# Progression of Decomposed Local-Effect Action Theories 

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#### Abstract

In many tasks related to reasoning about consequences of a logical theory, it is desirable to decompose the theory into a number of weakly-related or independent components. However, a theory may represent knowledge that is subject to change, as a result of executing actions that have effects on some of the initial properties mentioned in the theory. Having once computed a decomposition of a theory, it is advantageous to know whether a decomposition has to be computed again in the newly-changed theory (obtained from taking into account changes resulting from execution of an action). In the paper, we address this problem in the scope of the situation calculus, where a change of an initial theory is related to the notion of progression. Progression provides a form of forward reasoning; it relies on forgetting values of those properties, which are subject to change, and computing new values for them. We consider decomposability and inseparability, two component properties known from the literature, and contribute by 1) studying the conditions when these properties are preserved and 2) when they are lost wrt progression and the related operation of forgetting. To show the latter, we demonstrate the boundaries using a number of negative examples. To show the former, we identify cases when these properties are preserved under forgetting and progression of initial theories in local-effect basic action theories of the situation calculus. Our paper contributes to bridging two different communities in Knowledge Representation, namely research on modularity and research on reasoning about actions.


## 1 Introduction

Modularity of theories has been established as an important research topic in knowledge representation. It includes both theoretical and practical aspects of modularity of theories formulated in different logical languages $(\mathcal{L})$, ranging from weak (but practical) description logics (DLs) such as $\mathcal{E} \mathcal{L}$ and DL-Lite to more expressive logics [13,15,17,18,38,47], to cite a few. Surprisingly, this research topic is little explored in the context of reasoning about actions. More specifically, it is natural to decompose a large heterogeneous theory covering several loosely-coupled application domains into components that have little or no intersection in terms of signatures. Potentially, such decomposition can facilitate solving the projection problem, which requires answering whether a given logical formula is true after executing a sequence of actions (events). In cases, when a query is a logical formula composed from symbols occurring in only one of the components, the query can be answered more easily than in the case when the whole theory is required. This decomposition can help in solving other reasoning problems (e.g. planning or high-level program execution) that require a solution to the projection problem as a prerequisite. To the best of our knowledge, the only previous work that explored decomposition of logical theories for the purposes of solving the projection problem are the papers [1,2]. These papers investigate decomposition in the situation calculus
[34,43], a well-known logical formalism for representation of actions and their effects. The author proposed reasoning procedures for a situation calculus theory by dividing the whole theory syntactically into weakly-related partitions. Specifically, he developed algorithms that use local computation inside syntactically-identified partitions and message passing between partitions. We take a different approach in our paper: instead of decomposing the whole action theory into subsets, as in [1,2], we consider signature decompositions of an initial theory only. Our components are not necessarily syntactic subsets of the initial theory. We concentrate on foundations, and explore properties of components produced by our decomposition. Whenever possible, we try to formulate these properties in a general logical language $\mathcal{L}$, that is a fragment of second order logic; however, when necessary, we focus on a specific logic.

This paper considers the decomposability and inseparability properties of logical theories. These properties are well known in research on modularization in the area of knowledge representation [17,38,18,32], but have not been studied previously in the scope of the situation calculus. Both properties are concerned with subdividing theories into components to facilitate reasoning. Informally, decomposability of a theory means that the theory can be equivalently represented as a union of two (or several) theories sharing a given set ( $\Delta$ ) of signature symbols. Inseparability of theories wrt some signature $\Delta$ means that the theories have the same set of logical consequences in the signature $\Delta$. If a theory $(\mathcal{T})$ is $\Delta$-decomposable into $\Delta$-inseparable components, then (under certain restrictions on the underlying logic) each component of the decomposition contains all information from $\mathcal{T}$ in its own signature. This is an ideal case of decomposition, since in this case the problem of entailment from $\mathcal{T}$ can be reduced to entailment from components, which are potentially smaller than the theory $\mathcal{T}$.

In the area of reasoning about actions, an initial logical theory represents knowledge that is subject to change due to the effects of actions on some of the properties mentioned in the theory. It can be updated with new information caused by actions, while some other knowledge should be forgotten, as it is no longer true in the next situation. We consider two types of update operators: 1) forgetting in arbitrary theories and 2) progression of theories in the situation calculus. Forgetting is a well-known operation on theories first introduced by Fangzhen Lin and Ray Reiter in their seminal paper [25]. Forgetting a signature $\sigma$ in a theory $\mathcal{T}$ means obtaining a theory indistinguishable from $\mathcal{T}$ in the rest of the signature symbols $\operatorname{sig}(\mathcal{T}) \backslash \sigma$. In this sense, forgetting a signature is close to the well-known notion of uniform interpolation. Forgetting a ground atom $P(\bar{t})$ in a theory $\mathcal{T}$ results in a theory that implies all the consequences of $\mathcal{T}$ "modulo" the truth value of $P(\bar{t})$. The operation of forgetting is closely related to progression in basic action theories in the situation calculus.

The situation calculus [43] is a knowledge representation logical formalism, which has been designed for axiomatization of problems in planning and high-level program execution. The idea is to 1 ) axiomatize a set of initial states (as an initial theory), 2) axiomatize preconditions telling when actions can be performed, and then 3) add the axioms about the effects of actions on situationdependent properties. After these steps, one can reason about the consequences of sequences of actions to determine whether properties of interest hold in a given situation resulting from executing a sequence of actions and whether a certain sequence of actions is consecutively executable. In the situation calculus, the so-called basic action theories represent such axiomatizations [43]. Each basic action theory contains an initial theory that represents incomplete knowledge about an initial situation $S_{0}$. In a special case, when there is complete knowledge about a finite number of individuals having unique names, the initial theory can be implemented as a relational database [43,8]. Roughly, a basic action theory $\mathcal{D}$ is a union of an initial theory $\mathcal{D}_{S_{0}}$ with some theory $\mathcal{T}$, defining
transitions among situations, and a set of "canonical" axioms assumed to be true for all application problems represented in the situation calculus. Informally speaking, an update of the initial theory after execution of an action is called "progression of the initial theory wrt an action". More precisely, progression of $\mathcal{D}_{S_{0}}$ wrt some action $\alpha$ is a logical consequence of $\mathcal{D}$ which contains all information from $\mathcal{D}$ about the situation resulting from the execution of $\alpha$ in the situation $S_{0}$. Ideally, it is computed as updating $\mathcal{D}_{S_{0}}$ with some logical consequences of $\mathcal{T}$, once all information in $\mathcal{D}_{S_{0}}$, which is no longer true in the resulting situation, has been forgotten. Progression is important for practical agents with indefinite horizon since progression is the only feasible way of maintaining knowledge about the world. Exploiting modularity in the vast agent's knowledge is important to guarantee that progression of the agent's knowledge will be computationally feasible.

Historically, the "situation calculus" (earlier referred to as "situational logic") is the earliest logical framework developed in the area of artificial intelligence (AI). Having been developed in the 1960s by John McCarthy and his colleagues [33,34,14], it is one of the most popular logical frameworks for reasoning about actions; it is presented in most well-known textbooks on AI. It is worth mentioning that there are both conceptual and technical differences between the situation calculus, designed for reasoning about arbitrary actions, and the Floyd-Hoare logic, Dijkstra's predicate transformers, and dynamic logic, and other related formalisms, which have been developed for reasoning about the correctness of computer programs. For example, the latter category of formalisms would consider the operator assigning a new value to a variable in a program as a primitive action, while the former would consider as primitive the actions on higher level of abstraction, such as moving a book from its current location to the table. For this reason, the situation calculus is chosen as foundation for high-level programming languages in cognitive robotics [22]. In our paper, when we refer to the "situation calculus", we are following the axiomatic approach and notation proposed by R.Reiter [43] who developed a general approach to axiomatizing direct effects and non-effects of actions. It has been observed for a long time that in practical applications, real-world actions have no effect on most properties. However, it was Reiter who first proposed an elegant axiomatization that represents compactly non-effects of actions. Reiter's book covers several extensions of the situation calculus to reasoning about concurrent actions, instantaneous actions, processes extended in time, interaction between action and knowledge, stochastic actions, as well as high-level programming languages based on the situation calculus. In our paper, we will focus on the cases of situation calculus when actions are sequential, atemporal, and deterministic. Despite this focus, our results can be subsequently adapted to characterize more general classes of actions. The main limitation of our work is in concentrating on direct effects only. Indirect effects of actions are beyond the scope of the present study and will be considered in future work.

In this paper, we are interested in the case when the initial theory is decomposed into inseparable components, studying which conditions guarantee preservation of decomposability and inseparability of components after forgetting or progression. We would like to avoid recomputing a decomposition of an updated initial theory after executing an action. Moreover, we would like to know whether the components remain inseparable after progression. Such invariance of decomposability and inseparability wrt progression is important since progression may continue indefinitely as long as new actions are being executed. If decomposability and inseparability are always preserved, then it would suffice to compute a decomposition of the initial theory once - this decomposition will remain "stable" after progression wrt any arbitrary sequence of actions. Additionally, if an executed action has effects only on one component of the initial theory, then we would like to be able to compute progression using only this part instead of the whole initial theory. This leads to the question of
when the decomposability and inseparability properties are preserved under progression and under forgetting. To answer this question we have to better understand the properties of these two operations. In our study, for brevity, when we refer to "decomposability" and "inseparability" properties of components, we will use the phrase component properties.

This paper contributes to the general understanding of forgetting and progression in the literature, since new results on them are needed for the purposes of our investigation. Not surprisingly, both forgetting and progression have intricate interactions with properties of decomposed components. We will demonstrate that, in general, it is very difficult to guarantee the preservation of decomposability and inseparability, because there is a certain conceptual distance between these notions on one hand, and forgetting and progression on the other - we provide examples witnessing this. Nevertheless, we will identify cases when these properties remain invariant. Our results show that some of these cases have a practically important formulation. An important contribution of the paper is in formulating clear negative examples that demonstrate cases when decomposability and inseparability are lost under progression. Thus, the paper contributes to understanding the limits of the component approach based on these properties. In particular, our examples demonstrate that there is little hope to preserve inseparability if the different components share a fluent. Decomposability turns out to be also a fragile property that can be easily lost after executing just one action in a simple basic action theory. Overall, this paper contributes by advancing the study of forgetting and progression, and also by carrying out a thorough and comprehensive study of when decomposability and inseparability are preserved and when they are lost.

We start in Section 2 by introducing basic notations and then provide a survey on decomposability and inseparability, the two component properties of theories considered in this paper. Then in Section 3 we introduce the basics of the situation calculus, proceeding to the component properties of forgetting in Section 3 and progression in Section 4. The last section, Section 5, includes a summary of the obtained results. A preliminary shorter version of this paper (without proofs) appeared in the proceedings of AAAI-13 conference [40]. This extended version of our paper includes new results not mentioned in the conference version as well as proofs and a detailed background material and discussion of previously published results about forgetting and progression, in order to make this paper self-contained.

## 2 Background

### 2.1 Conventions and Notations

Let $\mathcal{L}$ be a logic (possibly many-sorted), which is a fragment (a set of sentences) of second-order logic (either by syntax or by translation of formulas), and has the standard model-theoretic Tarskian semantics. We call the signature a subset of non-logical symbols of $\mathcal{L}$ (and treat equality as a logical symbol). If $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ are two many-sorted structures and $\Delta$ is a signature then we say that $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ agree on $\Delta$ if they have the same domains for each sort and the same interpretation of every symbol from $\Delta$. If $\mathcal{M}$ is a structure and $\sigma$ is a subset of predicate and function symbols from $\mathcal{M}$, then we denote by $\left.\mathcal{M}\right|_{\sigma}$ the reduct of $\mathcal{M}$ to $\sigma$, i.e., the structure with predicate and function names from $\sigma$, where every symbol of $\sigma$ names the same entity as in $\mathcal{M}$. The structure $\mathcal{M}$ is called expansion of $\left.\mathcal{M}\right|_{\sigma}$. For a set of formulas $\mathcal{T}$ in $\mathcal{L}$, we denote by $\operatorname{sig}(\mathcal{T})$ the signature of $\mathcal{T}$, i.e. the set of all non-logical symbols which occur in $\mathcal{T}$. We will use the same notation $\operatorname{sig}(\varphi)$ for the signature of a formula $\varphi$ in $\mathcal{L}$. If $t$ is a term in the logic $\mathcal{L}$ then the same notation $\operatorname{sig}(t)$ will be used for the set of all non-logical symbols occurring in $t$. Throughout this paper, we use the notion
of theory as a synonym for a set of formulas in $\mathcal{L}$, which are sentences when translated into secondorder logic. Whenever we mention a set of formulas, it is assumed that this set is in $\mathcal{L}$, if the context is not specified. For two theories, $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$, the notation $\mathcal{T}_{1} \equiv \mathcal{T}_{2}$ will be the abbreviation for the semantic equivalence. If $\mathcal{T}$ is a set of formulas in $\mathcal{L}$ and $\Delta$ is a signature, then Cons $(\mathcal{T}, \Delta)$ will denote the set of semantic consequences of $\mathcal{T}$ (in $\mathcal{L}$ ) in the signature $\Delta$, i.e. Cons $(\mathcal{T}, \Delta)=\{\varphi \in$ $\mathcal{L} \mid \mathcal{T} \models \varphi$ and $\operatorname{sig}(\varphi) \subseteq \Delta\}$. We emphasize that this is a notation for a set of formulas in $\mathcal{L}$, because $\mathcal{T}$ may semantically entail formulas that are outside of $\mathcal{L}$.

### 2.2 Basic Facts about Decomposability and Inseparability

In the area of theory modularization, a module (or component) is usually understood as a set of theory consequences that satisfy certain properties. The latter are determined by requirements to a module in the context of application. Some approaches follow the idea that a module should be a syntactic subset of the axioms of a given theory. For instance, a theory can be partitioned into subsets of axioms meeting certain requirements of balance among the partition. Thereafter, reasoning wrt the initial theory can be reduced to reasoning within the obtained components via a message passing algorithm, which communicates between the partitions to find information needed to answer a query [3]. As a rule, the information to be communicated relates to the signatures shared between the partitions. The advantage of this approach is that the partitioning algorithm can be relatively simple and rely on syntactic analysis of theory axioms, thereby circumventing semantics. On the other hand, it may not be possible to eliminate some dependences between the partitions, if they are induced by syntactic form of the axioms. For instance, if a theory $\mathcal{T}$ consists of the axioms $\{\forall x P(x), \quad \forall x(P(x) \leftrightarrow Q(x))\}$, then it may not be possible to infer that it can be represented as the union of two components $\{\forall x P(x)\}$ and $\{\forall x Q(x)\}$, which do not share any signature symbols. In other words, a theory with syntactic dependencies may have an axiomatization that yields a partitioning into components, which either do not have symbols in common, or in a more general case, share a fixed signature (given as a parameter of decomposition).

In our paper, we adopt the following notion that was introduced in [38] and applied to the study of modularity in [17].

Definition 2.1 ( $\Delta$-decomposability property) Let $\mathcal{T}$ be a theory in $\mathcal{L}$ and $\Delta \subseteq \operatorname{sig}(\mathcal{T})$ a subsignature. We call $\mathcal{T} \Delta$-decomposable, if there are theories $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ in $\mathcal{L}$ such that

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\(-\operatorname{sig}\left(\mathcal{T}_{1}\right) \cap \operatorname{sig}\left(\mathcal{T}_{2}\right)=\Delta\), but \(\operatorname{sig}\left(\mathcal{T}_{1}\right) \neq \Delta \neq \operatorname{sig}\left(\mathcal{T}_{2}\right)\);
\(-\operatorname{sig}\left(\mathcal{T}_{1}\right) \cup \operatorname{sig}\left(\mathcal{T}_{2}\right)=\operatorname{sig}(\mathcal{T})\);
- \(\mathcal{T} \equiv \mathcal{T}_{1} \cup \mathcal{T}_{2}\).
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The pair $\left\langle\mathcal{T}_{1}, \mathcal{T}_{2}\right\rangle$ is called $\Delta$-decomposition of $\mathcal{T}$ and the theories $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are called $\Delta-$ decomposition components of $\mathcal{T}$. We will sometimes omit the word "decomposition" and call the sets $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ simply components of $\mathcal{T}$, when the signature $\Delta$ is clear from the context. The sets $\operatorname{sig}\left(\mathcal{T}_{1}\right) \backslash \Delta$ and $\operatorname{sig}\left(\mathcal{T}_{2}\right) \backslash \Delta$ are called signature ( $\Delta$-decomposition) components of $\mathcal{T}$.

The notion of $\Delta$-decomposition is defined using a pair of theories, but it can be easily extended to the case of a family of theories. It is important to realize that $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are not necessarily subsets of axioms of $\mathcal{T}$ in the above definition. We only require that $\mathcal{T} \equiv \mathcal{T}_{1} \cup \mathcal{T}_{2}$. Clearly, if $\mathcal{L}$ satisfies compactness and $\mathcal{T}$ is a finite $\Delta$-decomposable theory in $\mathcal{L}$ for a signature $\Delta$, then there is a $\Delta$ decomposition $\left\langle\mathcal{T}_{1}, \mathcal{T}_{2}\right\rangle$ of $\mathcal{T}$, where $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are finite.

Note that the axioms of theory $\mathcal{T}$ given before Definition 2.1 can not be (syntactically) partitioned into subsets having no signature symbols in common. However $\mathcal{T}$ has a different axiomatization given by $\forall x P(x)$ and $\forall x Q(x)$ and hence, is $\varnothing$-decomposable. Thus, in general $\Delta$-decomposition can be finer than a syntactic partitioning based on a particular axiomatization of a theory.

According to the definition of decomposability, computing a decomposition means finding another (equivalent) representation of a theory, which defines the required components. This means that a decomposition procedure must employ logical reasoning. Therefore, potentially it is more computationally complex than syntactic partitioning, which splits a theory into syntactic subsets of axioms. However, the research on algorithmic properties of decomposability (see e.g., $[39,9,17,35,38]$ ) shows that deciding whether a theory is $\Delta$-decomposable turns out to be not harder than deciding the entailment in the underlying logic. Studying the complexity of decomposability in different logics is an ongoing research topic. An algorithm for computing decomposition components can be obtained, e.g., from a procedure of computing uniform interpolants (if the logic enjoys efficient uniform interpolation, see Proposition 2 in [38] and [35]), or by applying the technique of eliminating non- $\Delta$-symbols from the axioms of a theory. The technique is described in [17] for the logics $\mathcal{E} \mathcal{L}$ and DL-Lite, which is further studied in [41] and can be extended to more expressive Description Logics. Using any form of equivalent rewriting means that the obtained axiomatization may be of a size larger than the set of axioms of the original theory. In particular, finding a decomposition may imply computing explicit definitions, in which case the size of the components depends on the complexity of such definitions in the underlying logic. For instance, a decomposition component may be of size exponentially larger than the original theory, which is evidenced by Example 28 in [17]. It is known that in general there is no upper bound on the complexity of explicit definitions in first-order logic $[11,36]$ and computing them is usually harder than entailment in FOL fragments (see e.g. [44]). On the other hand, one can take control over the growth of the component sizes by carefully choosing which signature $\Delta$ can be shared between the components. Using Example 28 from [17], for instance, it possible to describe a situation when tuning up $\Delta$ can exponentially reduce the component sizes. In general however, this question motivates research on the succinctness of explicit definitions and uniform interpolants in different logics.

An important requirement often considered in the literature is that a module must contain all information about a signature of interest $\Delta$, which is typically a subset of the signature of the module. In other words, it is required that a module must entail the same consequences in signature $\Delta$, as the source theory. Having fixed a signature $\Delta$, the ability to see differences between two theories strongly depends on the logic being used as "lens" for their examination: the more expressive power the logic employed has, the more differences it is possible to see. Probably the most powerful tool in measuring similarity of theories is the language of second-order logic. If two theories have the same sets of second-order consequences in a signature $\Delta$, then the classes of reducts of their models onto $\Delta$ coincide, i.e. both theories "define" the same semantics for $\Delta$-symbols. Keeping in mind that a module is usually understood as a set of consequences of a source theory, it is important to note the following model-theoretic fact, which will be helpful for grasping the results of this paper. It says that if a logic $\mathcal{L}$ is weaker than second-order, then in general, a set of consequences of a theory $\mathcal{T}$ in $\mathcal{L}$ may not be able to capture the intended semantics of symbols from a subsignature $\Delta$, as defined by $\mathcal{T}$.

Fact 2.2 If $\mathcal{T}$ is a theory in $\mathcal{L}$ and $\Delta$ a signature, then some models of $\operatorname{Cons}(\mathcal{T}, \Delta)$ may not have an expansion to a model of $\mathcal{T}$.

Indeed, let $\mathcal{L}$ be first-order logic and $\{P, f\}$ be a signature, where $P$ is a unary predicate and $f$ is a unary function. Let $\mathcal{T}$ be a theory saying that $f$ is a bijection between the interpretation of $P$ and its complement. Thus, $\mathcal{T}$ axiomatizes the class of models, where the interpretation of $P$ and its complement are of the same cardinality. Let $\mathcal{M}$ be a model from this class and let $\mathcal{N}$ be a model of the same signature $\{P\}$ in which the interpretation of $P$ is a countable set, but the complement is uncountable. The models $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent, i.e., no formula in signature $\{P\}$ can distinguish between these two models. For instance, this can be shown by using the fact (e.g., see [10]) that every sentence in signature $\{P\}$ is equivalent to a boolean combination of formulas $\exists \geqslant m P$ and $\exists \geqslant m \neg P$, where an integer $m>0$, which mean " $P$ (respectively, $\neg P$ ) holds on at least $m$ distinct elements". Therefore, $\mathcal{N}$ is a model of $\operatorname{Cons}(\mathcal{T},\{P\})$, but clearly, it has no expansion to a model of $\mathcal{T}$.

As will be noted in Section 3, forgetting is an operation, which gives a set of (second-order) consequences axiomatizing the same class of models, as the original theory, modulo forgotten signature/ground atom.

It is known that in general, a set of consequences of a theory may not be finitely axiomatizable in the logic, in which the theory is formulated. For instance, the following example is widely known in the literature on Description Logics (e.g., see Section 3.2 in [31]).

Fact 2.3 If $\mathcal{T}$ is a theory in $\mathcal{L}$ and $\Delta$ a signature, then $\operatorname{Cons}(\mathcal{T}, \Delta)$ may not be finitely axiomatizable in $\mathcal{L}$.

Let $\mathcal{T}$ be the first-order theory axiomatized by the following two axioms:
$\forall x[A(x) \rightarrow B(x)]$
$\forall x[B(x) \rightarrow \exists y(R(x, y) \wedge B(y))]$
Consider the signature $\Delta=\{A, R\}$. Then it is not hard to verify that $\operatorname{Cons}(\mathcal{T}, \Delta)$ is equivalent to the following infinite set of formulas:

$$
\begin{aligned}
& \forall x A(x) \rightarrow \exists y R(x, y) \\
& \forall x A(x) \rightarrow \exists y \exists u[R(x, y) \wedge R(y, u)] \\
& \forall x A(x) \rightarrow \exists y \exists u \exists v[R(x, y) \wedge R(y, u) \wedge R(u, v)]
\end{aligned}
$$

By compactness, this theory is not finitely axiomatizable in first-order logic.
A well-known concept used to characterize similarity of two theories wrt a signature is inseparability. This notion has also appeared in the context of entailment in Description Logics, e.g., see [18,32].

Definition 2.4 ( $\Delta$-inseparability) Theories $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ in $\mathcal{L}$ are called $\Delta$-inseparable, for a signature $\Delta$, if Cons $\left(\mathcal{T}_{1}, \Delta\right)=\operatorname{Cons}\left(\mathcal{T}_{2}, \Delta\right)$.

In other words, $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are $\Delta$-inseparable if for any $\mathcal{L}$-formula $\psi$ in signature $\Delta, \mathcal{T}_{1}$ entails $\psi$ iff $\mathcal{T}_{2}$ does. That is, in the language $\mathcal{L}$, no query in signature $\Delta$ separates $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ from each other. If Cons () is augmented with the third parameter specifying a logic, in which consequences are taken, then inseparability gives rise to a variety of notions of similarity between theories. As informally noted, two theories may be inseparable wrt a $\operatorname{logic} \mathcal{L}$, but entail different consequences wrt a language more expressive than $\mathcal{L}$. For the purpose of this paper, we consider the non-parametrized notion of inseparability, assuming that the language of interest is the underlying logic $\mathcal{L}$, in which
the theories are formulated. This assumption is natural if one is only interested in entailment of $\mathcal{L}$-formulas.

Inseparability plays an important role for decompositions. Assume that we have a theory $\mathcal{T}$ that is $\Delta$-decomposable into some components $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$. Although, the union $\mathcal{T}_{1} \cup \mathcal{T}_{2}$ must entail all consequences of $\mathcal{T}$ in the signature $\Delta$, the components $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ may not be $\Delta$-inseparable, if we demand them to be finite. For example, the set of $\Delta$-consequences of $\mathcal{T}_{2}$ may not be finitely axiomatizable in $\mathcal{L}$ by axioms of $\mathcal{T}_{1}$. This easily follows from Fact 2.3 which shows that this phenomenon is already possible in weak languages such as the sub-boolean description logic $\mathcal{E L}$. On the other hand, $\Delta$-inseparability of decomposition components can always be obtained if the underlying logic $\mathcal{L}$ has uniform interpolation (cf. Proposition 2 in [38]). Both $\Delta$-decomposition and $\Delta$-inseparability are required to achieve modularity. Without $\Delta$-inseparability the components are not self-sufficient, since a component may not entail some of the consequences in the shared vocabulary $\Delta$. The ideal case is when a theory $\mathcal{T}$ has $\Delta$-decomposition into finite $\Delta$-inseparable components, as noted in Fact 2.6 further in this section.

In contrast to decomposability, deciding $\Delta$-inseparability of theories is usually harder, than deciding entailment in the logic in which the theories are formulated, as proved by the results in [18], [32], and the results on the complexity of deciding conservative extensions [29,12]. However, there are practical cases in which this property is guaranteed to hold for any decomposition components of a given theory wrt a certain signature $\Delta$. For example, if the theory without equality is a set of ground atoms, then such theory is $\Delta$-decomposable iff there exist two subsets of atoms having only $\Delta$-symbols in common and containing at least one non- $\Delta$-symbol. This property is easy to check by computing syntactic connectedness of the signature symbols. It is straightforward to verify that if $\Delta$ does not contain predicate symbols, then the obtained decomposition components are guaranteed to be $\Delta$-inseparable. In other words, a set of ground atoms can be easily decomposed into inseparable components, if they share constants only with no common predicate symbols. A practically important generalization of theories consisting of ground atoms is proper ${ }^{+}$theories [21,27]. Developing computationally tractable techniques for decomposition of proper ${ }^{+}$theories into inseparable components is of particular interest. We note the importance of having inseparable decomposition components below.

The well-known property of logics related to signature decompositions of theories is the Parallel Interpolation Property (PIP) first considered in a special form in [20] and studied later in a more general form in [17].

Definition 2.5 (Parallel Interpolation Property) A logic $\mathcal{L}$ is said to have the parallel interpolation property (PIP) if for any theories $\mathcal{T}_{1}, \mathcal{T}_{2}$ in $\mathcal{L}$ with $\operatorname{sig}\left(\mathcal{T}_{1}\right) \cap \operatorname{sig}\left(\mathcal{T}_{2}\right)=\Delta$ and any formula $\varphi$ in $\mathcal{L}$, the condition $\mathcal{T}_{1} \cup \mathcal{T}_{2} \models \varphi$ yields the existence of sets of formulas $\mathcal{T}_{1}^{\prime}$ and $\mathcal{T}_{2}^{\prime}$ in $\mathcal{L}$ such that:

- $\mathcal{T}_{i} \models \mathcal{T}_{i}^{\prime}$, for $i=1,2$, and $\mathcal{T}_{1}^{\prime} \cup \mathcal{T}_{2}^{\prime} \models \varphi$;
$-\operatorname{sig}\left(\mathcal{T}_{i}^{\prime}\right) \backslash \Delta \subseteq\left(\operatorname{sig}\left(\mathcal{T}_{i}\right) \cap \operatorname{sig}(\varphi)\right) \backslash \Delta$.
Note that PIP is closely related to Craig's interpolation [5,6]. In fact, PIP can be understood as an iterated version of Craig's interpolation in the logics that have compactness and deduction theorem (see Lemma 1 in [38]). Many logics known to have Craig interpolation - e.g., second- and first-order logics, numerous modal logics, and some description logics, also have PIP. It is easy to note that, in the presence of PIP, decomposing a set $\mathcal{T}$ of formulas into inseparable components wrt a signature $\Delta$ gives a family of theories that imply all the consequences of $\mathcal{T}$ in their own subsignatures.

Fact 2.6 Let $\mathcal{L}$ have PIP, $\mathcal{T}$ be a theory in $\mathcal{L}$, and $\Delta$ a signature. Let $\left\langle\mathcal{T}_{1}, \mathcal{T}_{2}\right\rangle$ be a $\Delta$-decomposition of $\mathcal{T}$, with $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ being $\Delta$-inseparable. Then for any formula $\varphi$ with $\operatorname{sig}(\varphi) \subseteq \operatorname{sig}\left(\mathcal{T}_{i}\right)$, for some $i=1,2$, we have $\mathcal{T} \models \varphi$ iff $\mathcal{T}_{i} \models \varphi$.

Proof. Assume $\operatorname{sig}(\varphi) \subseteq \operatorname{sig}\left(\mathcal{T}_{1}\right)$. If $\mathcal{T}_{1} \models \varphi$ then $\mathcal{T} \models \varphi$ by definition of $\Delta$-decomposability. If $\mathcal{T} \models \varphi$ then $\mathcal{T}_{1} \cup \mathcal{T}_{2} \models \varphi$ and by PIP, there are $\mathcal{T}_{1}^{\prime}$ and $\mathcal{T}_{2}^{\prime}$ such that $\mathcal{T}_{1} \models \mathcal{T}_{1}^{\prime}, \mathcal{T}_{2} \models \mathcal{T}_{2}^{\prime}, \mathcal{T}_{1}^{\prime} \cup \mathcal{T}_{2}^{\prime} \models \varphi$, and $\operatorname{sig}\left(\mathcal{T}_{2}^{\prime}\right) \subseteq \Delta$. As $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are $\Delta$-inseparable, we obtain $\mathcal{T}_{1} \models \mathcal{T}_{2}^{\prime}$ and conclude that $\mathcal{T}_{1} \models \varphi$.

In other words, in the presence of PIP, inseparable decomposition components can be used instead of the original theory for checking the entailment of formulas in the corresponding subsignatures. This is the reason for our interest in the inseparability property in connection with decompositions. As shown in $[1,2,3]$, a decomposition of a theory can be beneficial even without inseparability thanks to applying the known methods of distributed reasoning via message passing between components. However, if components are inseparable, then the reasoner can avoid message passing completely.

### 2.3 Basics of the Situation Calculus

The language of the situation calculus $\mathcal{L}_{s c}$ has the first-order syntax over three sorts action, situation, object. It is provided with the standard model-theoretic semantics. It is defined over the countably infinite alphabet $A_{s c}=\left\{d o, \preceq, S_{0}\right.$, Poss $\} \cup \mathcal{A} \cup \mathcal{F} \cup \mathcal{O} \cup \mathcal{P}$, where do is a binary function symbol of sort situation; $\preceq$ is a binary relation on situations; $S_{0}$ is the constant of sort situation; $\operatorname{Poss}(a, s)$ is a binary predicate (saying whether $a$ is possible in $s$ ) with the first argument of sort action and the second one of sort situation; $\mathcal{A}$ is a set of action functions with arguments of sort object, $\mathcal{F}$ is a set of so-called fluents, i.e., predicates having as arguments a tuple (vector) of sort object and one last argument of sort situation; $\mathcal{O}$ is a set of constants of sort object; and $\mathcal{P}$ is a set of static predicates and functions, i.e., those that only have objects as arguments. A symbol $v \in A_{s c}$ (predicate or function) is called situation-independent if $v \in \mathcal{A} \cup \mathcal{O} \cup \mathcal{P}$. A ground term is of sort situation iff it is either the constant $S_{0}$ or a term $\operatorname{do}(A(\bar{t}), S)$, where $A(\bar{t})$ is a ground action term and $S$ is a ground situation term. For instance, a term $d o\left(A_{2}\left(\overline{t_{2}}\right)\right.$, do $\left.\left(A_{1}\left(\overline{t_{1}}\right), S_{0}\right)\right)$ denotes the situation resulting from executing actions $A_{1}\left(\overline{t_{1}}\right)$ and $A_{2}\left(\overline{t_{2}}\right)$ consecutively from the initial situation $S_{0}$. Informally, static predicates specify object properties that never change no matter what actions are executed and fluents describe those object properties that are situation-dependent. The language of the situation calculus is used to formulate basic action theories $(\mathcal{B A} \mathcal{T}$ s); they may serve as the formal specifications of planning problems. Every $\mathcal{B} \mathcal{A} \mathcal{T}$ consists of a set of foundational axioms $\Sigma$, which specify constraints on how the function $d o$ and fluents must be understood, a theory $D_{\text {una }}$ stating the unique name assumption for action functions and objects, an initial theory $D_{S_{0}}$ describing knowledge about the initial situation $S_{0}$, a theory $D_{a p}$ specifying preconditions of action execution, and a theory $D_{s s}$ (the set of successor-state axioms, SSAs for short) which contains definitions of fluents in the next situation in terms of static predicates and the values of fluents in the previous situation. A detailed example of a $\mathcal{B A T}$ is given at the end of this section.

Example 1 (The Blocks World). We illustrate some of the syntactic definitions using the well-known Blocks World example. The domain of objects in this example consists of blocks that can form towers such that a block can be on the top of only one other block and conversely only one block can be staying on the top of another block. The unary predicate Block holds for objects. The towers of blocks
can be described using the fluents $O n(x, y, s)$, a block $x$ is on $y$ in situation $s$, and Clear $(x, s)$, a block $x$ is clear in $s$ meaning that there is no block on top of $x$ in situation $s$. The first fluent applies to pairs of blocks in a tower, while the second fluent characterizes the top block. An initial theory $D_{S_{0}}$ may include axioms about the initial configuration of blocks named using object constants $A, B, C$, e.g., $O n\left(A, B, S_{0}\right)$, the block $A$ is on $B$ initially, $\neg \exists x O n\left(x, A, S_{0}\right)$ and $\neg \exists x O n\left(x, C, S_{0}\right)$, i.e., there are no blocks on top of blocks $A$ and $C$. Notice that both fluents are predicates with situation as the last argument. A theory $D_{\text {una }}$ includes axioms saying that all blocks $A, B, C$ are pairwise distinct. The function $\operatorname{move}(x, y, z)$ maps blocks $x, y, z$ into a separate sort action that represents moving block $x$ staying on top of block $y$ from block $y$ onto another block $z$. The precondition axioms $D_{a p}$ characterize when this action is possible, e.g., move $(A, B, C)$ is possible in the initial situation $S_{0}$, because both $A$ and $C$ are clear, but move $(B, A, C)$ is not possible in $S_{0}$, because the block $B$ is not clear, and it is not staying on $A$ in $S_{0}$. The situation $\operatorname{do}\left(\right.$ move $\left.(A, B, C), S_{0}\right)$ results from executing action move $(A, B, C)$ in the initial situation $S_{0}$. This action has effects on the fluents in the sense that the fluent predicates about $S_{0}$ may change their truth values in the situation do $\left(\right.$ move $\left.(A, B, C), S_{0}\right)$. Observe however that in do(move $(A, C, B)$, do(move $\left.\left.(A, B, C), S_{0}\right)\right)$ fluents are true iff they are true in $S_{0}$, since $\operatorname{move}(A, C, B)$ is inverse wrt move $(A, B, C)$ when these actions executed consecutively. The following successor state axiom characterizes all effects of all actions on the fluent $O n$ :

$$
\forall x, y, z, a, s O n(x, z, d o(a, s)) \leftrightarrow \exists y(a=\operatorname{move}(x, y, z)) \vee O n(x, z, s) \wedge \neg \exists y(a=\operatorname{move}(x, z, y))
$$

More specifically, block $x$ is on block $z$ after doing an action $a$ in situation $s$ iff the last action $a$ was moving $x$ from some other block $y$ to $z$, or if $x$ was already on $z$ in $s$, and the last action $a$ did not move it elsewhere. Subsequently, we do not write the $\forall$-quantifiers explicitly at front of the axioms.

In every basic action theory $\mathcal{D}$ over a signature $\sigma \subseteq A_{s c}$, the set of foundational axioms $\Sigma$ consists of the following formulas [42] (note the axiom schema for induction):

```
\(\forall a_{1}, a_{2}, s_{1}, s_{2}\left[\operatorname{do}\left(a_{1}, s_{1}\right)=\operatorname{do}\left(a_{2}, s_{2}\right) \rightarrow a_{1}=a_{2} \wedge s_{1}=s_{2}\right]\)
\(\forall s \neg\left(s \preceq S_{0} \wedge s \neq S_{0}\right)\)
\(\forall s_{1}, s_{2}\left[s_{1} \preceq s_{2} \leftrightarrow \exists a\left(d o\left(a, s_{1}\right) \preceq s_{2}\right) \vee s_{1}=s_{2}\right]\)
\(\forall P P\left(S_{0}\right) \wedge \forall a, s[P(s) \rightarrow P(d o(a, s))] \rightarrow \forall s P(s)\)
```

Reiter observed in [42] that foundational axioms $\Sigma$ generalize a single successor function over natural numbers to the case of multiple successors over situations. The second order induction axiom serves to exclude non-standard trees as models.

For every pair of distinct action functions $\left\{A, A^{\prime}\right\} \subseteq \sigma$ and every pair $\langle a, b\rangle$ of distinct object constants from $\sigma$, a theory $D_{\text {una }}$ contains axioms of the form:
$a \neq b$
$\forall \bar{x}, \bar{y} A(\bar{x}) \neq A^{\prime}(\bar{y})$
$\forall \bar{x}, \bar{y} A\left(x_{1}, \ldots, x_{n}\right)=A\left(y_{1}, \ldots, y_{n}\right) \rightarrow x_{1}=y_{1} \wedge \ldots \wedge x_{n}=y_{n}$ if $A$ is n-ary.
No other axioms are in $D_{u n a}$.
To define the remaining subtheories of $\mathcal{B \mathcal { A }}$, we need to introduce the following syntactic notion (taken from [37,43]).

Definition 2.7 A formula $\varphi$ in language $\mathcal{L}_{s c}$ is called uniform in a situation term $S$ if:

1. it does not contain quantifiers over variables of sort situation;
2. it does not contain equalities between situation terms;
3. the predicates Poss, $\preceq$ do not occur in $\varphi:\{$ Poss,$\preceq\} \cap \operatorname{sig}(\varphi)=\varnothing$;
4. for every fluent $F \in \operatorname{sig}(\varphi)$, the term in the situation argument of $F$ is $S$.

A set $\mathcal{T}$ of formulas in $\mathcal{L}_{s c}$ is called uniform in a situation term $S$ if every formula of $\mathcal{T}$ is uniform in $S$.

By definition, a set $\mathcal{T}$ of formulas uniform in a situation term $S$ either does not contain any situation terms (and hence, fluents), or the only situation term is $S$ which occurs as the situation argument of each fluent from $\operatorname{sig}(\mathcal{T})$. In the example above, the formula on the right hand side of the SSA is a formula uniform in $s$. If $\mathcal{T}$ is a set of sentences uniform in situation term $S$, i.e., $\mathcal{T}$ has no free variables, and $S$ occurs in formulas of $\mathcal{T}$, then by items (1), (2) of the definition, $S$ must be ground and thus, it must either be the constant $S_{0}$, or have the form $d o\left(A(\bar{t}), S^{\prime}\right)$, where $S^{\prime}$ is a ground situation term. Note that if the constant $S_{0}$ or the binary function symbol $d o$ is present in $\operatorname{sig}(\mathcal{T})$ and $\mathcal{T}$ is uniform in $S$, then necessarily $S_{0} \in \operatorname{sig}(S)$, or $d o \in \operatorname{sig}(S)$, respectively. By items (1) and (2), such theory $\mathcal{T}$ does not restrict the interpretation of the term $S$ and the cardinality of the sort situation, so the observations above lead to the following property of uniform theories, which informally can be summarized by saying that in sentences of a theory $\mathcal{T}$ uniform in a ground situation term $S$, we can understand this situation term as playing a role of an index that can remain implicit. Whenever we change the interpretation of $S$ (e.g., by choosing a different interpretation for $d o$ and $S_{0}$ ) in a model of $\mathcal{T}$, it suffices to "move" interpretations of fluents to this new point to obtain again a model for $\mathcal{T}$.

Lemma 2.8 Let $\mathcal{T}$ be a set of sentences uniform in a ground situation term $S$. Let $\mathcal{M}=\langle$ Act $\cup$ Sit $\cup O b j$, do, $\left.\mathbf{S}_{\mathbf{0}}, \mathbf{F}_{\mathbf{1}}, \ldots, \mathbf{F}_{\mathbf{n}}, \mathcal{I}\right\rangle$ be a model of $\mathcal{T}$, where Act, Sit, and Obj are domains for the corresponding sorts action, situation, and object, do and $\mathbf{S}_{\mathbf{0}}$ are the interpretations of the function do and constant $S_{0}$, respectively, $\mathbf{F}_{\mathbf{1}}, \ldots, \mathbf{F}_{\mathbf{n}}$ are the interpretations of fluents from $\operatorname{sig}(\mathcal{T})$, and $\mathcal{I}$ is the interpretation of the rest of symbols from $\operatorname{sig}(\mathcal{T})$. For example, $\mathbf{F}_{\mathbf{i}}$ is a set of tuples $\left\langle u_{1}, \ldots, u_{m-1}, \mathbf{S}\right\rangle$, where $\mathbf{S}$ is the interpretation of the ground term $S$ in $\mathcal{M}$.

Consider the structure $\mathcal{M}^{\prime}=\left\langle\right.$ Act $\cup$ Sit $\left.\cup O b j, \mathbf{d o}^{\prime}, \mathbf{S}_{\mathbf{0}}{ }^{\prime}, \mathbf{F}_{\mathbf{1}}{ }^{\prime}, \ldots, \mathbf{F}_{\mathbf{n}}{ }^{\prime}, \mathcal{I}\right\rangle$, where Sit is an arbitrary set, the domain for sort situation, do' and $\mathbf{S}_{\mathbf{0}}{ }^{\prime}$ are arbitrary interpretations of do and $S_{0}$ on Sit', respectively, and for $i \leqslant n, \mathbf{F}_{\mathbf{i}}{ }^{\prime}$ denotes the interpretation of the fluent $F_{i}$ as a set of tuples $\left\langle u_{1}, \ldots, u_{m-1}, \mathbf{S}^{\prime}\right\rangle$, with $\mathbf{S}^{\prime}$ being the interpretation of term $S$ in $\mathcal{M}^{\prime}$ and $\left\langle u_{1}, \ldots, u_{m-1}, \mathbf{S}\right\rangle \in \mathbf{F}_{\mathbf{i}}$.

Then, $\mathcal{M}^{\prime}$ is a model of $\mathcal{T}$. By definition, the interpretation of situation-independent predicates and functions is the same in $\mathcal{M}^{\prime}$ and $\mathcal{M}$.

This lemma can be easily proved by induction over possible syntactic form of sentences in $\mathcal{T}$. If $S$ and $S^{\prime}$ are two situation terms and $\mathcal{T}$ is a set of formulas uniform in $S$, then we denote by $\mathcal{T}\left(S^{\prime} / S\right)$ the set of formulas obtained from $\mathcal{T}$ by replacing every occurrence of $S$ with $S^{\prime}$. This notation will be extensively used in Section 4. Obviously, $\mathcal{T}\left(S^{\prime} / S\right)$ is uniform in $S^{\prime}$.

The initial theory $\mathcal{D}_{S_{0}}$ of $\mathcal{D}$ is defined as an arbitrary set of sentences in the signature $\sigma$ that are uniform in the situation constant $S_{0}$. Throughout the paper, we assume that $\mathcal{D}_{S_{0}}$ is a theory in (any fragment of) second-order logic that can be translated into a set of sentences of first-order logic uniform in $S_{0}$. In particular, $\mathcal{D}_{S_{0}}$ can include both an ABox and a TBox in an appropriate Description Logic, as argued in $[16,48]$.

Next, for every n-ary action function $A \in \sigma$, a theory $\mathcal{D}_{\text {ap }}$ includes an axiom of the form

$$
\forall \bar{x}, s\left(\operatorname{Poss}(A(\bar{x}), s) \leftrightarrow \Pi_{A}(\bar{x}, s)\right)
$$

where $\Pi_{A}(\bar{x}, s)$ is a formula uniform in $s$ with free variables among $\bar{x}$ and $s$. Informally, $\Pi_{A}(\bar{x}, s)$ characterizes preconditions for executing the action $A$ in the situation $s$. No other formulas are in $\mathcal{D}_{\text {ap }}$.

Example 1 (continuation). The following is the precondition axiom for move $(x, y, z)$ :

$$
\begin{array}{r}
\operatorname{Poss}(\operatorname{move}(x, y, z), s) \leftrightarrow \operatorname{Block}(x) \wedge \operatorname{Block}(y) \wedge \operatorname{Block}(z) \wedge O n(x, y, s) \wedge \\
C l e a r(x, s) \wedge \operatorname{Clear}(z, s) \wedge x \neq z
\end{array}
$$

The action move $(x, y, z)$ is possible iff $x, y, z$ are blocks, $x$ is located on $y$ in situation $s$, and both the block $x$ that is to be moved, and a destination block $z$ are not occupied by any other blocks. Notice the preconditions do not allow moving a block back to the same location where it was before.

Finally, for every fluent $F \in \sigma$, a theory $D_{s s}$ contains an axiom of the form

$$
\forall \bar{x}, a, s\left(F(\bar{x}, d o(a, s)) \leftrightarrow \gamma_{F}^{+}(\bar{x}, a, s) \vee F(\bar{x}, s) \wedge \neg \gamma_{F}^{-}(\bar{x}, a, s)\right)
$$

specifying a condition $\gamma_{F}^{+}(\bar{x}, a, s)$ when fluent $F$ becomes true in situation $d o(a, s)$, or when $F$ remains true in situation $d o(a, s)$ if it is true in $s$, unless another condition $\left.\gamma_{F}^{-}(\bar{x}, a, s)\right]$ holds. Here, $\gamma_{F}^{+}$is a disjunction of formulas of the form $[\exists \bar{y}]\left(a=A^{+}(\bar{t}) \wedge \phi^{+}(\bar{x}, \bar{y}, s)\right)$, where $A^{+}$is an action function, $\bar{t}$ is a (possibly empty) vector of object terms with variables at most among $\bar{x}$ and $\bar{y}$, and $\phi^{+}$is a formula uniform in $s$ with variables at most among $\bar{x}, \bar{y}$, and $s$. We write $[\exists \bar{y}]$ to show that $\exists \bar{y}$ is optional; it is present only if $\bar{t}$ includes $\bar{y}$ or if $\phi$ has an occurrence of $\bar{y}$. The formula $\phi^{+}$is called a positive context condition meaning that $A^{+}(\bar{t})$ makes the fluent $F$ true if this context condition holds in $s$, but otherwise, $A^{+}(\bar{t})$ has no effect on $F$. Similarly, $\gamma_{F}^{-}$is a disjunction of formulas of the form $[\exists \bar{z}]\left(a=A^{-}\left(\overline{t^{\prime}}\right) \wedge \phi^{-}(\bar{x}, \bar{z}, s)\right)$, where $A^{-}$is an action function, $\overline{t^{\prime}}$ is a (possibly empty) vector of object terms with variables at most among $\bar{x}$ and $\bar{z}$, and $\phi^{-}$is a formula uniform in $s$ with variables at most among $\bar{x}, \bar{z}$, and $s$. The formula $\phi^{-}$is called a negative context condition meaning that $A^{-}(\bar{t})$ makes the fluent $F$ false if this context condition holds in $s$, but otherwise, $A^{-}(\bar{t})$ has no effect on $F$. In the definition above, we assume that the empty disjunction is equal to false. No other formulas are in $\mathcal{D}_{s s}$. This completes the definition of $\mathcal{D}_{s s}$. Subsequently, the following will be useful.

Definition 2.9 (SSA and active position of an action) The axioms of $\mathcal{D}_{\text {ss }}$ in the form above are called successor state axioms (SSAs) of a basic action theory $\mathcal{D}$.

An action function $f$ is said to be in active position of some $\operatorname{SSA} \varphi \in \mathcal{D}_{\text {ss }}$ if $f$ occurs either as $A^{+}$, or $A^{-}$in the definition of $\mathcal{D}_{\text {ss }}$ above.

We say that $\varphi \in \mathcal{D}_{\text {ss }}$ is SSA for the fluent $F$ if $F$ is the fluent from the left-hand side of $\varphi$.
Example 1 (continuation). The following is the SSA for the fluent $E H(x, s)$ meaning the height of a block $x$ is even, i.e., the number of blocks under $x$ is odd:

$$
\begin{aligned}
E H(x, \operatorname{do}(a, s)) \leftrightarrow & \exists y, z(a=\operatorname{move}(x, y, z) \wedge \neg E H(z, s)) \vee \\
& E H(x, s) \wedge \neg \exists y, z(a=\operatorname{move}(x, y, z) \wedge E H(z, s)) .
\end{aligned}
$$

Then formula $\neg E H(z, s)$ is a positive context condition. If it holds in $s$, i.e., if the height of block $z$ is not even in a situation $s$, then in the situation that results from moving $x$ from $y$ to a block $z$, the height of $x$ becomes even. But if the positive context condition does not hold in $s$, then $\operatorname{move}(x, y, z)$ does not make the height of block $x$ even. Also, if the height of $x$ is even in $s$, then it remains even unless a block $x$ is moved from $y$ on top of $z$ and the height of $z$ is even in $s$. The
formula $\operatorname{EH}(z, s)$ is a negative context condition, i.e., if the height of block $z$ is even in $s$, then the action $\operatorname{move}(x, y, z)$ has a negative conditional effect on the fluent $E H(x, s)$ in the sense that this fluent becomes false in the situation that results from doing move $(x, y, z)$ in $s$. In this SSA, an action function move occurs both as $A^{+}$and $A^{-}$on the right hand side of this SSA.

Following the consistency requirement on SSAs by Reiter (see Proposition 3.2.6 in [43]), we require that if an action function $f$ occurs in active position in some SSA for a fluent $F$, then $f$ is not in active position in either $\gamma_{F}^{+}$, or $\gamma_{F}^{-}$. Informally, this means that an action cannot have both positive and negative effects on $F$.

Each SSA for a fluent $F$ completely defines the truth value of $F$ in the situation $d o(a, s)$ in terms of what holds in situation $s$. Also, SSA compactly represents non-effects by quantifying $\forall a$ over variables of sort action. Only action terms that occur explicitly on the right-hand side of SSA for a fluent $F$ have effects on this fluent, while all other actions have no effect.

We note that the original version of Reiter's situation calculus admits functional fluents, e.g. functions having a vector of arguments of sort object and one last argument of sort situation. Reiter defines the notion of SSA for functional fluents [43]. Without loss of generality, we omit functional fluents in this paper.

The following fundamental result, which will be used in our Theorem 4.4, says that the initial theory together with the UNA is the core of any basic action theory, while the rest of the constituent theories may be considered as add-ons.

Proposition 2.10 (Theorem 1 in [37]) A basic action theory $\Sigma \cup D_{u n a} \cup D_{S_{0}} \cup D_{a p} \cup D_{s s}$ is satisfiable iff $D_{\text {una }} \cup D_{S_{0}}$ is satisfiable.

Suppose $\mathcal{A}_{1}, \cdots, \mathcal{A}_{n}$ is a sequence of ground action terms, and $\varphi(s)$ is a formula with one free variable $s$ of sort situation which is uniform in $s$. One of the most important reasoning tasks in the situation calculus is the projection problem: that is, to determine whether

$$
\mathcal{D} \models \varphi\left(d o\left(\mathcal{A}_{n}, \operatorname{do}\left(\mathcal{A}_{n-1}, d o\left(\cdots, \operatorname{do}\left(\mathcal{A}_{1}, S_{0}\right)\right)\right)\right)\right)
$$

Informally, $\varphi$ represents some property of interest and entailment holds iff this property is true in the situation resulting from performing the sequence of actions $\mathcal{A}_{1}, \cdots, \mathcal{A}_{n}$ starting from $S_{0}$.

Another basic reasoning task is the executability problem. Let

$$
\operatorname{executable}\left(\operatorname{do}\left(\mathcal{A}_{n}, \operatorname{do}\left(\mathcal{A}_{n-1}, \operatorname{do}\left(\cdots, \operatorname{do}\left(\mathcal{A}_{1}, S_{0}\right)\right)\right)\right)\right)
$$

be an abbreviation of the formula

$$
\operatorname{Poss}\left(\mathcal{A}_{1}, S_{0}\right) \wedge \bigwedge_{i=2}^{n} \operatorname{Poss}\left(\mathcal{A}_{i}, \operatorname{do}\left(\mathcal{A}_{1}, d o\left(\cdots, \operatorname{do}\left(\mathcal{A}_{i-1}, S_{0}\right)\right)\right)\right.
$$

Then, the executability problem is to determine whether

$$
\mathcal{D} \models \operatorname{executable}\left(d o\left(\mathcal{A}_{n}, \operatorname{do}\left(\mathcal{A}_{n-1}, d o\left(\cdots, d o\left(\mathcal{A}_{1}, S_{0}\right)\right)\right)\right)\right),
$$

i.e. whether it is possible to perform the sequence of actions starting from $S_{0}$.

Planning and high-level program execution are two important settings, where the executability and projection problems arise naturally. Regression is a central computational mechanism that forms the basis for an automated solution to the executability and projection tasks in the situation calculus ([43]). Regression requires reasoning backwards: a given formula

$$
\varphi\left(d o\left(\mathcal{A}_{n}, d o\left(\mathcal{A}_{n-1}, d o\left(\cdots, d o\left(\mathcal{A}_{1}, S_{0}\right)\right)\right)\right)\right)
$$

is recursively transformed into a logically equivalent formula by using SSAs until the resulting formula has only occurrences of the situation term $S_{0}$. It is easy to see that regression becomes computationally intractable if the sequence of actions grows indefinitely [16]. In this case, an alternative to regression is progression, which provides forward-style reasoning. The initial theory $\mathcal{D}_{S_{0}}$ is updated to take into account the effects of an executed action. Computing the progression of a given theory
$\mathcal{D}_{S_{0}}$ requires forgetting facts in $\mathcal{D}_{S_{0}}$ which are no longer true after executing an action. The closely related notions of progression and forgetting are discussed in the next sections.

Definition 2.11 (local-effect SSA and $\mathcal{B A} \mathcal{T}$ ) An SSA $\varphi \in \mathcal{D}_{\text {ss }}$ for the fluent $F$ is called local-effect if the set of arguments of every action function in active position of $\varphi$ contains all object variables from $F$. A basic action theory is said to be local-effect if every axiom of $\mathcal{D}_{\text {ss }}$ is a local-effect SSA.

Local-effect $\mathcal{B} \mathcal{A} \mathcal{T}$ s are a well-known ${ }^{3}$ class of theories, for which the operation of progression (Section 4) can be computed effectively [27], without regard to decidability of the underlying theory $\mathcal{D}_{S_{0}}$. They are special in the sense that the truth value of each fluent defined by a local-effect SSA can change only for objects explicitly named as arguments of the executed action. Therefore, in local-effect $\mathcal{B} \mathcal{A} \mathcal{T}$ s, each action can change only finitely many ground fluent atoms. This allows for computing forgetting (the operation considered in Section 3) efficiently. Informally speaking, forgetting erases from $\mathcal{D}_{S_{0}}$ those finitely many fluent atoms which changed after executing an action.

Example 1 (continuation). Observe that in the Blocks World example considered above, the action move has only local effects on the fluents $O n$ and $C l e a r$. As an informal example of an action that has global effects, consider the action $\operatorname{drive}\left(t, l_{1}, l_{2}\right)$ of driving a truck $t$ loaded with boxes from one location $l_{1}$ to another location $l_{2}$. Consider also the fluent $A t(x, l, s)$ that holds if an object $x$ is at a location $l$ in $s$. Observe that this action would have a global effect on location of all boxes loaded on the truck since these boxes are not named explicitly in the action function $\operatorname{drive}\left(t, l_{1}, l_{2}\right)$, but the SSA for $A t(x, l, s)$ would have a $\forall$-quantifier over the object argument $x$. Therefore, the truth value of $A t(x, l, s)$ changes not only for $t$, but also for other objects not mentioned in drive $\left(t, l_{1}, l_{2}\right)$. It would be awkward to include all the boxes loaded in $t$ as arguments of this action. For this reason, axioms for the logistics domain should include actions with global effects on the fluents.

Before we proceed to a discussion of component properties under forgetting (Section 3) and to progression of initial theories (Section 4), we consider an example that helps to illustrate the notion of $\mathcal{B A} \mathcal{T}$ and the advantages of decomposition of its initial theory. Our example combines the simplified Blocks World (BW) with a kind of Stacks World. A complete axiomatization of BW modelled as a finite collection of finite chains can be found in [4]. In this example, and subsequently, we resort to the common situation calculus convention that free variables in $\mathcal{B \mathcal { A }} \mathcal{T}$ axioms are implicitly taken to be universally quantified at front.

Example 2 (A running example of $\mathcal{B A} \mathcal{T}$ ). The blocks-and-stacks-world consists of a finite set of blocks and a finite set of other entities. Blocks can be located on top of each other, while other entities can be either in a heap of unlimited capacity, or can be organized in stacks. There is an unnamed manipulator that can move a block from one block to another, provided that there is nothing on the top of the blocks. It can also put an entity from the heap upon a stack with a named top element, or move the top element of a stack into the heap. For stacking/unstacking operations we adopt the push/pop terminology and use the unary predicate Block to distinguish between blocks and other entities. We use the following action functions and relational fluents to axiomatize this example as a local-effect $\mathcal{B A T}$ in SC .

## Actions

- move $(x, y, z)$ : Move block $x$ from block $y$ onto block $z$, provided both $x$ and $z$ are clear.

[^0]- push $(x, y)$ : Stack entity $x$ from the heap on top of entity $y$.
- $\operatorname{pop}(x)$ : Unstack entity $x$ into the heap, provided $x$ is the top element and is not in the heap.


## Fluents

- $O n(x, z, s)$ : Block $x$ is on block $z$, in situation $s$.
- Clear $(x, s)$ : Block $x$ has no other blocks on top of it in $s$.
- $\operatorname{Top}(x, s)$ : Entity $x$ is the top element of a stack in $s$.
- Inheap $(x, s)$ : Entity $x$ is in the heap in situation $s$.
- Under $(x, y, s)$ : Entity $y$ is directly under $x$ in a stack in situation $s$.

The sub-theories of the basic action theory are defined as follows.
Successor state axioms (theory $\mathcal{D}_{s s}$ )

```
\(\operatorname{On}(x, z, \operatorname{do}(a, s)) \leftrightarrow \exists y(a=\operatorname{move}(x, y, z)) \vee \operatorname{On}(x, z, s) \wedge \neg \exists y(a=\operatorname{move}(x, z, y))\)
\(\operatorname{Clear}(x, \operatorname{do}(a, s)) \leftrightarrow \exists y, z(a=\operatorname{move}(y, x, z) \wedge\)
    \(\operatorname{On}(y, x, s)) \vee \operatorname{Clear}(x, s) \wedge \neg \exists y, z(a=\operatorname{move}(y, z, x))\)
\(\operatorname{Inheap}(x, \operatorname{do}(a, s)) \leftrightarrow a=\operatorname{pop}(x) \vee \operatorname{Inheap}(x, s) \wedge \neg \exists y(a=\operatorname{push}(x, y))\)
\(\operatorname{Top}(x, \operatorname{do}(a, s)) \leftrightarrow \exists y(a=\operatorname{push}(x, y)) \vee \exists y(a=\operatorname{pop}(y) \wedge U n d e r(y, x, s)) \vee\)
    \(\operatorname{Top}(x, s) \wedge a \neq \operatorname{pop}(x) \wedge \neg \exists y(a=\operatorname{push}(y, x))\)
\(\operatorname{Under}(x, y, \operatorname{do}(a, s)) \leftrightarrow a=\operatorname{push}(x, y) \vee U n d e r(x, y, s) \wedge a \neq \operatorname{pop}(x)\)
```

The first axiom is saying that a block $x$ will be on top of a block $z$ after moving $x$ from another block $y$ onto $z$, or if $x$ was already on $z$ and it was not moved elsewhere. The second axiom is saying that $x$ will become clear, i.e., there will be no blocks on top of $x$ after moving the block $y$ that was previously on top of $x$ onto another block $z$. Otherwise, if a block $x$ was already clear, it remains clear unless some block $y$ will be moved from the block $z$ onto the block $x$. The third axiom asserts that an entity $x$ is in a heap once it has been removed from a stack, or if it was already in a heap, and it was not stacked on top of another entity $y$. In the fourth axiom, when an entity $x$ is stacked upon an entity $y, x$ become the new top. Also, it becomes the top, when $x$ was located under some $y$ that was removed into a heap. Otherwise, an entity $x$ remains on the top unless it was unstacked or buried under by stacking another entity $y$ onto $x$. In the last fifth axiom, an entity $y$ will be under another entity $x$ after stacking $x$ on top of $y$, or $y$ remains under $x$ after any action that does not remove $x$ into a heap. It is easy to observe that all these SSAs are local-effect, and we will exploit this fact later in our paper.
Action precondition axioms (theory $\mathcal{D}_{a p}$ )
$\operatorname{Poss}(\operatorname{move}(x, y, z), s) \leftrightarrow \operatorname{Block}(x) \wedge \operatorname{Block}(y) \wedge \operatorname{Block}(z) \wedge O n(x, y, s) \wedge$
$\operatorname{Clear}(x, s) \wedge \operatorname{Clear}(z, s) \wedge x \neq z$
$\operatorname{Poss}(\operatorname{push}(x, y), s) \leftrightarrow \neg \operatorname{Block}(x) \wedge \neg \operatorname{Block}(y) \wedge \operatorname{Top}(y, s) \wedge \operatorname{Inheap}(x, s)$
$\operatorname{Poss}(\operatorname{pop}(x), s) \leftrightarrow \neg \operatorname{Block}(x) \wedge \operatorname{Top}(x, s)$
The precondition axioms are self-explanatory. The action move $(x, y, z)$ is possible in any situation $s$ where a block $x$ is located on top of a block $y$, both $x$ and a destination block $z$ are clear (i.e., not obstructed by any blocks on top of them) and $x$ is different from $z$. The last condition precludes moving $x$ on top of itself. According to the second precondition axiom, it is possible to stack $x$ on $y$ in any situation $s$, if $x$ and $y$ are entities which are not blocks, $y$ is the top of a stack, and $x$ is in a heap. The opposite operation of unstacking $x$ is possible if and only if $x$ is a top entity in situation $s$.

Initial Theory $\left(\mathcal{D}_{S_{0}}\right)$ is defined as the set of axioms ${ }^{4}$ using object constants $\{A, B, C\}$ :

$$
\begin{aligned}
& \neg \exists y O n\left(y, x, S_{0}\right) \wedge \exists y O n\left(x, y, S_{0}\right) \wedge \neg \operatorname{Inheap}\left(x, S_{0}\right) \rightarrow \operatorname{Clear}\left(x, S_{0}\right) \\
& \exists y \operatorname{On}\left(x, y, S_{0}\right) \rightarrow \operatorname{Block}(x) \\
& \left(\operatorname{Top}\left(x, S_{0}\right) \vee \operatorname{Inheap}\left(x, S_{0}\right)\right) \rightarrow \neg \operatorname{Block}(x) \\
& \operatorname{On}\left(A, B, S_{0}\right) \wedge \operatorname{Block}(B) \wedge \operatorname{Block}(C) \wedge \operatorname{Clear}\left(A, S_{0}\right) \wedge \operatorname{Clear}\left(C, S_{0}\right)
\end{aligned}
$$

Unique names axioms for actions and objects (theory $\mathcal{D}_{\text {una }}$ ) is the set of unique names axioms for all pairs of object constants and action functions used above.
Then $\Sigma \cup \mathcal{D}_{u n a} \cup \mathcal{D}_{a p} \cup \mathcal{D}_{s s} \cup \mathcal{D}_{S_{0}}$ is the resulting local-effect basic action theory.
Notice that all fluents are syntactically related in $\mathcal{D}_{S_{0}}$, so purely syntactic techniques fail to decompose $\mathcal{D}_{S_{0}}$ into components sharing no fluents. However, $\mathcal{D}_{s s}$ is the union of two theories with the intersection of signatures equal to $\{d o\}$. The set of precondition axioms is also union of two theories - the first axiom by itself is one of them, and the conjunction of the second and third axioms is another one - with the intersection of signatures equal to $\{$ Poss, Block $\}$. At the same time, the initial theory $\mathcal{D}_{S_{0}}$ is $\Delta$-decomposable for $\Delta=\left\{\right.$ Block, $\left.S_{0}\right\}$ into two distinct $\Delta$-inseparable components:

```
\(\neg \exists y \operatorname{On}\left(y, x, S_{0}\right) \wedge \exists y \operatorname{On}\left(x, y, S_{0}\right) \rightarrow C l e a r\left(x, S_{0}\right)\)
\(\exists y \operatorname{On}\left(x, y, S_{0}\right) \rightarrow \operatorname{Block}(x)\)
\(\operatorname{On}\left(A, B, S_{0}\right) \wedge \operatorname{Block}(B) \wedge \operatorname{Block}(C) \wedge C l e a r\left(A, S_{0}\right) \wedge C l e a r\left(C, S_{0}\right)\)
    and
\(\left(\operatorname{Top}\left(x, S_{0}\right) \vee \operatorname{Inheap}\left(x, S_{0}\right)\right) \rightarrow \neg \operatorname{Block}(x)\)
\(\exists x \operatorname{Block}(x)\)
```

This example is continued after Theorem 4.5 in Section 4, where we will show that the progression for $\mathcal{B} \mathcal{A} \mathcal{T}$ s of this kind preserves both decomposability and inseparability of the decomposition components.

## 3 Properties of Forgetting

There are two basic types of forgetting considered in the literature: forgetting a signature and forgetting a ground atom. As will be explained in Section 4, progression of $\mathcal{B A} \mathcal{T}$ s is closely related to forgetting. In particular, computing progression of a local-effect $\mathcal{B A \mathcal { A }}$ involves forgetting a set of ground atoms representing facts that are no longer true after an action execution. Thus, in order to understand the behavior of the component properties of theories under progression, one needs to first examine their relationship to forgetting, which is the purpose of this section. Although we are focused on forgetting ground atoms, the counterpart results for signature forgetting often come for free and are therefore included into this section. Moreover, they help to see the difference between the two types of forgetting, which contributes to a better understanding of this operation wrt the component properties, which we believe would be of interest to a broader audience in the literature. To emphasize broader applicability of these results, we consider a general first- and second-order logic setting in the remainder of this section.

[^1]Let us define a relation on structures as follows. Let $\sigma$ be a signature or a ground atom and $\mathcal{M}$, $\mathcal{M}^{\prime}$ be two many-sorted structures. Then we set $\mathcal{M} \sim_{\sigma} \mathcal{M}^{\prime}$ if:

- $\mathcal{M}$ and $\mathcal{M}^{\prime}$ have the same domain for each sort;
- $\mathcal{M}$ and $\mathcal{M}^{\prime}$ interpret all symbols which do not occur in $\sigma$ identically;
- if $\sigma$ is a ground atom $P(\bar{t})$ then $\mathcal{M}$ and $\mathcal{M}^{\prime}$ agree on interpretation $\bar{u}$ of $\bar{t}$ and for every vector of elements $\bar{v} \neq \bar{u}$, we have $\mathcal{M} \models P(\bar{v})$ iff $\mathcal{M}^{\prime} \models P(\bar{v})$.

Obviously, $\sim_{\sigma}$ is an equivalence relation.
The following notion summarizes the well-known Definitions 1 and 7 in [25].
Definition 3.1 (Forgetting an atom or signature) Let $\mathcal{T}$ be a theory in $\mathcal{L}$ and $\sigma$ be either a signature, or some ground atom. A set $\mathcal{T}^{\prime}$ of formulas in a fragment of second-order logic is called the result of forgetting $\sigma$ in $\mathcal{T}$ (denoted by forget $(\mathcal{T}, \sigma)$ ) if for any structure $\mathcal{M}^{\prime}$, we have $\mathcal{M}^{\prime} \models \mathcal{T}^{\prime}$ iff there is a model $\mathcal{M} \vDash \mathcal{T}$ such that $\mathcal{M} \sim{ }_{\sigma} \mathcal{M}^{\prime}$.

It is known that forget $(\mathcal{T}, \sigma)$ always exists, i.e. it is second-order definable, for a finite set of formulas $\mathcal{T}$ in $\mathcal{L}$ and a finite signature or a ground atom $\sigma$ (see [25], or Section 2.1 in [27]). On the other hand, the definition yields $\mathcal{T} \models$ forget $(\mathcal{T}, \sigma)$; thus, forget $(\mathcal{T}, \sigma)$ is a set of second-order consequences of $\mathcal{T}$ which suggests that it may not always be definable in the logic, where $\mathcal{T}$ is formulated, and it may not be finitely axiomatizable in this logic, even if $\mathcal{T}$ is so.

Fact 3.2 (Basic properties of forgetting) If $\sigma$ and $\pi$ are signatures or ground atoms and $\mathcal{T}, \mathcal{T}^{\prime}$ are theories in $\mathcal{L}$ then:

```
- forget \((\mathcal{T}, \sigma \cup \pi) \equiv\) forget \((f o r g e t(\mathcal{T}, \sigma), \pi)\) (if \(\sigma\) and \(\pi\) are signatures)
- forget \((\operatorname{forget}(\mathcal{T}, \sigma), \pi) \equiv\) forget \((f o r g e t ~(\mathcal{T}, \pi), \sigma)\)
- forget \((\) forget \((\mathcal{T}, \sigma), \sigma) \equiv \operatorname{forget}(\mathcal{T}, \sigma)\)
\(-\operatorname{forget}(\mathcal{T}, \sigma) \equiv \mathcal{T}\) if \(\sigma\) is a signature with \(\sigma \cap \operatorname{sig}(\mathcal{T})=\varnothing\), or a ground atom with predicate
    not contained in \(\operatorname{sig}(\mathcal{T})\)
\(-\operatorname{forget}\left(\mathcal{T} \cup \mathcal{T}^{\prime}, \sigma\right) \not \equiv \operatorname{forget}(\mathcal{T}, \sigma) \cup\) forget \(\left(\mathcal{T}^{\prime}, \sigma\right)(\) see Example 5)
\(-\operatorname{forget}(\varphi \vee \psi, \sigma) \equiv \operatorname{forget}(\varphi, \sigma) \vee \operatorname{forget}(\psi, \sigma)\) (if \(\varphi, \psi\) are formulas in \(\mathcal{L}\) ).
```

These properties either follow immediately from the definition, or from the results proven in [25].
Proposition 3.3 (Signature of forget $(\mathcal{T}, \sigma)$ ) Let $\mathcal{T}$ be a theory in $\mathcal{L}$, $\sigma$ a signature (or a ground atom) and let $\operatorname{forget}(\mathcal{T}, \sigma)$ be a set of formulas in a language $\mathcal{L}^{\prime}$, a fragment of second-order logic with PIP. Then $\operatorname{forget}(\mathcal{T}, \sigma)$ is logically equivalent in $\mathcal{L}^{\prime}$ to a set offormulas in the signature $\operatorname{sig}(\mathcal{T}) \backslash \sigma(\operatorname{sig}(\mathcal{T})$, respectively $)$.

Proof. We consider the case when $\sigma$ is a signature; the case of a ground atom being proved analogously. Assume that $\sigma \cap \operatorname{sig}(\operatorname{forget}(\mathcal{T}, \sigma)) \neq \varnothing$. Denote by $\operatorname{forget}(\mathcal{T}, \sigma)^{*}$ a "copy" of the set of formulas $\operatorname{forget}(\mathcal{T}, \sigma)$, where each symbol from $\sigma \cup[\operatorname{sig}(\operatorname{forget}(\mathcal{T}, \sigma)) \backslash \operatorname{sig}(\mathcal{T})]$ is uniquely replaced with a fresh symbol, not present in $\operatorname{sig}(\operatorname{forget}(\mathcal{T}, \sigma))$. We claim that forget $(\mathcal{T}, \sigma)^{*} \models_{\mathcal{L}^{\prime}}$ forget $(\mathcal{T}, \sigma)$. There is nothing to prove if forget $(\mathcal{T}, \sigma)^{*}$ is unsatisfiable. Note that, by definition of forgetting, forget $(\mathcal{T}, \sigma)^{*}$ and $\operatorname{forget}(\mathcal{T}, \sigma)$ are satisfiable iff $\mathcal{T}$ is. Let us assume that $\mathcal{T}$ is satisfiable. Take an arbitrary model $\mathcal{M}^{*} \models \operatorname{forget}(\mathcal{T}, \sigma)^{*}$; then there exists a model $\mathcal{M}^{\prime} \models \operatorname{forget}(\mathcal{T}, \sigma)$ which agrees on $\operatorname{sig}\left(\operatorname{forget}(\mathcal{T}, \sigma)^{*}\right)$ with $\mathcal{M}^{*}$ and interprets symbols
from $\sigma \cup[\operatorname{sig}(\operatorname{forget}(\mathcal{T}, \sigma)) \backslash \operatorname{sig}(\mathcal{T})]$ equally to the interpretation of the corresponding fresh symbols in $\mathcal{M}^{*}$. Therefore, we may assume that $\mathcal{M}^{*} \sim_{\sigma} \mathcal{M}^{\prime}$. By definition of forgetting, there is a model $\mathcal{M} \models \mathcal{T}$ such that $\mathcal{M}^{\prime} \sim_{\sigma} \mathcal{M}$, hence $\mathcal{M}^{*} \sim_{\sigma} \mathcal{M}$ and $\mathcal{M}^{*} \models \operatorname{forget}(\mathcal{T}, \sigma)$. We have $\operatorname{forget}(\mathcal{T}, \sigma)^{*} \models_{\mathcal{L}^{\prime}} \operatorname{forget}(\mathcal{T}, \sigma)$ and $\operatorname{sig}\left(\operatorname{forget}(\mathcal{T}, \sigma)^{*}\right) \cap \operatorname{sig}(\operatorname{forget}(\mathcal{T}, \sigma)) \subseteq \operatorname{sig}(\mathcal{T}) \backslash$ $\sigma$. By PIP, there is a set of formulas $\Theta$ in signature $\operatorname{sig}(\mathcal{T}) \backslash \sigma$ such that $\operatorname{forget}(\mathcal{T}, \sigma)^{*} \models_{\mathcal{L}^{\prime}} \Theta$ and $\Theta \models_{\mathcal{L}^{\prime}}$ forget $(\mathcal{T}, \sigma)$. Note that forget $(\mathcal{T}, \sigma)^{*} \models_{\mathcal{L}^{\prime}} \Theta$ yields forget $(\mathcal{T}, \sigma) \models_{\mathcal{L}^{\prime}} \Theta$, because every model of forget $(\mathcal{T}, \sigma)$ can be expanded to a model of $\operatorname{forget}(\mathcal{T}, \sigma)^{*}$ and the reduct of this model onto (a subset of) $\operatorname{sig}(\mathcal{T}) \backslash \sigma$ suffices to satisfy $\Theta$. Thus, we conclude that forget $(\mathcal{T}, \sigma)$ is equivalent to $\Theta$.

Corollary 3.4 Let $\mathcal{T}$ be a theory in $\mathcal{L}$ having PIP and $\sigma$ a signature. Then $\mathcal{T} \equiv \operatorname{forget}(\mathcal{T}, \sigma)$ iff $\mathcal{T}$ is equivalent to a set of formulas in the signature $\operatorname{sig}(\mathcal{T}) \backslash \sigma$.

We note that the similar statement does not hold when $\sigma$ is a ground atom. It follows from Proposition 3.3 that in case $\sigma$ is a signature, forget $(\mathcal{T}, \sigma)$ axiomatizes the class of reducts of models of $\mathcal{T}$ onto the signature $\operatorname{sig}(\mathcal{T}) \backslash \sigma$. Clearly, if $\mathcal{T}$ is a theory in language $\mathcal{L}$, then forget $(\mathcal{T}, \sigma)$ may not be in $\mathcal{L}$, however it is always expressible in second-order logic if $\mathcal{T}$ is finitely axiomatizable (we note that second-order logic has PIP). For the case when $\sigma$ is a signature, $\operatorname{forget}(\mathcal{T}, \sigma)$ is known as $\operatorname{sig}(\mathcal{T}) \backslash \sigma$-uniform interpolant of $\mathcal{T}$ wrt the language $\mathcal{L}$ and second-order queries, that is, wrt the pair ( $\mathcal{L}$, second-order logic), see Definition 13 in [18] and Lemma 39 in [32] for a justification. In other words, $\mathcal{T}$ and forget $(\mathcal{T}, \sigma)$ semantically entail the same second-order formulas in signature $\mathcal{T} \backslash \sigma$.

If $\sigma$ is a ground atom $P(\bar{t})$ then, by definition, for any model $\mathcal{M} \models \mathcal{T}$, forget $(\mathcal{T}, \sigma)$ must have two "copies" of $\mathcal{M}$ : a model with the value of $P(\bar{t})$ false and a model where this value is true. Let $\mathcal{L}$ be first-order logic. In contrast to forgetting a signature, for any recursively axiomatizable theory $\mathcal{T}$ in $\mathcal{L}$ and a ground atom $\sigma$, one can effectively construct the set of formulas forget $(\mathcal{T}, \sigma)$ in $\mathcal{L}$ such that forget $(\mathcal{T}, \sigma)$ is finitely axiomatizable iff $\mathcal{T}$ is. This follows from Theorem 4 in [25], where it is shown that forgetting a ground atom $P(\bar{t})$ in a theory $\mathcal{T}$ can be computed by simple syntactic manipulations:

- for an axiom $\varphi \in \mathcal{T}$, denote by $\varphi[P(\bar{t})]$ the result of replacing every occurrence of atom $P\left(\overline{t^{\prime}}\right)$ (with $\overline{t^{\prime}}$ a term) by formula $\left[\bar{t}=\bar{t}^{\prime} \wedge P(\bar{t})\right] \vee\left[\bar{t} \neq \overline{t^{\prime}} \wedge P\left(\overline{t^{\prime}}\right)\right]$
- denote by $\varphi^{+}[P(\bar{t})]$ the formula $\varphi[P(\bar{t})]$ with every occurrence of the ground atom $P(\bar{t})$ replaced with true and similarly, denote by $\varphi^{-}[P(\bar{t})]$ the formula $\varphi[P(\bar{t})]$ with $P(\bar{t})$ replaced with false
- then $\operatorname{forget}(\mathcal{T}, P(\bar{t}))$ is equivalent to $\left(\bigwedge_{\varphi \in \mathcal{T}} \varphi^{+}[P(\bar{t})]\right) \bigvee\left(\bigwedge_{\varphi \in \mathcal{T}} \varphi^{-}[P(\bar{t})]\right)$.

The disjunction corresponds to the union of two classes of models obtained from models of $\mathcal{T}$, with the ground atom $P(\bar{t})$ interpreted as true and false, respectively. This fact is important for effective computation of progression for local-effect $\mathcal{B} \mathcal{A} \mathcal{T}$ s mentioned in Section 4.

Example 3 (Forgetting a ground atom). Consider a theory $\mathcal{T}=\{\varphi\}$, where $\varphi=\neg P(c)$, i.e., $P(c)$ is false in every model of $\mathcal{T}$. Consider forgetting $P(c)$ in $\mathcal{T}$. By the (semantic) definition of forgetting, the set of models of forget $(\mathcal{T}, P(c))$ consists of models of $\mathcal{T}$ and those models, in which $P(c)$ is true. Therefore, any structure is a model of forget $(\mathcal{T}, P(c))$. Now consider the syntactic definition of forgetting given above. We have

$$
\varphi[P(c)]=\neg([c=c \wedge P(c)] \vee[c \neq c \wedge P(c)])
$$

thus, $\varphi^{+}[P(c)] \equiv$ false, $\varphi^{-}[P(c)] \equiv$ true and hence, $\operatorname{forget}(\mathcal{T}, P(c)) \equiv \varphi^{+}[P(c)] \vee \varphi^{-}[P(c)]$ is a tautology.

Now consider forgetting $P(c)$ in the theory $\mathcal{T}=\{\varphi\}$, where $\varphi=\forall x P(x)$. By the (semantic) definition of forgetting, any structure, in which $P$ is true on every element, except possibly, the interpretation of $c$, is a model of forget $(\mathcal{T}, P(c))$. By the syntactic definition of forgetting we have:

$$
\varphi[P(c)]=\forall x([x=c \wedge P(c)] \vee[x \neq c \wedge P(x)])
$$

thus, $\varphi^{+}[P(c)]=\forall x(x=c \vee[x \neq c \wedge P(x)]), \varphi^{-}[P(c)]=\forall x(x \neq c \wedge P(x))$. Since $\varphi^{-}[P(c)] \equiv$ false, we obtain forget $(\mathcal{T}, P(c)) \equiv \varphi^{+}[P(c)]$ and hence, forget $(\mathcal{T}, P(c)) \equiv$ $\forall x(x \neq c \rightarrow P(x))$.

We note that in case a theory $\mathcal{T}$ is finitely axiomatizable, computing forget $(\mathcal{T}, P(\bar{t}))$ in the way above doubles the size of theory in the worst case, due to the disjunction. It is sometimes necessary to consider forgetting of some set $S$ of ground atoms in a theory $\mathcal{T}$. This is equivalent to iterative computation of forgetting of atoms from $S$ starting from the theory $\mathcal{T}$ (the order on atoms can be chosen arbitrary as noted in Fact 3.2). However, it is important to note that the size of the resulting theory is $O\left(2^{|S|} \times|\mathcal{T}|\right)$, where $|S|$ is the number of atoms in $S$ and $|\mathcal{T}|$ is the size of $\mathcal{T}$.

Proposition 3.5 (Interplay of forgetting and entailment) Let $\mathcal{T}$ and $\mathcal{T}_{1}$ be two sets of formulas in $\mathcal{L}$, with $\mathcal{T} \models \mathcal{T}_{1}$, and $\sigma$ be a signature or a ground atom. Then the following holds:


Proof. Follow the diagram starting from the top-left column. By definition of forgetting, every model of $\mathcal{T}$ is a model of forget $(\mathcal{T}, \sigma)$, so we have $\mathcal{T} \models \operatorname{forget}(\mathcal{T}, \sigma)$ as shown in the left column of the diagram. Similarly, $\mathcal{T}_{1} \models$ forget $\left(\mathcal{T}_{1}, \sigma\right)$ in the right column of the diagram. To prove entailment at the bottom we rely on entailments in the columns and on the given entailment at the top, i.e., we navigate the diagram up from the bottom-left corner, then move right, and finally go down to the bottom-right expression. Let $\mathcal{M}^{\prime}$ be an arbitrary model of $\operatorname{forget}(\mathcal{T}, \sigma)$. Then there is a model $\mathcal{M} \models \mathcal{T}$ such that $\mathcal{M} \sim_{\sigma} \mathcal{M}^{\prime}$. Since $\mathcal{T} \models \mathcal{T}_{1}$, we have $\mathcal{M} \models \mathcal{T}_{1}$, so we conclude that $\mathcal{M}^{\prime} \models$ forget $\left(\mathcal{T}_{1}, \sigma\right)$, because $\mathcal{M}$ is a model satisfying the conditions of Definition 3.1 for $\mathcal{T}_{1}$ and $\mathcal{M}^{\prime}$. Thus, we proved entailment shown in the bottom row of the diagram.

Proposition 3.6 (Preservation of consequences under forgetting) Let $\mathcal{T}$ be a theory in $\mathcal{L}$ and $\sigma$ be either a signature or a ground atom. Let $\varphi$ be a formula such that either $\operatorname{sig}(\varphi) \cap \sigma=\varnothing$ (in case $\sigma$ is a signature), or which does not contain the predicate from $\sigma$ (if $\sigma$ is a ground atom). Then $\mathcal{T} \models \varphi$ iff $\operatorname{forget}(\mathcal{T}, \sigma) \models \varphi$.

Proof. From Proposition 3.5, we have $\mathcal{T} \models \operatorname{forget}(\mathcal{T}, \sigma)$, thus forget $(\mathcal{T}, \sigma) \models \varphi$ yields $\mathcal{T} \models \varphi$. Now let $\mathcal{T} \models \varphi$ and assume there is a model $\mathcal{M}^{\prime}$ of $\operatorname{forget}(\mathcal{T}, \sigma)$ such that $\mathcal{M}^{\prime} \not \models \varphi$. By definition of forgetting, there exists a model $\mathcal{M}$ of $\mathcal{T}$ such that $\mathcal{M} \sim_{\sigma} \mathcal{M}^{\prime}$, i.e. $\mathcal{M}$ and $\mathcal{M}^{\prime}$ have the same universe and may differ only on interpretation of signature $\sigma$ (ground atom $\sigma$ ). By the condition on signature of $\varphi$, then $\mathcal{M}$ is not a model of $\varphi$, which contradicts $\mathcal{T} \models \varphi$.

Now we answer the question when inseparability is preserved under forgetting. This is important for our research, since we are interested in preservation of inseparability under progression, the operation which relies on forgetting in local-effect $\mathcal{B} \mathcal{A} \mathcal{T}$ s. We demonstrate that it is important to distinguish between forgetting something in $\Delta$ (the common symbols of theories) or outside of the shared signature. While Proposition 3.6 shows that the situation is simple in the latter case, it is apriory unclear, whether the same holds in the former. Example 4 demonstrates that this is not true, while the accompanying Propositions 3.7, 3.8 describe the cases when this situation can be recovered. We believe that giving the accompanying positive results is important in order to provide a big picture to the reader. Proposition 3.7 shows that signature forgetting (the arguably more frequently used type of forgetting in the literature) preserves inseparability, while Proposition 3.8 tackles this question from another perspective. It shows that semantic inseparability, the property also well studied in the literature, is the stronger form of inseparability, which is invariant under forgetting.

Observe that by Proposition 3.6 and the first item of Fact 3.2, when studying preservation of $\Delta$-inseparability of two sets of formulas for a signature $\Delta$, it is sufficient to consider the case of forgetting a subset of $\Delta$ or a ground atom with the predicate from $\Delta$, respectively.

Proposition 3.7 (Preservation of $\Delta$-insep. under signature forgetting) Let $\mathcal{L}$ have PIP and $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ be two $\Delta$-inseparable sets of formulas in $\mathcal{L}$ with $\operatorname{sig}\left(\mathcal{T}_{1}\right) \cap \operatorname{sig}\left(\mathcal{T}_{2}\right)=\Delta$, for a signature $\Delta$. Let $\sigma$ be a subsignature of $\Delta$ and forget $\left(\mathcal{T}_{1}, \sigma\right)$ and $\operatorname{forget}\left(\mathcal{T}_{2}, \sigma\right)$ be sets of formulas of $\mathcal{L}$. Then forget $\left(\mathcal{T}_{1}, \sigma\right)$ and forget $\left(\mathcal{T}_{2}, \sigma\right)$ are $\Delta$-inseparable.

Proof. Let $\varphi$ be a formula with $\operatorname{sig}(\varphi) \subseteq \Delta$ such that forget $\left(\mathcal{T}_{1}, \sigma\right) \models \varphi$. By Proposition 3.3, we may assume that for $i=1,2$ the signature of $\operatorname{forget}\left(\mathcal{T}_{i}, \sigma\right)$ is a subset of $\operatorname{sig}\left(\mathcal{T}_{i}\right) \backslash \sigma$. We depict the direction of the proof in the figure below.


We start with the lower-left corner of the diagram and navigate up, then right, and finally down. As forget $\left(\mathcal{T}_{1}, \sigma\right) \models \varphi$, by PIP, there is a set of formulas $\mathcal{T}_{1}^{\prime}$ with $\operatorname{sig}\left(\mathcal{T}_{1}^{\prime}\right) \subseteq \operatorname{sig}\left(\right.$ forget $\left(\mathcal{T}_{1}\right.$, $\sigma)) \cap \operatorname{sig}(\varphi) \subseteq \Delta \backslash \sigma$ such that forget $\left(\mathcal{T}_{1}, \sigma\right) \models \mathcal{T}_{1}^{\prime}$ and $\mathcal{T}_{1}^{\prime} \models \varphi$. Then, by Proposition 3.6, we have $\mathcal{T}_{1} \models \mathcal{T}_{1}^{\prime}$. This proves entailment in the top-left corner. Since $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are $\Delta$ inseparable and $\operatorname{sig}\left(\mathcal{T}_{1}^{\prime}\right) \subseteq \Delta$, we obtain $\mathcal{T}_{2} \models \mathcal{T}_{1}^{\prime}$. Therefore, the top-right entailment holds. Again, since $\operatorname{sig}\left(\mathcal{T}_{1}^{\prime}\right) \cap \sigma=\varnothing$, by Proposition 3.6, we conclude that $\operatorname{forget}\left(\mathcal{T}_{2}, \sigma\right) \models \mathcal{T}_{1}^{\prime}$ and thus, forget $\left(\mathcal{T}_{2}, \sigma\right) \models \varphi$.

The following example demonstrates that a similar result does not hold under forgetting a ground atom with the predicate from $\Delta$.

Example 4 ( $\Delta$-inseparability lost under forgetting a ground atom). We give an example of a logic $\mathcal{L}$, sets of formulas $\mathcal{T}_{1}, \mathcal{T}_{2}$ in $\mathcal{L}$, and a signature $\Delta=\operatorname{sig}\left(\mathcal{T}_{1}\right) \cap \operatorname{sig}\left(\mathcal{T}_{2}\right)$ such that $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are $\Delta$-inseparable, but forget $\left(\mathcal{T}_{1}, R(c, c)\right)$ and forget $\left(\mathcal{T}_{2}, R(c, c)\right)$ are not, for a ground atom $R(c, c)$ with a predicate $R \in \Delta$. Let $\mathcal{L}$ be Description Logic $\mathcal{E} \mathcal{L} \mathcal{O}^{\perp}$, i.e. the sub-boolean logic $\mathcal{E} \mathcal{L}$ augmented with nominals and the bottom concept $\perp$. Let $\Sigma=\{R, a, c\}$ be signature, where $R$ is
a role name (binary predicate) and $a, c$ are nominals (i.e. constants). Define a set of formulas $\mathcal{T}_{1}$ in the signature $\Sigma$ as $\{\{a\} \sqcap\{c\} \sqsubseteq \perp,\{c\} \sqsubseteq \exists R .\{a\}, \top \sqsubseteq \exists R$.丁 $\rceil$. Set $\Delta=\{R, c\}$ and consider the set of formulas $\mathcal{T}_{2}=\{\top \sqsubseteq \exists R$. $\top$, $\operatorname{Taut}(c)\}$, where $\operatorname{Taut}(c)$ is a tautology with the nominal $c$ (e.g., the formula $\{c\} \sqsubseteq \top$ ). We have $\operatorname{sig}\left(\mathcal{T}_{1}\right) \cap \operatorname{sig}\left(\mathcal{T}_{2}\right)=\Delta$ and it is easy to check that $\mathcal{T}_{2}$ is equivalent to Cons $\left(\mathcal{T}_{1}, \Delta\right)$ in the logic $\mathcal{E} \mathcal{L O}^{\perp}$; thus, $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are $\Delta$-inseparable. Now consider forget $\left(\mathcal{T}_{1}, R(c, c)\right)$ and forget $\left(\mathcal{T}_{2}, R(c, c)\right)$ as sets of formulas in second-order logic (we assume the standard translation of formulas of $\mathcal{E} \mathcal{L} \mathcal{O}^{\perp}$ into the language of second-order logic). We verify that they are not $\Delta$-inseparable and the formula $\top \sqsubseteq \exists R$. $\top$ is the witness for this. By definition of $\mathcal{T}_{1}$, we have forget $\left(\mathcal{T}_{1}, R(c, c)\right) \models \mathcal{T}_{1}$, since any model of $\mathcal{T}_{1}$ with a changed truth value of the predicate $R$ on the pair $\langle c, c\rangle$ is still a model of $\mathcal{T}_{1}$. On the other hand, forget $\left(\mathcal{T}_{2}, R(c, c)\right) \not \models$ $\top \sqsubseteq \exists R$. $\top$, because $\mathcal{T}_{2}$ has the one-element model $\mathcal{M}$, where $R$ is reflexive (on the sole element corresponding to $c$ ). Hence, by definition of forgetting, the one-element model $\mathcal{M}^{\prime}$ with $R$ false on the pair $\langle c, c\rangle$ must be a model of forget $\left(\mathcal{T}_{2}, R(c, c)\right)$, but obviously, $\mathcal{M}^{\prime} \not \equiv \top \sqsubseteq \exists R$.丁.

It turns out that the preservation of inseparability under forgetting a ground atom requires rather strong model-theoretic conditions like $\left(^{*}\right.$ ) in Proposition 3.8 below. Specialists might notice that ${ }^{(*)}$ ) is equivalent to semantic $\Delta$-inseparability of the initial sets of formulas (see Definition 11 in [18]) which is very hard to decide from the computational point of view (see Theorem 3 in [30], Lemma 40 in [32]). Nevertheless, there are practically useful restrictions under which the complexity becomes feasible [19]. Semantic $\Delta$-inseparability is stronger than the notion of inseparability given in Definition 2.4: it means that the theories are indistinguishable by second-order formulas. On the other hand, Proposition 3.8 says that whenever there is a chance to satisfy $(*)$ for two given sets of formulas, one does not need to check it again after forgetting something in their common signature. To compare condition (*) with Example 4, note that the aforementioned one-element model of $\mathcal{T}_{2}$ does not expand to a model of $\mathcal{T}_{1} \cup \mathcal{T}_{2}$.

Proposition 3.8 (Preservation of $\Delta$-inseparability under forgetting) Let $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ be two sets of formulas in $\mathcal{L}$, with $\operatorname{sig}\left(\mathcal{T}_{1}\right) \cap \operatorname{sig}\left(\mathcal{T}_{2}\right)=\Delta$, for a signature $\Delta$, which satisfy the following condition ( ${ }^{*}$ ): for $i=1,2$, any model of $\mathcal{T}_{i}$ can be expanded to a model of $\mathcal{T}_{1} \cup \mathcal{T}_{2}$. Then:

- $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are $\Delta$-inseparable;
- for $\sigma$ a signature or a ground atom, forget $\left(\mathcal{T}_{1}, \sigma\right)$ and $\operatorname{forget}\left(\mathcal{T}_{2}, \sigma\right)$ satisfy $\left(^{*}\right)$ as well.

Proof. $\Delta$-inseparability is the immediate consequence of (*): if $\varphi$ is a formula with $\operatorname{sig}(\varphi) \subseteq \Delta$, $\mathcal{T}_{1} \models \varphi$, but $\mathcal{T}_{2} \not \models \varphi$, then there is a model $\mathcal{M}_{2}$ of $\mathcal{T}_{2}$ such that $\mathcal{M}_{2} \not \models \varphi$. Then there is an expansion $\mathcal{M}$ of $\mathcal{M}_{2}$ such that $\mathcal{M} \models \mathcal{T}_{1} \cup \mathcal{T}_{2},\left.\mathcal{M}\right|_{\text {sig }\left(\mathcal{T}_{1}\right)} \models \mathcal{T}_{1}$, but $\left.\mathcal{M}\right|_{\operatorname{sig}\left(\mathcal{T}_{1}\right)} \not \models \varphi$, a contradiction. Now let us verify that for $i=1,2$, any model of forget $\left(\mathcal{T}_{i}, \sigma\right)$ can be expanded to a model of forget $\left(\mathcal{T}_{1}, \sigma\right) \cup$ forget $\left(\mathcal{T}_{2}, \sigma\right)$. For instance, let $\mathcal{M}_{2}^{\prime}$ be a model of $\operatorname{forget}\left(\mathcal{T}_{2}, \sigma\right)$. Consider a model $\mathcal{M}_{2}$ of $\mathcal{T}_{2}$, such that $\mathcal{M}_{2} \sim_{\sigma} \mathcal{M}_{2}^{\prime}$, and expand it to a model $\mathcal{M}$ of $\mathcal{T}_{1} \cup \mathcal{T}_{2}$. Then by definition of forgetting, there must be a model $\mathcal{M}^{\prime} \models \operatorname{forget}\left(\mathcal{T}_{1}, \sigma\right)$ with $\mathcal{M}^{\prime} \sim_{\sigma} \mathcal{M}$, which agrees with $\mathcal{M}_{2}^{\prime}$ on $\sigma$ (if $\sigma$ is a signature), or on the predicate of $\sigma$ (if $\sigma$ is a ground atom). By construction, $\mathcal{M}^{\prime}$ is an expansion of $\mathcal{M}_{2}^{\prime}$ and thus a model for $f$ orget $\left(\mathcal{T}_{1}, \sigma\right) \cup$ forget $\left(\mathcal{T}_{2}, \sigma\right)$.

Let $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ be two sets of formulas in $\mathcal{L}$, with $\operatorname{sig}\left(\mathcal{T}_{1}\right) \cap \operatorname{sig}\left(\mathcal{T}_{2}\right)=\Delta$, for a signature $\Delta$, and let $\sigma$ be either a subsignature of $\Delta$ or a ground atom with the predicate from $\Delta$. It is known that in general, forgetting $\sigma$ may not be distributive over union of sets of formulas. The entailment forget $\left(\mathcal{T}_{1} \cup \mathcal{T}_{2}, \sigma\right) \models \operatorname{forget}\left(\mathcal{T}_{1}, \sigma\right) \cup \operatorname{forget}\left(\mathcal{T}_{2}, \sigma\right)$ holds by Proposition 3.5, but Example

5 below easily shows that even strong semantic conditions related to modularity do not guarantee the reverse entailment. On the other hand, forgetting something outside of the common signature of $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ is distributive over union, as formulated in Corollary 3.10 which is a consequence of the criterion in Proposition 3.9 and is used in the proof of one of our main results, Theorem 4.5.
Example 5 (Failure of componentwise forgetting in $\Delta$ ). Let $\mathcal{L}$ be first-order logic and $\Delta=\{P, c\}$ be the signature consisting of a unary predicate $P$ and a constant $c$. Define theories $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ as: $\mathcal{T}_{1}=$ $\{A \rightarrow P(c)\}, \mathcal{T}_{2}=\{P(c) \rightarrow B\}$, where $A, B$ are nullary predicate symbols. We have sig $\left(\mathcal{T}_{1}\right) \cap$ $\operatorname{sig}\left(\mathcal{T}_{2}\right)=\Delta$ and for $i=1,2$, any model of $\mathcal{T}_{i}$ can be expanded to a model of $\mathcal{T}_{1} \cup \mathcal{T}_{2}$. Clearly, $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are $\Delta$-inseparable and for $i=1,2$, Cons $\left(\mathcal{T}_{i}, \Delta\right)$ is the set of tautologies in $\Delta$. By definition of forgetting, for $i=1,2$, forget $\left(\mathcal{T}_{i}, P(c)\right)$ is a set of tautologies and thus, forget $\left(\mathcal{T}_{1}, P(c)\right) \cup$ forget $\left(\mathcal{T}_{2}, P(c)\right) \neq$ forget $\left(\mathcal{T}_{1} \cup \mathcal{T}_{2}, P(c)\right)$, because forget $\left(\mathcal{T}_{1} \cup \mathcal{T}_{2}, P(c)\right) \models A \rightarrow B$ (by Proposition 3.6). For the case of forgetting a signature, say a nullary predicate $P$, it suffices to consider $\Delta=\{P\}$ and theories $\mathcal{T}_{1}=\{A \rightarrow P\}, \mathcal{T}_{2}=\{P \rightarrow B\}$, where $A, B$ are nullary predicates.

Proposition 3.9 (A criterion for componentwise forgetting) Let $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ be two sets offormulas and $\sigma$ either a signature or a ground atom. Then the following statements are equivalent:
$-\operatorname{forget}\left(\mathcal{T}_{1}, \sigma\right) \cup \operatorname{forget}\left(\mathcal{T}_{2}, \sigma\right) \models \operatorname{forget}\left(\mathcal{T}_{1} \cup \mathcal{T}_{2}, \sigma\right)$

- for any two models $\mathcal{M}_{1} \models \mathcal{T}_{1}$ and $\mathcal{M}_{2} \models \mathcal{T}_{2}$, with $\mathcal{M}_{1} \sim_{\sigma} \mathcal{M}_{2}$, there exists a model $\mathcal{M} \models$ $\mathcal{T}_{1} \cup \mathcal{T}_{2}$ such that $\mathcal{M} \sim_{\sigma} \mathcal{M}_{i}$, for some $i=1,2$.

Proof. Note in the second condition, the requirement $\mathcal{M} \sim_{\sigma} \mathcal{M}_{i}$ for some $i=1,2$ is equivalent to $\mathcal{M} \sim_{\sigma} \mathcal{M}_{i}$ for all $i=1,2$, by transitivity of $\sim_{\sigma} .(\Rightarrow)$ : Let $\mathcal{M}_{1} \models \mathcal{T}_{1}$ and $\mathcal{M}_{2} \models \mathcal{T}_{2}$ be models with $\mathcal{M}_{1} \sim_{\sigma} \mathcal{M}_{2}$. Then there are models $\mathcal{M}_{1}^{\prime}$ and $\mathcal{M}_{2}^{\prime}$ such that for $i=1,2, \mathcal{M}_{i}^{\prime} \models \operatorname{forget}\left(\mathcal{T}_{i}, \sigma\right)$ and $\mathcal{M}_{i}^{\prime} \sim_{\sigma} \mathcal{M}_{i}$. Then, by transitivity of $\sim_{\sigma}$, for all $i, j=1,2$ we have $\mathcal{M}_{i}^{\prime} \sim_{\sigma} \mathcal{M}_{j}$ and thus, $\mathcal{M}_{i}^{\prime} \models$ forget $\left(\mathcal{T}_{j}, \sigma\right)$. Then $\mathcal{M}_{1}^{\prime} \models \operatorname{forget}\left(\mathcal{T}_{1} \cup \mathcal{T}_{2}, \sigma\right)$, so there exists a model $\mathcal{M} \models \mathcal{T}_{1} \cup \mathcal{T}_{2}$ such that $\mathcal{M} \sim_{\sigma} \mathcal{M}_{1}^{\prime}$ and hence, $\mathcal{M} \sim_{\sigma} \mathcal{M}_{1} .(\Leftarrow)$ : Let $\mathcal{M}^{\prime}$ be a model of $\operatorname{forget}\left(\mathcal{T}_{1}, \sigma\right) \cup \operatorname{forget}\left(\mathcal{T}_{2}, \sigma\right)$. There exist models $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ such that for $i=1,2, \mathcal{M}_{i} \models \mathcal{T}_{i}$ and $\mathcal{M}_{i} \sim_{\sigma} \mathcal{M}^{\prime}$. Then $\mathcal{M}_{1} \sim_{\sigma}$ $\mathcal{M}_{2}$, hence, there must be a model $\mathcal{M}$ of $\mathcal{T}_{1} \cup \mathcal{T}_{2}$ with $\mathcal{M} \sim_{\sigma} \mathcal{M}_{i}$ for some $i=1,2$. Then we obtain that $\mathcal{M} \sim_{\sigma} \mathcal{M}^{\prime}$ and thus, by definition of forgetting, $\mathcal{M}^{\prime}$ is a model of $\operatorname{forget}\left(\mathcal{T}_{1} \cup \mathcal{T}_{2}, \sigma\right)$.

To compare this criterion with Example 5, observe that there exist models $\mathcal{M}_{1} \models \mathcal{T}_{1}$ and $\mathcal{M}_{2} \models$ $\mathcal{T}_{2}$ with common domain such that $\mathcal{M}_{1} \models A \wedge P(c) \wedge \neg B$ and $\mathcal{M}_{2} \models A \wedge \neg P(c) \wedge \neg B$. Thus, $\mathcal{M}_{1} \sim_{P(c)} \mathcal{M}_{2}$, however, there does not exist a model $\mathcal{M}$ of $\mathcal{T}_{1} \cup \mathcal{T}_{2}$ such that $\mathcal{M} \sim_{P(c)} \mathcal{M}_{i}$, for some $i=1,2$. Neither $\mathcal{M}_{1}$, nor $\mathcal{M}_{2}$ is a model for $\mathcal{T}_{1} \cup \mathcal{T}_{2}$.

Corollary 3.10 (Forgetting in the scope of one component) Let $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ be two sets of formulas, with $\operatorname{sig}\left(\mathcal{T}_{1}\right) \cap \operatorname{sig}\left(\mathcal{T}_{2}\right)=\Delta$, for a signature $\Delta$, and $\sigma$ be either a subsignature of $\operatorname{sig}\left(\mathcal{T}_{1}\right) \backslash \Delta$ or a ground atom with the predicate from $\operatorname{sig}\left(\mathcal{T}_{1}\right) \backslash \Delta$. Then $\operatorname{forget}\left(\mathcal{T}_{1} \cup \mathcal{T}_{2}, \sigma\right)$ is equivalent to forget $\left(\mathcal{T}_{1}, \sigma\right) \cup \mathcal{T}_{2}$. Moreover, if $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are $\Delta$-inseparable, then so are forget $\left(\mathcal{T}_{1}, \sigma\right)$ and $\mathcal{T}_{2}$.

Proof. Note that by the choice of $\sigma, \mathcal{T}_{2}$ is equivalent to forget $\left(\mathcal{T}_{2}, \sigma\right)$ and thus, by Proposition 3.5, it suffices to verify the entailment $\operatorname{forget}\left(\mathcal{T}_{1}, \sigma\right) \cup \operatorname{forget}\left(\mathcal{T}_{2}, \sigma\right) \models \operatorname{forget}\left(\mathcal{T}_{1} \cup \mathcal{T}_{2}, \sigma\right)$. If there are models $\mathcal{M}_{1} \models \mathcal{T}_{1}$ and $\mathcal{M}_{2} \models \mathcal{T}_{2}$, with $\mathcal{M}_{1} \sim_{\sigma} \mathcal{M}_{2}$, then in fact, $\mathcal{M}_{1} \models \mathcal{T}_{1} \cup \mathcal{T}_{2}$, by the choice of $\sigma$ and definition of $\sim_{\sigma}$. Thus, the criterion from Proposition 3.9 obviously yields the required entailment. It remains to note that $\Delta$-inseparability of $\operatorname{forget}\left(\mathcal{T}_{1}, \sigma\right)$ and forget $\left(\mathcal{T}_{2}, \sigma\right)$ follows from the choice of $\sigma$, Proposition 3.6, and $\Delta$-inseparability of $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$.

In general, the results of this section prove that the operation of forgetting does not behave well wrt the modularity properties of the input, since they are logic-dependent. Stronger model-theoretic conditions on the input are needed due to the model-theoretic nature of forgetting.

## 4 Properties of Progression

We have considered some component properties of forgetting. It turns out that the operation of progression is closely related to forgetting in initial theories. However, in case of progression, we can not restrict ourselves to working with initial theories only; we need also to take into account information from successor state axioms. The aim of this section is to study component properties of progression wrt different forms of SSAs and common signatures $\Delta \mathrm{s}$ (deltas) of components of initial theories. We will consider local-effect SSAs discussed in [27] and deltas, which do not contain fluents.

We use the following notations further in the paper. For a ground action term $\alpha$ in the language of the situation calculus, we denote by $S_{\alpha}$ the situation term $d o\left(\alpha, S_{0}\right)$. To define progression, we introduce an equivalence relation on many-sorted structures in the situation calculus signature. For two structures $\mathcal{M}, \mathcal{M}^{\prime}$ and a ground action $\alpha$, we set $\mathcal{M} \sim_{S_{\alpha}} \mathcal{M}^{\prime}$ if:

- $\mathcal{M}$ and $\mathcal{M}^{\prime}$ have the same sorts for action and object;
- $\mathcal{M}$ and $\mathcal{M}^{\prime}$ interpret all situation-independent predicate and function symbols identically;
- $\mathcal{M}$ and $\mathcal{M}^{\prime}$ agree on interpretation of all fluents at $S_{\alpha}$, i.e., for every fluent $F$ and every variable assignment $\theta$, we have $\mathcal{M}, \theta \models F\left(\bar{x}, S_{\alpha}\right)$ iff $\mathcal{M}^{\prime}, \theta \models F\left(\bar{x}, S_{\alpha}\right)$.

That is, if $\mathcal{M} \sim_{S_{\alpha}} \mathcal{M}^{\prime}$ then the structures $\mathcal{M}$ and $\mathcal{M}^{\prime}$ are allowed to differ in sorts for situation and interpretations of fluents at situation terms, not equal to $S_{\alpha}$.

Note that a similar notation with $\sim$ is used to denote the equivalence relation on models from Definition 3.1 of forgetting. The two notations are easily distinguished depending on the context and are standard in the literature, therefore we adopt both of them in our paper.

Definition 4.1 (Progression, modified Definition 9.1.1 in [43]) Let $\mathcal{D}$ be a basic action theory with unique name axioms $\mathcal{D}_{\text {una }}$ and the initial theory $\mathcal{D}_{S_{0}}$, and let $\alpha$ be a ground action term. A set $\mathcal{D}_{S_{\alpha}}$ of formulas in a fragment of second-order logic is called progression of $\mathcal{D}_{S_{0}}$ wrt $\alpha$ if it is uniform in the situation term $S_{\alpha}$ and for any structure $\mathcal{M}, \mathcal{M}$ is a model of $\Sigma \cup \mathcal{D}_{s s} \cup \mathcal{D}_{\text {ap }} \cup \mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}$ iff there is a model $\mathcal{M}^{\prime}$ of $\mathcal{D}$ such that $\mathcal{M} \sim_{S_{\alpha}} \mathcal{M}^{\prime}$.

Below, we use $\mathcal{D}_{S_{\alpha}}$ to denote progression of the initial theory wrt the action term $\alpha$, if the context of $\mathcal{B A} \mathcal{T}$ is clear. We sometimes abuse terminology and call progression not only the theory $\mathcal{D}_{S_{\alpha}}$, but also the operation of computing this theory (when the existence of an effective operation is implicitly assumed). It can be seen (Theorem 2 in [26] and Theorem 2.10 in [27]) that progression always exists, i.e., is second-order definable, if the signature of $\mathcal{B A \mathcal { T }}$ is finite and the initial theory $\mathcal{D}_{S_{0}}$ is finitely axiomatizable. On the other hand, by the definition, for any $\mathcal{B} \mathcal{A} \mathcal{T} \mathcal{D}$, we have $\mathcal{D} \models$ $\mathcal{D}_{S_{\alpha}}$ and, similarly to the operation of forgetting, it is possible to provide an example (see Definition 2, Conjecture 1, and Theorem 2 in [45]), when the progression $\mathcal{D}_{S_{\alpha}}$ is not definable (even by an infinite set of formulas) in the logic in which $\mathcal{D}$ is formulated.

To understand the notion of progression intuitively, note the following. The progression $\mathcal{D}_{S_{\alpha}}$ is a set of consequences of $\mathcal{B} \mathcal{A} \mathcal{T}$ that are uniform in the situation term $S_{\alpha}$; it can be viewed as
the strongest postcondition of the precondition $\mathcal{D}_{S_{0}}$ wrt the action $\alpha$. Thus, informally, $\mathcal{D}_{S_{\alpha}}$ is all the information about the situation $S_{\alpha}$ implied by $\mathcal{B A} \mathcal{T}$. This is guaranteed by the model-theoretic property with the relation $\sim_{S_{\alpha}}$ in the definition. Recall that the initial theory of $\mathcal{B A T}$ describes information in the initial situation $S_{0}$ and SSAs are essentially the rules for computing the new truth values of fluents that change after performing actions. Thus, progression $\mathcal{D}_{S_{\alpha}}$ can be viewed as minimal "modification" of the initial theory obtained after executing the action $\alpha$. In particular, the initial theory of $\mathcal{B A} \mathcal{A}$ can be replaced with $\mathcal{D}_{S_{\alpha}}\left(S_{\alpha} / S_{0}\right)$ (recall the notation from Section 2.3) which gives a new $\mathcal{B A} \mathcal{T}$, with $S_{\alpha}$ as the new initial situation. Let $\varphi(s)$ be a formula uniform in a situation variable $s$. To solve the projection problem for $\varphi\left(S_{\alpha}\right)$, i.e., to find whether $\varphi\left(S_{\alpha}\right)$ holds in the situation $S_{\alpha}$ wrt $\mathcal{B A} \mathcal{T} \mathcal{D}$, one might wish to compute progression $\mathcal{D}_{S_{\alpha}}$ and then check whether $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}} \models \varphi\left(S_{\alpha}\right)$ holds (or equivalently, whether $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right) \models \varphi\left(S_{0}\right)$ holds). By Proposition 2.10, this is equivalent to $\mathcal{D} \models \varphi\left(S_{\alpha}\right)$, so this progression-based approach solves the projection problem for $\varphi\left(S_{\alpha}\right)$. This helps to demonstrate why progression may be useful.

Consequently, of interest are cases when progression can be computed effectively as a theory in the same logic that is used to formulate underlying $\mathcal{D}_{S_{0}}$. The well-known approach is to consider the local-effect $\mathcal{B A} \mathcal{T}$ s (recall Definition 2.11) in which progression can be obtained by just a syntactic modification of the initial theory $\mathcal{D}_{S_{0}}$ with respect to SSAs. This approach is based on effective forgetting of a finite set of ground atoms (extracted from SSAs) in the initial theory of $\mathcal{B A} \mathcal{A}$. Recall the well-known observation from Section 3 that, given a theory $\mathcal{T}$ (in an appropriate logic $\mathcal{L}$ ), forgetting a finite set of ground atoms in $\mathcal{T}$ can be computed effectively by straightforward syntactic manipulations with the axioms of $\mathcal{T}$. Thus, the cornerstone of computing progression in the localeffect case is to extract effectively the set of ground atoms from SSAs that need to be forgotten. Subsequently, in $\mathcal{D}_{S_{\alpha}}$, they are replaced with new values of fluents, which are computed from SSAs. An interested reader may consult the whole paper [27], while here we only introduce necessary notations from Definition 3.4 of [27], which will be used in Theorem 4.5.

Let $\mathcal{D}$ be a $\mathcal{B A} \mathcal{T}$ with a set $\mathcal{D}_{s s}$ of SSAs, an initial theory $\mathcal{D}_{S_{0}}$, and a unique name theory $\mathcal{D}_{\text {una }}$, and let $\alpha$ be a ground action term. Take a generic SSA ( $\dagger$ ) for the fluent $F$ (see Section 2.3) and replace there an action variable $a$ with the action term $\alpha$. Then, use unique name axioms for actions to replace equalities (or negations of equalities) between action functions with equalities (or negations of equalities, respectively) between object arguments. After that, apply the usual FO logic equivalences to eliminate existential quantifiers inside $\gamma_{F}^{+}(\bar{x}, \alpha, s), \gamma_{F}^{-}(\bar{x}, \alpha, s)$, if any. Recall these are formulas uniform in $s$ that appear on the right-hand side of a generic SSA ( $\dagger$ ). Observe that in a local-effect SSA, when one substitutes a ground action term $A\left(\overline{b_{x}}, \overline{b_{z}}\right)$ for a variable $a$ in the formula $[\exists \bar{z}] . a=A(\bar{x}, \bar{z}) \wedge \phi(\bar{x}, \bar{z}, s)$, applying UNA for actions yields $[\exists \overline{]}] . \bar{x}=\overline{b_{x}} \wedge \bar{z}=\overline{b_{z}} \wedge \phi(\bar{x}, \bar{z}, s)$, and applying $\exists z(z=b \wedge \phi(z)) \equiv \phi(b)$ repeatedly results in the logically equivalent formula $\bar{x}=\overline{b_{x}} \wedge$ $\phi\left(\bar{x}, \bar{b}_{z}, s\right)$. In a transformed SSA that is obtained after doing all these simplifications, it is convenient to consider all object constants appearing in equalities between object variables and constants. These represent values where the fluent $F$ changes. To compute the new value of the fluent it is sufficient to instantiate object variables of $F$ with the corresponding constants. Denote

$$
\begin{aligned}
& \Delta_{F}=\left\{\bar{t} \mid \bar{x}=\bar{t} \text { appears in } \gamma_{F}^{+}(\bar{x}, \alpha, s) \text { or } \gamma_{F}^{-}(\bar{x}, \alpha, s)\right. \text { in a transformed SSA } \\
&\left.\quad \text { for } F \text { instantiated with } \alpha \text { and equivalently rewritten wrt } \mathcal{D}_{\text {una }}\right\}, \\
& \Omega(s)=\left\{F(\bar{t}, s) \mid \bar{t} \in \Delta_{F}\right\} .
\end{aligned}
$$

Consider $\Omega(s)$ and notice that $\Omega\left(S_{0}\right)$ is a finite set of ground atoms to be forgotten. According to Fact 3.2, forgetting several ground atoms can be accomplished consecutively in any order.

An instantiation of $\mathcal{D}_{s s}$ wrt $\Omega\left(S_{0}\right)$, denoted by $\mathcal{D}_{s s}\left[\Omega\left(S_{0}\right)\right]$, is the set of formulas of the form:

$$
F\left(\bar{t}, d o\left(\alpha, S_{0}\right)\right) \leftrightarrow \gamma_{F}^{+}\left(\bar{t}, \alpha, S_{0}\right) \vee F\left(\bar{t}, S_{0}\right) \wedge \neg \gamma_{F}^{-}\left(\bar{t}, \alpha, S_{0}\right)
$$

These formulas represent instantiations of the transformed SSAs with object constants where the fluents change. Observe that $\mathcal{D}_{s s}\left[\Omega\left(S_{0}\right)\right]$ effectively defines new values for those fluents, which are affected by the action $\alpha$. However, these definitions use fluents wrt $S_{0}$, which may include fluents to be forgotten. For this reason, forgetting should be performed not only in $\mathcal{D}_{S_{0}}$, but in $\mathcal{D}_{s s}\left[\Omega\left(S_{0}\right)\right]$ as well.

Proposition 4.2 (Theorem 3.6 in [27]) In the notations above, the following is a progression of $\mathcal{D}_{S_{0}}$ wrt $\alpha$ in the sense of Definition 4.1:

$$
\mathcal{D}_{S_{\alpha}}=\operatorname{forget}\left(\mathcal{D}_{s s}\left[\Omega\left(S_{0}\right)\right] \cup \mathcal{D}_{S_{0}}, \Omega\left(S_{0}\right)\right)\left(S_{\alpha} / S_{0}\right)
$$

This formula demonstrates that progression is a set of formulas obtained after forgetting old values of fluents in the initial theory and in instantiation of transformed SSAs that provide new values of fluents, and then replacing $S_{0}$ with $S_{\alpha}$. Thus, computing a progression in a local-effect $\mathcal{B} \mathcal{A T}$ is an effective syntactic transformation of the initial theory, which leads to the unique form of the updated theory $\mathcal{D}_{S_{\alpha}}$. This fact will be used in Theorem 4.5. It is important to realize that this transformation can lead to an exponential blow-up of the initial theory, as noted after Theorem 3.6 in [27], due to the possible exponential blow-up after forgetting a set of ground atoms. This is not a surprise, because even in propositional logic, forgetting a symbol in a formula is essentially the elimination of a "middle term" (introduced by Boole), which results in the disjunction of two instances of the input formula [23]. As a consequence, forgetting may result in a formula that is roughly twice as long as the input formula. It is important to realize that the exponential blowup is not inevitable in the case of progression. As shown in [27], there are practical classes of the initial theories for which there is no blow-up and the size of the progressed theory is actually linear wrt the size of the initial theory.

Example 1 (continuation). As was discussed before, all SSAs in this example are local effect. Instantiate the action variable $a$ in the SSAs with a ground action move $\left(C_{1}, C_{2}, C_{3}\right)$. Then, we get:

$$
\begin{aligned}
\operatorname{Clear}\left(x, \operatorname{do}\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right), s\right)\right) \leftrightarrow & \exists y, z\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right)=\operatorname{move}(y, x, z)\right) \vee \\
& \operatorname{Clear}(x, s) \wedge \neg \exists y, z\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right)=\operatorname{move}(y, z, x)\right), \\
\operatorname{On}\left(x, y, \operatorname{do}\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right), s\right)\right) \leftrightarrow & \exists z\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right)=\operatorname{move}(x, z, y)\right) \vee \\
& \operatorname{On}(x, y, s) \wedge \neg \exists z\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right)=\operatorname{move}(x, y, z)\right) .
\end{aligned}
$$

Applying UNA for actions yields the following axioms:

$$
\begin{aligned}
\operatorname{Clear}\left(x, \operatorname{do}\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right), s\right)\right) \leftrightarrow & \exists y, z\left(y=C_{1} \wedge x=C_{2} \wedge z=C_{3}\right) \vee \\
& C l e a r(x, s) \wedge \neg \exists y, z\left(y=C_{1} \wedge z=C_{2} \wedge x=C_{3}\right), \\
\operatorname{On}\left(x, y, \operatorname{do}\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right), s\right)\right) \leftrightarrow & \exists z\left(x=C_{1} \wedge z=C_{2} \wedge y=C_{3}\right) \vee \\
& O n(x, y, s) \wedge \neg \exists z\left(x=C_{1} \wedge y=C_{2} \wedge z=C_{3}\right) .
\end{aligned}
$$

Doing the equivalent first order simplifications yields the transformed SSAs:

$$
\begin{aligned}
& \operatorname{Clear}\left(x, \operatorname{do}\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right), s\right)\right) \leftrightarrow\left(x=C_{2}\right) \vee \operatorname{Clear}(x, s) \wedge \neg\left(x=C_{3}\right) \\
& \operatorname{On}\left(x, y, \operatorname{do}\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right), s\right)\right) \leftrightarrow\left(x=C_{1} \wedge y=C_{3}\right) \vee \operatorname{On}(x, y, s) \wedge \neg\left(x=C_{1} \wedge y=C_{2}\right) .
\end{aligned}
$$

The argument set $\Delta_{F}$ for the fluent $F$ wrt a ground action $\alpha$ is a set of constants appearing in the transformed SSA for $F$ instantiated with $\alpha$. For example, the set $\Delta_{\text {clear }}$ for the fluent $C l e a r(x, s)$ wrt a ground action move $\left(C_{1}, C_{2}, C_{3}\right)$ is $\left\{C_{2}, C_{3}\right\}$. For the fluent $O n(x, y, s)$ this argument set $\Delta_{\text {on }}$ is $\left\{\left\langle C_{1}, C_{3}\right\rangle,\left\langle C_{1}, C_{2}\right\rangle\right\}$. The characteristic set $\Omega$ of a ground action move $\left(C_{1}, C_{2}, C_{3}\right)$ is a set of all ground atoms subject to change by this action. Therefore

$$
\Omega(s)=\left\{\operatorname{Clear}\left(C_{2}, s\right), \operatorname{Clear}\left(C_{3}, s\right), \operatorname{On}\left(C_{1}, C_{3}, s\right), \operatorname{On}\left(C_{1}, C_{2}, s\right)\right\}
$$

Notice that if block $C_{3}$ is clear at $s$, it no longer remains clear after doing move $\left(C_{1}, C_{2}, C_{3}\right)$ action, but block $C_{2}$ will become clear. However, this action has no effect on the property of $C_{1}$ being clear, and for this reason, $C_{1}$ is not in $\Delta_{\text {clear }}$ and not in the characteristic set $\Omega$.

Using these atoms to instantiate the transformed SSA, i.e., by replacing object arguments with constants from $\Delta_{F}$, we obtain the set $\mathcal{D}_{s s}[\Omega]$ of formulas representing new values of fluents, e.g.,

$$
\begin{aligned}
& \operatorname{On}\left(C_{1}, C_{3}, \operatorname{do}\left(\operatorname{move}\left(C_{1}, C_{2}, C_{3}\right), S_{0}\right)\right) \leftrightarrow \\
& \qquad C_{1}=C_{1} \wedge C_{3}=C_{3} \vee \operatorname{On}\left(C_{1}, C_{3}, S_{0}\right) \wedge \neg\left(C 2=C_{3} \wedge C_{3}=C_{2}\right) .
\end{aligned}
$$

After the equivalent simplifications using UNA, the instantiated SSAs wrt $\Omega\left(S_{\alpha}\right)$, where $S_{\alpha}=$ do(move $\left.\left(C_{1}, C_{2}, C_{3}\right), S_{0}\right)$ will be the following set:

$$
\left\{\operatorname{Clear}\left(C_{2}, S_{\alpha}\right), \neg \operatorname{Clear}\left(C_{3}, S_{\alpha}\right), \operatorname{On}\left(C_{1}, C_{3}, S_{\alpha}\right), \neg \operatorname{On}\left(C_{1}, C_{2}, S_{\alpha}\right)\right\}
$$

Note that in this example $\mathcal{D}_{s s}[\Omega]$ are very simple, but in a general case, if a SSA includes context conditions, these axioms may include fluents wrt $S_{0}$. Finally, according to Proposition 4.2, to compute a progression $\mathcal{D}_{S_{\alpha}}$ of an initial theory $\mathcal{D}_{S_{0}}$ for BW , we have to forget all old values of the fluents from $\Omega\left(S_{0}\right)$ in the theory $\mathcal{D}_{s s}[\Omega] \cup \mathcal{D}_{S_{0}}$, and subsequently replace the situation $S_{\alpha}$ with $S_{0}$.

Now we are ready to formulate the results on component properties of progression in terms of decomposability and inseparability. We start with negative examples in which every $\mathcal{B} \mathcal{A} \mathcal{T}$ is localeffect and the initial theories are formulated in first-order logic. As the progression $\mathcal{D}_{S_{\alpha}}$ is a set of formulas uniform in some situation term $S_{\alpha}$, which may occur in every formula of $\mathcal{D}_{S_{\alpha}}$ (thus potentially spoiling decomposability), we consider the mentioned decomposability and inseparability properties regarding the theory $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ instead of $\mathcal{D}_{S_{\alpha}}$. Otherwise, in every result we would have to speak of $\Delta \cup \operatorname{sig}\left(S_{\alpha}\right)$-decomposability of progression, since the symbols from sig $\left(S_{\alpha}\right)=$ $\operatorname{sig}\left(d o\left(\alpha, S_{0}\right)\right)=\left\{d o, S_{0}\right\} \cup \operatorname{sig}(\alpha)$ may occur in all components.

Consider a $\mathcal{B A} \mathcal{T} \mathcal{D}$ with $\Delta$-decomposable initial theory $\mathcal{D}_{S_{0}}$ for a signature $\Delta$. The general definition of a successor state axiom gives enough freedom to design examples showing (non)preservation of the decomposability property of $\mathcal{D}_{S_{0}}$ or inseparability of its components. Note that an SSA may contain symbols that are not even present in $\operatorname{sig}\left(\mathcal{D}_{S_{0}}\right)$, or symbols from both components of $\mathcal{D}_{S_{0}}$ (if decomposition exists). Therefore, it makes sense to restrict our study to those $\mathcal{B} \mathcal{A} \mathcal{T}$ s, where SSAs have one of the well-studied forms, e.g., to local-effect theories. It turns out that this form is still general enough to easily formulate negative results demonstrating that the aforementioned properties are not preserved without stipulations.

First, we provide an example showing that the decomposability property of the initial theory can be easily lost under progression. Next, we show that $\Delta$-inseparability of components of the initial theory $\mathcal{D}_{S_{0}}$ can be easily lost when fluents are present in $\Delta$ (see Example 7). The third observation is that even if there are no fluents in $\Delta$, some components of $\mathcal{D}_{S_{0}}$ can split after progression into theories which are no longer inseparable (see Example 8). All observations hold already for localeffect $\mathcal{B A} \mathcal{T}$ s and follow from the fact that some new information from SSAs can be added to the initial theory after progression, which spoils its component properties. We only need to provide a
combination of an initial theory with a set of SSAs that are appropriate for this purpose. The aim of Theorem 4.4 following these negative examples is to prove that if $\Delta$ does not contain fluents and the components of $\mathcal{D}_{S_{0}}$ do not split after progression, then $\Delta$-inseparability is preserved after progression under a slight stipulation which is caused only by generality of the theorem and the nonuniqueness of progression in the general case. This stipulation is avoided in Theorem 4.5, where we consider the class of local-effect $\mathcal{B A} \mathcal{T}$ s. Recall that all free variables in axioms of $\mathcal{B A} \mathcal{T}$ s are assumed to be universally quantified.

Example 6 (Decomposability lost under progression). Consider basic action theory $\mathcal{D}$, with $\{F, P, A, c\} \subseteq$ $\operatorname{sig}(\mathcal{D})$, where $F$ is a fluent, $P$ a predicate, $A$ an action function, and $c$ an object constant. Let the theory $\mathcal{D}_{s s}$ consist of the single axiom

$$
F(x, d o(a, s)) \leftrightarrow(a=A(x)) \wedge P(x) \vee F(x, s)
$$

and let the initial theory $\mathcal{D}_{S_{0}}$ consist of two axioms $\neg F\left(c, S_{0}\right)$ and $\exists x P(x)$. Clearly, $\mathcal{D}_{S_{0}}$ is a $\varnothing-$ decomposable.

Consider action $\alpha=A(c)$ and let us compute progression of $\mathcal{D}_{S_{0}}$ wrt $\alpha$. We apply Proposition 4.2, since $\mathcal{D}$ is local-effect. The instantiation of the SSA from $\mathcal{D}_{s s}$ with $\alpha$ has the form

$$
F(x, d o(A(c), s)) \leftrightarrow(A(c)=A(x)) \wedge P(x) \vee F(x, s)
$$

for which equivalent rewriting wrt $\mathcal{D}_{\text {una }}$ gives

$$
F(x, d o(A(c), s)) \leftrightarrow(x=c) \wedge P(x) \vee F(x, s)
$$

Hence, we have $\Omega\left(S_{0}\right)=\left\{F\left(c, S_{0}\right)\right\}$ and $\mathcal{D}_{s s}\left[\Omega\left(S_{0}\right)\right]=F\left(c, S_{\alpha}\right) \leftrightarrow P(c) \vee F\left(c, S_{0}\right)$.
Since $\neg F\left(c, S_{0}\right) \in \mathcal{D}_{S_{0}}$, the theory $\mathcal{D}_{s s}\left[\Omega\left(S_{0}\right)\right] \cup \mathcal{D}_{S_{0}}$ is equivalent to $\left\{F\left(c, S_{\alpha}\right) \leftrightarrow P(c)\right\} \cup$ $\mathcal{D}_{S_{0}}$. By Proposition 4.2, forgetting the ground atom $F\left(c, S_{0}\right)$ in this theory and substituting $S_{0}$ with $S_{\alpha}$ gives the theory $\mathcal{D}_{S_{\alpha}}$, the progression of $\mathcal{D}_{S_{0}}$ wrt $\alpha$. By using the definition of forgetting, it is easy to confirm that $\mathcal{D}_{S_{\alpha}}$ is equivalent to $\left\{F\left(c, S_{\alpha}\right) \leftrightarrow P(c), \exists x P(x)\right\}$. One can verify that $\mathcal{D}_{S_{\alpha}}$ (and also $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ ) is not $\Delta$-decomposable theory, for any $\Delta$. Notice that decomposability is lost, because fluent $F$ and predicate $P$ from different components of $\mathcal{D}_{S_{0}}$ become related to each other after progression.

For a signature $\Delta$, with $S_{0} \in \Delta$, and an action $A(c)$, we now give an example of a local-effect basic action theory $\mathcal{D}$ with $\mathcal{D}_{S_{0}}$, an initial theory $\Delta$-decomposable into finite $\Delta$-inseparable components. This example shows that progression $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ of $\mathcal{D}_{S_{0}}$ wrt $A(c)$ (with term $S_{\alpha}$ substituted with $S_{0}$ ) is finitely axiomatizable and $\Delta$-decomposable, but the decomposition components are no longer $\Delta$-inseparable, unless we allow them to be infinite.

Example 7 ( $\Delta$-inseparability is lost when fluents are in $\Delta$ ). Consider a basic action theory $\mathcal{D}$ with $\{F, P, R, A, b, c\} \subseteq \operatorname{sig}(\mathcal{D})$, where $F$ is a fluent, $P, R$ are predicates, $A$ an action function, and $b, c$ object constants. Let $\Delta=\left\{F, R, S_{0}, c\right\}$ and define subtheories of $\mathcal{D}$ as follows:

$$
\begin{aligned}
\mathcal{D}_{s s}= & \{F(x, d o(a, s)) \leftrightarrow(a=A(x)) \wedge P(x) \vee F(x, s)\} \text { (i.e. as in the previous example) } \\
\mathcal{D}_{S_{0}}= & \mathcal{D}_{1} \cup \mathcal{D}_{2}, \text { with } \\
& \mathcal{D}_{1}=\left\{\operatorname{Taut}\left(F, R, S_{0}, b\right), \neg F\left(c, S_{0}\right)\right\}, \text { where } \operatorname{Taut}\left(F, R, S_{0}, b, c\right) \text { is a } \\
& \text { tautological formula in the signature }\left\{F, R, S_{0}, b, c\right\}, \text { which is uniform in } S_{0} \\
& \mathcal{D}_{2}=\left\{P(x) \rightarrow \exists y(R(x, y) \wedge P(y)), \neg F\left(c, S_{0}\right)\right\} .
\end{aligned}
$$

By the syntactic form, $\mathcal{D}_{S_{0}}$ is $\Delta$-decomposable: we have $\mathcal{D}_{S_{0}}=\mathcal{D}_{1} \cup \mathcal{D}_{2}, \operatorname{sig}\left(\mathcal{D}_{1}\right) \cap \operatorname{sig}\left(\mathcal{D}_{2}\right)=$ $\Delta$, $\operatorname{sig}\left(\mathcal{D}_{1}\right) \backslash \Delta=\{b\}$, and $\operatorname{sig}\left(\mathcal{D}_{2}\right) \backslash \Delta=\{P\}$. It is also easy to confirm that $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are $\Delta$-inseparable.

By Proposition 4.2 it is easy to verify that the union of $\left\{\operatorname{Taut}\left(F, R, S_{0}, b, c\right)\right\}$ and $\mathcal{D}_{2}^{\prime}=\left(\mathcal{D}_{2} \backslash\right.$ $\left.\left\{\neg F\left(c, S_{0}\right)\right\}\right) \cup\{\varphi\}$, where $\varphi=F\left(c, S_{\alpha}\right) \leftrightarrow P(c)$ is a progression $\left(\mathcal{D}_{S_{\alpha}}\right)$ of $\mathcal{D}_{S_{0}}$ wrt $\alpha=A(c)$.

By the syntactic form, $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is a $\Delta$-decomposable theory. On the other hand, we have $\varphi \models F\left(c, S_{\alpha}\right) \rightarrow P(c)$, thus

$$
\mathcal{D}_{2}^{\prime}\left(S_{0} / S_{\alpha}\right) \models\left\{F\left(c, S_{0}\right) \rightarrow \exists y R(c, y), F\left(c, S_{0}\right) \rightarrow \exists y \exists z[R(c, y) \wedge R(y, z)], \ldots\right\}
$$

This is an infinite set of formulas in signature $\Delta$. It follows from Fact 2.3 that this theory is not finitely axiomatizable by formulas of first-order logic in signature $\Delta$ and it is easy to verify that $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ can not have a decomposition into finite $\Delta$-inseparable components.

Note that in the example above, the initial theory $\mathcal{D}_{S_{0}}$ is in fact $\varnothing$-decomposable with one signature component equal to $\{b\}$ and the other component containing the rest of the symbols. It is easy to see that the progression of $\mathcal{D}_{S_{0}}$ wrt $A(c)$ is $\varnothing$-decomposable as well. We use tautologies in the example just to illustrate the idea that information from SSA can propagate to the initial theory after progression, thus making the components lose the inseparability property. There is a plenty of freedom to formulate similar examples with the help of non-tautological formulas which syntactically "bind" symbols $F, R, S_{0}, b$ in the theory $\mathcal{D}_{1}$. We appeal to a similar observation in Example 8.
Example 8 (Split of a component and loss of $\Delta$-inseparability). Consider $\mathcal{B} \mathcal{A} \mathcal{T} \mathcal{D}$, with $\left\{F_{1}, F_{2}, D, B, P, R, A, c\right\} \subseteq \operatorname{sig}(\mathcal{D})$, where $F_{1}, F_{2}$ are fluents, $D, B, P, R$ predicates, $A$ an action function, and $c$ an object constant. Let $\Delta=\left\{D, R, S_{0}\right\}$ and define the subtheories of $\mathcal{D}$ as follows:

$$
\begin{aligned}
\mathcal{D}_{s s}=\{ & \left.F_{1}(x, d o(a, s)) \leftrightarrow F_{1}(x, s) \wedge \neg(a=A(x)), \quad F_{2}(x, d o(a, s)) \leftrightarrow F_{2}(x, s)\right\} \\
\mathcal{D}_{S_{0}}= & \mathcal{D}_{1} \cup \mathcal{D}_{2}, \text { where } \mathcal{D}_{1} \text { is the set of formulas with occurrences of } D, R, S_{0}: \\
& D(x) \vee R(x, y) \rightarrow F_{1}\left(c, S_{0}\right) \\
& D(x) \rightarrow P(x) \\
& P(x) \rightarrow \exists y(R(x, y) \wedge P(y))
\end{aligned}
$$

and $\mathcal{D}_{2}$ consists of the following three formulas (which also mention $D, R, S_{0}$ ):

$$
\begin{aligned}
& D(x) \rightarrow B(x) \\
& B(x) \rightarrow \exists y(R(x, y) \wedge B(y))
\end{aligned}
$$

$\operatorname{Taut}\left(F_{2}, S_{0}\right)$, a tautology in the signature $\left\{F_{2}, S_{0}\right\}$, uniform in $S_{0}$. Here, $F_{2}$ is an auxiliary fluent introduced to have an occurrence of $S_{0}$ in $\mathcal{D}_{2}$.

By definition, $\mathcal{D}_{S_{0}}$ is $\Delta$-decomposable into $\Delta$-inseparable components $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$. Note that $\mathcal{D}_{s s} \models \neg F_{1}\left(c, d o\left(A(c), S_{0}\right)\right)$, which is the result of substitution of the ground action $A(c)$, situation constant $S_{0}$, and object constant $c$ in SSA.

Consider progression of $\mathcal{D}_{S_{0}}$ wrt the action $\alpha=A(c)$. By Proposition 4.2, it is equivalent to the theory $\mathcal{D}_{S_{\alpha}}=\mathcal{D}_{1}^{\prime} \cup \mathcal{D}_{1}^{\prime \prime} \cup \mathcal{D}_{2}^{\prime}$, where $\mathcal{D}_{1}^{\prime}$ is the set of the following formulas:
$\neg F_{1}\left(c, d o\left(A(c), S_{0}\right)\right)$
$\operatorname{Taut}(D, R)$, a tautological formula in the signature $\{D, R\}$ which is uniform in $S_{\alpha}$,
$\mathcal{D}_{1}^{\prime \prime}$ is the set of formulas:

$$
\begin{aligned}
& D(x) \rightarrow P(x) \\
& P(x) \rightarrow \exists y(R(x, y) \wedge P(y))
\end{aligned}
$$

$\operatorname{Taut}\left(F_{2}, S_{\alpha}\right)$, a tautological formula in the signature $\left\{F_{2}, d o, A, c, S_{0}\right\}$ which is uniform in $S_{\alpha}$
and $\mathcal{D}_{2}^{\prime}$ is the theory $\mathcal{D}_{2}$ with every occurrence of $S_{0}$ substituted with $S_{\alpha}$.
Clearly, $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is $\Delta$-decomposable. Note that after progression the component $\mathcal{D}_{1}$ is "split" into $\mathcal{D}_{1}^{\prime}\left(S_{0} / S_{\alpha}\right)$ and $\mathcal{D}_{1}^{\prime \prime}\left(S_{0} / S_{\alpha}\right)$ and these theories are not $\Delta$-inseparable (similarly, $\mathcal{D}_{1}^{\prime}\left(S_{0} / S_{\alpha}\right)$ and $\mathcal{D}_{2}^{\prime}\left(S_{0} / S_{\alpha}\right)$ ). By Fact 2.3 , it can be shown that they can not be made $\Delta$-inseparable while remaining finitely axiomatizable.

To formulate the theorems below, we let $\mathcal{D}$ denote a $\mathcal{B A} \mathcal{T}$ with the initial theory $\mathcal{D}_{S_{0}}$, the set of successor state axioms $\mathcal{D}_{s s}$, and the unique name axioms $\mathcal{D}_{u n a}$. Example 7 has resulted in the following definition.

Definition 4.3 (Fluent-free signature) A signature $\Delta$ is called fluent-free if no fluent (from the alphabet of situation calculus) is contained in $\Delta$.

Theorem 4.2 complements Examples 4.3 and 4.4, which identify properties of decomposed actions theories causing loss of inseparability of components after progression. The theorem shows that if these properties are absent then inseparability is preserved. As we have already seen in Example 7, the initial theory and progression may differ in consequences involving symbols of fluents. Thus in general, preservation of $\Delta$-inseparability can be guaranteed only for fluent-free signatures $\Delta$, which is reflected in the conditions of the theorem. Besides, by the model-theoretic Definition 4.1, progression is not uniquely defined - there is no restriction on occurrences of the unique name axioms in progression, which may easily lead to loss of inseparability of the components. In other words, progression may logically imply unique name axioms even if the initial theory did not imply them. Some decomposition components of progression may imply such formulas, while the others may not. For this reason, we speak of inseparability "modulo" theory $\mathcal{D}_{\text {una }}$ in the first point of the theorem below. In particular, we have to make the assumption that not only the components $\left\{D_{i}\right\}_{i \in I \subseteq \omega}$ of the initial theory are pairwise $\Delta$-inseparable, but so are the theories $\left\{\mathcal{D}_{\text {una }} \cup D_{i}\right\}_{i \in I}$.

For fluent-free deltas, the progression entails exactly those $\Delta$-formulas, which are entailed already by the initial theory (together with UNA-axioms), and the question is how these formulas can be "distributed" between the components. The second point of the theorem rules out the case (described in Example 8), when $\Delta$-consequences are split between the components of progression. Note that in the theorem we do not specify how the progression was obtained (cf. Theorem 4.5) and the only condition that relates the components of progression with those of the initial theory says about containment of $\Delta$-consequences.

Theorem 4.4 (Preservation of $\Delta$-insep. for fluent-free $\Delta$ ) Let $\mathcal{L}$ have PIP and $\mathcal{D}$ be a $\mathcal{B A} \mathcal{T}$ in which $\mathcal{D}_{S_{0}}$ and $\mathcal{D}_{\text {una }}$ are theories in $\mathcal{L}$. Let $\sigma \subseteq \operatorname{sig}\left(\mathcal{D}_{S_{0}}\right)$ be a fluent-free signature and denote $\Delta=\operatorname{sig}\left(\mathcal{D}_{\text {una }}\right) \cup \sigma$. Suppose the following:

- $\mathcal{D}_{S_{0}}$ is $\sigma$-decomposable with some components $\left\{D_{i}\right\}_{i \in I \subseteq \omega}$ such that the theories from $\left\{\mathcal{D}_{\text {una }} \cup\right.$ $\left.D_{i}\right\}_{i \in I}$ are pairwise $\Delta$-inseparable;
- $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is equivalent to the union of theories $\left\{D_{j}^{\prime}\right\}_{j \in J \subseteq \omega}$ such that for every $j \in J$ and some $i \in I$, Cons $\left(\mathcal{D}_{\text {una }} \cup D_{j}^{\prime}, \Delta\right) \supseteq \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup D_{i}, \Delta\right)$.
Then the theories from $\left\{\mathcal{D}_{\text {una }} \cup D_{j}^{\prime}\right\}_{j \in J \subseteq \omega}$ are pairwise $\Delta$-inseparable.
Proof. Let us demonstrate that for all $j \in J$ we have $\operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup D_{j}^{\prime}, \Delta\right)=\operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup\right.$ $\mathcal{D}_{S_{0}}, \Delta$ ), from which the statement of the theorem obviously follows. Essentially, we prove the following inclusions (the corresponding points of the proof are marked with circles):

(4)

1) Note that for any $i \in I, \mathcal{D}_{S_{0}}$ is $\sigma$-decomposable with components $D_{i}$ and $\bigcup_{k \in I \backslash\{i\}} D_{k}$. We claim that $\mathcal{D}_{u n a} \cup D_{i}$ and $\mathcal{D}_{u n a} \cup \bigcup_{k \in I \backslash\{i\}} D_{k}$ are $\Delta$-inseparable. Let $\varphi$ be a formula in signature $\Delta$. If $\mathcal{D}_{\text {una }} \cup D_{i} \models \varphi$ then clearly, $\mathcal{D}_{\text {una }} \cup \bigcup_{k \in I \backslash\{i\}} D_{k} \models \varphi$ by $\Delta$-inseparability from the condition of the theorem. On the other hand, if $\mathcal{D}_{\text {una }} \cup \bigcup_{k \in I \backslash\{i\}} D_{k} \models \varphi$ then by PIP we have $\mathcal{T}_{\text {una }} \cup$ $\bigcup_{k \in I \backslash\{i\}} \mathcal{T}_{k} \models \varphi$, where $\mathcal{D}_{\text {una }} \models \mathcal{T}_{\text {una }}, \operatorname{sig}\left(\mathcal{T}_{\text {una }}\right) \subseteq \operatorname{sig}\left(\mathcal{D}_{\text {una }}\right)$ and $D_{k} \models \mathcal{T}_{k}$ for $k \in I \backslash\{i\}$, $\operatorname{sig}\left(\mathcal{T}_{k}\right) \subseteq \Delta$. Again, by $\Delta$-inseparability, for each $k \in I \backslash\{i\}$ we have $\mathcal{D}_{\text {una }} \cup D_{i} \models \mathcal{T}_{k}$ and thus, $\mathcal{D}_{\text {una }} \cup D_{i} \models \varphi$.

Therefore, if $\varphi \in \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{0}}, \Delta\right)$, then for every $i \in I,\left[\mathcal{D}_{\text {una }} \cup \bigcup_{k \in I \backslash\{i\}} D_{k}\right] \cup\left[\mathcal{D}_{\text {una }} \cup\right.$ $\left.D_{i}\right] \models \varphi$ and then by PIP and inseparability shown above, $\mathcal{D}_{\text {una }} \cup D_{i} \models \varphi$. Since $\mathcal{D}_{S_{0}} \models \bigcup_{i \in I} D_{i}$ by decomposability, we obtain Cons $\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{0}}, \Delta\right)=\operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup D_{i}, \Delta\right)$ for all $i \in I$.
2) Let us show that $\operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}, \Delta\right) \subseteq \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{0}}, \Delta\right)$. First, take a formula $\psi \in$ Cons $\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}, \Delta\right)$, which does not contain situation terms. From the definition of progression, every model of $\mathcal{D}$ is a model of $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}$, so $\mathcal{D} \models \mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}$ and hence, $\mathcal{D} \models \psi$. If $\mathcal{D}_{\text {una }} \cup$ $\mathcal{D}_{S_{0}} \not \vDash \psi$, then $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{0}} \cup\{\neg \psi\}$ is satisfiable and since $\psi$ is a uniform formula, by Proposition 2.10, $\mathcal{D} \cup\{\neg \psi\}$ is satisfiable, which contradicts $\mathcal{D} \models \psi$. Therefore, $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{0}} \models \psi$.

It remains to verify that the set Cons $\left(\mathcal{D}_{u n a} \cup \mathcal{D}_{S_{\alpha}}, \Delta\right)$ is axiomatized by sentences which do not contain situation terms. We have $\Delta=\operatorname{sig}\left(\mathcal{D}_{\text {una }}\right) \cup \sigma \subseteq \operatorname{sig}\left(\mathcal{D}_{\text {una }}\right) \cup \operatorname{sig}\left(\mathcal{D}_{S_{0}}\right)$, so $\{d o, \preceq$ , Poss $\} \cap \Delta=\varnothing$, by definition of $\mathcal{D}_{\text {una }}$ and $\mathcal{D}_{S_{0}}$. As $\sigma$ if fluent-free by the condition of the theorem (and $\operatorname{sig}\left(\mathcal{D}_{\text {una }}\right)$ is fluent-free by definition of $\left.\mathcal{B} \mathcal{A} \mathcal{T}\right), \Delta$ may contain only situation-independent predicates and functions. Thus, any formula $\varphi \in \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}, \Delta\right)$ may contain situation terms only in equalities, where each term is either the constant $S_{0}$ (in case $S_{0} \in \sigma$ ) or a bound variable of sort situation. Suppose that this is the case and there is no $\psi \in \operatorname{Cons}\left(\mathcal{D}_{u n a} \cup \mathcal{D}_{S_{\alpha}}, \Delta\right)$ such that $\psi \models \varphi$ and $\psi$ does not contain situation terms. By the syntax of $\mathcal{L}_{s c}$ and the choice of $\Delta$, then $\varphi$ is a boolean combination of formulas without situation terms and sentences over signature $\left\{S_{0}\right\}$ stating that $\varphi$ has a model with cardinality $\mid$ Sit $\mid$ of sort situation lying in the interval $[n, m]$ for $n \in \omega$ and $m \in \omega \cup\{\infty\}$. We denote sentences of this form by $\exists^{[n, m]} \theta_{=}$. We may assume that $\varphi$ is in conjunctive normal form and that there is a formula $\xi$, a boolean combination of $\exists^{[n, m]} \theta=$ such that $\not \vDash \xi$, $\nLeftarrow \neg \xi$, either $\xi$ or $\xi \vee \eta$ is a conjunct of $\varphi$, and $\eta \notin \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}, \Delta\right)$, $\operatorname{sig}(\eta) \subseteq \Delta$, is
a formula without situation terms. As $\not \models \xi$ and $\not \models \neg \xi$, there are $n, m \in \omega$ such that $\xi$ does not have a model with $|S i t|=n$ and $\neg \xi$ does not have a model with $\mid$ Sit $\mid=m$. Then by Lemma 2.8, we conclude that $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}} \not \vDash \xi$ and $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}} \not \vDash \neg \xi$. In particular, $\xi$ can not be a conjunct of $\varphi$. If $\xi \vee \eta$ is a conjunct, then there exists a model $\mathcal{M}$ of $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}$ such that $\mathcal{M} \vDash \xi$ and $\mathcal{M} \not \models \eta$. Then, by applying Lemma 2.8 again, there must be a model $\mathcal{M}^{\prime}$ of $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}$ with $\mid$ Sit $\mid=n$ where the interpretation of situation-independent predicates and functions is the same as in $\mathcal{M}$. Thus, $\mathcal{M}^{\prime} \not \vDash \xi$ and since $\eta$ does not contain situation terms, $\mathcal{M}^{\prime} \not \vDash \eta$, which contradicts $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}} \models \varphi$.
3) Now let us demonstrate that $\operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right), \Delta\right) \subseteq \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}, \Delta\right)$. Note that $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is uniform in $S_{0}$. Following the above proved, assume that there is a formula $\varphi \in$ Cons $\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right), \Delta\right)$ such that $\varphi$ does not contain situation terms and $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}} \not \models \varphi$. Take a model $\mathcal{M}$ of $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}$ such that $\mathcal{M} \not \vDash \varphi$. Then, by Lemma 2.8, there exists a model $\mathcal{M}^{\prime}$ of $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}$ such that the domain for sort situation in $\mathcal{M}^{\prime}$ is a singleton set (i.e., the interpretation of terms $S_{0}$ and $S_{\alpha}$ coincide in $\mathcal{M}^{\prime}$ ) and the interpretation of situation-independent symbols is the same in $\mathcal{M}$ and $\mathcal{M}^{\prime}$. Then $\mathcal{M}^{\prime} \not \models \varphi$, but clearly $\mathcal{M}^{\prime} \models \mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ which contradicts the assumption $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}} \not \vDash \varphi$.
4) Finally, by the condition of the theorem, for all $j \in J$, we have $\mathcal{D}_{\text {una }} \cup D_{j}^{\prime} \subseteq \mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}$ $\left(S_{0} / S_{\alpha}\right)$ and from points 1-3 above we obtain Cons $\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right), \Delta\right) \subseteq \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup\right.$ $\left.\mathcal{D}_{S_{0}}, \Delta\right)$. Hence, for all $j \in J$ we have Cons $\left(\mathcal{D}_{\text {una }} \cup D_{j}^{\prime}, \Delta\right) \subseteq \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{0}}, \Delta\right)$. On the other hand, we also have Cons $\left(\mathcal{D}_{\text {una }} \cup D_{i}, \Delta\right) \subseteq \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup D_{j}^{\prime}, \Delta\right)$ from the condition of the theorem. Therefore from the inclusion $\forall i \in I$ Cons $\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{0}}, \Delta\right) \subseteq \operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup D_{i}, \Delta\right)$ of point 1 we conclude that Cons $\left(\mathcal{D}_{\text {una }} \cup D_{j}^{\prime}, \Delta\right)=\operatorname{Cons}\left(\mathcal{D}_{\text {una }} \cup \mathcal{D}_{S_{0}}, \Delta\right)$ for all $j \in J$.

The next theorem provides a result on local-effect $\mathcal{B A} \mathcal{T} s$ with initial theories in first-order logic for which progression becomes more concrete, since it can be computed by syntactic manipulations. In contrast to Theorem 4.4, this allows us to judge about inseparability without the theory $\mathcal{D}_{\text {una }}$ in background. Recall that, in general, a $\mathcal{B A} \mathcal{T}$ includes non-trivial precondition axioms. On the righthand side of each precondition axiom, there is a formula $\Pi_{A}(\bar{x}, s)$ that is a formula uniform in $s$ with free variables among $\bar{x}$ and $s$. However, any $\mathcal{B A} \mathcal{T}$ can be transformed into an action theory without precondition axioms by introducing the right hand side formulas from the precondition axioms as conjuncts of context conditions for each corresponding active position of an action term in a SSA. Therefore, without loss of generality, and for simplicity of presentation, we subsequently consider the $\mathcal{B A} \mathcal{T}$ s where all precondition axioms are trivial.

Essentially, the conditions of the theorem are defined to guarantee componentwise computation of progression for a decomposable initial theory. A finite set $\mathcal{D}_{s s}$ of the SSAs is considered to be syntactically divided into the union of $|I|$ sub-theories sharing some fluent-free signature $\Delta_{1}$ (which may include actions, static predicates, and object constants), as well as function do (which occurs in every SSA). Informally, each of $|I|$ sub-theories is about a different set of properties, e.g., one of them could be about the blocks world, while another could be about the logistics world, with the two theories possibly sharing some constants, such as a box name, and situation-independent predicates, such as shapes of the boxes. The initial theory $\mathcal{D}_{S_{0}}$ is $\Delta_{2}$-decomposable, for a fluent-free signature $\Delta_{2}$, into $|J|$ components. To distinguish visually components $D_{j}^{\prime}$ of $\mathcal{D}_{S_{0}}$ from the components $D_{i}$ of $\mathcal{D}_{s s}$, we write $D_{j}^{\prime}$ with apostrophe when we mean components of $D_{S_{0}}$, and $D_{i}$ without apostrophe when we mean groups of SSAs. Informally, each component $D_{j}^{\prime}$ is about a separate aspect of the initial theory. The syntactic form of the initial theory may not reveal the components readily, but they can be discovered through decomposition. Naturally, it is expected that independent components of
the initial theory should remain independent after doing any actions. This imposes a condition that each component from the initial theory should be related with its own group of SSAs.

The last two conditions of the theorem enforce that the subtheories of $\mathcal{D}_{s s}$ are aligned with the components of $\mathcal{D}_{S_{0}}$ via syntactic occurrences of fluents. The second to last condition says that every fluent mentioned in a SSA must also occur in the initial theory $\mathcal{D}_{S_{0}}$. It is easy to satisfy by adding tautologies with the corresponding fluents to $\mathcal{D}_{S_{0}}$. Together with the last condition it guarantees that for every $\operatorname{SSA} \varphi$ containing fluents $F_{1}, \ldots, F_{n}$ there is a corresponding component of $\mathcal{D}_{S_{0}}$, which describes the initial interpretation of these fluents, and this is the component that must be updated upon executing an action mentioned in active position of $\varphi$. The last condition also enforces that actions, static predicates, or object constants separated by the decomposition of $\mathcal{D}_{S_{0}}$ must be also separated by the subtheories of $\mathcal{D}_{s s}$, whenever they occur in SSAs. This guarantees that these symbols do not become connected after progression (as opposed to the situation presented in Example 6). Thus, the theory $\mathcal{D}_{s s} \cup \mathcal{D}_{S_{0}}$ can be divided into parts (consisting of successor-state axioms and statements about the initial situation) which may mention common actions, static predicates and constants, but talk about different fluents. In other words, these subtheories define independent sets of situation-related properties, which is natural for a composite action theory describing a single domain of objects from a number of different perspectives. Note that it is allowed for a single action to have effects on groups of fluents (possibly, all the fluents at once, without regard to distribution of the fluents between the subtheories), which is reflected in the theorem condition that $\Delta_{1}$ is just fluent-free, but not action free. We impose stronger restriction in Corollary 4.6, which describes a class of $\mathcal{B A} \mathcal{T} s$ representing composite subject domains, like the one mentioned in the running example from Section 2.3.

For the reader's convenience, we stress that in the formulation of the theorem, the indices $i$ and $j$ vary over components of $\mathcal{D}_{s s}$ and $\mathcal{D}_{S_{0}}$, respectively. The signatures $\Delta_{1}$ and $\Delta_{2}$ are the sets of allowed common symbols between the components of $\mathcal{D}_{s s}$ and $\mathcal{D}_{S_{0}}$, respectively. We recall that $\mathcal{A}(\mathcal{F}$, respectively) denotes the set of action functions (the set of fluents, respectively) from the alphabet of the language of the situation calculus.

Theorem 4.5 (Preservation of components in local-effect $\mathcal{B A T}$ ) Let $\mathcal{D}$ be a local-effect $\mathcal{B A T}$, with $\mathcal{D}_{S_{0}}$ an initial theory in first-order logic. Let $\Delta_{1}, \Delta_{2}$ be fluent-free signatures, do $\notin \Delta_{1}, \Delta_{2}$, and $\alpha=A(\bar{c})$ a ground action term. Denote $\Delta=\Delta_{1} \cup \Delta_{2} \cup\left\{c_{1}, \ldots, c_{k}\right\}$, if $\bar{c}=\left\langle c_{1}, \ldots, c_{k}\right\rangle$, and suppose the following:

- $\mathcal{D}_{s s}$ is the union of theories $\left\{D_{i}\right\}_{i \in I}$, with $\operatorname{sig}\left(D_{n}\right) \cap \operatorname{sig}\left(D_{m}\right) \subseteq \Delta_{1} \cup\{d o\}$ for all $n, m \in$ $I \neq \varnothing, n \neq m$;
- $\mathcal{D}_{S_{0}}$ is $\Delta_{2}$-decomposable into finite components $\left\{D_{j}^{\prime}\right\}_{j \in J}$ uniform in $S_{0}$ such that $\operatorname{sig}\left(D_{j}^{\prime}\right) \backslash$ $\Delta \neq \varnothing$, for all $j \in J$;
$-\operatorname{sig}\left(\mathcal{D}_{s s}\right) \cap \mathcal{F} \subseteq \operatorname{sig}\left(\mathcal{D}_{S_{0}}\right)$;
- for every $i \in I$, there is $j \in J$ such that $\operatorname{sig}\left(D_{i}\right) \cap \operatorname{sig}\left(\mathcal{D}_{S_{0}}\right) \subseteq \operatorname{sig}\left(D_{j}^{\prime}\right)$.

Then $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is $\Delta$-decomposable. If the components $\left\{D_{j}^{\prime}\right\}_{j \in J}$ are pairwise $\Delta$-inseparable, then so are the components of $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ in the corresponding decomposition.

Proof. The proof consists of two parts, both of which rely on the constructive definition of progression for local-effect $\mathcal{B A} \mathcal{T}$ s from Section 4 and component properties of forgetting discussed in Section 3. In the first part, we show $\Delta$-decomposability of $D_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ by constructing its components explicitly and in the second part we prove that these components are $\Delta$-inseparable.

1) By definition of $\mathcal{B A} \mathcal{A}$, for every $i \in I$, we have $\operatorname{sig}\left(D_{i}\right) \cap \mathcal{F} \neq \varnothing$ and thus, from the conditions of the theorem, $\operatorname{sig}\left(D_{i}\right) \cap \operatorname{sig}\left(\mathcal{D}_{S_{0}}\right) \neq \varnothing, \operatorname{sig}\left(\mathcal{D}_{s s}\right) \cap \mathcal{F}=\operatorname{sig}\left(\mathcal{D}_{S_{0}}\right) \cap \mathcal{F}$. Hence, for every $i \in I$ there is $j \in J$ such that $\operatorname{sig}\left(D_{i}\right) \cap \mathcal{F} \subseteq \operatorname{sig}\left(D_{j}^{\prime}\right)$. Moreover, such $j \in J$ is unique for every $i \in I$, because otherwise there would exist $n, m \in J, n \neq m$, such that $\operatorname{sig}\left(D_{n}^{\prime}\right) \cap$ $\operatorname{sig}\left(D_{m}^{\prime}\right) \cap \mathcal{F} \neq \varnothing$, which contradicts the condition that $\Delta_{2}$ is fluent-free. Therefore, there is a map $f: I \rightarrow J$ such that for every $i \in I, \operatorname{sig}\left(D_{i}\right) \cap \mathcal{F} \subseteq \operatorname{sig}\left(D_{f(i)}^{\prime}\right)$. Note that there may exist $j \in J$ such that $\operatorname{sig}\left(D_{j}^{\prime}\right) \cap \mathcal{F}=\varnothing$ and in this case $j$ is the image of no $i \in I$. Let us denote the image of $f$ by $\tilde{J}$ (so, $\tilde{J} \subseteq J$ ).

Now, for every $i \in I$, consider the set of formulas $D_{i}[\Omega]$, the instantiation of $D_{i}$ w.r.t. $\Omega\left(S_{0}\right)$, and for each $j \in \tilde{J}$, denote $\widetilde{D}_{j}=\left[\bigcup_{i \in f^{-1}(j)}\left(D_{i}[\Omega]\right)\right] \cup D_{j}^{\prime}$. Then, by Proposition 4.2, $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ (progression of $\mathcal{D}_{S_{0}}$ wrt $\alpha$, with term $S_{\alpha}$ substituted with $S_{0}$ ) is logically equivalent to

$$
\left[\operatorname{forget}\left(\bigcup_{j \in \tilde{J}} \widetilde{D}_{j}, \Omega\left(S_{0}\right)\right) \cup \bigcup_{j \in J \backslash \tilde{J}} D_{j}^{\prime}\right]\left(S_{0} / S_{\alpha}\right)
$$

As $\Delta_{1}$ and $\Delta_{2}$ are fluent-free, the signatures $\left\{\operatorname{sig}\left(\widetilde{D}_{j}\right)\right\}_{j \in \tilde{J}}$ do not have fluents in common and thus, by Corollary 3.10, $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is equivalent to

$$
\left[\bigcup_{j \in \tilde{J}} \operatorname{forget}\left(\widetilde{D}_{j},\left.\Omega\left(S_{0}\right)\right|_{j}\right) \cup \bigcup_{j \in J \backslash \tilde{J}} D_{j}^{\prime}\right]\left(S_{0} / S_{\alpha}\right)
$$

where for $j \in \tilde{J},\left.\Omega\left(S_{0}\right)\right|_{j}$ is the subset of ground atoms from $\Omega\left(S_{0}\right)$ with fluents from $\operatorname{sig}\left(D_{j}^{\prime}\right)$. For all $j \in J \backslash \tilde{J}$, we have $\operatorname{sig}\left(D_{j}^{\prime}\right) \cap \mathcal{F}=\varnothing$ and $D_{j}^{\prime}$ is uniform in $S_{0}$, so it follows that $S_{0} \notin$ $\operatorname{sig}\left(D_{j}^{\prime}\right)$ and thus, $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is equivalent to the union

$$
\left[\bigcup_{j \in \tilde{J}} \operatorname{forget}\left(\widetilde{D}_{j},\left.\Omega\left(S_{0}\right)\right|_{j}\right)\right]\left(S_{0} / S_{\alpha}\right) \cup \bigcup_{j \in J \backslash \tilde{J}} D_{j}^{\prime}
$$

For every $j \in J$, let $D_{j}^{\prime \prime}$ be the set of formulas (forget $\left.\left(\widetilde{D}_{j},\left.\Omega\left(S_{0}\right)\right|_{j}\right)\right)\left(S_{0} / S_{\alpha}\right)$ (in case $\left.j \in \tilde{J}\right)$ or the set of formulas $D_{j}^{\prime}$ (if $j \in J \backslash \tilde{J}$ ). So $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is equivalent to $\bigcup_{j \in J} D_{j}^{\prime \prime}$. By the definition of forgetting a set of ground atoms one can assume that $\operatorname{sig}\left(D_{j}^{\prime}\right) \subseteq \operatorname{sig}\left(D_{j}^{\prime \prime}\right)$ and $\operatorname{sig}\left(D_{j}^{\prime \prime}\right) \backslash$ $\operatorname{sig}\left(D_{j}^{\prime}\right) \subseteq \operatorname{sig}\left(\mathcal{D}_{s s}\right) \cup\left\{c_{1}, \ldots, c_{k}\right\}$, for all $j \in J$.

Let us show that $\left[\operatorname{sig}\left(D_{i}^{\prime \prime}\right) \cap \operatorname{sig}\left(D_{j}^{\prime \prime}\right)\right] \subseteq \Delta$, for all distinct $i, j \in J$. Assume there are distinct $i, j \in J$ such that $\left[\operatorname{sig}\left(D_{i}^{\prime \prime}\right) \cap \operatorname{sig}\left(D_{j}^{\prime \prime}\right)\right] \backslash \Delta=\Sigma \neq \varnothing$, for a signature $\Sigma$. Then $d o \notin \Sigma$, since both $D_{i}^{\prime \prime}$ and $D_{j}^{\prime \prime}$ are uniform in $S_{0}$. If there is a single subtheory $D_{m}$ of $\mathcal{D}_{s s}, m \in I$, such that $\Sigma \subseteq \operatorname{sig}\left(D_{m}\right)$, then the last two conditions of the theorem yield $\Sigma \subseteq \Delta_{2}$, which is a contradiction, because we have assumed $\Sigma \cap \Delta=\varnothing$. If there are distinct subtheories $D_{m}$ and $D_{n}$ of $\mathcal{D}_{s s}, m, n \in$ $I$, such that $\Sigma \subseteq \operatorname{sig}\left(D_{m}\right) \cap \operatorname{sig}\left(D_{n}\right)$, then $\Sigma \subseteq \Delta_{1}$, and we again arrive at contradiction.

It follows that the pairwise intersection of any signatures from $\left\{\operatorname{sig}\left(D_{j}^{\prime \prime}\right)\right\}_{j \in J}$ is a subset of $\Delta$ and it follows from the second condition of the theorem that $\operatorname{sig}\left(D_{j}^{\prime \prime}\right) \backslash \Delta \neq \varnothing$. Then $\left\{D_{j}^{\prime \prime} \cup\right.$ $\operatorname{Taut}(\Delta, j)\}_{j \in J}$ is $\Delta$-decomposition of $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$, where for each $j \in J, \operatorname{Taut}(\Delta, j)$ is a set of tautologies in signature $\Delta \backslash \operatorname{sig}\left(D_{j}^{\prime \prime}\right)$ which are uniform in $S_{0}$.
2) Now let us verify that the sets of formulas from $\left\{D_{j}^{\prime \prime}\right\}_{j \in J}$ are pairwise $\Delta$-inseparable, if so are the components of $\mathcal{D}_{S_{0}}$.
a) First, consider the sets from the union

$$
\bigcup_{j \in \tilde{J}} \widetilde{D}_{j} \cup \bigcup_{j \in J \backslash \tilde{J}} D_{j}^{\prime}
$$

The pairwise intersection of their signatures is contained in $\Delta \cup \operatorname{sig}\left(S_{\alpha}\right)$. We claim that the sets from this union are pairwise $\Delta$-inseparable.

By our definition, for all $j \in \tilde{J}$ we have $D_{j}^{\prime} \subseteq \widetilde{D}_{j}$ and hence, $\operatorname{Cons}\left(D_{j}^{\prime}, \Delta\right) \subseteq \operatorname{Cons}\left(\widetilde{D}_{j}, \Delta\right)$, so let us check that $\operatorname{Cons}\left(\widetilde{D}_{j}, \Delta\right) \subseteq \operatorname{Cons}\left(D_{j}^{\prime}, \Delta\right)$ for every $j \in \tilde{J}$. Each formula in $D_{i}[\Omega]$, for $i \in f^{-1}(j), j \in \tilde{J}$, has the form

$$
\begin{equation*}
F\left(\bar{c}, d o\left(A\left(c_{1}, \ldots, c_{k}\right), S_{0}\right)\right) \leftrightarrow\left(\varepsilon_{1} \wedge \phi^{+}\right) \vee\left(F\left(\bar{c}, S_{0}\right) \wedge \varepsilon_{2} \wedge \phi^{-}\right) \tag{*}
\end{equation*}
$$

where $F$ is a fluent from $\operatorname{sig}\left(D_{j}^{\prime}\right), \bar{c}$ is a vector of constants from $\left\{c_{1}, \ldots, c_{k}\right\}, \phi^{+}, \phi^{-}$are sentences uniform in $S_{0}$, and each $\varepsilon_{1}, \varepsilon_{2}$ equals true or false (the parameters to summarize different cases of this formula). This is a definition of ground atom $F\left(\bar{c}, d o\left(A\left(c_{1}, \ldots, c_{k}\right), S_{0}\right)\right.$ via fluents at situation $S_{0}$ and situation-independent predicates and functions. Therefore, since $\Delta$ is fluent-free and for all $j \in \tilde{J}, D_{j}^{\prime}$ is uniform in $S_{0}$, every model $\mathcal{M}$ of $D_{j}^{\prime}$ can be transformed into a model $\mathcal{M}^{\prime}$ of $\widetilde{D}_{j}$ which agrees with $\mathcal{M}$ on $\Delta$. The model $\mathcal{M}^{\prime}$ is obtained in two steps. First, we expand $\mathcal{M}$ with an arbitrary interpretation of function $d o$ and situation-independent predicates and functions from $\operatorname{sig}\left(D_{i}[\Omega]\right) \backslash \operatorname{sig}\left(D_{j}^{\prime}\right)$ for every $i \in f^{-1}(j)$. Then we continue with this expanded model and modify the truth value of each fluent $F$ at the interpretation of the tuple $\left\langle\bar{c}, \operatorname{do}\left(A\left(c_{1}, \ldots, c_{k}\right), S_{0}\right)\right\rangle$ according to the obtained truth value of the formula in the definition of $F\left(\bar{c}, \operatorname{do}\left(A\left(c_{1}, \ldots, c_{k}\right), S_{0}\right)\right.$ above. This gives us the model $\mathcal{M}^{\prime}$. Hence, if $\varphi \in \operatorname{Cons}\left(\widetilde{D}_{j}, \Delta\right)$ and $\varphi \notin \operatorname{Cons}\left(D_{j}^{\prime}, \Delta\right)$, then there is a model $\mathcal{M}$ of $D_{j}^{\prime}$ such that $\mathcal{M} \not \vDash \varphi$, but then $\mathcal{M}^{\prime} \models \widetilde{D}_{j}$ and $\mathcal{M}^{\prime} \not \models \varphi$, a contradiction. Therefore, we conclude that for all $j \in \tilde{J}$, Cons $\left(\widetilde{D}_{j}, \Delta\right)=\operatorname{Cons}\left(D_{j}^{\prime}, \Delta\right)$ and, by pairwise $\Delta$-inseparability of the components of $\mathcal{D}_{S_{0}}$, the sets from the union ( $\ddagger$ ) are $\Delta$-inseparable.
b) Since $\Delta$ is fluent-free and $\Omega\left(S_{0}\right)$ consists only of ground atoms with fluents, from Corollary 3.10 we conclude that the sets from the following union are $\Delta$-inseparable:

$$
\bigcup_{j \in \tilde{J}} \operatorname{forget}\left(\widetilde{D}_{j},\left.\Omega\left(S_{0}\right)\right|_{j}\right) \cup \bigcup_{j \in J \backslash \tilde{J}} D_{j}^{\prime} \text {. }
$$

Now we are ready to prove that the sets from $\left\{D_{j}^{\prime \prime}\right\}_{j \in J}$ are pairwise $\Delta$-inseparable. For every $j \in \tilde{J}$, let us denote $G_{j}=\operatorname{forget}\left(\widetilde{D}_{j},\left.\Omega\left(S_{0}\right)\right|_{j}\right)$. We will demonstrate that for every $j \in \tilde{J}$ it holds Cons $\left(G_{j}\left(S_{0} / S_{\alpha}\right), \Delta\right)=$ Cons $\left(G_{j}, \Delta\right)$, from which the statement follows. First, let us verify that Cons $\left(G_{j}\left(S_{0} / S_{\alpha}\right), \Delta\right) \subseteq \operatorname{Cons}\left(G_{j}, \Delta\right)$. Assume that for some $j \in \tilde{J}$ (we fix this $j$ for the following) there is a formula $\varphi \in \operatorname{Cons}\left(G_{j}\left(S_{0} / S_{\alpha}\right), \Delta\right)$ and a model $\mathcal{M}$ of $G_{j}$ such that $\mathcal{M} \not \vDash \varphi$, and arrive at contradiction.

By the syntactic definition of forgetting a ground atom, the term $S_{\alpha}$ occurs in $G_{j}$ only in subformulas obtained from the definitions $(*)$, so let us consider such a definition for a ground atom $F\left(\bar{c}, S_{\alpha}\right)$ with some fluent $F$. Let us recall that $G_{j}$ is the result of forgetting a set of ground atoms with fluents having $S_{0}$ as situation argument. Since $\bar{c}$ is the vector of object arguments in the definition of $F\left(\bar{c}, S_{\alpha}\right)$ in $(*)$, we have $\left.F\left(\bar{c}, S_{0}\right) \in \Omega\left(S_{0}\right)\right|_{j}$. Therefore, if $\mathcal{M} \models^{\varepsilon} F\left(\bar{c}, S_{0}\right)(\varepsilon$ denotes the optional negation in front of atom), then there is a model $\mathcal{M}^{\prime} \models \neg^{\varepsilon} F\left(\bar{c}, S_{0}\right)$ such that $\mathcal{M}^{\prime} \sim_{\sigma} \mathcal{M}$,
with $\sigma=F\left(\bar{c}, S_{0}\right)$, and hence, $\mathcal{M}^{\prime} \notin \varphi$ (since $\Delta$ is fluent-free) and the truth value of $F\left(\bar{c}, S_{\alpha}\right)$ in $\mathcal{M}$ and $\mathcal{M}^{\prime}$ is the same. Hence, either in $\mathcal{M}$ or $\mathcal{M}^{\prime}$ the truth values of $F\left(\bar{c}, S_{\alpha}\right)$ and $F\left(\bar{c}, S_{0}\right)$ coincide. The similar argument applies to the whole set of definitions $(*)$ from $\widetilde{D}_{j}$ under forgetting the set $\left.\Omega\left(S_{0}\right)\right|_{j}$. Therefore we may assume that in $\mathcal{M}$ or $\mathcal{M}^{\prime}$, for each fluent $F \in \operatorname{sig}\left(G_{j}\right)$ the values of $F\left(\bar{c}, S_{\alpha}\right)$ and $F\left(\bar{c}, S_{0}\right)$ coincide. So $\mathcal{M} \models G_{j}\left(S_{0} / S_{\alpha}\right)$ or $\mathcal{M}^{\prime} \models G_{j}\left(S_{0} / S_{\alpha}\right)$ which is a contradiction, because $\varphi$ holds in neither of these models.

To prove the reverse inclusion Cons $\left(G_{j}, \Delta\right) \subseteq$ Cons $\left(G_{j}\left(S_{0} / S_{\alpha}\right), \Delta\right)$, observe that $G_{j}\left(S_{0} / S_{\alpha}\right)$ is uniform in $S_{0}$. Hence, by an observation similar to Lemma 2.8, every model $\mathcal{M}$ of $G_{j}\left(S_{0} / S_{\alpha}\right)$ can be expanded to a model $\mathcal{M}^{\prime}$, where the interpretation of function $d o$ is such that the values of terms $S_{\alpha}$ and $S_{0}$ in $\mathcal{M}^{\prime}$ coincide. Then $\mathcal{M}^{\prime} \models G_{j}$ and thus, there is no formula $\varphi \in \operatorname{Cons}\left(G_{j}, \Delta\right)$ such that $\varphi \notin$ Cons $\left(G_{j}\left(S_{0} / S_{\alpha}\right), \Delta\right)$.

We note that a result similar to Theorem 4.5 can be proved in a more general case, for progression of not-necessarily local-effect $\mathcal{B A} \mathcal{T}$ s, by considering progression as a set of consequences of $\mathcal{D}_{\text {una }} \cup \mathcal{D}_{s s} \cup \mathcal{D}_{S_{0}}$ uniform in $S_{\alpha}$ or using the second-order definition of progression from Theorem 2.10 in [27]. Since both definitions of progression are non-constructive, one would have to deal with background theories such as $\mathcal{D}_{\text {una }}$, when reasoning about decomposition of the initial theory. Although it would be possible to define a more general notion of decomposability wrt a background theory by following this direction, this study would take us too far away from the goals of this paper, and it would not be illuminating.

The proof of the theorem uses Proposition 4.2 and the component properties of forgetting from Section 3. The important observation behind this result is that in order to compute progression of an initial theory wrt an action having effects only on fluents from one decomposition component, it suffices to compute forgetting only in this component. Given a decomposition of the initial theory into inseparable components, the rest of the conditions in the theorem are purely syntactical and easy to check. For example, these conditions would naturally hold if one merges weakly-related action theories, as illustrated in the running example (continued below). SSAs can be grouped into $|I|$ components by drawing a graph with fluent names as vertices, and an edge from the fluent on the left-hand side of each SSA going to each fluent occurring on the right-hand side of the same SSA. Similarly, it is easy to check the last condition of the Theorem that guarantees alignment of groups of axioms in SSAs with decomposition components of $\mathcal{D}_{S_{0}}$.

In the above conditions, observe that if an action $A$ occurs in active position of SSAs from two different sub-theories of $\mathcal{D}_{s s}$, then computing progression may involve forgetting in two corresponding components of $\mathcal{D}_{S_{0}}$. This can potentially lead to appearance of new common $\Delta_{1}$-symbols in the components of progression. As a consequence, $\Delta_{2}$-decomposability of progression may be destroyed, but it is desirable to preserve it. A practically important class of $\mathcal{B} \mathcal{A} \mathcal{T}$ s, for which this interference can be avoided, is described in the corollary below. Note the first condition in the corollary that every action mentioned in $\mathcal{B A T}$ can have effects on fluents only from one component of $\mathcal{D}_{s s}$. Together with the second condition this guarantees preservation of $\Delta_{2}$-decomposability and inseparability of the initial theory after progression. The third condition in the corollary guarantees preservation of all the conditions of Theorem 4.5 for the $\mathcal{B A T}$ obtained after progression and thus, one can compute progression for arbitrary long sequences of actions while preserving decomposability of $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ and inseparability of its components.

Corollary 4.6 (Strong preservation of components in local-effect $\mathcal{B A T} \mathbf{s}$ ) In the conditions and notations of Theorem 4.5, let $\alpha=A(\bar{c})$ be a ground action term, where $\bar{c}=\left\langle c_{1}, \ldots, c_{k}\right\rangle$ is a tuple of constants, and let the following conditions hold:

- no action function is in $\Delta_{1}$;
- $\left\{c_{1}, \ldots, c_{k}\right\} \subseteq \operatorname{sig}\left(D_{j}^{\prime}\right)$, for some $j \in J$, whenever $A$ is in active position in a SSA for a fluent $F \in \operatorname{sig}\left(D_{j}^{\prime}\right)$;
- it holds that $\Delta_{1} \subseteq \Delta_{2}$.

Then $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is $\Delta_{2}$-decomposable into $\Delta_{2}$-inseparable components. Moreover, all the conditions of Theorem 4.5 hold for the $\mathcal{B} \mathcal{A} \mathcal{T}$ with the initial theory $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ obtained after progression.

Proof. By the first condition, action $A$ can be in active position of SSAs of a single subtheory $D_{i}$ of $\mathcal{D}_{s s}$. Then, due to the componentwise computation of progression shown in the proof of Theorem 4.5, progression can affect the single corresponding component $D_{f(i)}^{\prime}$ of $\mathcal{D}_{S_{0}}$. The second condition of the corollary guarantees that $\left\{c_{1}, \ldots, c_{k}\right\} \subseteq \operatorname{sig}\left(D_{f(i)}^{\prime}\right)$ and together with the third condition this yields that $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is $\Delta_{2}$-decomposable into $\Delta_{2}$-inseparable components, just like $\mathcal{D}_{S_{0}}$ is.

Computing the progression of $\mathcal{D}_{S_{0}}$ wrt $\alpha$ is essentially a syntactic modification of $D_{f(i)}^{\prime}$ which may introduce signature symbols from context conditions of $D_{i}$ only into $D_{f(i)}^{\prime}$ and into no other components of $\mathcal{D}_{S_{0}}$. Denote by $D_{f(i)}^{\prime \prime}$ the theory obtained from $D_{f(i)}^{\prime}$ in this way.

Let us verify that all the conditions of Theorem 4.5 are preserved for the $\mathcal{B A} \mathcal{T}$ with the initial theory $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ obtained after progression. The first condition of the theorem holds by default. By the definition of forgetting ground atoms, one can assume that sig $\left(D_{f(i)}^{\prime}\right) \subseteq \operatorname{sig}\left(D_{f(i)}^{\prime \prime}\right)$. Since $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is $\Delta_{2}$-decomposable and, by the definition of $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$, all the components of $\mathcal{D}_{S_{0}}$ except $D_{f(i)}^{\prime}$ remain unchanged after progression, the second and third conditions of the theorem hold. To show the last condition suppose the opposite, i.e. there is $k \in I$, for which the condition does not hold. Then $k \neq i$, since $\operatorname{sig}\left(D_{f(i)}^{\prime}\right) \subseteq \operatorname{sig}\left(D_{f(i)}^{\prime \prime}\right)$, and there is a subsignature $\Sigma \subseteq$ $\operatorname{sig}\left(D_{f(i)}^{\prime \prime}\right) \backslash \operatorname{sig}\left(D_{f(i)}^{\prime}\right)$ such that $\Sigma \subseteq \operatorname{sig}\left(D_{k}\right)$. By the definition of $D_{f(i)}^{\prime \prime}$, we may assume that $\Sigma \subseteq \operatorname{sig}\left(D_{i}\right)$. As $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is a set of formulas uniform in $S_{0}$, we have do $\notin \Sigma$ and thus, $\Sigma \subseteq$ $\Delta_{1} \subseteq \Delta_{2}$. Let $D_{j}^{\prime}$ be the component of $\mathcal{D}_{S_{0}}$, for which the condition $\operatorname{sig}\left(D_{k}\right) \cap \mathcal{D}_{S_{0}} \subseteq \operatorname{sig}\left(D_{j}^{\prime}\right)$ holds. Since $\Sigma \subseteq \Delta_{2}$ and $\mathcal{D}_{S_{0}}$ is $\Delta_{2}$-decomposable, we have $\Sigma \subseteq \operatorname{sig}\left(D_{j}^{\prime}\right)$ and thus $D_{j}^{\prime}$ is the required component for $D_{k}$, a contradiction.

Example 2 (continuation). Note that the $\mathcal{B} \mathcal{A} \mathcal{T}$ considered in the example satisfies the conditions of the corollary with fluent-free signatures $\Delta_{1}=\varnothing$ and $\Delta_{2}=\left\{B l o c k, S_{0}\right\}$. The theory $\mathcal{D}_{s s}$ is a union of two theories, with the intersection of signatures equal to $\{d o\}$. As already noted in the example, the initial theory $\mathcal{D}_{S_{0}}$ is $\Delta_{2}$-decomposable into $\Delta_{2}$-inseparable components. Now, consider the ground action $\alpha=\operatorname{move}(A, B, C)$. By Corollary 3.10 and Proposition 4.2, in order to compute the theory $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ (the progression of $\mathcal{D}_{S_{0}}$ wrt $\alpha$, with the term $S_{\alpha}$ substituted with $S_{0}$ ), it suffices to forget the ground atoms $\operatorname{On}\left(A, B, S_{0}\right)$ and $C l e a r\left(C, S_{0}\right)$ in the first decomposition component of $\mathcal{D}_{S_{0}}$ and update it with the ground atoms $\operatorname{On}\left(A, C, S_{0}\right)$ and $C l e a r\left(B, S_{0}\right)$. The second component of $\mathcal{D}_{S_{0}}$ remains unchanged. One can check that $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is the union of the following theories:

```
\varphi}\psi\psi\wedge(x\not=C)->Clear (x, S S )
\psi}->\mathrm{ Block(x)
Block(B)^Block (C)^On (A,C,S S ) ^ \negOn (A,B,S S )
Clear (A, S S ) ^Clear (B, S0 ) ^\negClear (C, So)
```

and

$$
\begin{aligned}
& \left(\operatorname{Top}\left(x, S_{0}\right) \vee \operatorname{Inheap}\left(x, S_{0}\right)\right) \rightarrow \neg \operatorname{Block}(x) \\
& \exists x \operatorname{Block}(x),
\end{aligned}
$$

where $\varphi$ and $\psi$, respectively, stand for
$(x \neq B) \wedge \neg \exists y\left((y \neq A \vee x \neq B) \wedge O n\left(y, x, S_{0}\right)\right)$,
$(x=A) \vee \exists y\left((x \neq A \vee B \neq y) \wedge O n\left(x, y, S_{0}\right)\right)$.
The theory $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ is $\Delta_{2}$-decomposable by the syntactic form and there is no need to compute a decomposition again after progression. Corollary 4.6 guarantees that the obtained components are $\Delta_{2}$-inseparable. It is important that in this case we can compute progression for arbitrary long sequences of actions while preserving both decomposability of $\mathcal{D}_{S_{\alpha}}\left(S_{0} / S_{\alpha}\right)$ and inseparability of its components.

## 5 Summary and Future Work

We have considered the impact of the theory update operations, such as forgetting and progression on preserving the component properties of theories, such as decomposability and inseparability. Forgetting and progression have a "semantic nature", since the input and the output of these transformations are related to each other by using restrictions on the classes of models. On the contrary, the decomposability and inseparability properties are defined using entailment in a particular logic. As logics (weaker than second-order) may not distinguish the needed classes of models, the conceptual "distance" between these two kinds of notions is potentially immense. This can be somewhat bridged by the choice of either an appropriate logic, or appropriate theories in the input. We have identified conditions that should be imposed on the components of input theories to match these notions more closely. Also, the Parallel Interpolation Property (PIP) was shown to be a relevant property of logics in our investigations. The results can be briefly summarized in the tables below. For brevity, we use $\sigma$ to denote a signature or a ground atom. We slightly abuse notation and consider $\sigma$ as a set of symbols even in the case of a ground atom implying that in the latter case $\sigma$ consists of the single predicate symbol from the atom. We assume that the input of operations of forgetting and progression is a union of theories $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$, with $\operatorname{sig}\left(\mathcal{T}_{1}\right) \cap \operatorname{sig}\left(\mathcal{T}_{2}\right)=\Delta$, for a signature $\Delta$.

| Property | Condition | Result | Reference |
| :---: | :---: | :---: | :---: |
| Preservation of $\Delta$-inseparability of $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ under forgetting $\sigma$ | $\sigma \cap \Delta=\varnothing$ | YES | Corollary 3.10 |
|  | $\sigma \subseteq \Delta$ and $\sigma$ is a ground atom | NO | Example 4 |
|  | $\sigma \subseteq \Delta$ and $\sigma$ is a signature | YES, if logic has PIP | Proposition 3.7 |
|  | $\overline{\sigma \subseteq \Delta}$ and $\mathcal{T}_{1}, \mathcal{T}_{2}$ are semantically inseparable | YES | Proposition 3.8 |
| Distributivity of forgetting $\sigma$ over union of $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ | $\sigma \cap \Delta=\varnothing$ | YES | Corollary 3.10 |
|  | $\sigma \subseteq \Delta$ | NO, even if $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ are semantically inseparable | Example 5 |
|  | $\overline{\mathcal{T}_{1}}$ and $\mathcal{T}_{2}$ are semantically inseparable "modulo $\sigma$ " | YES | Proposition 3.9 |


| Property | Condition | Preservation | Reference |
| :---: | :---: | :---: | :---: |
| $\left\|\begin{array}{l} \Delta- \\ \text { inseparability } \\ \text { of compo- } \\ \text { nents of initial } \\ \text { theory under } \\ \text { progression } \end{array}\right\|$ | at least one fluent is present in $\Delta$ | NO | Example 7 |
|  | $\Delta$ is fluent-free and some components of initial theory split under progression | NO | Example 8 |
|  | $\Delta$ is fluent-free and components of initial theory do not split under progression | YES, <br> modulo the unique name assumption theory | Theorem 4.4 |
|  | $\mathcal{B A T}$ is local-effect, $\Delta$ is fluent-free and components of initial theory do not split under progression | YES | Theorem 4.5 |
| $\Delta-$ <br> decomposabi- <br> lity and preservation of signature components of an initial theory under progression wrt action term $\alpha$ | Unconditionally, in particular for local-effect $\mathcal{B A} \mathcal{T}$ s and fluent-free $\Delta$ 's | NO | Example 6 |
|  | $\mathcal{B A T}$ is local-effect, $\Delta$ is fluent-free, and components of $\mathcal{D}_{S_{0}}$ are aligned with components of $\mathcal{D}_{s s}$ | YES, <br> modulo common symbols of the components of $\mathcal{D}_{s s}$ and constants in term $\alpha$ | Theorem 4.5 |
|  | if additionally the constants in term $\alpha$ are contained in a single $\Delta$-decomposition component of $\mathcal{D}_{S_{0}}$ | YES | Corollary 4.6 |

The examples and Lemmas given in the paper demonstrate that the sufficient conditions for invariance of decomposability and inseparability wrt progression in local-effect action theories cannot be relaxed. Our research has required new understanding of progression and the related notion of forgetting wrt modularity of theories. The new results about forgetting are general and may find applicability outside of reasoning about actions. Given a decomposition of the initial theory into inseparable components, the rest of the conditions in Theorem 4.5 and Corollary 4.6 are purely syntactical and therefore are easy to check. The important practical observation behind these results is that in order to compute the progression of an initial theory wrt an action having effects only on fluents from one decomposition component, it suffices to compute forgetting only in this component. As illustrated by the running example, non-interacting dynamic systems may share only some common names or static entities, such as location. The fact that the dynamic systems share no fluents can be obscured by the way they are presented, whereas decomposition would make it explicit. We believe that our positive results are applicable to a large and general class of basic action theories. The significant contribution of the paper is in exploring the important connections between research on
modularity and reasoning about action. The paper starts bridging the gap between these two different research communities in knowledge representation.

There are several directions where future work may proceed. In this paper, we concentrate on local-effect action theories only. However, recently [7] defined a new broad class of action theories called bounded situation calculus action theories, in which actions may have non-local, but bounded effects. Moreover, for these action theories, one can find cases when progression is effectively computable [46]. Therefore, it is natural to explore when decomposability and inseparability remain invariant wrt progression in bounded action theories. Additionally, we noted that there is a realistic case of initial theories, for which the size of a progressed theory with local effects does not grow exponentially. The initial theories of this kind are known as proper ${ }^{+}$theories [21,27]. Therefore, it is worth while to develop computationally tractable techniques for decomposition of proper ${ }^{+}$theories.

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[^0]:    ${ }^{3}$ The phrase local-effect actions first appeared in [28], but it was motivated by actions with simple effects defined in the paper [24], where simple effects are understood similar to the Def. 2.11.

[^1]:    ${ }^{4}$ Some of these axioms, e.g., the second axiom, remain true after executing any of the possible actions, but this fact is irrelevant to the purposes of this example.

