

Self-organized Segregation on the Grid

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Abstract

We consider an agent-based model with exponentially distributed waiting times in which two types of agents interact locally over a graph, and based on this interaction and on the value of a common intolerance threshold τ , decide whether to change their types. This is equivalent to a zero-temperature Ising model with Glauber dynamics, an Asynchronous Cellular Automaton (ACA) with extended Moore neighborhoods, or a Schelling model of self-organized segregation in an open system, and has applications in the analysis of social and biological networks, and spin glasses systems. Some rigorous results were recently obtained in the theoretical computer science literature, and this work provides several extensions. We enlarge the intolerance interval leading to the expected formation of large segregated regions of agents of a single type from the known size $\epsilon > 0$ to size ≈ 0.134 . Namely, we show that for $0.433 < \tau < 1/2$ (and by symmetry $1/2 < \tau < 0.567$), the expected size of the largest segregated region containing an arbitrary agent is exponential in the size of the neighborhood. We further extend the interval leading to expected large segregated regions to size ≈ 0.312 considering “almost segregated” regions, namely regions where the ratio of the number of agents of one type and the number of agents of the other type vanishes quickly as the size of the neighborhood grows. In this case, we show that for $0.344 < \tau \leq 0.433$ (and by symmetry for $0.567 \leq \tau < 0.656$) the expected size of the largest almost segregated region containing an arbitrary agent is exponential in the size of the neighborhood. This behavior is reminiscent of supercritical percolation, where small clusters of empty sites can be observed within any sufficiently large region of the occupied percolation cluster. The exponential bounds that we provide also imply that complete segregation, where agents of a single type cover the whole grid, does not occur with high probability for $p = 1/2$ and the range of intolerance considered.

I. INTRODUCTION

A. Background

A basic observation made by Thomas Schelling while studying the mechanisms leading to social segregation in the United States [1], [2] was that individuals in a social network have interactions with their friends and neighbors rather than with the entire population, and this often triggers global effects that were not originally intended, nor desired. Schelling proposed a simple stochastic model to predict these global outcomes, which has become popular in the social sciences. Two types of agents are randomly placed at the vertices of a two-dimensional grid and interact with a small subset of nodes located in their local neighborhood. Based on these interactions, the boolean state of each agent is determined as follows. All agents have a common intolerance threshold, indicating the minimum fraction of agents of their same type that must be located in their neighborhood to make their state happy. Unhappy agents randomly move to vacant locations where they will be happy. A peculiar effect observed by simulating several variants of this model is that when the system reaches a stable state, large areas of segregated agents of the same type are observed, for a wide range of the intolerance threshold value. Individuals, Schelling concluded, tend to spontaneously self-segregate. See Figure 1 for a simulation of this behavior.

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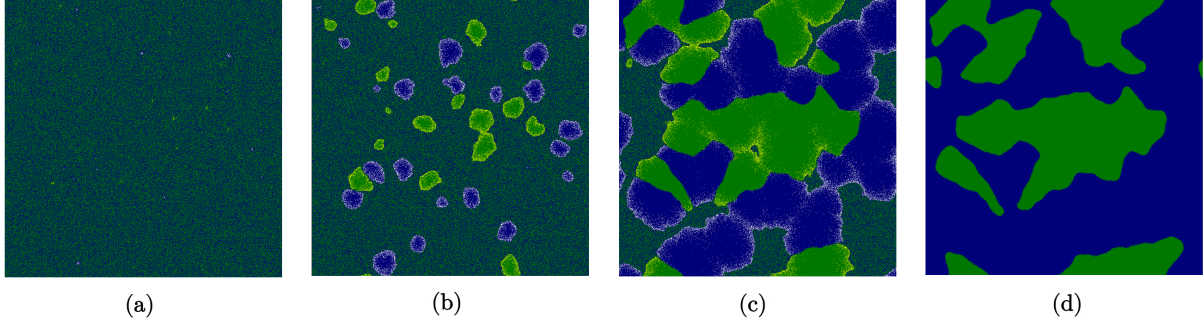


Fig. 1. Self-segregation arising over time for a value of the intolerance $\tau = 0.42$ on a grid of size 1000×1000 and neighborhood size 441. Green and blue indicate areas of “happy” agents of type (+1) and (-1), respectively. White and yellow indicate areas of “unhappy” agents of type (+1) and (-1) respectively. Initial configuration (a), intermediate configurations (b)-(c), final configuration (d). When the process terminates all agents are happy but large segregated regions can be observed.

Similar models have been considered in the statistical physics literature well before Schelling’s observation. For an intolerance value of $1/2$, for example, agents take the same value of the majority of their neighbors, and self-organized segregation in the Schelling model corresponds to spontaneous magnetization in the Ising model with zero temperature, where spins align along the direction of the local field [3], [4]. In computation theory, mathematics, physics, complexity theory, theoretical biology and microstructure modeling, the model is known as a two-dimensional, two-state Asynchronous Cellular Automaton (ACA) with extended Moore neighborhoods and exponential waiting times [5]. Other related models appeared in epidemiology [6], [7], economics [8], engineering and computer sciences [9], [10]. Mathematically, all of these models fall in the general area of interacting particle systems, or contact processes, and exhibit phase transitions [11], [12].

Schelling-type models can be roughly divided into two classes. A *Kawasaki dynamic* model assumes there are no vacant positions in the underlying graph, and a pair of unhappy agents swap their locations if this will make both of them happy. A *Glauber dynamic* model assumes single agents to simply flip their types if this makes them happy. This flipping action indicates that the agent has moved out of the system and a new agent has occupied its location. While in a Kawasaki model the system is “closed” and the number of agents of the same type is fixed, in a Glauber model the system is “open” and the number of agents of the same type may change over time. Sometimes the model dynamics are defined as having unhappy agents swap (or flip) regardless of whether this makes them happy or not. We assume throughout Glauber dynamics and agents to flip only if this makes them happy. Another possible variant is to assume that agents have a small probability of acting differently than what the general rule prescribes, other variants also consider having multiple intolerance levels, multiple agent types, different agent distributions, and time-varying intolerance [13]–[22].

B. Contribution

We focus on the case of two types of agents placed uniformly at random on a two-dimensional grid according to a Bernoulli distribution of parameter $p = 1/2$ and having a single intolerance level $0 < \tau < 1$, and study the range of intolerance leading to the formation of large segregated regions. Even for the one-dimensional version of this problem rigorous results appeared only recently.

Brandt et al. [23] considered a ring graph for the Kawasaki model of evolution. In this setting, letting the neighborhood of an agent be the set of nearby agents that is used to determine whether the agent is happy or not, they showed that for an intolerance level $\tau = 1/2$, the expected size of the largest segregated region containing an arbitrary agent in steady state is polynomial in the size of the neighborhood. Barmpalias

et al. [24] showed that there exists a value of $\tau^* \approx 0.35$, such that for all $\tau < \tau^*$ the initial configuration remains static with high probability (w.h.p.), while for all $\tau^* < \tau < 1/2$ the size of the largest segregated region in steady state becomes exponential in the size of the neighborhood w.h.p. On the other hand, for all $\tau > 1/2$ the system evolves w.h.p. towards a state with only two segregated components. For the Glauber model the behavior is similar, but symmetric around $\tau = 1/2$, with a first transition from a static configuration to exponential segregation occurring at $\tau \approx 0.35$, a special point $\tau = 1/2$ with the largest segregated region of expected polynomial size, then again exponential segregation until $\tau \approx 0.65$, and finally a static configuration for larger values of τ .

In a two-dimensional grid graph on a torus, the case $\tau = 1/2$ is open. Immorlica et al. [25] have shown for the Glauber model the existence of a value $\tau^* < 1/2$, such that for all $\tau^* < \tau < 1/2$ the expected size of the largest segregated region is exponential in the size of the neighborhood. This shows that segregation is expected in the small interval $\tau \in (1/2 - \epsilon, 1/2)$. Note that this does not imply exponential segregation w.h.p., but only expected segregated regions of exponential size. Barmpalias et al. [26] considered a model in which each type of agent has a different intolerance, i.e., τ_1 and τ_2 . For the special case of $\tau_1 = \tau_2 = \tau$, they have shown that when $\tau > 3/4$, or $\tau < 1/4$, the initial configuration remains static w.h.p.

Our main contribution is depicted in Figure 2. We consider the Glauber model for the two-dimensional grid graph on a torus. First, we enlarge the intolerance interval that leads to the formation of large segregated regions from the known size $\epsilon > 0$ to size ≈ 0.134 , namely we show that when $0.433 < \tau < 1/2$ (and by symmetry $1/2 < \tau < 0.567$), the expected size of the largest segregated region is exponential in the size of the neighborhood. Second, we further extend the interval leading to large segregated regions to size ≈ 0.312 . In this case, the main contribution is that we consider “almost segregated” regions, namely regions where the ratio of the number of agents of one type and the number of agents of the other type quickly vanishes as the size of the neighborhood grows, and show that for $0.344 < \tau \leq 0.433$ (and by symmetry for $0.567 \leq \tau < 0.656$) the expected size of the largest almost segregated region is exponential in the size of the neighborhood.

As shown for the one dimensional case in [24] and conjectured for the two-dimensional case in [26], we show that as the intolerance parameter gets farther from one half, in both directions, the average size of both the segregated and almost segregated regions gets larger: higher tolerance in our model does not necessarily lead to less segregation. On the contrary, it can increase the size of the segregated areas. This result is depicted in Figure 3. The intuitive explanation is that highly tolerant agents are seldom unhappy in the initial configuration, and the segregated regions of opposite types that unhappy agents may ignite are likely to start from far apart, and may grow larger before meeting at their boundaries.

Finally, the exponential upper bound that we provide on the expected size of the largest segregated region implies that complete segregation, where agents of a single type cover the whole grid, does not occur w.h.p. for the range of intolerance considered. In contrast, Fontes et al. [27] have shown the existence of a critical probability $1/2 < p^* < 1$ for the initial Bernoulli distribution of the agents such that for $\tau = 1/2$ and $p > p^*$ the Glauber model on the d -dimensional grid converges to a state where only one type of agents are present. This shows that complete segregation occurs w.h.p. for $\tau = 1/2$ and $p \in (1 - \epsilon, 1)$. Morris [28] has shown that p^* converges to $1/2$ as $d \rightarrow \infty$. Caputo and Martinelli [29] have shown the same result for d -regular trees, while Kanoria and Montanari [30] derived it for d -regular trees in a synchronous setting where flips occur simultaneously, and obtained lower bounds on $p^*(d)$ for small values of d . The case $d = 1$ was first investigated by Erdős and Ney [31], and Arratia [32] has proven that $p^*(1) = 1$.

C. Techniques

Our proofs are based on a typicality argument showing a self-similar structure of the neighborhoods in the initial state of the process, and on the identification of geometric configurations igniting a cascading

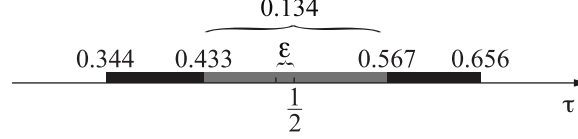


Fig. 2. We enlarge the width of the intolerance interval for which the expected size of the largest segregated region containing an arbitrary agent is exponential in the size of the neighborhood from the known value $\epsilon > 0$ to ≈ 0.134 (grey region). We also show that the expected size of the largest almost segregated region containing an arbitrary agent is exponential in the size of the neighborhood for an intolerance interval of width ≈ 0.312 (grey plus black region).

process leading to segregation. We make extensive use of tools from percolation theory, including the exponential decay of the radius of the open cluster below criticality [33], concentration bounds on the passage time [34] (see also [35], [36]), and on the chemical distance between percolation sites [37]. We also make frequent use of renormalization, and correlation inequalities for contact processes [38]. In this framework, we provide an extension of the Fortuin-Kasteleyn-Ginibre (FKG) inequality in a dynamical setting that can be of independent interest.

The paper is organized as follows. In section II we introduce the model, state our results, and give a summary of the proof construction. In section III we study the initial configuration and derive some properties of the sub-neighborhoods of the unhappy agents. In section IV we study the dynamics of the segregation process and derive the main results. Concluding remarks are given in section V.

II. MODEL AND MAIN RESULTS

A. The Model

Initial Configuration. We consider an $n \times n$ grid graph G_n embedded on a torus $\mathbb{T} = [0, n) \times [0, n)$, an integer $w \in O(\sqrt{\log n})$ called *horizon*, and a rational $0 \leq \tau \leq 1$ called *intolerance*. All arithmetic operations over the coordinates are performed modulo n , i.e., $(x, y) = (x + n, y) = (x, y + n)$. We place an agent at each node of the grid and choose its type independently at random to be (+1) or (-1) according to a Bernoulli distribution of parameter $p = 1/2$.

A *neighborhood* is a connected sub-graph of G_n . A *neighborhood of radius ρ* is the set of all agents with l_∞ distance at most ρ from a central node, and is denoted by \mathcal{N}_ρ . The *size* of a neighborhood is the number of agents in it. The *neighborhood of an agent u* is a neighborhood of radius equal to the horizon and centered at u , and is denoted by $\mathcal{N}(u)$.

Dynamics. We let the rational τ called *intolerance* be $\lceil \tilde{\tau} N \rceil / N$, where $\tilde{\tau} \in [0, 1]$ and $N = (2w + 1)^2$ is the size of the neighborhood of an agent. The integer τN represents the minimum number of agents of the same type as u that must be present in $\mathcal{N}(u)$ to make u *happy*. More precisely, for every agent u , we let $s(u)$ be the ratio between the number of agents of the same type as u in its neighborhood and the size of the neighborhood. At any point in continuous-time, if $s(u) \geq \tau$ then u is labeled *happy*, otherwise it is labeled *unhappy*. We assign independent and identical Poisson clocks to all agents, and every time a clock rings the type of the agent is flipped if and only if the agent is unhappy and this flip will make the agent happy. Two observations are now in order. First, for $\tau < 1/2$ flipping its type will always make an unhappy agent happy, but this is not the case for $\tau > 1/2$. Second, the process dynamics are equivalent to a discrete-time model where at each discrete time step one unhappy agent is chosen uniformly at random and its type is flipped if this will make the agent happy.

Termination. The process continues until there are no unhappy agents left, or there are no unhappy agents that can become happy by flipping their type. By defining a Lyapunov function to be the sum

over all agents u of the number of agents of the same type as u present in its neighborhood, it is easy to argue that the process indeed terminates.

Segregation. The *monochromatic region* of an agent u is the neighborhood with largest radius containing agents of a single type and that also contains u when the process stops. Let $\epsilon > 0$ and $N = (2w + 1)^2$. The *almost monochromatic region* of an agent u , is the neighborhood with largest radius such that the ratio of the number of agents of one type and the number of agents of the other type is bounded by e^{-N^ϵ} and that also contains u when the process stops.

Throughout the paper we use the terminology *with high probability* (w.h.p.) meaning that the probability of an event approaches one as N approaches infinity.

B. The Results

To state our results, we let $\tau_1 \approx 0.433$ be the solution of

$$\frac{3}{4} \left[1 - H \left(\frac{4}{3} \tau_1 \right) \right] - [1 - H(\tau_1)] = 0, \quad (1)$$

where H is the binary entropy function

$$H(\tau_1) = -\tau_1 \log_2 \tau_1 - (1 - \tau_1) \log_2 (1 - \tau_1), \quad (2)$$

and $\tau_2 \approx 0.344$ be the solution of

$$1024\tau_2^2 - 384\tau_2 + 11 = 0 \quad (3)$$

We also let M and M' be the sizes of the monochromatic and almost monochromatic regions of an arbitrary agent, respectively.

We consider values of the intolerance $\tau \in (\tau_2, 1 - \tau_2) \setminus \{1/2\}$. Most of the work is devoted to the study of the intervals $(\tau_2, \tau_1]$ and $(\tau_1, 1/2)$, a symmetry argument extends the analysis to the intervals $(1/2, 1 - \tau_1)$ and $[1 - \tau_1, 1 - \tau_2)$. The following theorems show that segregation occurs for values of τ in the grey region of Figure 2, where we expect an exponential monochromatic region, and in the black region of Figure 2, where we expect an exponential almost monochromatic region.

Theorem 1. *For all $\tau \in (\tau_1, 1 - \tau_1) \setminus \{1/2\}$ and for sufficiently large N , we have*

$$2^{a(\tau)N - o(N)} \leq \mathbb{E}[M] \leq 2^{b(\tau)N + o(N)}, \quad (4)$$

where a and b are decreasing functions of τ for $\tau < 1/2$ and increasing for $\tau > 1/2$.

Theorem 2. *For all $\tau \in (\tau_2, \tau_1] \cup [1 - \tau_1, 1 - \tau_2)$ and for sufficiently large N , we have*

$$2^{a(\tau)N - o(N)} \leq \mathbb{E}[M'] \leq 2^{b(\tau)N + o(N)}, \quad (5)$$

where a and b are decreasing functions of τ for $\tau < 1/2$ and increasing for $\tau > 1/2$.

The numerical values for $a(\tau)$ and $b(\tau)$ derived in the proofs of the above theorems are plotted in Figure 3. For $\tau \in (\tau_1, 1 - \tau_1) \setminus \{1/2\}$, as the intolerance gets farther from one half in both directions, larger monochromatic regions are expected.

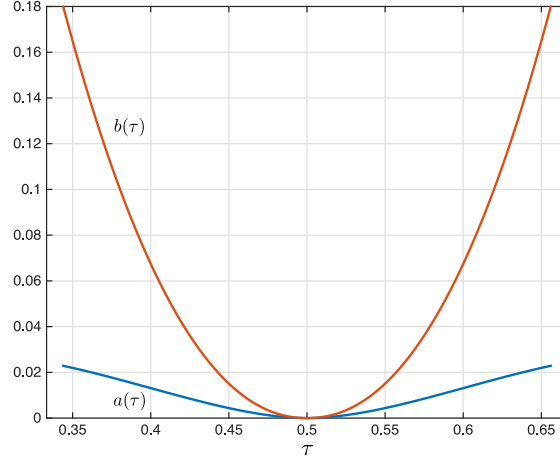


Fig. 3. Exponent multipliers $a(\tau)$ and $b(\tau)$ for the lower bound and upper bounds on the expected size of the largest segregated region $\mathbb{E}[M]$, and the expected size of the largest almost segregated region $\mathbb{E}[M']$.

C. Proof Outline

The main idea of the proof is to identify a local initial configuration that can potentially trigger a cascading process leading to segregation. We then bound the probability of occurrence of such a configuration in the initial state, and of the conditions to trigger segregation.

To identify this local configuration, we study the relationship between the typical neighborhood of an unhappy agent and the sub-neighborhoods contained within this neighborhood, showing a self-similar structure. Namely, the fraction of agents of the same type, when scaled by the size of the neighborhood, remains roughly the same (Proposition 1). We then define a *radical region* that contains a nucleus of unhappy agents (Lemma 4), and using the self-similar structure of the neighborhoods we construct a geometric configuration where a sequence of flips can lead to the formation of a neighborhood of agents of the same type inside a radical region (Lemma 5). Finally, we provide a lower bound for the probability of occurrence of this configuration in the initial state of the system (Lemma 6), which can initiate the segregation process.

The second part of the proof is concerned with the process dynamics, and shows a cascading effect ignited by the radical regions that leads to the formation of exponentially large segregated areas. We consider an indestructible and impenetrable structure around a radical region called a *firewall* and show that once formed it remains static and protects the radical region inside it from vanishing (Lemma 9). Conditioned on certain events occurring in the area surrounding the radical region, including the formation of the initial configuration described in the first part of the proof, we show that an agent close to the radical region will be trapped w.h.p. inside an exponentially large firewall whose interior becomes monochromatic (Lemma 10), see Figure 4(a). We then obtain a lower bound on the joint probability of the conditioning events and this leads to a lower bound on the probability that an agent is eventually contained in a monochromatic region of exponential size. Since the lower bound holds for both type of agents, we expect to have both types of exponential monochromatic regions in a large area by the end of the process. This leads to an exponential upper bound on the expected size of the largest monochromatic region of each type. To perform our computations, we rely on a bound on the passage time on the square lattice [34] to upper bound the rate of spread of other monochromatic regions outside the firewall, and ensure that they do not interfere with its formation during the dynamics of the process.

The construction described above works for all $\tau_1 < \tau < 1/2$. For smaller values of τ , agents are more tolerant and this may cause the construction of a firewall to fail, since tolerant agents do not easily

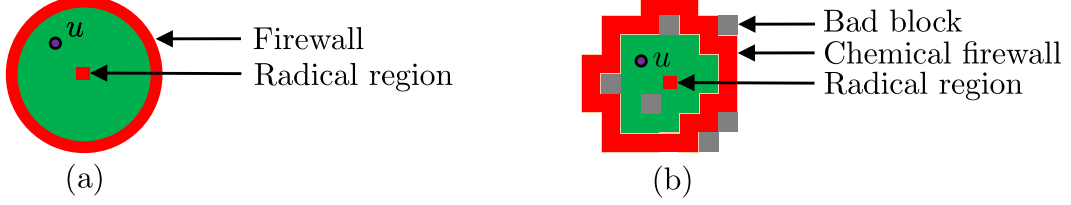


Fig. 4. An arbitrary agent u that is close to a radical region will be trapped inside a firewall of exponential size whose interior will eventually become monochromatic (a), or almost monochromatic (b).

become unhappy and flip their types igniting the cascading process. In order to overcome this difficulty, we introduce a *chemical firewall* through a comparison with a Bernoulli site percolation model, see Figure 4(b). This firewall is constructed through renormalization and is initially made of *good* blocks that occur independently and with probability above the critical threshold for site percolation on the square grid. Using a theorem in [37] on the chemical distance between good blocks, we show that they form a large cycle that, once it becomes monochromatic, isolates its interior. Finally, using the exponential decay of the size of the clusters of bad blocks [33], we show that the region inside the chemical firewall becomes *almost monochromatic*, namely for all $\tau_2 < \tau \leq \tau_1$, we expect the formation of exponentially large regions where the ratio of number of agents of one type and the number of agents of the other type quickly vanishes.

All results are extended to the interval $1/2 < \tau < 1 - \tau_2$ using a symmetry argument.

Compared to the proof in [25], our derivation differs in the following aspects. The definition of radical region is fundamentally different from the viral nodes considered in [25], and the identification of the radical regions gives us an immediate understanding of the arrangement of the agents in the initial configuration in terms of self-similarity arising at different scales. Our definition of an annular firewall that forms quickly enough eliminates the need for additional arguments from first passage percolation that are used in [25], it allows for a wider range of intolerance parameters, and it is easily generalized to the notion of chemical firewall using the results from [37]. The renormalization of the grid for the study of the growth of the monochromatic regions is also different from [25] and works for a wider range of the intolerance. The idea of considering almost monochromatic regions is new, and so are the approaches that we use from percolation theory to argue the existence of the chemical firewall and the size of the minority clusters. Finally, we rigorously apply a variation of the FKG inequality to show positive correlation of certain events, while in [25] it is often informally argued that similar correlations exist in their setting.

III. TRIGGERING CONFIGURATION

We start our analysis considering the initial configuration of the system. Proposition 1 shows a similarity relationship between the neighborhood of an agent and its sub-neighborhoods. This relationship is exploited in Lemma 5 to construct an initial configuration of agents that can trigger the segregation process. Lemma 6 provides a bound on the probability of occurrence of this triggering configuration.

Let $\mathcal{N}(u)$ be the neighborhood of an arbitrary agent u containing N agents. Consider a sub-neighborhood $\mathcal{N}'(u) \subset \mathcal{N}(u)$ containing N' agents and let γ be the scaling factor N'/N . Let W and W' be the random variables representing the number of (-1) agents in $\mathcal{N}(u)$ and $\mathcal{N}'(u)$ respectively. The following proposition shows that, conditioned on W being less than τN , W' is very close to the rescaled quantity $\gamma \tau N$, with overwhelming probability as $N \rightarrow \infty$.

Proposition 1. *For any $\epsilon \in (0, 1/2)$ and $c \in \mathbb{R}^+$ there exists $c' \in \mathbb{R}^+$ such that for all $N \geq 1$*

$$P\left(|W' - \gamma \tau N| < cN^{1/2+\epsilon} \mid W < \tau N\right) \geq 1 - e^{-c'N^{2\epsilon}}.$$

To prove this proposition, where the two constants ϵ and c are introduced for technical convenience in its later applications, we need the following three lemmas.

Lemma 1. *Let \mathcal{N} be a set of $(+1)$ and (-1) arbitrary agents in the grid such that it has exactly K agents of type (-1) and $N - K$ agents of type $(+1)$. Then, if we choose a set \mathcal{N}' of size N' of agents uniformly at random from \mathcal{N} , we have*

$$P(W' \geq \gamma K + t) \leq e^{\frac{-t^2}{2N'}}, \quad (6)$$

and

$$P(W' \leq \gamma K - t) \leq e^{\frac{-t^2}{2N'}}, \quad (7)$$

where W' is the random variable indicating the number of (-1) agents in \mathcal{N}' , and $\gamma = N'/N$.

Proof. Let W'_i be a random variable indicating the type of the i 'th agent in \mathcal{N}' , namely W'_i is one if the type is (-1) and zero otherwise. Let $\mathcal{F}_i = \sigma(W'_1, \dots, W'_i)$, where $\sigma(X)$ denotes the sigma field generated by random variable X . It is easy to see that for all $n \in \{1, \dots, N'\}$, $M_n = \mathbb{E}[W' | \mathcal{F}_n]$ is a martingale. It is also easy to see that $M_0 = \mathbb{E}[W'] = \gamma N'$, and $M_{N'} = W'$. For all $n \in \{1, 2, \dots, N'\}$, we also have

$$\begin{aligned} |M_n - M_{n-1}| &= \left| \mathbb{E} \left(\sum_{i=1}^{N'} W'_i \mid \mathcal{F}_n \right) - \mathbb{E} \left(\sum_{i=1}^{N'} W'_i \mid \mathcal{F}_{n-1} \right) \right| \\ &= \left| W'_n + \frac{K - \sum_{i=1}^n W'_i}{N - n} (N' - n) - \frac{K - \sum_{i=1}^{n-1} W'_i}{N - (n-1)} [N' - (n-1)] \right| \\ &\leq 1. \end{aligned}$$

Now, using Azuma's inequality [39], we have

$$P(W'_i \geq \gamma K + t) = P(M_{N'} \geq M_0 + t) \leq e^{\frac{-t^2}{2N'}}.$$

With the same argument we can derive (7). ■

Lemma 2. *Let $\epsilon \in (0, 1/2)$ and $c \in \mathbb{R}^+$. There exists $c' \in \mathbb{R}^+$ such that for all $N \geq 1$*

$$P\left(W' < \gamma \tau N + cN^{1/2+\epsilon} \mid W < \tau N\right) \geq 1 - e^{-c'N^{2\epsilon}}.$$

Proof. Let us denote $cN^{1/2+\epsilon}$ by $v(N)$. We let

$$\begin{aligned} p_w &= P\left(W' \geq \gamma \tau N + v(N) \mid W < \tau N\right) \\ &\leq P\left(W' \geq \gamma \tau N + v(N) \mid W \leq \tau N\right) \\ &\leq P\left(W' \geq \gamma \tau N + v(N) \mid W = \tau N\right) \end{aligned}$$

The first inequality is trivial. The second inequality follows from

$$P\left(W' \geq \gamma \tau N + v(N) \mid W \leq \tau N\right)$$

being the probability of choosing $W' \geq \gamma \tau N + v(N)$ agents from a set with $W \leq \tau N$. It is easy to see that this probability can only increase if we have $W = \tau N$. The result follows by applying Lemma 1. ■

Let $\mathcal{N}''(u) = \mathcal{N}(u) \setminus \mathcal{N}'(u)$. Let us denote the number of agents in $\mathcal{N}''(u)$ by N'' . Let W'' denote the random variable representing the number of (-1) agents in $\mathcal{N}''(u)$.

Lemma 3. Let $\epsilon \in (0, 1/2)$ and $c \in \mathbb{R}^+$. There exist $c' \in \mathbb{R}^+$ such that for all $N \geq 1$

$$P\left(W' > \gamma\tau N - cN^{1/2+\epsilon} \mid W < \tau N\right) \geq 1 - e^{-c'N^{2\epsilon}}.$$

Proof. Let us denote $cN^{1/2+\epsilon}$ by $v(N)$, and $\tau N - 1$ by N_τ . Let

$$\begin{aligned} p_w &= P(W' \leq \tau\gamma N - v(N) \mid W < \tau N) \\ &= P(W' \leq \tau N' - v(N) \mid W' + W'' < \tau N) \\ &\leq \frac{P(W' \leq \tau N' - v(N), W' + W'' \leq N_\tau)}{P(W \leq N_\tau)} \\ &\leq \frac{\sum_{k=0}^{\lfloor \tau N' - v(N) \rfloor} P(W' = k) \sum_{m=0}^{\min\{N_\tau - k, N''\}} P(W'' = m)}{P(W \leq N_\tau)} \\ &= \frac{\sum_{k=0}^{\lfloor \tau N' - v(N) \rfloor} \binom{N'}{k} \sum_{m=0}^{\min\{N_\tau - k, N''\}} \binom{N''}{m}}{\sum_{n=0}^{N_\tau} \binom{N}{n}}. \end{aligned} \tag{8}$$

We use the following inequality, valid for all $a \in (0, 0.5)$

$$\binom{N}{aN} \leq \sum_{m=0}^{aN} \binom{N}{m} \leq \frac{1-a}{1-2a} \binom{N}{aN}.$$

Since $\tau < 1/2$, it follows that $\binom{N}{N_\tau}$ is a lower bound for the denominator of (8). We also have the following upper bound for the numerator

$$\begin{aligned} \sum_{k=0}^{\lfloor \tau N' - v(N) \rfloor} \binom{N'}{k} \sum_{m=0}^{\min\{N_\tau - k, N''\}} \binom{N''}{m} &\leq \\ \sum_{k=0}^{\lfloor \tau N' - v(N) \rfloor} c^k \binom{N'}{k} \binom{N''}{\min\{N_\tau - k, \lfloor N''/2 \rfloor\}}, \end{aligned}$$

where $\{c^k\}$ are positive constants for $k = 0, 1, \dots, \lfloor \tau N' - v(N) \rfloor$. Since for all $l \in \{0, 1, \dots, \lfloor \tau N' - v(N) \rfloor\}$, we have

$$\frac{\binom{N'}{\lfloor \tau N' - v(N) \rfloor} \binom{N''}{\min\{N_\tau - \lfloor \tau N' - v(N) \rfloor, \lfloor N''/2 \rfloor\}}}{\binom{N'}{\lfloor \tau N' - v(N) \rfloor - l} \binom{N''}{\min\{N_\tau - \lfloor \tau N' - v(N) \rfloor + l, \lfloor N''/2 \rfloor\}}} \geq 1,$$

it follows that there exist a constant $c_1 \in \mathbb{R}^+$ such that

$$c_1 N \binom{N'}{\lfloor \tau N' - v(N) \rfloor} \binom{N''}{N_\tau - \lfloor \tau N' - v(N) \rfloor}$$

is an upper bound for the numerator. Putting things together, we have

$$\begin{aligned} p_w &\leq c_1 N \frac{\binom{N'}{\lfloor \tau N' - v(N) \rfloor} \binom{N''}{N_\tau - \lfloor \tau N' - v(N) \rfloor}}{\binom{N}{N_\tau}} \\ &\leq c_1 N P(W' \leq \tau N' - v(N) \mid W = N_\tau). \end{aligned}$$

Using the same argument as in Lemma 2, we now have

$$p_w \leq e^{-c'N^{2\epsilon}},$$

where $c' \in \mathbb{R}^+$ is a constant. ■

Proposition 1. Let

$$\begin{aligned} A &= \left\{ \tau\gamma N - cN^{1/2+\epsilon} < W' \right\}, \\ B &= \left\{ W' < \tau\gamma N + cN^{1/2+\epsilon} \right\}, \\ C &= \{W < \tau N\}. \end{aligned}$$

By Lemmas 2 and 3 there exist constants $c_1, c_2 > 0$ such that we have

$$\begin{aligned} P(A \cap B | C) &= 1 - P\left(A^C \cup B^C \mid C\right) \\ &\geq 1 - \left(P\left(A^C \mid C\right) + P\left(B^C \mid C\right)\right) \\ &\geq 1 - \left(e^{-c_1N^{2\epsilon}} + e^{-c_2N^{2\epsilon}}\right). \end{aligned}$$

Hence, there exists a constant $c' \in \mathbb{R}^+$ such that

$$P\left(A \cap B \mid C\right) \geq 1 - e^{-c'N^{2\epsilon}},$$

and the proof is complete. ■

We now identify a configuration that has the potential to trigger a cascading process. We show that a neighborhood that is slightly larger than the neighborhood of an agent and that contains a fraction of same type agents that is slightly less than τ has the desired configuration. For any $\epsilon, \epsilon' \in (0, 1/2)$ let $\hat{\tau} = \tau[1 - 1/(\tau N^{1/2-\epsilon})]$ and define a *radical region* $\mathcal{N}_{(1+\epsilon')w}$ to be a neighborhood of radius $(1 + \epsilon')w$ containing less than $\hat{\tau}(1 + \epsilon')^2N$ agents of type (-1). We also define an *unhappy region* $\mathcal{N}_{\epsilon'w}$ to be a neighborhood of radius $\epsilon'w$, containing at least $\lfloor \tau\epsilon'^2N - N^{1/2+\epsilon} \rfloor$ unhappy agents of type (-1).

Lemma 4. A radical region $\mathcal{N}_{(1+\epsilon')w}$ contains an unhappy region $\mathcal{N}_{\epsilon'w}$ at its center w.h.p.

Proof. Let $\epsilon \in (0, 1/2)$. We show that w.h.p. the region $\mathcal{N}_{\epsilon'w}$ co-centered with $\mathcal{N}_{(1+\epsilon')w}$ has at least $\lfloor \tau\epsilon'^2N - N^{1/2+\epsilon} \rfloor$ agents of type (-1) such that all of them are unhappy. Let A be the event that there are less than $\tau\epsilon'^2N - N^{1/2+\epsilon}$ agents of type (-1) in $\mathcal{N}_{\epsilon'w}$, which has N' agents. By Proposition 1, there exists $c_1, c_2 > 0$ such that

$$P(A) \leq P\left(W' \leq \hat{\tau}N' - c_1N^{1/2+\epsilon} \mid W_{(1+\epsilon')w} < (1 + \epsilon')^2\hat{\tau}N\right) \leq e^{-c_2N^{2\epsilon}},$$

where $W_{(1+\epsilon')w}$ represents the number of (-1) agents in $\mathcal{N}_{(1+\epsilon')w}$. Let \mathcal{I} denote the set of the positions of all the agents in $\mathcal{N}_{\epsilon'w}$, and let B_i be the event that a (-1) agent positioned at $i \in \mathcal{I}$ is happy. By Proposition 1, there exists $c_3 > 0$ such that, for all $i \in \mathcal{I}$

$$P(B_i) = P\left(W_i \geq \hat{\tau}N + c_uN^{1/2+\epsilon} \mid W_{(1+\epsilon')w} < (1 + \epsilon')^2\hat{\tau}N\right) \leq e^{-c_3N^{2\epsilon}},$$

where W_i is the number of (-1) agents in the neighborhood of i and $c_u > 0$ is chosen so that the threshold for being happy is met. It follows that there exists $c > 0$ such that

$$P\left(A \cap B_1^C \cap \dots \cap B_{|\mathcal{I}|}^C\right) \geq 1 - Ne^{-cN^{2\epsilon}},$$

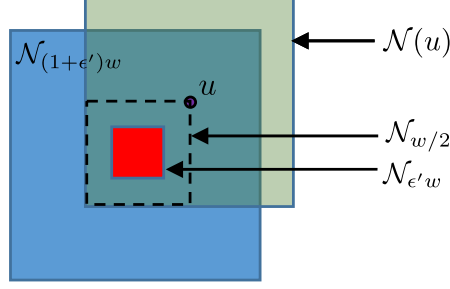


Fig. 5. Regions discussed in Lemma 5. $\mathcal{N}_{\epsilon'w}$ is an unhappy region w.h.p., the dashed box is $\mathcal{N}_{w/2}$, u is a corner agent in $\mathcal{N}_{w/2}$, and finally $\mathcal{N}(u)$ is the neighborhood of agent u .

where $|\mathcal{I}|$ denotes the cardinality of \mathcal{I} . ■

A radical region is *expandable* if there is a sequence of at most $(w+1)^2$ possible flips inside it that can make the neighborhood $\mathcal{N}_{w/2}$ at its center monochromatic.

We consider a geometric configuration where a radical region, and neighborhoods $\mathcal{N}_{\epsilon'w}$, $\mathcal{N}_{w/2}$ and \mathcal{N}_ρ with $\rho > 3w$, are all co-centered. We consider the process dynamics and let u^+ denote an arbitrary (+1) agent and

$$T(\rho) = \inf \left\{ t : \exists v \in \mathcal{N}_\rho, u^+ \text{ would be unhappy at the location of } v \right\}. \quad (9)$$

The next lemma shows that the radical region in this configuration is expandable w.h.p., provided that ϵ' is large enough and no (+1) agent at the location of any agent in \mathcal{N}_ρ is unhappy. The main idea is that the (-1) agents in the unhappy region at the center of the radical region can trigger a process that leads to a monochromatic (+1) region of radius $w/2$.

Lemma 5. *For all $\epsilon' > f(\tau)$, where*

$$f(\tau) = \frac{3(\tau - 0.5) + \sqrt{9(\tau - 0.5)^2 - 7(\tau - 0.5)(3\tau + 0.5)}}{2(3\tau + 0.5)}, \quad (10)$$

there exists w.h.p. a sequence of at most $(w+1)^2$ possible flips in $\mathcal{N}_{(1+\epsilon')w}$ such that if they happen before $T(\rho)$, then all the agents inside $\mathcal{N}_{w/2}$ will become of type (+1).

Proof. Let $\epsilon \in (0, 1/2)$. Let us denote the neighborhood with radius $\epsilon'w$ and co-centered with the radical region by $\mathcal{N}_{\epsilon'w}$, see Figure 5. By Lemma 4, with probability at least $1 - e^{-O(N^{2\epsilon})}$ there are at least $\lfloor \tau \epsilon'^2 N - N^{2\epsilon} \rfloor$ agents of type (-1) inside this neighborhood such that all of them are unhappy. Next, we show that if these unhappy agents flip before $T(\rho)$, all the agents inside the neighborhood $\mathcal{N}_{w/2}$ will be unhappy w.h.p., which gives the desired result.

First, we notice that if there is a flip of an unhappy (-1) agent in $\mathcal{N}_\rho \setminus \mathcal{N}_{w/2}$ it can only increase the probability of the existence of the sequence of flips we are looking for, hence conditioned on having these flips before $T(\rho)$, the worst case is when these flips occur with the initial configuration of $\mathcal{N}_\rho \setminus \mathcal{N}_{w/2}$. Since a corner agent in $\mathcal{N}_{w/2}$ shares the least number of agents with the radical region, it is more likely for it to have the largest number of (+1) agents in its neighborhood compared to other agents in $\mathcal{N}_{w/2}$. Hence, as a worst case, we may consider a corner agent in $\mathcal{N}_{w/2}$ which is co-centered with the radical region.

Let us assume that $\epsilon' \in (0, 1/2)$, in this case $\mathcal{N}_{\epsilon'w}$ is completely contained in the neighborhood of each of the agents in $\mathcal{N}_{w/2}$. Let us denote the neighborhood shared between the neighborhood of the

agent u at the corner of $\mathcal{N}_{w/2}$ and the radical region by $\mathcal{N}''(u)$. Also, let us denote the scaling factor corresponding to this shared neighborhood by γ'' . We have

$$\gamma'' = \frac{(3/2 + \epsilon')^2}{4(1 + \epsilon')^2} \pm O\left(\frac{1}{\sqrt{N}}\right).$$

By Proposition 1 it follows that with probability at least $1 - e^{-O(N^{2\epsilon})}$ there are at most

$$\frac{(3/2 + \epsilon')^2 \tau}{4} N + o(N),$$

agents of type (-1) in $\mathcal{N}''(u)$. Hence, we can conclude that, for any agent in $\mathcal{N}_{w/2}$, w.h.p., there are at most this many (-1) agents in the intersection of the neighborhood of this agent and the radical region.

Also, using Lemma 18 of the Appendix, with probability at least $1 - e^{-O(N^{2\epsilon})}$ we have at most

$$\frac{1}{2} \left(1 - (3/2 + \epsilon')^2/4\right) N + o(N),$$

agents of type (-1) in the part of the neighborhood of the corner agent u in $\mathcal{N}_{w/2}$ that is also not in the radical region. Combining the above results, we can conclude that with probability at least $1 - e^{-O(N^{2\epsilon})}$ there are at most

$$\frac{(3/2 + \epsilon')^2 \tau}{4} N + \frac{1}{2} \left(1 - \frac{(3/2 + \epsilon')^2 \tau}{4}\right) N + o(N),$$

agents of type (-1) in the neighborhood of an agent in $\mathcal{N}_{w/2}$. Let us denote this event for the corner agent u by A_1 . Let us denote the events of having at most this many (-1) agents in the neighborhoods of other agents in $\mathcal{N}_{w/2}$ by $A_2, \dots, A_{|\mathcal{N}_{w/2}|}$, where $|\mathcal{N}_{w/2}|$ denotes the number of agents in $\mathcal{N}_{w/2}$. We have

$$\begin{aligned} P(A_1 \cap \dots \cap A_{(w+1)^2}) &\geq 1 - P(A_1^C \cup \dots \cup A_{|\mathcal{N}_{w/2}|}^C) \\ &\geq 1 - (w+1)^2 P(A_1^C) \\ &\geq 1 - e^{-O(N^{2\epsilon})}. \end{aligned}$$

The goal is now to find the range of ϵ' for which $\mathcal{N}_{\epsilon'w}$ is large enough that once all of its unhappy agents flip, all the agents in $\mathcal{N}_{w/2}$ become unhappy w.h.p. It follows that we need

$$\frac{(3/2 + \epsilon')^2 \tau}{4} N + \frac{1}{2} \left(1 - \frac{(3/2 + \epsilon')^2 \tau}{4}\right) N - \tau \epsilon'^2 N + o(N) < \tau N,$$

to hold w.h.p. Dividing by N , and letting N go to infinity, after some algebra it follows that

$$\epsilon' > \frac{3(\tau - 0.5) + \sqrt{9(\tau - 0.5)^2 - 7(\tau - 0.5)(3\tau + 0.5)}}{2(3\tau + 0.5)} = f(\tau), \quad (11)$$

where $f(\tau) < 1/2$ for $\tau \in (\tau_2, 1/2)$, as desired. ■

Figure 6 depicts $f(\tau)$ as a function of τ . When τ is close to one half, it is sufficient to have an ϵ' close to zero to potentially trigger a segregation process. In this case, a small number of agents located in a small unhappy region are needed to flip in order to make other agents in the radical region unhappy. However, as τ decreases and agents become more tolerant, a larger number of agents must make a flip in the unhappy region in order to make other agents in the radical region unhappy, and hence larger values of ϵ' are needed.

Using Lemma 5, we obtain an exponential bound on the probability of having an expandable radical region inside a sufficiently large neighborhood. This shows that the probability that an expandable radical region is sufficiently close to an arbitrary agent u in the initial configuration, is not too small.

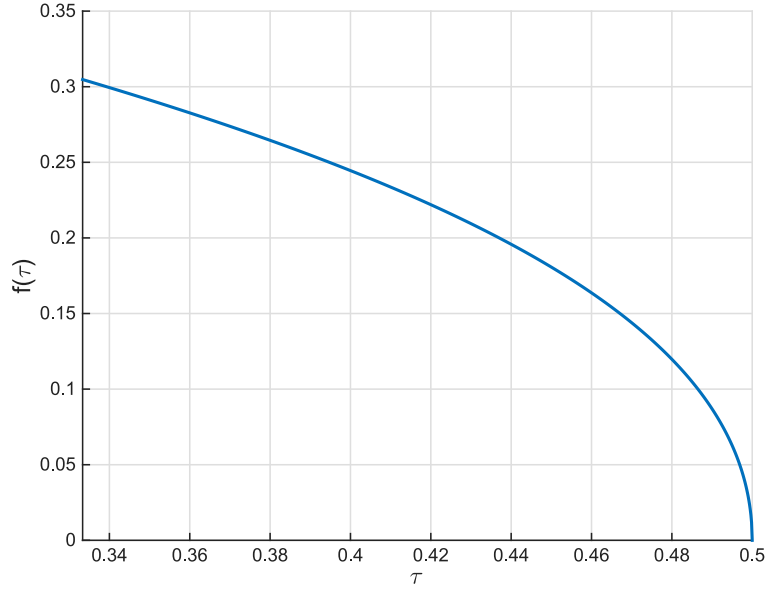


Fig. 6. The infimum of ϵ' to potentially trigger a cascading process.

Lemma 6. Let $r = 2^{\lceil 1-H(\tau') \rceil N/2 - o(N)}$, where $\tau' = (\tau N - 2)/(N - 1)$. Let

$$C = \{\mathcal{N}_r \text{ contains an expandable radical region at } t = 0\}.$$

For all $\epsilon' > f(\tau)$ and sufficiently large N , we have

$$P(C) \geq 2^{-\lceil 1-H(\tau') \rceil (2\epsilon' + \epsilon'^2)N - o(N)}.$$

Proof. Let \mathcal{N}_r be an arbitrary neighborhood of radius $r = 2^{\lceil 1-H(\tau') \rceil N/2 - o(N)}$ and let \mathcal{N}_ρ be a neighborhood of radius $\rho = r + w$ and with the same center as \mathcal{N}_r . Let

$$A = \{\forall v \in \mathcal{N}_\rho, u^+ \text{ would be happy at the location of } v \text{ at time } t = 0\},$$

$$C = \{\mathcal{N}_r \text{ contains an expandable radical region at time } t = 0\},$$

$$S_{\epsilon'} = \{\mathcal{N}_r \text{ contains a radical region of radius } (1 + \epsilon')w \text{ at time } t = 0\}.$$

We have

$$\begin{aligned} P(C) &\geq P(C \cap S_{\epsilon'} \cap A) \\ &= P\left(C \mid A, S_{\epsilon'}\right) P(S_{\epsilon'} \cap A). \end{aligned}$$

Using the FKG inequality and since $S_{\epsilon'}$ and A are increasing events, we have

$$P(C) \geq P\left(C \mid A, S_{\epsilon'}\right) P(S_{\epsilon'})P(A).$$

By Lemma 5 we have that $P(C|A, S_{\epsilon'})$ occurs w.h.p. By Lemmas 21 and 22 of the Appendix we have that

$$P(S_{\epsilon'}) \geq 2^{-\lceil 1-H(\tau') \rceil [2\epsilon' + \epsilon'^2]N - o(N)}.$$

Finally, $P(A)$ tends to one as $N \rightarrow \infty$ which leads to the desired result. \blacksquare

So far, we have identified a local configuration (radical region) that can lead to the formation of a small monochromatic neighborhood w.h.p. In the following section we show that this monochromatic neighborhood is in fact capable of making a large region monochromatic or almost monochromatic.

IV. THE SEGREGATION PROCESS

We now consider the dynamics of the segregation process and show that for all $\tau \in (\tau_1, 1/2)$ the expected size of the monochromatic region in steady state is exponential, while for all $\tau \in (\tau_2, \tau_1]$ the expected size of the almost monochromatic region is exponential.

A. Monochromatic region

We need the following definitions and preliminary results for proving the first part of Theorem 1. A *firewall* of radius r and center u is a set of agents of the same type contained in an annulus

$$A_r(u) = \left\{ y : r - \sqrt{2}w \leq \|u - y\| \leq r \right\},$$

where $\|\cdot\|$ denotes Euclidean distance and $r \geq 3w$. By Lemma 9, once formed a firewall of sufficiently large radius remains static, and since its width is $\sqrt{2}w$ the agents inside the inner circle are not going to be affected by the configurations outside the firewall.

We now call a neighborhood with radius $w/2$ a w -block. Consider the grid graph G_n . Let us renormalize this grid into w -blocks and denote the resulting graph by G'_n where each vertex of it is a w -block. Consider i.i.d. random variables $\{t(v) : v \in G'_n\}$, each attached to a vertex of G'_n . Let F denote the common distribution of these random variables and assume $F(0^-) = 0$, $\int_{[0,\infty)} xF(dx) < \infty$, and that F is not concentrated on one point. Consider a path η consisting of the vertices $v_1, \dots, v_k \in G'_n$ and define the passage time of this path

$$T^*(\eta) = \sum_{i=1}^k t(v_i).$$

We also define

$$T_k = \inf_{\eta \in (0 \leftrightarrow k\zeta_1)} \{T^*(\eta)\},$$

where ζ_1 is a coordinate vector and $(0 \leftrightarrow k\zeta_1)$ indicates the set of paths between the origin and $k\zeta_1$.

The following theorem, originally stated for bond percolation, also holds for site percolation and appears as Theorem 1 in [34].

Theorem 3 (Kesten). *Let $F(0) < p_c(\mathbb{Z}^d)$ where p_c is the critical probability for site percolation on \mathbb{Z}^d , and $\int e^{\gamma x} F(dx) < \infty$ for some $\gamma > 0$. Then, there exist $c_1, c_2, c_3, c_4 \in \mathbb{R}^+$ independent of k and such that*

$$P\left(|T_k - \mathbb{E}[T_k]| > x\sqrt{k}\right) < c_1 e^{-c_2 x},$$

for $x < c_3 k$ and $c_4 k^{-2} \leq \mathbb{E}[T_k]/k - \mu$ where $\mu = \lim_{k \rightarrow \infty} T_k/k$.

Using the above theorem, we obtain the following lower bound on the conditional probability that the spread of unhappy agents takes a sufficiently large amount of time.

Lemma 7. *Let \mathcal{N}_ρ be a neighborhood with radius $\rho > N^3$ and let u^+ denote an arbitrary (+1) agent. Let*

$$A = \left\{ \forall v \in \mathcal{N}_\rho, u^+ \text{ would be happy at the location of } v \text{ at time } t = 0 \right\}.$$

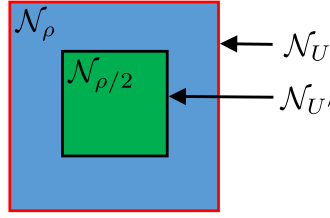


Fig. 7. Neighborhoods described in the proof of Lemma 7.

There exist constants $c, c', c'' \in \mathbb{R}^+$ independent of N , such that for all $N \geq 1$,

$$P\left(T(\rho/2) > c'' \frac{\rho}{N^{3/2}} \mid A\right) > 1 - c\rho^2 e^{-c'\rho^{1/3}},$$

where $T(\rho)$ is defined in (9).

Proof. We renormalize the grid into w -blocks starting with the block at the center of \mathcal{N}_ρ and construct G'_n as described above. Let \mathcal{N}_U be the set of all the w -blocks on the outside boundary of \mathcal{N}_ρ (these are the blocks that are connected to \mathcal{N}_ρ in G'_n). In order to find an upper bound for the speed of the spread of the unhappy agents, assume that all the (+1) agents in a w -block will become unhappy with a single flip in one of its eight l_∞ closest neighboring w -blocks. Also assume that all the agents in \mathcal{N}_U are unhappy of type (+1). Finally, denote the w -blocks on the outside boundary of $\mathcal{N}_{\rho/2}$ with $\mathcal{N}_{U'}$.

We show that the speed of the spread of unhappy blocks, i.e., w -blocks containing unhappy agents, is independent of the configuration of the agents outside the neighborhood $\mathcal{N}_\rho \cup \mathcal{N}_U$ and then use Theorem 3 to obtain the final result.

Consider G'_n in which each vertex is a w -block as described above. Here we attach i.i.d. random variables $\{t(v) : v \in G'_n\}$ to each vertex. Let these random variables have a common exponential distribution with mean $1/N$. Consider a path η consisting of the vertices v_1, \dots, v_k and the passage time $T^*(\eta) = \sum_{i=1}^k t(v_i)$. Let

$$T' = \inf_{\eta \in (\mathcal{N}_U \leftrightarrow \mathcal{N}_{U'})} T^*(\eta),$$

where $(\mathcal{N}_U \leftrightarrow \mathcal{N}_{U'})$ is the set of paths connecting \mathcal{N}_U to $\mathcal{N}_{U'}$. It is easy to see that $T' \leq T(\rho/2)$.

We now argue that regardless of the configuration of agents in the blocks of the graph G'_n containing $\mathcal{N}_\rho \cup \mathcal{N}_U$, the path with the smallest $T^*(\eta)$ consists only of w -blocks inside $\mathcal{N}_\rho \cup \mathcal{N}_U$. Assume that this is not the case, then a w -block is in $T^*(\eta)$ but it is not in $\mathcal{N}_\rho \cup \mathcal{N}_U$. There needs to be a path from this block to a block in $\mathcal{N}_{U'}$. This path has to cross the \mathcal{N}_U , and as a result there is another path from \mathcal{N}_U to $\mathcal{N}_{U'}$ that is at least as short as η . It follows that the shortest path from \mathcal{N}_U to $\mathcal{N}_{U'}$ only consists of blocks from \mathcal{N}_ρ .

Now we can assume that $\mathcal{N}_\rho \cup \mathcal{N}_U$ is in an infinite lattice of blocks \mathbb{L} , where i.i.d. random variables $\{t(v) : v \in \mathbb{L}\}$ are attached to its nodes. Let B_U and $B_{U'}$ be two blocks in \mathcal{N}_U and $\mathcal{N}_{U'}$ that have the minimum l_1 distance. We let

$$T'' = \inf_{\eta \in (B_U \leftrightarrow B_{U'})} T^*(\eta).$$

By Theorem 3 and since the neighborhood is divided into w -blocks so that k is proportional to ρ/\sqrt{N} , we conclude that there exist a constant $c'' \in \mathbb{R}^+$ such that for any pair of w -blocks in \mathcal{N}_U and $\mathcal{N}_{U'}$, there exist constants $c, c' \in \mathbb{R}^+$ such that for all $N \geq 1$

$$\begin{aligned} P\left(T'' \leq c'' \frac{\rho}{N^{3/2}} \mid A\right) &\leq P\left(T'' \leq \frac{\rho}{N^{1/2}} \frac{\mu}{N} - x \sqrt{\frac{\rho}{\sqrt{N}}} \mid A\right) \\ &\leq P\left(T'' \leq \mathbb{E}[T''] - x \sqrt{\frac{\rho}{\sqrt{N}}} \mid A\right) \\ &\leq ce^{-c'(\rho)^{1/3}}, \end{aligned}$$

where $x = \rho^{1/3}$ and we have used the fact that if for a first passage percolation process with exponential distribution with unit mean we have $\lim_{n \rightarrow \infty} T_n/n = \mu$, then for the passage times of our process, which is assumed to be exponential with mean $1/N$, we have $\lim_{n \rightarrow \infty} T_n/n = \mu/N$. Finally, by the union bound, the probability that any of the unhappy agents in \mathcal{N}_U affects an agent in $\mathcal{N}_{U'}$ before or at time $c''\rho/N^{3/2}$ is at most $c(4\rho)(8\rho)e^{-c'(\rho)^{1/3}}$. Hence, we have

$$\begin{aligned} P\left(T(\rho/2) > c'' \frac{\rho}{N^{3/2}} \mid A\right) &\geq P\left(T' > c'' \frac{\rho}{N^{3/2}} \mid A\right) \\ &> 1 - c(4\rho)(8\rho)e^{-c'(\rho)^{1/3}}, \end{aligned}$$

which tends to one as $N \rightarrow \infty$. ■

Call a *region of expansion* any neighborhood whose configuration is such that by placing a neighborhood $\mathcal{N}_{w/2}$ of type (+1) agents anywhere inside it, all the (-1) agents on the outside boundary of $\mathcal{N}_{w/2}$ become unhappy with probability one.

Lemma 8. *Let $\tau \in (\tau_1, 1/2)$ and let \mathcal{N}_{4r} be a neighborhood of radius $4r = 2^{\lceil 1-H(\tau') \rceil} N^{2-o(N)}$ such that $\rho > 8r$. Let*

$$D = \left\{ \forall t < T(\rho/2), \mathcal{N}_{4r} \text{ is a region of expansion} \right\},$$

then D occurs w.h.p.

Proof. Since D is increasing in a flip of a (-1) agent, we can focus on the case when the initial configuration is preserved. In this case, for the configuration to be expandable we need to make sure that any agent right outside the boundary of a monochromatic w -block will be unhappy. We obtain a lower bound for the probability of this event. With the same argument as in the proof of Lemma 19 of the Appendix, a lower bound for the probability that a given agent right outside the boundary of a monochromatic neighborhood $\mathcal{N}_{w/2}$ is unhappy, is

$$1 - 2^{-\lceil 1-H(\frac{4}{3}\tau) \rceil \frac{3}{4}N - o(N)}.$$

Let us denote the latter event for the (-1) agents right outside the boundary of $\mathcal{N}_{w/2}$ by A_1, \dots, A_L , where L is the number of (-1) agents right outside the boundary of $\mathcal{N}_{w/2}$. It is easy to see that these are all increasing events and using the FKG inequality we conclude that

$$P(A_1 \cap \dots \cap A_L) \geq P(A_1)P(A_2) \dots P(A_L) \geq \left(1 - 2^{-\lceil 1-H(\frac{4}{3}\tau) \rceil \frac{3}{4}N - o(N)}\right)^L.$$

Now, for any $v \in \mathcal{N}_{4r}$ let B_v be the event that all the (-1) agents outside $\mathcal{N}_{w/2}$ centered at v are unhappy. It is also easy to see that B_v 's are increasing events. Hence, with another application of the FKG inequality we have

$$P\left(\bigcap_{v \in \mathcal{N}_{4r}} B_v\right) \geq \left(1 - 2^{-\lceil 1-H(\frac{4}{3}\tau) \rceil \frac{3}{4}N - o(N)}\right)^{2^{\lceil 1-H(\tau) \rceil} N + o(N)},$$

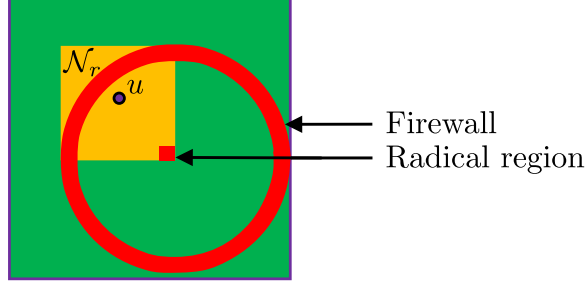


Fig. 8. Neighborhoods described in the proof of Lemma 10.

where we have used the fact that $L < N$. ■

Consider a disc of radius r , centered at an agent such that all the agents inside the disc are of the same type. It is easy to see that if r is sufficiently large then all the agents inside the disc will remain happy regardless of the configuration of the agents outside the disc. Lemma 6 in [25] shows that for $r > w^3$ this would be the case for sufficiently large w . Here we state a similar lemma but for an annulus, i.e., a firewall, without proof.

Lemma 9. *Let $A_r(u)$ be the set of agents contained in an annulus of outer radius $r \geq w^3$ and of width $\sqrt{2}w$ centered at u . For all $\tau \in (\tau_2, 1/2)$ and for a sufficiently large constant w , if $A_r(u)$ is monochromatic at time t , then it will remain monochromatic at all times $t' > t$.*

Lemma 10. *Let \mathcal{N}_ρ , $\mathcal{N}_{\rho/2}$, \mathcal{N}_{4r} , and \mathcal{N}_r be all centered at u with $\rho = 2^{[1-H(\tau')]N/2}$ and $r = 2^{[1-H(\tau')]N/2-o(N)}$, $r < \rho/8$. Let u^+ denote an arbitrary (+1) agent, $T(\rho)$ be as defined in (9), and κ be such that $\kappa r N^{1/2}$ is the sum of the number of agents in a firewall with radius $2r$ and the number of agents in a line of width $w + 1$ that connects the center to the boundary of the firewall and includes $\mathcal{N}_{w/2}$ at its center. Conditioned on the following events, w.h.p. the monochromatic region of u will have at least radius r .*

- 1) $A = \{\forall v \in \mathcal{N}_\rho, u^+ \text{ would be happy at the location of } v \text{ at } t = 0\}$,
- 2) $B = \{T(\rho/2) > 2\kappa r N^{1/2}\}$,
- 3) $C = \{\mathcal{N}_r \text{ contains an expandable radical region at } t = 0\}$,
- 4) $D = \{\forall t < T(\rho/2), \mathcal{N}_{4r} \text{ is a region of expansion}\}$.

Proof. Conditioned on events A, B, C , and D , an expandable radical region contained in \mathcal{N}_r can lead to the formation of a firewall of radius $2r$ centered at this region. Let $M(r)$ denote the event that the radius of the monochromatic region of u is at least r . Let T_f be the time at which this firewall forms, meaning that all the agents contained in the annulus become of the same type. We have

$$P\left(M(r) \mid A, B, C, D\right) \geq P\left(T_f < 2\kappa r \sqrt{N} \mid A, B, C, D\right)$$

Let T'_f be the sum of $\kappa r N^{1/2}$ exponential random variables with mean one. It is easy to see that T'_f is an upper bound for the time it takes until the firewall is formed, since the worst case scenario for the formation of the firewall is when the $\kappa r N^{1/2}$ agents flip to (+1), one by one. Hence, we have

$$P\left(T_f < 2\kappa r \sqrt{N} \mid A, B, C, D\right) \geq P\left(T'_f < 2\kappa r \sqrt{N}\right).$$

Next, we bound this probability. We have

$$P\left(T'_f \geq 2\kappa r \sqrt{N}\right) \leq P\left(|T'_f - \mathbb{E}[T'_f]| \geq \kappa r \sqrt{N}\right).$$

By Chebyshev's inequality, we have

$$P\left(T'_f \geq 2\kappa r\sqrt{N}\right) = O\left(\frac{\text{Var}(T'_f)}{(r\sqrt{N})^2}\right) = O\left(\frac{r\sqrt{N}}{(r\sqrt{N})^2}\right) = O\left(\frac{1}{r\sqrt{N}}\right).$$

It follows that w.h.p. agent u will be trapped inside a firewall together with an expandable radical region and the interior of the firewall will be a region of expansion until the end of the process. Hence this interior will eventually become monochromatic and, as a result, agent u will have a monochromatic region of size at least proportional to r^2 , as desired. ■

We can now give the proof for the first part of Theorem 1.

Proof of Theorem 1 (for $\tau_1 < \tau < 1/2$) First, we derive the lower bound in the theorem letting

$$a(\tau) = \left[1 - (2\epsilon' + \epsilon'^2)\right] [1 - H(\tau')], \quad (12)$$

where $\epsilon' > f(\tau)$, and $\tau' = (\tau N - 2)/(N - 1)$.

We consider neighborhoods \mathcal{N}_ρ , $\mathcal{N}_{\rho/2}$, and \mathcal{N}_r , with $\rho = 2^{[1-H(\tau')]N/2}$ and $r < \rho/8$, all centered at node u as depicted in Figure 9. We let u^+ be an arbitrary (+1) agent, and consider the following event in the initial configuration

$$A = \left\{ \forall v \in \mathcal{N}_\rho, u^+ \text{ would be happy at the location of } v \text{ at } t = 0 \right\}. \quad (13)$$

By Lemma 21 of the Appendix, we have

$$P(A) \rightarrow 1, \text{ as } N \rightarrow \infty. \quad (14)$$

We then consider a firewall of radius $2r$ centered anywhere inside \mathcal{N}_r , let $\kappa > 0$ so that $\kappa r N^{1/2}$ is the sum of the number of agents in it and the number of agents in a line of width $w + 1$ that connects its center to its boundary and includes $\mathcal{N}_{w/2}$ at its center. Consider the event

$$B = \left\{ T(\rho/2) > 2\kappa r N^{1/2} \right\},$$

where $T(\rho)$ is defined in (9). By Lemma 7, we can choose r proportional to $\rho/(N^2)$ so that

$$P(B|A) \rightarrow 1, \text{ as } N \rightarrow \infty. \quad (15)$$

With this choice, we also have

$$r = 2^{[1-H(\tau')]N/2 - o(N)},$$

and if we consider the event

$$C = \{ \mathcal{N}_r \text{ contains an expandable radical region at } t = 0 \},$$

by Lemma 6, we have for N sufficiently large

$$P(C) \geq 2^{-[1-H(\tau')](2\epsilon' + \epsilon'^2)N - o(N)}. \quad (16)$$

Consider a neighborhood \mathcal{N}_{4r} also centered at u and the event

$$D = \{ \forall t < T(\rho/2), \mathcal{N}_{4r} \text{ is a region of expansion} \}.$$

By Lemma 8, we have

$$P(D) \rightarrow 1, \text{ as } N \rightarrow \infty. \quad (17)$$

We now note that A, B, C, D are increasing events with respect to a partial ordering on their outcomes. More precisely, consider two outcomes of the sample space $\omega, \omega' \in \Omega$ such that $\omega, \omega' \in E$ where E is

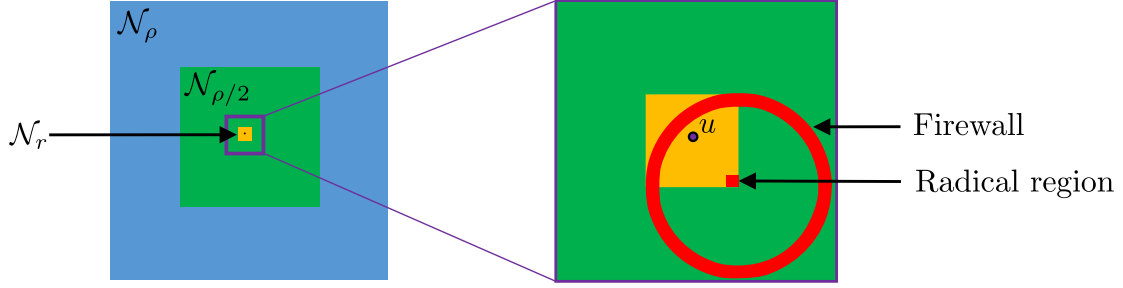


Fig. 9. Neighborhoods described in the proof of Theorem 1.

an event. We define a partial ordering on the outcomes such that $\omega' \geq \omega$ if for all time steps, the set of agents of type (+1) in ω is a subset of the set of agents of type (+1) in ω' . Event E is increasing if $1_E(\omega') \geq 1_E(\omega)$ where 1_E is the indicator function of the event E . According to this definition, A , B , C , D are increasing events. By combining (14), (15), (16), and (17), and using a version of the FKG inequality adapted to our dynamic process, stated in Lemma 23 of the Appendix, it follows that for N sufficiently large

$$\begin{aligned}
 P(A \cap B \cap C \cap D) &\geq P(A)P(B)P(C)P(D) \\
 &\geq P(A)P(B \cap A)P(C)P(D) \\
 &= P(B|A)[P(A)]^2 P(C)P(D) \\
 &= 2^{-[1-H(\tau')][2\epsilon' + \epsilon'^2]N - o(N)}.
 \end{aligned} \tag{18}$$

Since by Lemma 10 we have that conditioning on A, B, C , and D , at the end of the process w.h.p. agent u will be part of a monochromatic region with radius at least r , it follows that (18) is also a lower bound for the probability that the monochromatic neighborhood of agent u will have size of at least proportional to r^2 . The desired lower bound on the expected size of the monochromatic region now easily follows by multiplying (18) by the size of a neighborhood of radius r .

Next, we show the corresponding upper bound, letting

$$b(\tau) = \left\lceil \frac{3}{2}(1 + \epsilon')^2 \right\rceil [1 - H(\tau')],$$

and ϵ' and τ' as defined above. For any $\delta > 0$, consider a neighborhood $\mathcal{N}_{\rho'}$ such that

$$\rho' = 2^{(1+\epsilon')^2[1-H(\tau')]N/2 + \delta N/2},$$

and divide $\mathcal{N}_{\rho'}$ into blocks of size \mathcal{N}_ρ in the obvious way. Let M_{+1} and M_{-1} denote the events of $\mathcal{N}_{\rho'}$ being monochromatic of type (+1) and (-1) respectively. Also let E_{+1} and E_{-1} be the events of having a monochromatic region of type (+1) and (-1) inside a firewall of radius $2r$ centered anywhere inside $\mathcal{N}_{\rho'}$. We have that for N sufficiently large

$$\begin{aligned}
 P(M_{+1} \cup M_{-1}) &\leq P(M_{+1}) + P(M_{-1}) \\
 &= P(M_{+1} \cap E_{-1}^C) + P(M_{-1} \cap E_{+1}^C) \\
 &\leq P(E_{-1}^C) + P(E_{+1}^C) \\
 &= 2P(E_{-1}^C) \\
 &\leq 2(1 - 2^{-[1-H(\tau')](2\epsilon' + \epsilon'^2)N - o(N)})\rho'^2/\rho^2 \\
 &= e^{-2^{\delta N - o(N)}}.
 \end{aligned} \tag{19}$$

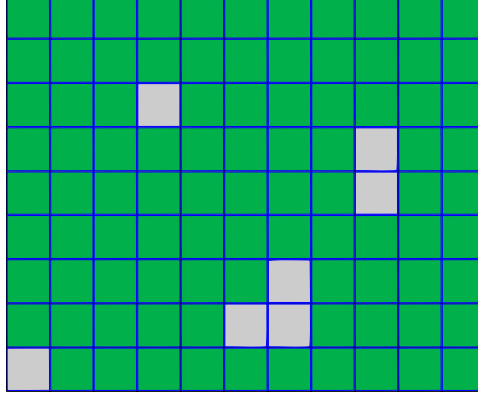


Fig. 10. Part of the grid renormalized into m -blocks. Green and gray indicate good and bad blocks respectively.

By considering the set of all the neighborhoods of radius ρ' sharing agent u , by the union bound the probability that at least one of them will be monochromatic of only one type is also bounded by (19). We now consider the expected size of the monochromatic region of agent u , that is bounded as

$$\mathbb{E}[M] \leq \sum_{m=1}^n m^2 p_m,$$

where p_m denotes the probability of having a monochromatic region of size m^2 containing u . We let

$$\rho'' = 2^{\lceil (1+\epsilon')^2(1-H(\tau'))N/2 + o(N) \rceil},$$

and divide the series into two parts

$$\begin{aligned} \mathbb{E}[M] &\leq \sum_{m=1}^{\rho''} m^2 p_m + \sum_{m=\rho''+1}^n m^2 p_m \\ &\leq 2^{\lceil \frac{3}{2}(1+\epsilon')^2(1-H(\tau'))N + o(N) \rceil} + \sum_{m=\rho''+1}^n m^2 p_m, \end{aligned} \quad (20)$$

where the first inequality follows from $p_m \leq 1$. Since by (19) for all $m \geq \rho'$, the probability of having a monochromatic region of size m^2 containing u has at most a double exponentially small probability, the tail of the remaining series in (20) converges to a constant, while for sufficiently large N the sum of the first $\rho' - \rho'' - 1$ terms is smaller than the first term of (20), and the proof is complete.

B. Almost monochromatic region

We now turn our attention to the case where $\tau \in (\tau_2, \tau_1]$. We define an m -block to be a neighborhood of radius $m/2$. Let \mathcal{I} be the collection of sets of agents in the possible intersections of a w -block with an m -block on the grid in the initial configuration. Also, let W_I be the random variable representing the number of (-1)'s in $I \in \mathcal{I}$, and N_I be the total number of agents in $I \in \mathcal{I}$.

Good block. For any $\epsilon \in (0, 1/2)$, a *good m -block* is an m -block such that for all $I \in \mathcal{I}$ we have $W_I - N_I/2 < N^{1/2+\epsilon}$. The m -blocks that do not satisfy this property are called *bad m -blocks* (see Fig. 10). It is easy to see that all the blocks contained in a good m -block are also good blocks.

For the following two definitions, we assume that the grid is renormalized into m -blocks. In this setting each m -block is horizontally or vertically adjacent to four other m -blocks.

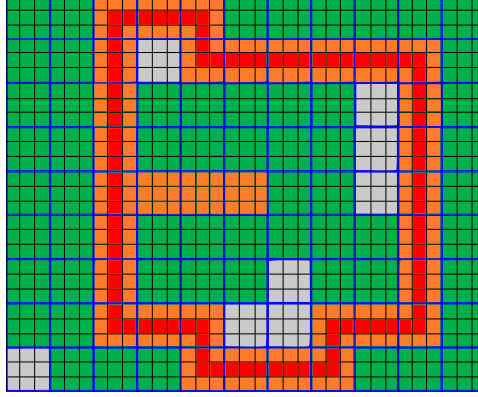


Fig. 11. Larger blocks are $6w^3$ -blocks and smaller ones are $2w^3$ -blocks. The red cycle indicates the chemical firewall which is in the cycle of an r -chemical path (orange).

m-path. An m -path is an ordered set of m -blocks such that each pair of consecutive m -blocks are either horizontally or vertically adjacent and no m -block appears more than once in the set. The *length* of the path is the number of m -blocks in the path. Two m -blocks are *connected* if there exists an m -path between them.

m-cycle. An m -cycle is a closed path in which the last m -block in its ordered set is adjacent to the first m -block. An m -cycle divides the m -blocks of the grid into two sets of m -blocks referred to as its *interior* and its *exterior*.

r-chemical path. Renormalize the grid into $6w^3$ -blocks starting from the block centered at agent u . To define an r -chemical path, consider two neighborhoods \mathcal{N}_{3r} and \mathcal{N}_r with radii $3r$ and r respectively and both centered at an agent u .

Let $r > 12w^3$. An r -chemical path centered at u , is the union of a $6w^3$ -cycle of good $6w^3$ -blocks contained in $\mathcal{N}_{3r} \setminus \mathcal{N}_r$ such that u is in its interior, and a path of good $6w^3$ -blocks from the $6w^3$ -block at the center of \mathcal{N}_r to a $6w^3$ -block in the $6w^3$ -cycle, such that the total length of the $6w^3$ -cycle and the $6w^3$ -path is proportional to $r/(6w^3)$ (see Fig. 11).

Chemical firewall. Renormalize the grid into $2w^3$ -blocks starting from the block centered at agent u and consider the r -chemical path defined above in this setting. A *chemical firewall* with radius r is a $2w^3$ -cycle contained in the cycle of the r -chemical path such that agent u is in its interior and all the agents in the $2w^3$ -cycle are of the same type (see Fig. 11).

Although the structure of a chemical firewall is very different from the annular firewall defined before, the size of the m -blocks are chosen such that it is easy to see that, with similar arguments given for Lemma 9, it acts as a firewall, i.e., the flips of the agents in its exterior cannot affect the agents in its interior.

An *r-expandable radical region* of type (-1) is a radical region such that it is expandable and it is located at the center of an r -chemical path.

Before proceeding with the first part of the proof of Theorem 2, we need the following results. The following lemma gives a lower bound for the probability that an arbitrary m -block with $m \leq N^3$ is a good m -block. Using this lemma, by renormalizing the grid into m -blocks we will argue that the probability that a block is a bad block can be arbitrary small for sufficiently large N .

Lemma 11. *Let $\epsilon \in (0, 1/2)$ and $m \leq N^3$. For all $I \in \mathcal{I}$ we have $W_I - N_I/2 < N^{1/2+\epsilon}$ with probability at least*

$$1 - e^{-cN^{2\epsilon} + o(N^{2\epsilon})}.$$

Proof. By Lemma 18 of the Appendix, for an arbitrary $I \in \mathcal{I}$ we have

$$P\left(W_I - N_I/2 \geq N^{1/2+\epsilon}\right) < e^{-cN^{2\epsilon}},$$

where $\epsilon \in (0, 1/2)$ and $c > 0$. Since there are less than N^3 elements in \mathcal{I} , we have

$$P\left(W_I - N_I/2 < N^{1/2+\epsilon} \text{ for all } I \in \mathcal{I}\right) \geq 1 - N^3 e^{-cN^{2\epsilon}}.$$

■

Let us consider a neighborhood consisting of exponentially large number of m -blocks where $m \leq N^3$. Based on the following lemma, the ratio between bad blocks and good blocks in this neighborhood is exponentially small w.h.p.

Lemma 12. *Let c be a positive constant and $\epsilon \in (0, 1/2)$. Let \mathcal{N}_ρ be a neighborhood consisting of m -blocks and with 2^{cN} agents. The ratio between bad blocks and good blocks is less than e^{-N^ϵ} w.h.p.*

Proof. By Lemma 11, the probability of having a bad block is less than $e^{-N^{2\epsilon} + o(N^{2\epsilon})}$. It is easy to show that the number of bad blocks is less than $2^{cN} e^{-N^{2\epsilon} + o(N^{2\epsilon})}$ w.h.p. Hence, the ratio between the number of bad blocks and the number of good blocks is less than e^{-N^ϵ} w.h.p., see Figure 10. ■

We now want to argue that the formation of a chemical firewall is likely. We first notice that a monochromatic w -block located inside a good $6w^3$ -block can make at least a $2w^3$ -block at the center of the good block monochromatic. This means that a monochromatic w -block at the center of the r -chemical path can create a chemical firewall (see Fig. 11). Our next goal is to show that the existence of an r -chemical path is likely. The critical step is to show that the length of the r -chemical path is proportional to $r/6w^3$.

We use a result from percolation theory [37] restated in the following. Consider site percolation on square lattice in the supercritical regime. Let $D(0, x) = \inf_\Gamma |\Gamma|$, where Γ is a path from the origin to the vertex x and $|\Gamma|$ is the number of vertices in the path. Let $0 \leftrightarrow x$ denote that 0 and x belong to the same connected component. The following is Theorem 1.4 from [37], and it asserts that the length of the shortest path between the origin and an arbitrary vertex x cannot be much different from its l_1 distance $\|x\|_1$, see Figure 12.

Theorem 4 (Garet and Marchand). *For all $\alpha > 0$, there exists $p'(\alpha) \in (p_c(d), 1)$ such that for all $p \in (p'(\alpha), 1]$, we have:*

$$\limsup_{\|x\|_1 \rightarrow +\infty} \frac{\ln P_p(0 \leftrightarrow x, D(0, x) \geq (1 + \alpha)\|x\|_1)}{\|x\|_1} < 0.$$

Now consider a two dimensional lattice which consists of good $6w^3$ -blocks and bad $6w^3$ -blocks. The probability of a site being good then, is at least the value computed in Lemma 11, hence for sufficiently large N we are dealing with a percolation problem in the super-critical regime. Let us denote a radical region with radius ϵ' by ϵ' -radical region.

Lemma 13. *W.h.p. an ϵ' -radical region is at the center of an r -chemical path at time $t = 0$ where $r < n/10$.*

Proof. Since an r -chemical path is contained in a neighborhood of radius $3r$, without loss of generality we can assume that this neighborhood is contained in a \mathbb{Z}^2 lattice. It is also clear that the flip of a (-1) agent,

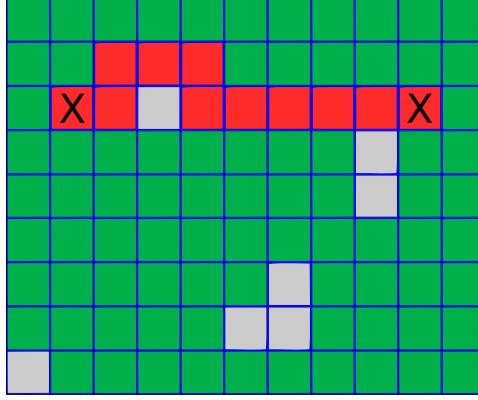


Fig. 12. The length of the shortest path of good blocks between two arbitrary vertices denoted by X is w.h.p. not much different from its l_1 -distance between them in the supercritical regime.

can only increase the probability of formation of the r -chemical path. Divide the resulting lattice into m -blocks such that the ϵ' -radical region is at the center of an m -block and call the resulting renormalized lattice \mathbb{L}' . Consider performing site percolation on this lattice by considering good $6w^3$ -blocks as open sites of \mathbb{L}' and bad $6w^3$ -blocks as its closed sites. As discussed above, for sufficiently large N we are dealing with a percolation problem in its super-critical regime. Consider two blocks containing agents $(2r, 2r)$ and $(-2r, 2r)$ in the original lattice denoted by 0 and x respectively. By Theorem 4 we conclude that for sufficiently large N there exists a constant $c > 0$ such that

$$P_p(0 \leftrightarrow x, D(0, x) \geq (1.25)\|x\|_1) \leq e^{-c\|x\|_1}$$

where $\|x\|_1$ is the l_1 distance of x from 0 and we have put $\alpha = 0.25$. By the union bound and the FKG inequality, we have

$$\begin{aligned} P_p(D(0, x) < 1.25\|x\|_1) &\geq P(0 \leftrightarrow x) - P(0 \leftrightarrow x, D(0, x) \geq (1.25)\|x\|_1) \\ &\geq \theta(p)^2 - e^{-c\|x\|_1}, \end{aligned}$$

where $\theta(p)$ is the probability that a node belongs to an infinite cluster and we have used the FKG inequality to conclude that $P(0 \leftrightarrow x) \geq \theta(p)^2$. Now, using Lemma 11 it is easy to see that for sufficiently large values of N this lower bound is as close as we want to one.

For each pair of corner agents of \mathcal{N}_{2r} on the same side the above argument holds. A similar argument also holds for the existence of a path from the center of \mathcal{N}_r to an arbitrary block on the boundary of \mathcal{N}_{3r} , i.e., a $6w^3$ -block which contains agents with l_∞ -distance of $3r$ from the center of \mathcal{N}_{3r} . It is also easy to see that these events are all increasing events, i.e., their indicator functions can only increase by changing a closed site to an open site, in this case, a bad $6w^3$ -block to a good $6w^3$ -block. Hence, by the FKG inequality, the joint probability of the existence of the above paths is at least their product which can be made arbitrary close to one for large values of N . ■

We need to show that w.h.p. the radical region located inside the firewall can make the interior of the firewall almost monochromatic by the end of the process. We show that there are no clusters of bad blocks of radius larger than a polynomial function of N in a neighborhood with exponential size in N . To show this we first restate a result from [33]. Let $S(k)$ be the ball of radius k with center at the origin, i.e., $S(k)$ is the set of all vertices x in \mathbb{Z}^2 for which $\Delta(0, x) \leq k$, where Δ denotes the l_1 distance. Let $\partial S(k)$ denote the surface of $S(k)$, i.e., the set of all x such that $\Delta(0, x) = k$. Let A_k be the event that

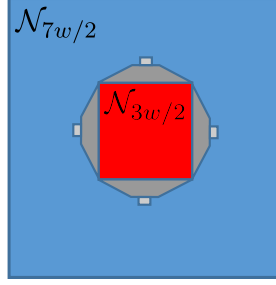


Fig. 13. Neighborhoods described in the proof of Lemma 16.

there exists an open path joining the origin to some vertex in $\partial S(k)$. Let the *radius* of a bad cluster be defined as

$$\sup\{\Delta(0, x) : x \in \text{bad cluster}\}.$$

The following result is Theorem 5.4 in [33].

Theorem 5 (Grimmett). (*Exponential tail decay of the radius of an open cluster.*) If $p < p_c$, there exists $\psi(p) > 0$ such that

$$P_p(A_k) < e^{-k\psi(p)}, \quad \text{for all } k.$$

Lemma 14. *W.h.p. there are no clusters of bad $6w^3$ -blocks with radius greater than N^2 blocks in a neighborhood with radius $4r = 2^{[1-H(\tau')]N/2-o(N)}$ at time $t = 0$.*

Proof. Let p , be the probability of having a bad $6w^3$ -block, and let $k = N^2$. By Theorem 5 it follows that w.h.p. there is no cluster of bad $6w^3$ -blocks containing a bad $6w^3$ -block with l_1 -distance from its center greater than N^2 $6w^3$ -blocks in a neighborhood with exponential radius in N . ■

It is easy to check that for $\tau > 3/8$, a monochromatic w -block in a good block can make the whole block monochromatic (except for possibly a margin of w at the borders). On the other hand, Lemma 15 shows that the same condition of Lemma 5 leads to the formation of a monochromatic $3w/2$ -block for $\tau \in (\tau_1, 3/8)$ because once the ϵ' -radical region leads to a monochromatic w -block at its center, it can as well lead w.h.p to a monochromatic $3w/2$ -block. Lemma 16 then shows that the spread of the monochromatic $3w/2$ -blocks is indeed possible.

Lemma 15. *Consider the \mathcal{N}_S neighborhood defined in Lemma 5 and co-centered with a neighborhood \mathcal{N}_ρ of radius $\rho > N$ with the property that no $(+1)$ agent inside \mathcal{N}_ρ will become unhappy until some time $T(\rho)$. Then w.h.p. there exists a set of flips with the following property: if they happen before $T(\rho)$ then all the agents inside a neighborhood with radius $3w/2$ concentric with \mathcal{N}_ρ will be of the same type.*

Proof. By Lemma 5, w.h.p. there exists a set of flips that if they happen before $T(\rho)$ will make a w -block at the center of \mathcal{N}_ρ unhappy. By Proposition 1, it follows that this monochromatic block will make all the (-1) agents in four identical trapezoids outside the w -block whose larger bases are the sides of the w -block unhappy, and hence monochromatic w.h.p. Now, with another application of Proposition 1 we have that for $\tau > \tau_1$, all the (-1) agents in a $3w/2$ -block with the same center as the w -block will be unhappy, hence the $3w/2$ -block can become monochromatic w.h.p. ■

Lemma 16. *Consider a good block at the center of \mathcal{N}_ρ with $\rho > m$. A $3w/2$ -block with $(+1)$ agents at the center of a $7w/2$ -block contained in the good block will make all (-1) agents right outside the*

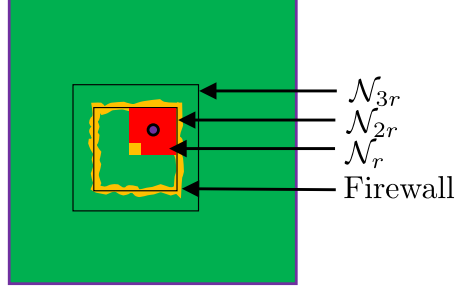


Fig. 14. Neighborhoods described in the proof of Lemma 17. Agent u is depicted by the circle in the red square and the ϵ' -radical region is depicted by the small orange square in the red square.

$3w/2$ -block unhappy with probability one and with at most $(3w/4 + 1)^2$ flips happening before $T(\rho)$, for sufficiently large N .

Proof. Consider four identical isosceles trapezoids outside the $3w/2$ -block whose larger bases are the sides of the $3w/2$ -block (see Figure 13). Let $\zeta = (3 - 8\tau)/2$ and $\nu = (16\tau - 5)/6$. Let the smaller bases of the above trapezoids be $2(3/4 - 2\zeta)w$ and their heights be $2\nu w$. For $\tau > 0.3463$, since these trapezoids are located inside a good block for sufficiently large N all the agents of type (-1) in these trapezoids will be unhappy with probability one. Consider the case where these trapezoids have become monochromatic after the flips of (-1) agents happening before $T(\rho)$. Now consider four identical rectangles located outside the trapezoids. Let one side of each of these rectangles be at the center of one of the smaller bases of each of the four trapezoids and of length $2(1/8 - \nu)w$ and let the other sides of the triangles be $w/4$. For $\tau > \tau_1$, all the agents of type (-1) located inside these rectangles will be unhappy. Now, as a worst case scenario, let us consider an agent outside the $3w/2$ -block and next to its corner which shares the smallest number of agents with the monochromatic regions. When the unhappy agents in the rectangles flip before $T(\rho)$, for this agent to be unhappy we need to have

$$\left[1 - \frac{1}{4} - \left(\frac{1}{4} + \frac{1}{2} - \zeta \right) \nu - \frac{1}{4} \left(\frac{1}{8} - \nu \right) \right] \frac{1}{2} + \frac{o(N)}{N} < \tau,$$

which can be simplified to (3). This means that for $\tau < \tau_1$ and for sufficiently large N this agent will be unhappy with probability one. Since all the other agents of type (-1) right outside the $3w/2$ -block share at least the same number of agents with the single-type regions, we have that for sufficiently large N , all the (-1) agents right outside the $3w/2$ -block will be unhappy with probability one. ■

The following lemma, which can be thought of as the counterpart of Lemma 10 for $\tau \in (\tau_2, \tau_1]$, shows that conditional on some events, the size of the almost monochromatic region of an arbitrary agent is exponential in N . Unless otherwise stated, by a good block we mean a good $6w^3$ -block and by a bad block we mean a bad $6w^3$ -block.

Lemma 17. Let \mathcal{N}_ρ , $\mathcal{N}_{\rho/2}$, \mathcal{N}_{4r} , and \mathcal{N}_r be all centered at u with

$$\begin{aligned} \rho &= 2^{\lceil 1-H(\tau') \rceil N/2}, \\ r &= 2^{\lceil 1-H(\tau') \rceil N/2 - o(N)}, \end{aligned}$$

and $r < \rho/8$. Let u^+ denote an arbitrary (+1) agent, $T(\rho)$ be as defined in (9), and $\kappa > 0$ be such that $\kappa r N^{3/2}$ is the total number of agents in a $2r$ -chemical path. Conditioned on the following events, w.h.p. the almost monochromatic region of u will have at least radius r .

- 1) $A = \{ \forall v \in \mathcal{N}_\rho, u^+ \text{ would be happy at the location of } v \text{ at } t = 0 \},$

- 2) $B = \{T(\rho/2) > 2\kappa r N^{3/2}\},$
- 3) $C = \{\mathcal{N}_r \text{ contains a } 2r\text{-expandable radical region at } t = 0\},$
- 4) $D = \{\nexists \text{ cluster of bad blocks with } l_1\text{-radius } r' > N^2 \text{ in } \mathcal{N}_{4r} \text{ at } t = 0\},$
- 5) $E = \{N_B/N_G < e^{-N^\epsilon} \text{ in } \mathcal{N}_r \text{ at } t = 0\}, \text{ where } N_B \text{ is the number of bad blocks and } N_G \text{ is the number of good blocks in } \mathcal{N}_r.$

Proof. Conditional on events $A, B,$ and $C,$ w.h.p. a $2r$ -expandable radical region will lead to the formation of a firewall that contains $\mathcal{N}_r.$ With additional conditioning on events D and E once the firewall is formed, the expandable radical region will turn all the interior of at least \mathcal{N}_r almost monochromatic by the end of the process. Let $M(r)$ denote the event that the radius of the almost monochromatic region of u is at least $r.$ Let T_f be the time at which the firewall forms, i.e., its agents become monochromatic. We have

$$P\left(M(r) \mid A, B, C, D, E\right) \geq P\left(T_f < 2\kappa r \sqrt{N} \mid A, B, C, D, E\right).$$

Let T'_f be the sum of $\kappa r N^{3/2}$ exponential random variables with mean one, where $\kappa r N^{3/2}$ is the total number of agents in the $2r$ -chemical path. It is easy to see that T'_f is an upper bound for the time it takes until the firewall is formed, i.e., all agents inside the firewall flip to (+1), one by one. Hence, we have

$$P\left(M(r) \mid A, B, C, D, E\right) \geq P\left(T'_f < 2\kappa r \sqrt{N}\right).$$

Next we bound this probability. We have

$$P\left(T'_f \geq 2\kappa r N^{3/2}\right) \leq P\left(|T'_f - \mathbb{E}[T'_f]| \geq \kappa r N^{3/2}\right).$$

By Chebyshev's inequality we have

$$P\left(T'_f \geq 2\kappa r N^{3/2}\right) = O\left(\frac{\text{Var} T'_f}{(r N^{3/2})^2}\right) = O\left(\frac{r \sqrt{N}}{(r N^{3/2})^2}\right) = O\left(\frac{1}{r N^{3/2}}\right),$$

leading to the desired result. ■

With the above definitions and results, we can proceed to the first part of the proof of Theorem 2 (for $\tau_2 < \tau \leq \tau_1$).

Proof of Theorem 2 (for $\tau_2 < \tau \leq \tau_1$): First, we derive the lower bound in the theorem letting

$$a(\tau) = \left[1 - (2\epsilon' + \epsilon'^2)\right] [1 - H(\tau')], \quad (21)$$

where $\epsilon' > f(\tau),$ and $\tau' = (\tau N - 2)/(N - 1).$

We consider neighborhoods $\mathcal{N}_\rho, \mathcal{N}_{\rho/2},$ and $\mathcal{N}_r,$ with $\rho = 2^{\lceil 1-H(\tau') \rceil N/2}$ and $r < \rho/8,$ all centered at node u as depicted in Figure 15. We let $\rho = 2^{\lceil 1-H(\tau') \rceil N/2},$ and u^+ be an arbitrary (+1) agent and consider the following event in the initial configuration

$$A = \{\forall v \in \mathcal{N}_\rho, u^+ \text{ would be happy at the location of } v \text{ at } t = 0\}.$$

By Lemma 21 of the Appendix, we have

$$P(A) \rightarrow 1, \text{ as } N \rightarrow \infty. \quad (22)$$

We then consider a chemical firewall of radius $2r$ centered anywhere inside $\mathcal{N}_r,$ let $\kappa > 0$ so that $\kappa r N^{3/2}$ is an upper bound on the total number of agents in the $2r$ -chemical path containing it, and consider the event

$$B = \{T(\rho/2) > 2\kappa r N^{3/2}\},$$

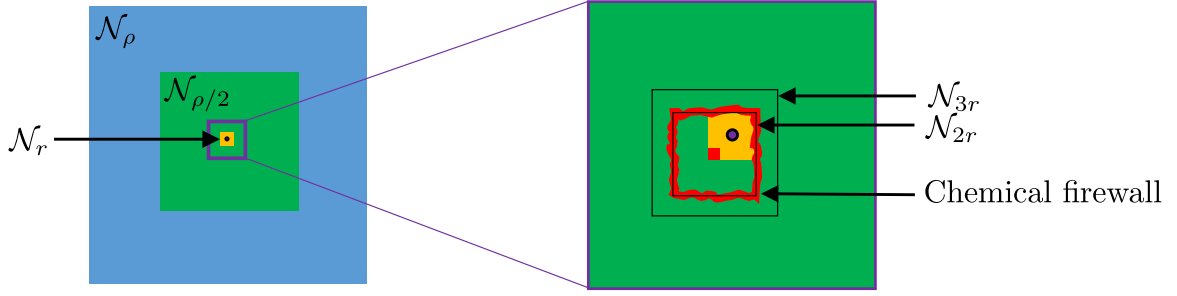


Fig. 15. Neighborhoods described in the proof of Theorem 2.

where $T(\rho)$ is defined in (9). By Lemma 7, we can choose r proportional to $\rho/(N^3)$ so that

$$P(B|A) \rightarrow 1, \text{ as } N \rightarrow \infty. \quad (23)$$

With this choice, we also have

$$r = 2^{[1-H(\tau')]N/2-o(N)},$$

and if we consider the event

$$C = \{\mathcal{N}_r \text{ contains a } 2r\text{-expandable radical region at } t = 0\},$$

by Lemma 6 and Lemma 13 and the FKG inequality, since $\epsilon' > f(\tau)$ we conclude that for sufficiently large N

$$P(C) \geq 2^{-[1-H(\tau')][2\epsilon' + (\epsilon')^2]N - o(N)}, \quad (24)$$

and there is a $2r$ -expandable radical region surrounding u . Let us divide the grid into m -blocks in the obvious way. Let the *radius* of a bad cluster be defined as

$$\sup\{\Delta(0, x) : x \in \text{bad cluster}\}.$$

where Δ denotes the l_1 distance. Let

$$D = \{\text{cluster of bad blocks with } l_1\text{-radius } r' > N^2 \text{ blocks in } \mathcal{N}_{4r} \text{ at } t = 0\}.$$

By Lemma 14, we have

$$P(D) \rightarrow 1, \text{ as } N \rightarrow \infty. \quad (25)$$

Finally, let $\epsilon \in (0, 1/2)$ and let N_B and N_G denote the total number of bad blocks sharing at least one agent with \mathcal{N}_r and good blocks contained in \mathcal{N}_r respectively and let

$$E = \{N_B/N_G < e^{-N^\epsilon} \text{ in } \mathcal{N}_r \text{ at } t = 0\}.$$

By an application of Lemma 12, also

$$P(E) \rightarrow 1, \text{ as } N \rightarrow \infty. \quad (26)$$

See Figure 15 for a visualization of the neighborhoods defined above.

Now it is easy to see that the events A , B , C , D , and E are increasing. By combining (22), (23), (24), (25), and (26), and using a version of the Fortuin-Kasteleyn-Ginibre (FKG) inequality adapted to our dynamic process described in Lemma 23 of the Appendix, it follows that for N sufficiently large

$$P(A \cap B \cap C \cap D \cap E) \geq P(A)P(B)P(C)P(D)P(E) \quad (27)$$

$$\begin{aligned} &\geq P(A)P(A \cap B)P(C)P(D)P(E) \\ &= P(B|A)[P(A)]^2P(C)P(D)P(E) \\ &= 2^{-[1-H(\tau')](2\epsilon' + (\epsilon')^2)N - o(N)}. \end{aligned} \quad (28)$$

Since by Lemma 17 we have that conditional on A, B, C, D, E , at the end of the process w.h.p. agent u will be part of an almost monochromatic region with radius at least r , it follows that (28) is also a lower bound for the probability that the monochromatic neighborhood of agent u will have size of at least proportional to r^2 . The desired lower bound on the expected size of the monochromatic region now easily follows by multiplying (28) by the size of a neighborhood of radius r . The second part of the proof follows the same argument as the second part of the proof of Theorem 1.

C. Extension to the interval $1/2 < \tau < 1 - \tau_2$

We call *super-unhappy agents* the unhappy agents that can potentially become happy once they flip their type. While for $\tau < 1/2$ unhappy agents can always become happy by flipping their type, for $\tau > 1/2$ this is only true for the super-unhappy agents. It follows that for $\tau > 1/2$ super-unhappy agents act in the same way as unhappy agents do for $\tau < 1/2$.

We let $\bar{\tau} = 1 - \tau + 2/N$. A *super-unhappy agent* of type (-1) is an agent for which $W < \bar{\tau}N$ where W is the number of (-1) agents in its neighborhood. The reason for adding the term $2/N$ in the definition is to account for the strict inequality that is needed for being unhappy and the flip of the agent at the center of the neighborhood which adds one agent of its type to the neighborhood. A *super-radical region* is a neighborhood \mathcal{N}_S of radius $S = (1 + \epsilon')w$ such that $W_S < \bar{\tau}'(1 + \epsilon')^2N$, where $\epsilon \in (0, 1/2)$ and

$$\bar{\tau}' = \left(1 - \frac{1}{\bar{\tau}N^{1/2-\epsilon}}\right) \bar{\tau}.$$

By replacing τ with $\bar{\tau}$, “unhappy agent” with “super-unhappy agent” and “radical region” with “super-radical region,” it can be checked that all proofs extend to the interval $1/2 < \tau < 1 - \tau_2$.

V. CONCLUDING REMARKS

The main lesson learned from our study is that even a small amount of intolerance can lead to segregation at the large scale. We remark, however, that the model is somewhat naturally biased towards segregation because agents can flip their type when a sufficiently large number of their neighbors are different from themselves, but they never flip when a large number of their neighbors are of their same type. Variations where agents could potentially flip in both situations, namely they are “uncomfortable” being both a minority or a majority in a largely segregated area, would be of interest. Another direction of further study could be the investigation of how the parameter of the initial distribution of the agents influences segregation, since it is only known that complete segregation occurs w.h.p. for $\tau = 1/2$ and $p \in (1 - \epsilon, 1)$, while we have shown that for $0.344 < \tau < 1/2$ and $p = 1/2$ the size of the monochromatic region is at most exponential in the size of its neighborhood, w.h.p. We also point out that for $\tau = 1/2$ and for $\tau \in [1/4, \tau_2] \cup [1 - \tau_2, 3/4]$ the behavior of the model is unknown. Finally, our results only show lower bounds on the expected size of the monochromatic region containing a given agent, but they do not show that in the steady state every agent ends up in an exponentially large monochromatic region with high probability. A possibility that is consistent with these results (but inconsistent with the simulation results) is that only an exponentially small fraction of the nodes are contained in large monochromatic regions at the end of the process, but that those regions are so large that the expected radius of the monochromatic region containing any node is exponentially large. Proving an exponential lower bound on the size of the monochromatic region w.h.p., rather than in expectation, would rule out this possibility.

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VI. APPENDIX

A. Concentration bound on the number of agents in the initial configuration

Lemma 18. *Let $\epsilon \in (0, 1/2)$, and let \mathcal{N} be an arbitrary neighborhood in the grid with N agents. There exist $c, c' \in \mathbb{R}^+$, such that*

$$P\left(|W - N/2| < cN^{1/2+\epsilon}\right) \geq 1 - 2e^{-c'N^{2\epsilon}}. \quad (29)$$

Proof. Let W_i be the random variable associated with the type of the i 'th agent in \mathcal{N} such that it is one whenever the type is (-1) and zero otherwise. Let $\mathcal{F}_i = \sigma(W_1, \dots, W_i)$. Then it is easy to see that $M_n = \mathbb{E}[W|\mathcal{F}_n]$ for $n = 1, \dots, N$ is a martingale. It is also easy to see that $M_0 = \mathbb{E}[W] = N/2$, and $M_N = W$. We also have

$$\begin{aligned} |M_n - M_{n-1}| &= \left| \mathbb{E}\left(\sum_{i=1}^N W_i | \mathcal{F}_n\right) - \mathbb{E}\left(\sum_{i=1}^N W_i | \mathcal{F}_{n-1}\right) \right| \\ &= \left| W_n + (N-n)/2 - [N - (n-1)]/2 \right| \\ &\leq \left| W_n - \frac{1}{2} \right| \leq 1/2, \end{aligned}$$

for $n = 1, 2, \dots, N$. Now using Azuma's inequality, there exist constants $c_1, c_2 \in \mathbb{R}^+$ such that

$$P\left(W - N/2 \geq cN^{1/2+\epsilon}\right) \leq e^{-c_1N^{2\epsilon}},$$

and

$$P\left(W - N/2 \leq -c'N^{1/2+\epsilon}\right) \leq e^{-c_2N^{2\epsilon}}.$$

It follows by an application of Boole's inequality that there exists a constant $c \in \mathbb{R}^+$ such that (29) holds. ■

B. Preliminary results for the proof of Theorem 1

First, we give a bound on the probability of having an unhappy agent in the initial configuration, we then extend this bound for a radical region.

Lemma 19. *Let p_u be the probability of being unhappy for an arbitrary agent in the initial configuration. There exist positive constants c_l and c_u which depend only on τ such that*

$$c_l \frac{2^{-[1-H(\tau')]N}}{\sqrt{N}} \leq p_u \leq c_u \frac{2^{-[1-H(\tau')]N}}{\sqrt{N}}.$$

where $\tau' = \frac{\tau N - 2}{N - 1}$, and H is the binary entropy function.

Proof. We have

$$p_u = \frac{1}{2^N} \sum_{k=0}^{\tau N - 2} \binom{N-1}{k} + \frac{1}{2^N} \sum_{k=0}^{\tau N - 2} \binom{N-1}{k}, \quad (30)$$

where the two unit reduction is to account for the strict inequality and the agent at the center of the neighborhood. Let $\tau' = \frac{\tau N - 2}{N - 1}$. After some algebra we have

$$\binom{N-1}{\tau'(N-1)} \leq \sum_{k=0}^{\tau'(N-1)} \binom{N-1}{k} \leq \frac{1 - \tau'}{1 - 2\tau'} \binom{N-1}{\tau'(N-1)},$$

and using Stirling's formula, there exist constants $c, c' \in \mathbb{R}^+$ such that

$$c \frac{2^{-[1-H(\tau')](N-1)}}{\sqrt{(N-1)\tau'(1-\tau')}} \leq \binom{N-1}{\tau'(N-1)} \leq c' \frac{2^{-[1-H(\tau')](N-1)}}{\sqrt{(N-1)\tau'(1-\tau')}}.$$

The result follows by combining the above inequalities. \blacksquare

Lemma 20. *There exist positive constants c_l and c_u which depend only on τ such that in the initial configuration, an arbitrary neighborhood with radius $(1 + \epsilon')w$ is a radical region with probability $p_{\epsilon'}$ where we have*

$$c_l 2^{-[1-H(\tau'')](1+\epsilon')^2 N - o(N)} \leq p_{\epsilon'} \leq c_u 2^{-[1-H(\tau'')](1+\epsilon')^2 N + o(N)},$$

where $\tau'' = (\lfloor \hat{\tau}(1 + \epsilon')^2 N \rfloor - 1) / (1 + \epsilon')^2 N$, $\hat{\tau} = (1 - 1/(\tau N^{1/2-\epsilon}))\tau$, and H is the binary entropy function.

Proof. The proof follows the same lines as in the proof of Lemma 19. \blacksquare

Lemma 21. *Let $\rho = 2^{[1-H(\tau')]N/2}$ and*

$$A = \left\{ \forall v \in \mathcal{N}_\rho, u^+ \text{ would be happy at the location of } v \text{ at } t = 0 \right\}.$$

Then A occurs w.h.p.

Proof. Let U_i for $i = 1, 2, \dots, |\mathcal{N}_\rho|$ be the event that agent u^+ would be happy at the location of i 'th agent of \mathcal{N}_ρ . It is easy to see that $P(U_i) = p_u$ (see (30)). Hence we have

$$\begin{aligned} P(A) &= P\left(U_1^C \cap \dots \cap U_{|\mathcal{N}_\rho|}^C\right) \\ &= 1 - P\left(U_1 \cup \dots \cup U_{|\mathcal{N}_\rho|}\right) \\ &\geq 1 - |\mathcal{N}_\rho| \frac{2^{-[1-H(\tau')]N}}{\sqrt{N}} \\ &\geq 1 - \frac{5}{\sqrt{N}} \end{aligned}$$

which tends to one as $N \rightarrow \infty$. \blacksquare

The following lemma gives a simple lower bound for the probability of having a radical region inside a neighborhood which has radius $r = 2^{[1-H(\tau')]N/2 - o(N)}$. We call a radical region with radius $(1 + \epsilon')w$ an ϵ' -radical region.

Lemma 22. *Any arbitrary neighborhood \mathcal{N}_r with radius $r = 2^{[1-H(\tau')]N/2 - o(N)}$ in the initial configuration has at least one ϵ' -radical region in it with probability at least $2^{-[1-H(\tau')](2\epsilon' + \epsilon'^2)N - o(N)}$.*

Proof. Divide the neighborhood into $2(1 + \epsilon')w$ -blocks, and let N_b denote the number of blocks in \mathcal{N}_r . Define the events

$$\begin{aligned} Q_i &= \{\text{The } i\text{-th block of } \mathcal{N}_r \text{ is an } \epsilon'\text{-radical region}\}, \\ Q &= \{\text{There is an } \epsilon'\text{-radical region in } \mathcal{N}_r\}. \end{aligned}$$

Using Lemma 20, it follows that

$$\begin{aligned}
P(Q) &\geq P(Q_1 \cup \dots \cup Q_{N_b}) \\
&= 1 - P(Q_1^C \cap \dots \cap Q_{N_b}^C) \\
&= \frac{4r^2}{(1+\epsilon')^2 N} 2^{-[1-H(\tau'')](1+\epsilon')^2 N - o(N)} \\
&= 2^{-[1-H(\tau')][2\epsilon'+\epsilon'^2]N - [H(\tau')-H(\tau'')](1+\epsilon')^2 N - o(N)} \\
&= 2^{-[1-H(\tau')][2\epsilon'+\epsilon'^2]N - o(N)}.
\end{aligned}$$

■

C. FKG-Harris inequality

The following is Theorem 4 in [38] which is originally by Harris [40]. Let σ_t be the configuration of the agents on the grid at time t . Let $\mathbb{E}^{\sigma_0}[X]$ be the expected value of the random variable X , when the initial state of the system is σ_0 . A probability distribution μ on $\{0, 1\}^{\mathbb{Z}^d}$ is said to be positively associated if for all increasing f and g we have

$$\mathbb{E}[f(\sigma)g(\sigma)] \geq \mathbb{E}[f(\sigma)]\mathbb{E}[g(\sigma)].$$

Theorem 6 (Harris). *Assume the process satisfies the following two properties: (a) Individual transitions affect the state at only one site. (b) For every continuous increasing function f and every $t > 0$, the function $\sigma_0 \rightarrow \mathbb{E}^{\sigma_0}[f(\sigma_t)]$ is increasing. Then, if the initial distribution is positively associated, so is the distribution at all later times.*

The following is a version of the FKG inequality [41] in our setting. The original inequality holds for a static setting and is extended here to our time-dynamic setting using Theorem 6.

Lemma 23 (FKG-Harris). *Let A and B be two increasing events defined on our process on the grid. We have*

$$P(A \cap B) \geq P(A)P(B).$$

Proof. Assume A and B are increasing random variables which depend only on the states of the sites v_1, v_2, \dots, v_k and first time step. We proceed by induction on k . First, let $k = 1$. Let $\omega(v_1)$ be the realization of the site v_1 . We also have

$$(1_A(\omega_1) - 1_A(\omega_2))(1_B(\omega_1) - 1_B(\omega_2)) \geq 0,$$

for all pairs of vectors ω_1 and ω_2 from the sample space. We have

$$\begin{aligned}
0 &\leq \sum_{\omega_1, \omega_2} (1_A(\omega_1) - 1_A(\omega_2))(1_B(\omega_1) - 1_B(\omega_2)) P(\omega(v_1) = \omega_1) P(\omega(v_1) = \omega_2) \\
&= 2(P(A \cap B) - P(A)P(B)),
\end{aligned}$$

as required. Assume now that the result is valid for values of n satisfying $k < n$. Then

$$\begin{aligned}
P(A \cap B) &= \mathbb{E} \left[P(A \cap B \mid \omega(v_1), \dots, \omega(v_{n-1})) \right] \\
&\geq \mathbb{E} \left[P(A \mid \omega(v_1), \dots, \omega(v_{n-1})) P(B \mid \omega(v_1), \dots, \omega(v_{n-1})) \right],
\end{aligned}$$

since, given $\omega(v_1), \dots, \omega(v_{n-1})$, 1_A and 1_B are increasing in the single variable $\omega(v_n)$. Now since $P(A|\omega(v_1), \dots, \omega(v_{n-1}))$ and $P(B|\omega(v_1), \dots, \omega(v_{n-1}))$ are increasing in the space of the $n-1$ sites, it follows from the induction hypothesis that

$$\begin{aligned} P(A \cap B) &\geq \mathbb{E} \left[P \left(A \mid \omega(v_1), \dots, \omega(v_{n-1}) \right) \right] \mathbb{E} \left[P \left(B \mid \omega(v_1), \dots, \omega(v_{n-1}) \right) \right] \\ &= P(A)P(B). \end{aligned} \tag{31}$$

Next, assume A and B are increasing random variables which depend only on the states of the sites in the first k time steps. We proceed by induction on $k < K$ such that K denotes the final time step over all the realizations. First, let $k = 0$. Let $\omega(t_0)$ be the configuration of the graph at the first time step. We have

$$P(A \cap B) \geq P(A)P(B),$$

by the above result. Assume now that the result is valid for all values of k satisfying $k < K$. Then, since our process satisfies the conditions of Theorem 6 and given $\omega(t_0), \dots, \omega(t_{K-1})$, 1_A and 1_B are increasing in $\omega(t_K)$, we have

$$\begin{aligned} P(A \cap B) &= \mathbb{E} \left[P \left(A \cap B \mid \omega(t_0), \dots, \omega(t_{K-1}) \right) \right] \\ &\geq \mathbb{E} \left[P \left(A \mid \omega(t_0), \dots, \omega(t_{K-1}) \right) P \left(B \mid \omega(t_0), \dots, \omega(t_{K-1}) \right) \right]. \end{aligned}$$

Now, since $P(A|\omega(t_0), \dots, \omega(t_{K-1}))$ and $P(B|\omega(t_0), \dots, \omega(t_{K-1}))$ are increasing in the space of the configurations of the graph in the first $K-1$ time steps, it follows from the induction hypothesis that

$$\begin{aligned} P(A \cap B) &\geq \mathbb{E} \left[P \left(A \mid \omega(t_0), \dots, \omega(t_{K-1}) \right) \right] \mathbb{E} \left[P \left(B \mid \omega(t_0), \dots, \omega(t_{K-1}) \right) \right] \\ &= P(A)P(B). \end{aligned}$$

■