\mathcal{P}|^{O(1 / \mu \cdot 1 / \delta \cdot \log (1 / \delta))}$.

Since the number of possible signature vectors is polynomial, the number of iterations Algorithm 2 will execute is also polynomial. Furthermore, as the running time of each iteration is also polynomial, this completes the proof of Theorem 1.1.

## 5 Conclusion

In this paper we have presented new ideas for local search algorithms leading to an improved approximation algorithm for the restricted max-min fair allocation problem. The obtained algorithm is also combinatorial and therefore bypasses the need of solving the exponentially large Configuration-LP.

Apart from further improving the approximation guarantee, we believe that an interesting future direction is to consider our techniques in the more abstract setting of matchings in hypergraphs. For example, Haxell [Hax95] proved, using an alternating tree algorithm, a sufficient condition for a bipartite hypergraph to admit a perfect matching.

Theorem 5.1 (Haxell's Condition). Consider an $(r+1)$-uniform bipartite hypergraph $\mathcal{H}=(\mathcal{P} \cup \mathcal{R}, E)$ such that for every edge $e \in E$, $|e \cap \mathcal{P}|=1$ and $|e \cap \mathcal{R}|=r$. For $C \subseteq \mathcal{P}$ let $H\left(E_{C}\right)$ denote the size of the smallest set $R \subseteq \mathcal{R}$ that hits all the edges in $\mathcal{H}$ that are incident to some vertex in $C$. If for every $C \subseteq \mathcal{P}$, $H\left(E_{C}\right)>(2 r-1)(|C|-1)$ then there exists a perfect matching in $\mathcal{H}$.

Note that Theorem 5.1 generalizes Hall's theorem for graphs. However, the proof of the statement does not lead to a polynomial time algorithm. In the conference version of this paper we had posed the question of whether a constructive analog of Theorem 5.1 can be obtained.

With the techniques presented here, we could prove the following weaker statement: there is a constant $C_{0}>0$ for which, given some $0<\varepsilon \leqslant 1$ and assuming $H\left(E_{C}\right) \geqslant C_{0}(1 / \varepsilon) r(|C|-1)$, there exists a polynomial time algorithm which assigns one edge $e_{p} \in E$ for every player $p \in \mathcal{P}$ such that it is possible to choose disjoint subsets $\left\{S_{p} \subseteq e_{p} \cap \mathcal{R}\right\}_{p \in \mathcal{P}}$ of size at least $(1-\varepsilon) r$.

Recently, the first author obtained such a constructivization answering our open question affirmatively [Ann16]. For some fixed $\varepsilon>0$ and $r$ he proved that, for $(r+1)$-uniform hypergraphs satisfying $H\left(E_{C}\right)>(2 r-1+\varepsilon)(|C|-1)$ a polynomial time algorithm exists for finding the perfect matching guaranteed by Theorem 5.1. However, the running time of this algorithm is exponential in both $r$ and $1 / \varepsilon$. It remains an open problem to find such an algorithm whose running time dependence on $r$ is polynomial.

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[^1]:    ${ }^{1}$ The edges of the algorithm naturally form layers as described in Section 3.1 and as depicted in Figure 2. The edges in $A_{i}$ are added so as to try to "replace" edges in $B_{i-1}$ in the matching $M . B_{i}$ are then the edges of $M$ that are blocking the edges in $A_{i}$.

[^2]:    ${ }^{2}$ The asymptotic formula for the number of partitions of $N$ is $\frac{1}{4 N \sqrt{3}} \exp \left(\pi \sqrt{\frac{2 N}{3}}\right)$ as $N \rightarrow \infty$ [HR18].

[^3]:    ${ }^{3}$ We note that this invariant says that each edge $(p, R) \in I$ has a subset $R^{\prime} \subseteq R$ so that $v_{p}\left(R^{\prime}\right) \geqslant \tau / \beta$ and $R^{\prime} \cap \mathcal{R}\left(A_{\leqslant \ell} \cup\right.$ $\left.B_{\leqslant \ell} \cup I \backslash\{(p, R)\}\right)=\emptyset$ since $B_{\leqslant \ell} \subseteq M$. In other words, the resources of $\left(p, R^{\prime}\right)$ are disjoint from all other resources in $\mathcal{L}$.

[^4]:    ${ }^{4}$ We remark that just as in the simpler algorithm, the set $B_{i}$ contains those edges that are blocking the edges in $A_{i}$. This follows from the definition of the build phase and Steps 3.c and 3.d that remove edges from $B_{i}$ when the matching has changed or when $A_{i}$ has changed. Furthermore, all edges in $A_{i}$ have all but at most $\tau / \beta$ resources blocked. Otherwise, the edge is added to $I$ in the build phase and if resources have been freed up later, the edge is removed from $A_{i}$ (and it may be added to $I$ ) during Step 3.c.

