

Hardness of approximation for strip packing

Anna Adamaszek* Tomasz Kociumaka† Marcin Pilipczuk‡ Michał Pilipczuk§

Abstract

Strip packing is a classical packing problem, where the goal is to pack a set of rectangular objects into a strip of a given width, while minimizing the total height of the packing. The problem has multiple applications, e.g. in scheduling and stock-cutting, and has been studied extensively.

When the dimensions of objects are allowed to be exponential in the total input size, it is known that the problem cannot be approximated within a factor better than $3/2$, unless $P = NP$. However, there was no corresponding lower bound for polynomially bounded input data. In fact, Nadiradze and Wiese [SODA 2016] have recently proposed a $(1.4 + \epsilon)$ approximation algorithm for this variant, thus showing that strip packing with polynomially bounded data can be approximated better than when exponentially large values in the input data are allowed. Their result has subsequently been improved to a $(4/3 + \epsilon)$ approximation by two independent research groups [FSTTCS 2016, arXiv:1610.04430]. This raises a question whether strip packing with polynomially bounded input data admits a quasi-polynomial time approximation scheme, as is the case for related two-dimensional packing problems like maximum independent set of rectangles or two-dimensional knapsack.

In this paper we answer this question in negative by proving that it is NP-hard to approximate strip packing within a factor better than $12/11$, even when admitting only polynomially bounded input data. In particular, this shows that the strip packing problem admits no quasi-polynomial time approximation scheme, unless $NP \subseteq DTIME(2^{\text{polylog}(n)})$.

1 Introduction

In the strip packing problem we are given a collection of rectangular items I where each item $i \in I$ is defined by its width $w_i \in \mathbb{N}$ and height $h_i \in \mathbb{N}$, together with a width parameter $W \in \mathbb{N}$. The goal is to pack all items of I into a strip of width W such that no two rectangles overlap, and the height of the packing is minimized. Formally, in a packing each item $i \in I$ corresponds to a rectangle $(x_i, x_i + w_i) \times (y_i, y_i + h_i)$, where $x_i, y_i \in \mathbb{N}$ and $x_i + w_i \leq W$, and the corresponding (open) rectangles are pairwise disjoint. The objective is to construct a packing (i.e., find values of x_i, y_i for all the input rectangles) minimizing the value of $H = \max_{i \in I} (y_i + h_i)$.

Motivation. The strip packing problem arises naturally in many settings. In the area of scheduling, it models scheduling of jobs where each job i requires a contiguous portion of w_i memory, where the total memory of the machine is W , over a period of time h_i . Minimizing the total height H of the solution corresponds to minimizing the makespan of the schedule. In the area of industrial manufacturing, we have the following cutting stock problem which also corresponds to strip packing. We want to cut rectangular pieces out of a sheet of material (e.g. cloth or wood) of fixed width, minimizing the total amount of material

*University of Copenhagen, Denmark, anad@di.ku.dk. Research supported by the Danish Council for Independent Research DFF-MOBILEX mobility grant.

†University of Warsaw, Poland, kociumaka@mimuw.edu.pl.

‡University of Warsaw, Poland, malcin@mimuw.edu.pl. Research supported by Polish National Science Centre grant UMO-2013/09/B/ST6/03136.

§University of Warsaw, Poland, michal.pilipczuk@mimuw.edu.pl. Research supported by Polish National Science Centre grant UMO-2013/11/D/ST6/03073. Author supported by Foundation for Polish Science (FNP) via the START stipend programme.

used. Here we do not allow the rectangles to be rotated (because of the constraints like patterns on the material or the grain of the wood). Recently, strip packing has also been applied in electricity allocation and peak demand reductions in smart-grid (see [5]).

Related work. The strip packing problem has been studied extensively since 1980, when Baker et al. [4] provided an algorithm with an asymptotic approximation ratio of 3. Subsequently, in a series of papers [14, 18, 17, 19, 9, 8] better approximation algorithms have been presented. Currently, the best result is a $5/3 + \epsilon$ approximation ratio by Harren et al. [8]. The best known lower bound for approximating the problem is $3/2$, and it can be shown by a straightforward reduction from PARTITION.

The asymptotic approximation ratio for strip packing has also been studied in [14, 7, 3]. There, the lower bound of $3/2$ does not hold. In fact, there is an asymptotic FPTAS given by Kenyon and Rémila [15] with an additive constant of $O(h_{\max}/\epsilon^2)$, and an asymptotic PTAS given by Jansen and Solis-Oba [11] with an additive constant of h_{\max} , where h_{\max} is the maximum height of a rectangle in the input instance. Strip packing has also been studied in the setting where rotations of the items by 90 degrees are allowed. In this setting, Jansen and van Stee [12] provided an asymptotic FPTAS.

Recently, pseudo-polynomial time algorithms for strip packing have been considered. Nadiradze and Wiese [16] have given an algorithm achieving an approximation ratio of $1.4 + \epsilon$, which has then been refined to a $4/3 + \epsilon$ -approximation algorithm by Gálvez et al. [5] and by Jansen and Rau [10]. The running time of these algorithms is polynomial when input data is polynomially bounded, that is, when all the numbers W , w_i and h_i are bounded polynomially in the number of items.

Strip packing is related to the *geometric knapsack* problem. There, given a collection of rectangular items I , where each item $i \in I$ is defined by its width $w_i \in \mathbb{N}$, height $h_i \in \mathbb{N}$, and weight $\omega_i \in \mathbb{R}^+$, and a rectangular box of size $W \times H$, the goal is to pack a subcollection of items of maximum total weight into the given box so that no two items overlap. For this problem there is a $(2 + \epsilon)$ -approximation algorithm by Jansen and Zhang [13], and a QPTAS by Adamaszek and Wiese [2].

Our results. The QPTAS for geometric knapsack [2], and also the pseudo-polynomial time algorithms for strip packing [16, 5, 10], are based on the machinery introduced by Adamaszek and Wiese [1] for designing a quasi-polynomial time approximation scheme for maximum independent set of rectangles. This raises a question, asked in [5, 10], whether there is also a quasi-polynomial time approximation scheme for strip packing, possibly based on the same machinery, when the input data is polynomially bounded.

We give a negative answer to this question, by showing APX-hardness of strip packing. This shows that strip packing behaves differently with respect to approximation than the related maximum independent set of rectangles and two-dimensional knapsack problems.

Theorem 1. *For every $\epsilon > 0$, it is NP-hard to approximate STRIP PACKING within a factor of $12/11 - \epsilon$, even if the dimensions of the rectangles are given in unary.*

At the heart of our reduction in the proof of Theorem 1 lies an example showing that a rearrangement argument, being the core engine of the approach of Nadiradze and Wiese [16] (cf. Section 4 of [16]), fails to work if there are three rows of rectangles, not two as in [16]. Figure 1 illustrates this example: if we pick a and b to be such integers that any nontrivial integral solution to the equation $ax + by + z = 0$ requires integers of magnitude much larger than the number of rectangles, then one can argue that the presented packing in a strip of height 11 is essentially the only one possible, and any rearrangement requires a strip of height of at least 12. Our reduction exploits this figure by chopping every $b \times 1$ rectangle vertically into three pieces, embedding a 3-PARTITION instance into the picture.

At the intuitive level, it is the combination of hard constraints—the requirement of packing all the items and the inability of widening the strip—that makes the problem hard to approximate. The issue lies in relatively tall items that have to “stick out” by a significant portion of their height in case they cannot be packed optimally into some prescribed space. In fact, Nadiradze and Wiese [16] have given a PTAS for the case when a constant (depending on ϵ) number of items can be dropped, even when the input data is exponentially bounded.

2 Hardness of approximation for strip packing

In this section, we prove Theorem 1 by a reduction from the 3-PARTITION problem, which is known to be strongly NP-complete [6].

Assume we are given an instance of the 3-PARTITION problem: a multiset $S = \{s_1, s_2, \dots, s_{3n}\}$ of $3n$ integers, summing up to zero; the goal is to partition S into n triples, each summing up to zero. Furthermore, let $M = 1 + \sum_{i=1}^{3n} |s_i|$. Since 3-PARTITION is strongly NP-hard, we can assume that all integers s_i are given in unary; that is, the running time bound, the dimensions of the rectangles, and the size of the output instance in our reduction can depend polynomially on both n and M .

Construction. We start by choosing two positive integers a, b using the following standard claim.

Claim 1. *Given positive integers n and M , one can in time polynomial in $n + M$ find two positive integers a, b with the following properties:*

- (I) $a, b > 3M$ and b is divisible by 3;
- (II) for every three integers x, y, z with $|x|, |y|, |z| \leq \max(9n, 3M)$, if $ax + by + z = 0$, then $x = y = z = 0$;
- (III) a and b are bounded polynomially in n and M .

Proof. Let us take $b = \max(9n, 3M) + 3$ and $a = b^2$. Let x, y, z, w be integers such that $ax + by + z = 0$ and $|x|, |y|, |z| \leq \max(9n, 3M)$. Consequently, we have $xb^2 + yb + z = 0$ and $|x|, |y|, |z| < b$. Taking the equality modulo b , we see that z is divisible by b and thus $z = 0$. Then, analyzing it modulo b^2 , we show that y is divisible by b and hence $y = 0$. Finally, we conclude that $x = 0$. \square

Armed with Claim 1, we can now construct rectangles in the output STRIP PACKING instance. First, we set the width of the strip to

$$W := 2(a + b) \cdot n. \tag{1}$$

We will now define a set of rectangles of total area $11W$, such that the rectangles can be packed into a rectangle $W \times 11$ if and only if the input 3-PARTITION instance is a yes-instance. As every rectangle in our construction will have integral height, if the input 3-PARTITION instance is a no-instance, then we will need a strip of width at least 12 to accommodate all rectangles. Consequently, such a construction would prove Theorem 1 due to (strong) NP-completeness of 3-PARTITION.

The output instance consists of the following rectangles:

(middle rectangles) We construct in total $6n$ *middle rectangles* as follows:

- $2n$ rectangles of height 2 and width a ;
- n rectangles of height 3 and width b ;
- for every $1 \leq i \leq 3n$, a rectangle of height 1 and width $b/3 + s_i$ (called *solution rectangles*).

(side rectangles) We construct in total $4n + 1$ *side rectangles* as follows:

- $2n$ rectangles of height 4 and width $a + b$;
- $2n - 1$ rectangles of height 5 and width $a + b$;
- one rectangle of height 5 and width a , and one rectangle of height 5 and width b .

Note that $b > 3M$ implies that all $b/3 > s_i$ for $1 \leq i \leq 3n$ and thus all rectangles are well-defined.

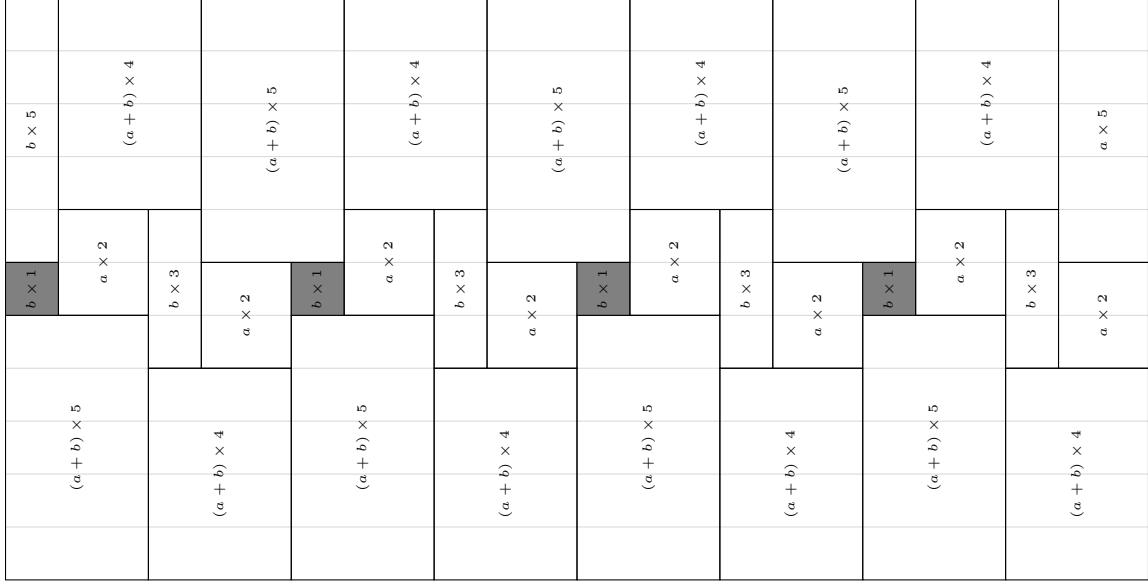


Figure 1: Canonical packing of the rectangles. Every gray $b \times 1$ rectangle consists of three solution rectangles, corresponding to a set in the solution to the input 3-PARTITION instance.

From partition to packing. We now show that if the input 3-PARTITION instance is a yes-instance, then we can pack all rectangles into a strip of size $W \times 11$. Let us group solution rectangles into n triples, as in the solution to the input 3-PARTITION instance. In this manner, we replace $3n$ solution rectangles with n middle rectangles, each of width b and height 1. Such rectangles can be arranged as in Figure 1:

1. We put n side rectangles of dimension $(a+b) \times 4$ and n side rectangles of dimension $(a+b) \times 5$ on the bottom of the $W \times 11$ strip in an alternating fashion, starting from a rectangle of height 5.
2. We put n side rectangles of dimension $(a+b) \times 4$, $n-1$ side rectangles of dimension $(a+b) \times 5$, and the two remaining side rectangles of dimension $a \times 5$ and $b \times 5$ on the top of the $W \times 11$ strip, in the following order: first the $b \times 5$ rectangle, then an alternating sequence of $(a+b) \times 4$ and $(a+b) \times 5$ rectangles, starting from an $(a+b) \times 4$ rectangle, and the $a \times 5$ rectangle at the end.
3. In the middle, we put the middle rectangles, in the following order of heights: 1, 2, 3, 2, \dots , 1, 2, 3, 2.

A direct check shows that the middle rectangles fit exactly in the spaces left by the side rectangles.

From packing to partition. Assume now that all rectangles can be packed into a strip of size $W \times 11$; let us fix such a packing. Our goal is to show that the input 3-PARTITION instance is a yes-instance, and the route to proving this is by verifying that the arrangement of non-solution rectangles must be exactly as depicted in Figure 1. A direct calculation shows that the total area of all rectangles is $11W$. Consequently, the $W \times 11$ strip is completely covered by the rectangles.

We say that a vertical line is *in general position* if it intersects the strip but does not contain any side of a rectangle. We say that a set \mathcal{R} of rectangles is *tightly packed* if every vertical line in general position intersects exactly one rectangle of \mathcal{R} . In other words, \mathcal{R} is tightly packed if there is exactly one rectangle of \mathcal{R} touching the left side of the strip, exactly one touching the right side, and when we scan the strip from left to right, then a rectangle from \mathcal{R} ends at some x -coordinate within the strip if and only if a new one rectangle starts at exactly the same x -coordinate.

We now show the following.

Claim 2. *Every vertical line in general position intersects exactly three rectangles: two side ones and one middle one. Consequently, the set of middle rectangles is tightly packed, and one can partition the set of side rectangles into two parts \mathcal{R}_1 and \mathcal{R}_2 such that \mathcal{R}_t is tightly packed for $t = 1, 2$.*

Proof. Since every rectangle has a height of at most 5, any two rectangles have a total height of at most 10, and thus every vertical line in general position intersects at least three rectangles. Since the total width of all rectangles is exactly $3W$, every vertical line in general position intersects exactly three rectangles. However, since any three side rectangles have a total height of at least 12, every vertical line in general position can intersect only two of them. As the side rectangles have a total width of exactly $2W$, every vertical line in general position intersects exactly two side rectangles, and thus exactly one middle rectangle.

For the second claim, we can construct \mathcal{R}_1 by taking one of the two side rectangles at the left side of the strip, and build \mathcal{R}_1 from left to right by picking a side rectangle starting at the x -coordinate where the previously picked side rectangle ends. There will be always such a side rectangle, because the vertical line in general position just after this x -coordinate again intersects two side rectangles. \lrcorner

Let \mathcal{R}_1 and \mathcal{R}_2 be as in Claim 2. Observe that, in each set \mathcal{R}_t , the total width of all rectangles equals $W = 2(a+b)n$. Since there are only $4n+1$ side rectangles, from Claim 1, property (II), it easily follows that one of the sets \mathcal{R}_t (w.l.o.g., say it is \mathcal{R}_1), contains n rectangles $(a+b) \times 4$ and n rectangles $(a+b) \times 5$, and the other one contains the remaining $2n+1$ rectangles, including the two rectangles $a \times 5$ and $b \times 5$. By potentially taking a symmetrical image of the strip, we may assume that the $b \times 5$ rectangle appears to the left of the $a \times 5$ rectangle in the strip.

Assume that the left end of the strip is at x -coordinate 0, and the right end is at x -coordinate W .

Claim 3. *Let x_0 be an x -coordinate where a side rectangle from \mathcal{R}_t ($t \in \{1, 2\}$) ends. Then $x_0 = a \cdot n_a + b \cdot n_b$ for some integers $0 \leq n_a, n_b \leq 2n$ with $n_b - n_a \in \{0, 1\}$.*

Proof. Since every side rectangle has width $a+b$, a , or b , and the rectangles of \mathcal{R}_t are tightly packed, we can express $x_0 = (a+b)m_{a+b} + am_a + bm_b$, where m_z is the number of rectangles of width z that are in \mathcal{R}_t and to the left of x_0 . By taking $n_a = m_{a+b} + m_a$ and $n_b = m_{a+b} + m_b$, we obtain $x_0 = a \cdot n_a + b \cdot n_b$ and $0 \leq n_a, n_b \leq 2n$.

Furthermore, $n_b - n_a = m_b - m_a$. Since there is only one rectangle of width b and one of width a , we have $m_a, m_b \in \{0, 1\}$. Since we have assumed that the $b \times 5$ one is to the left of the $a \times 5$ one, it cannot hold that $m_b = 0$ and $m_a = 1$. Consequently, $m_b - m_a \in \{0, 1\}$. \lrcorner

Claim 4. *If we order the middle rectangles from left to right (recall that they are tightly packed), then every maximal consecutive segment of solution rectangles consists of three rectangles of total width exactly b .*

Proof. Let $R_{i_1}, R_{i_2}, \dots, R_{i_k}$ be a maximal consecutive segment of solution rectangles in the left-to-right ordering of all middle rectangles. Assume that the width of R_{i_j} is $b/3 + s_{i_j}$. We shall prove that $k = 3$ and that $\sum_{j=1}^k s_{i_j} = 0$.

Let x_0 be the x -coordinate of the left side of R_{i_1} and x'_0 be the x -coordinate of the right side of R_{i_k} . Since the sequence $R_{i_1}, R_{i_2}, \dots, R_{i_k}$ is maximal, other middle rectangles have different heights, and the whole $W \times 11$ strip is covered by rectangles, both at x_0 and at x'_0 a side rectangle ends. By Claim 3 we have $x_0 = an_a + bn_b$ and $x'_0 = an'_a + bn'_b$ for some integers n_a, n_b, n'_a, n'_b with $\delta := n_b - n_a \in \{0, 1\}$ and $\delta' := n'_b - n'_a \in \{0, 1\}$. Furthermore, $x'_0 - x_0 = kb/3 + \sum_{j=1}^k s_{i_j}$, that is,

$$kb/3 + \sum_{j=1}^k s_{i_j} = x'_0 - x_0 = a(n'_a - n_a) + b(n'_b - n_b),$$

i.e.,

$$0 = a(3n'_a - 3n_a) + b(3n'_b - 3n_b - k) - 3 \sum_{j=1}^k s_{i_j}.$$

Note that $|3n'_a - 3n_a| \leq 6n \leq \max(9n, 3M)$, $|3n'_b - 3n_b - k| \leq 9n \leq \max(9n, 3M)$, and $|3 \sum_{j=1}^k s_{i_j}| \leq 3M \leq \max(9n, 3M)$. Thus, by Claim 1, property (II), we easily obtain $n'_a = n_a$, $n'_b - n_b = k/3$, and $\sum_{j=1}^k s_{i_j} = 0$. However, recall that $\delta = n_b - n_a \in \{0, 1\}$ and $\delta' = n'_b - n'_a \in \{0, 1\}$ while $k \geq 1$. Then $\delta' = \delta + k/3$, so $\delta, \delta' \in \{0, 1\}$ and $k \geq 1$ implies that the only possibility is $k = 3$, $\delta = 0$, and $\delta' = 1$. \lrcorner

Consequently, the solution rectangles are partitioned into triples of rectangles of total width b . Such a partition of the solution rectangles induces a solution to the input 3-PARTITION instance. This finishes the proof of Theorem 1.

References

- [1] Anna Adamaszek and Andreas Wiese. Approximation schemes for maximum weight independent set of rectangles. In *FOCS 2013*, pages 400–409. IEEE, 2013.
- [2] Anna Adamaszek and Andreas Wiese. A quasi-PTAS for the two-dimensional geometric knapsack problem. In *SODA 2015*, pages 1491–1505, 2015.
- [3] Brenda S. Baker, Donna J. Brown, and Howard P. Katseff. A $5/4$ algorithm for two-dimensional packing. *J. Algorithms*, 2(4):348–368, 1981.
- [4] Brenda S. Baker, Edward G. Coffman Jr., and Ronald L. Rivest. Orthogonal packings in two dimensions. *SIAM J. Comput.*, 9(4):846–855, 1980.
- [5] Waldo Gálvez, Fabrizio Grandoni, Salvatore Ingala, and Arindam Khan. Improved pseudo-polynomial-time approximation for strip packing. In *FSTTCS 2016*, 2016. To appear.
- [6] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.
- [7] Igal Golan. Performance bounds for orthogonal oriented two-dimensional packing algorithms. *SIAM J. Comput.*, 10(3):571–582, 1981.
- [8] Rolf Harren, Klaus Jansen, Lars Prädell, and Rob van Stee. A $(5/3 + \epsilon)$ -approximation for strip packing. *Comput. Geom.*, 47(2):248–267, 2014.
- [9] Rolf Harren and Rob van Stee. Improved absolute approximation ratios for two-dimensional packing problems. In *APPROX 2009 and RANDOM 2009*, pages 177–189, 2009.
- [10] Klaus Jansen and Malin Rau. Improved approximation for two dimensional strip packing with polynomial bounded width. *CoRR*, abs/1610.04430, 2016.
- [11] Klaus Jansen and Roberto Solis-Oba. Rectangle packing with one-dimensional resource augmentation. *Discrete Optimization*, 6(3):310–323, 2009.
- [12] Klaus Jansen and Rob van Stee. On strip packing with rotations. In *STOC 2005*, pages 755–761, 2005.
- [13] Klaus Jansen and Guochuan Zhang. Maximizing the total profit of rectangles packed into a rectangle. *Algorithmica*, 47(3):323–342, 2007.
- [14] Edward G. Coffman Jr., M. R. Garey, David S. Johnson, and Robert Endre Tarjan. Performance bounds for level-oriented two-dimensional packing algorithms. *SIAM J. Comput.*, 9(4):808–826, 1980.
- [15] Claire Kenyon and Eric Rémy. A near-optimal solution to a two-dimensional cutting stock problem. *Math. Oper. Res.*, 25(4):645–656, 2000.
- [16] Giorgi Nadiradze and Andreas Wiese. On approximating strip packing with a better ratio than $3/2$. In *SODA 2016*, pages 1491–1510. SIAM, 2016.
- [17] Ingo Schiermeyer. Reverse-fit: A 2-optimal algorithm for packing rectangles. In *ESA 1994*, pages 290–299, 1994.
- [18] Daniel Dominic Sleator. A 2.5 times optimal algorithm for packing in two dimensions. *Inf. Process. Lett.*, 10(1):37–40, 1980.
- [19] A. Steinberg. A strip-packing algorithm with absolute performance bound 2. *SIAM J. Comput.*, 26(2):401–409, 1997.