About the Polynomial Solutions of Homogeneous Linear Differential Equations Depending on Parameters

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Abstract

The aim of this paper is to decide whether a linear differential equation with polynomial coefficients depending on parameters has got polynomial solutions. More precisely we want to construct a finite set T of necessary and sufficient algebraic and arithmetic conditions such that there is a polynomial solution if and only if the parameters belong to T. The presence of Diophantine equations makes the general problem undecidable. We get such a set T when the recurrence relation associated to the equation (in an appropriate basis) has got two terms. Using hypergeometric sequences we also succeed in constructing sufficient conditions for a family of equations.

1 Introduction

Let $L = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \cdots + a_0(x)$ be a linear differential operator with coefficients in $K[x]$.

Let us assume first that K is $\bar{\mathbb{Q}}$. The main problem in differential Galois theory is to determine the differential Galois group of the linear differential homogeneous equation $L(y) = 0$ that is to say the algebraic relations between the solutions of the equation. No algorithm exists to handle this general problem ([18]). However one can characterize the liouvillian solutions, that is to say solutions that are built up using integration, exponentiations, algebraic functions and composition ([18]). The question of deciding when the equation $L(y) = 0$ has such solutions has been particularly discussed for a long time ([17], [21], [24]). M.F. Singer has proved that one can decide whether or not the equation $L(y) = 0$ has got liouvillian solutions ([17]). When the order of the equation is equal to two, Kovacic effectively computes them ([11]). Their computation is reduced to the computation of polynomial, rational or exponential solutions of differential linear equations ([22]). Many algorithms have been constructed to perform the computation of these three types of solutions ([1], [2], [15], [20], [23],...) .

Let us assume now that K is $\bar{\mathbb{Q}}(M_1, \ldots, M_s)$ where $M_1, \ldots, M_s$ are parameters lying in $\mathbb{C}$. The main question is to find which parameters lead to a liouvillian solution.

In 1992, A. Duval and M. Loday-Richaud have studied the liouvillian solutions of some particular families of equations of order two depending upon some parameters (Schwarz and Heun equations, [5]). At first sight, one could think that the computation of the polynomial solutions would be easy and that the main difficulties would be encountered within the computation of the exponential solutions. One can show that the presence of the parameters does not modify the method used in the computation of the exponential parts ([3]). So the problem of describing the liouvillian solutions can be algorithmically reduced to describing the values of the parameters such that some linear differential equations have polynomial solutions. It is this last problem that A. Duval and M. Loday-Richaud have encountered while describing the liouvillian solutions (of the Schwarz and Heun equations) and that we focus on in this article.

In section 2, we prove that there is no algorithm which decides for which values of the parameters any given linear differential equation depending on the parameters has polynomial solutions. We note that if we fix a nonnegative integer d then the set of the parameters leading to a polynomial solution of degree less than or equal to d is a constructible set. The main difficulty is when the degree of the polynomial solution actually depends upon the parameters. Our aim is to construct a finite set of necessary and sufficient algebraic and arithmetic conditions on the parameters leading to a polynomial solution whose degree may not be numerically fixed. This last problem remains open in full generality, so we propose a 'tool box' that can solve many cases. The key to our approach is the link between polynomial solutions to linear differential equations and finite solutions to linear recurrence relations.

In section 3, we consider two terms recurrence relations. We construct a finite set of necessary and sufficient conditions on the parameters leading to a finite solution to such a recurrence relation.

In section 4, we study three terms recurrence relations of order two. We prove that, under some hypotheses, for all nonnegative integer d, the set of the parameters leading to a polynomial solution of degree less than or equal to d is a non empty constructible set. Then we generalize Hautot's method ([8], [7], [9]) and give a finite set of sufficient conditions on the parameters leading to a finite solution (whose degree may depend on the parameters) to the recurrence relation.

Lastly, in section 5, we adapt Petkovsek's method ([14]) to find finite hypergeometric solutions to any parameterized recurrence relation. We detail the case of a family of recurrence relations for which we construct finite non empty sets of arithmetic or algebraic sufficient conditions on the parameters leading to finite hypergeometric solutions (whose degree may yet depend on the parameters).

With this approach (and with Hautot's method), I show how to handle one of the unsolved equations met in [5].

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2 Necessary conditions and undecidability

Let $\underline{M} = (M_1, \ldots, M_s)$ denote a tuple of parameters and let $K_{\underline{M}} = \bar{\mathbb{Q}}(\underline{M})$. Let $L_{\underline{M}} = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \cdots + a_0(x)$ be a linear differential operator with coefficients in $K_{\underline{M}}[x]$ where $\partial = \frac{d}{dx}$.

This paper is concerned with effectively finding all the values $\underline{m} = (m_1, \ldots, m_s)$ in $\mathbb{C}^s$ such that $L_{\underline{m}}(y) = 0$ has polynomial solutions. In the sequel of the paper, L both designates $L_{\underline{m}}$ and $L_{\underline{M}}$.

Let $y = \sum_{i=0}^{+\infty} y_i x^i$ be a formal series. It satisfies $L(y) = 0$ if and only if its coefficients y_i satisfy a recurrence relation (see [1] for more details) :

$$\alpha_0(i)y_i + \cdots + \alpha_b(i)y_{i+b} = 0 \tag{1}$$

where b is a fixed nonnegative integer,

$\forall j \in \{0, \ldots, b\}, \alpha_j(t) \in K_{\underline{M}}[t]$, $\forall i < 0, y_i = 0$.

We call the recurrence relation (1) the recurrence relation associated to the operator L in the basis $(x^i)_{i \in \mathbb{N}}$.

Note that we can write y in any other basis (P_i) provided the polynomials P_i satisfy some additional properties (see [1]).

Definition 1 *The number b is called the* order *of the recurrence relation.*

Definition 2 *A sequence $(y_i)_{i \in \mathbb{Z}}$ satisfying a recurrence relation and such that :*

$$\exists d \in \mathbb{N}, \exists v \in \mathbb{N}/\forall i \in \mathbb{Z}, (i < v \text{ or } i > d) \Rightarrow y_i = 0$$

is called a finite solution *to the recurrence relation.*

The aim is to find a finite set of necessary and sufficient algebraic conditions on the parameters such that there exists a nonzero finite solution to the recurrence relation (1). that is to say such that there exists two nonnegative integers v and d ($v \leq d$) such that the recurrence relation (1) has a solution of the type

$$(\ldots, 0, y_v, \ldots, y_d, 0, \ldots)$$

with $y_v \neq 0$ and $y_d \neq 0$.

Let us first give some necessary conditions on the degree and the valuation of the sought polynomial solution.

Let us take $i = d$ in (1) then $\alpha_0(d)y_d = 0$. But $y_d \neq 0$ so $\alpha_0(d) = 0$.

Let $n_0 = min\{j \in \{0, \ldots, b\}/\alpha_j(t) \in K_{\underline{M}}[t] \setminus K_{\underline{M}}\}$.

A first necessary condition is

$$\alpha_0 \equiv \cdots \equiv \alpha_{n_0-1} \equiv 0 \text{ and } \exists d \in \mathbb{N}/\alpha_{n_0}(d - n_0) = 0.$$

This last equation is called the 'right indicial equation' or 'infinity indicial equation' which gives the possibilities for the degree d of the polynomial solution. We encounter the problem of deciding when a polynomial of $K_{\underline{M}}[t]$ has an integer root. We see later in this section that we do not try to solve this problem as it can be written with a finite number of arithmetic and algebraic conditions.

In the same way one can construct the 'left indicial equation'. Let $n_b = max\{j \in \{0, \ldots, b\}/\alpha_j(t) \in K_{\underline{M}}[t] \setminus K_{\underline{M}}\}$. Then a second necessary condition is :

$$\alpha_{n_b+1} \equiv \cdots \equiv \alpha_b \equiv 0 \text{ and } \exists v \in \mathbb{N}/\alpha_{n_b}(v - n_b) = 0.$$

Example 1 *Consider the following linear differential equation :*

$$2x^3y'' + ((3 - 2M_2)x^2 + x)y' - (P(M_1, M_2)x^2 + M_2x + M_1)y = 0$$

where P is a polynomial with integer coefficients. Its associated recurrence relation in the basis $(x^i)_{i \in \mathbb{N}}$ is :

$$-P(M_1, M_2)y_i - (3 + 2i)(-i - 1 + M_2)y_{i+1} + (-M_1 + 2 + i)y_{i+2} = 0$$

So a first necessary condition is

$$P(m_1, m_2) = 0.$$

Then the new recurrence relation is

$$(1 + 2i)(i - m_2)y_i + (i + 1 - m_1)y_{i+1} = 0$$

and the necessary condition becomes

$$\exists m_1, m_2 \in \mathbb{N}/(m_1 \leq m_2 \text{ and } P(m_1, m_2) = 0).$$

We note that this last equation is a Diophantine one.

The presence of arithmetic conditions may lead us to solve Diophantine equations in terms of integer solutions (like in the previous example). Furthermore, in 1970, Y. Matiyasevich proved Hilbert's Tenth Problem which states that there is no universal method for solving Diophantine equations in terms of integer solutions ([12]). Using this last result one can state the same type of theorem for linear differential equations with coefficients in $K_{\underline{M}}[x]$.

Theorem 1 *(J.-A. Weil) There exists no algorithm that, given any linear differential homogeneous equation depending on parameters, decides for which values of the parameters this equation has got a rational solution or not.* **Proof**

Let us consider the following differential linear equation :

$$y' - y\left(\frac{M_1}{x-1} + \cdots + \frac{M_s}{x-s} + P(M_1, \ldots, M_s)\right) = 0$$

where $M_1, \ldots, M_s$ are parameters and P is any polynomial in $\mathbb{Z}[M_1, \ldots, M_s]$.

This equation is equivalent to :

$$y = c(x-1)^{M_1} \ldots (x-s)^{M_s} \exp(P(M_1, \ldots, M_s)x)$$

So it has rational solutions if and only if there exists $\underline{m} = (m_1, \ldots, m_s)$ in $\mathbb{Z}^s$ such that $P(\underline{m}) = 0$. However, by Matyasevich's theorem ([12]), there is no algorithm which can solve the latter for any P in $\mathbb{Z}[X_1, \ldots, X_s]$; so there is no algorithm which can determine whether a given linear differential equation has got rational solutions or not. $\square$

In the sequel of this paper we are looking for a finite set of necessary and sufficient algebraic and arithmetic conditions on the parameters leading to a polynomial solution. So we do not care neither about the possible presence of Diophantine equations, nor about the problem of deciding when a polynomial of $K_{\underline{M}}[t]$ has an integer root.

When we apply the recurrence relation (1) to $i = -b, \ldots, d$, then we get a system $\mathcal{S}_d$ of $d+b+1$ linear equations with $d+1$ unknowns.

If we fix a nonnegative integer d (not depending upon parameters) then the size of the matrix $\mathcal{M}_d$ of the linear system $\mathcal{S}_d$ is numerically known. It suffices to decide for which values of the parameters the rank of this matrix is less than or equal to d. We can even compute the solutions of the linear system $\mathcal{S}_d$ ([19]).

Lemma 1 *Let d be a fixed nonnegative integer. Then the set of the parameters leading to a polynomial solution of degree less than or equal to d is an algebraic set.*

If we do not fix d and if the indicial equation has a solution depending on the parameters then the size of the matrix associated to the recurrence relation also depends on the parameters and so it is difficult to find a finite number of conditions on the parameters leading to a nonzero solution of the linear system.

3 Two terms recurrence relations

We give here a finite set of necessary and sufficient algebraic or arithmetic conditions on the parameters leading to a polynomial solution of $L(y) = 0$ when the recurrence relation associated to it (in a suitable basis) has two terms (see [10] for a study of such recurrence relations).

Theorem 2 *Assume that the recurrence relation associated to L (in a suitable given basis) has two terms :*

$$\alpha_0(i)y_i + \alpha_b(i)y_{i+b} = 0.$$

The equation $L(y) = 0$ has got a nonzero polynomial solution if and only if there exists two nonnegative integers v and d such that $v \leq d$, $v \equiv d \pmod b$, $\alpha_b(v-b) = 0$ and $\alpha_0(d) = 0$. **Proof**

Suppose that $L(y) = 0$ has a nonzero polynomial solution $y = \sum_{i=0}^{d} y_i x^i$ with $y_d \neq 0$. Then $\alpha_0(d) = 0$. Let j in $\{0, \ldots, b-1\}$ and k_d in $\mathbb{N}$ such that $d = bk_d + j$. Suppose that $\alpha_b(j + kb) \neq 0$ for all $-1 \leq k \leq k_d - 1$. Then one can check that $y_{j+(k+1)b} = 0$ for all $k \in \{-1, \ldots, k_d - 1\}$. In particular one must have $y_d = y_{j+bk_d} = 0$, which is false, so there exists $k_v \in \{-1, \ldots, k_d - 1\}$ such that $\alpha_b(j + bk_v) = 0$. To conclude let $v = j + b(1 + k_v)$, then $\alpha_b(v - b) = 0$, $v \equiv d \pmod b$ and $v \leq d$.

Suppose that there exists two nonnegative integers v and d such that $v \leq d$, $v \equiv d \pmod b$, $\alpha_b(v-b) = 0$ and $\alpha_0(d) = 0$. Let $\tilde{d}$ be the smallest integer such that $v \leq \tilde{d} \leq d$, $\alpha_0(\tilde{d}) = 0$ and $\tilde{d} \equiv d \pmod b$. We shall prove that $L(y) = 0$ has a polynomial solution of degree $\tilde{d}$. For this, put $y_{\tilde{d}} = 1$. Then $y_{\tilde{d}-b}, y_{\tilde{d}-2b}, \ldots, y_v$ can be (uniquely) determined using the recurrence relation $y_{j+kb} = -\frac{\alpha_b(j+kb)}{\alpha_0(j+kb)} y_{j+(k+1)b}$. Now set $y_i = 0$ for $i \notin \{\tilde{d} - b, \ldots, v\}$. Then one can verify that $y = \sum_{i=0}^{\infty} y_i x^i$ is a nonzero polynomial solution of our equation. $\square$

4 Three terms recurrence relations of order two

In this section we assume that the recurrence relation associated to the operator L (in a suitable basis) is the following one :

$$\alpha_0(i)\, y_i + \alpha_1(i)\, y_{i+1} + \alpha_2(i)\, y_{i+2} = 0 \qquad (2)$$

where α_0 and α_2 are non constant polynomials with coefficients in $K_{\underline{M}}$. We have seen that α_2 must have an integer root greater than or equal to -2. For notational convenience, we assume that $\alpha_2(-2) = 0$. The matrix associated to the recurrence relation is then a square matrix and in this particular case one can state the following lemma which gives a necessary and sufficient non algebraic condition :

Lemma 2 *Let d be a nonnegative integer such that $\alpha_0(d) = 0$. For i in $\mathbb{N}$, let $\mathcal{M}_i$ be the square matrix associated to the recurrence (2) and let Δ_i be its determinant : $\mathcal{M}_i =$*

$$\begin{bmatrix} \alpha_1(-1) & \alpha_2(-1) & & & 0 \\ \alpha_0(0) & \alpha_1(0) & \alpha_2(0) & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_0(i-2) & \alpha_1(i-2) & \alpha_2(i-2) \\ 0 & & & \alpha_0(i-1) & \alpha_1(i-1) \end{bmatrix}$$

Then the differential equation $L(y) = 0$ has got a nonzero polynomial solution of degree less than or equal to d if and only if

$$\Delta_d = 0.$$

Proof

If $\sum_{k=0}^{d} y_k x^k$ is a nonzero solution of $L(y) = 0$ then, by construction of $\mathcal{M}_d$, the linear system $\mathcal{M}_d Y = 0$ has a nonzero solution, so the determinant Δ_d of the matrix $\mathcal{M}_d$ cancels. If $\Delta_d = 0$ then the linear system $\mathcal{M}_d Y = 0$ has a nonzero solution $Y = {}^t(y_0, \ldots, y_d)$. Furthermore $\alpha_0(d) = \alpha_2(-2) = 0$, so $(\ldots, 0, y_0, \ldots, y_d, 0, \ldots)$ is a solution to the recurrence relation. $\square$

4.1 Existence of a polynomial solution for all degrees

In lemma 1 we show that, for any fixed d in $\mathbb{N}$, the set of the $\underline{m}$ leading to a polynomial solution is a constructible set. In this section, we explain, on a family of equations, a strategy to study whether this set is non empty for all d in $\mathbb{N}$.

We use the recurrence relation satisfied by (Δ_i) :

Propriety 1 *Let $(\Delta_i)_{i \in \mathbb{N}}$ be the sequence defined in the lemma 2. It also satisfies the following recursion :*

$$\begin{cases} \Delta_0 &= \alpha_1(-1) \\ \Delta_1 &= \alpha_1(-1)\alpha_1(0) - \alpha_2(-1)\alpha_0(0) \\ \forall i \in \mathbb{N} \setminus \{0, 1\}, & \\ \Delta_i &= \alpha_1(i-1)\Delta_{i-1} - \alpha_0(i-1)\alpha_2(i-2)\Delta_{i-2} \end{cases}$$

Proof
This can be easily seen by developing Δ_i along the last line. □

Proposition 1 *Assume that*

$$(H_0)\left\{\begin{array}{l}\alpha_1(i)=\Phi(i)M_1^{\alpha}+\tilde{\alpha_1}(i)\\ \alpha\in\mathbb{N}^*\\ \alpha_0(i),\tilde{\alpha_1}(i),\alpha_2(i)\in\bar{\mathbb{Q}}(M_2,\ldots,M_s)[i]\\ \forall i\in\{-1,0,1,\ldots\},\ \Phi(i)\in\mathbb{Q}^*\end{array}\right.$$

Let d be a nonnegative integer. If there exists $(\tilde{m}_2,\ldots,\tilde{m}_s)$ such that $\alpha_0(d)=0$, then there exists $(m_1,\ldots,m_s)$ leading to a nonzero finite solution of degree less than or equal to d of (2).

Proof
Using the recursion satisfied by Δ_i, one proves by induction on i that

$$\Delta_i=(\prod_{j=-1}^{i-1}\Phi(j))M_1^{(i+1)\alpha}+\sum_{k=0}^{i}g_{i,k}M_1^{k\alpha}$$

where $g_{i,k}\in\bar{\mathbb{Q}}(M_2,\ldots,M_s)$.
Let d be a nonnegative integer and let $(m_2,\ldots,m_s)$ such that $\alpha_0(d)=0$. Then Δ_d can be seen as a polynomial in $F[M_1]$ where F is in the algebraic closure of $\mathbb{Q}$. So there exists m_1 cancelling Δ_d.
Using lemma 2 one concludes that $(m_1,m_2,\ldots,m_s)$ leads to a nonzero finite solution to (2). □

Example 2 *The following linear differential operator L_h comes from the proposition 14 page 240 of [5]. It is one of the equations met in [5] while studying the liouvillian solutions of the non hypergeometric confluent Heun equation. One wants to characterize the values of $(\alpha,\beta,\gamma,\mu,\nu)$ in $\mathbb{C}^5$ such that $L_h(y)=0$ has polynomial solutions where*

$$\begin{array}{l}L_h=2x(1-x)\frac{d^2}{dx^2}+2(1-\beta+(-\alpha+\beta+\gamma-2)x+\alpha x^2)\frac{d}{dx}\\ \quad+(1-2\nu-(1-\beta)(1+\alpha-\gamma)-2\alpha\mu x)=0.\end{array}$$

The associated recurrence relation in the basis $(x^i)_{i\in\mathbb{N}}$ is the following one

$$h_0(i)y_i+h_1(i)y_{i+1}+h_2(i)y_{i+2}=0 \quad (3)$$

where

$$\left\{\begin{array}{lcl}h_0(i)&=&2\alpha(i-\mu)\\ h_1(i)&=&-2\nu-4-3\alpha+3\gamma+\beta\alpha+3\beta-\beta\gamma+\\ &&2(-\alpha+\beta+\gamma-2)i-2i(i+1)\\ h_2(i)&=&2(i+2)(2-\beta+i)\end{array}\right.$$

We note that $h_2(-2)=0$. Let d be any numerically fixed positive integer. A necessary condition to get a polynomial solution of degree d is $\mu=d$.
We wonder whether there exists parameters satisfying $\Delta_d=0$. But according to proposition 1,

$$\forall i\in\{-1,\ldots,d\},\ \Delta_i=(-2\nu)^i+\sum_{k=0}^{i-1}g_{i,k}(\alpha,\beta,\gamma,\mu)\nu^k$$

where $g_{i,k}\in\bar{\mathbb{Q}}[\alpha,\beta,\gamma,\mu]$.
So

$$\Delta_d=0\Leftrightarrow(-2\nu)^d+\sum_{k=0}^{d-1}g_{d,k}(\alpha,\beta,\gamma,d)=0$$

Necessarily there exists α,β,γ,ν satisfying $\Delta_d=0$. So for each positive integer d, there exists values $(\alpha,\beta,\gamma,\mu,\nu)$ in $\mathbb{C}^5$ of the parameters such that the equation $L_h(y)=0$ has a nonzero polynomial solution of degree d.

4.2 The sufficient conditions of A. Hautot

The following proposition is a generalization of A. Hautot's idea that he applied to three particular linear differential equations ([8], [7], [9]). We construct finite sets V of algebraic and arithmetic conditions on the parameters such that for any $\underline{m}$ in V the equation $L_{\underline{m}}(y)=0$ has a polynomial solution whose degree may depend on the parameters.

Proposition 2 *Let i_0 be any numerically fixed integer in $\{-1,0,\ldots\}$ and V_{i_0} be the set of the $(m_1,\ldots,m_s)$ satisfying*

$$\left\{\begin{array}{l}\alpha_2(-2)=\alpha_2(i_0)=0\\ \Delta_{i_0+1}=0\\ \exists d\in\{i_0+1,i_0+2,\ldots\}/\alpha_0(d)=0.\end{array}\right..$$

If V_{i_0} is non empty then for each $(m_1,\ldots,m_s)$ in V_{i_0} there is a nonzero finite solution of degree less than or equal to d to the recurrence relation (2).

Proof
Let $(m_1,\ldots,m_s)\in V_{i_0}$ and let d be an integer greater than i_0 such that $\alpha_0(d)=0$. As $\alpha_2(i_0)$ is assumed to be zero, the matrix $\mathcal{M}_d$ can be written like a diagonal block matrix :

$$\mathcal{M}_d=\begin{pmatrix}\mathcal{M}_{i_0+1}&0\\ \mathcal{B}_d&\mathcal{C}_d\end{pmatrix}$$

As $det(\mathcal{M}_{i_0+1})=\Delta_{i_0+1}=0$, we get $\Delta_d=0$, which proves the proposition. □

Example 3 *The differential equation $L_h=0$ from the first example belongs to the family studied by A. Hautot in [8]. If α is equal to zero then we get a two term recursion*

$$(-2\nu-4+3\gamma+3\beta-\beta\gamma+2(\beta+\gamma-2)(i-1)-2i(i-1))y_i+$$
$$2(i+1)(1-\beta+i)y_{i+1}=0$$

The parameters leading to a polynomial solution for $L_h(y)=0$ are those such that the left indicial equation has integer solutions.
To simplify the exposition, we now assume that α is equal to 1. Then

$$*\left\{\begin{array}{lcl}h_0(i)&=&i-\mu\\ h_1(i)&=&-\nu-\frac{7}{2}+\frac{3}{2}\gamma+2\beta-\frac{1}{2}\beta\gamma+\\ &&(\gamma+\beta-4)i-i^2\\ h_2(i)&=&(i+2)(2-\beta+i)\end{array}\right.$$

Let us take $i_0=1$, then $\alpha_2(i_0)=0\Leftrightarrow\beta=3$ and

$$\mathcal{M}_2=\begin{bmatrix}\frac{5}{2}-\nu-\gamma&-2&0\\ -\mu&\frac{5}{2}-\nu&-2\\ 0&-\mu+1&\frac{1}{2}-\nu+\gamma\end{bmatrix}$$

$$V_2=\{(\beta,\gamma,\mu,\nu)/\beta=3,\ \Delta_2=0\text{ and }\mu\in\mathbb{N}\backslash\{0,1\}\}$$

For each (β,γ,ν,μ) in V_2, the differential equation has got a polynomial solution of degree μ.

5 The hypergeometric solutions

In this section we adapt the method of Petkovsek for computing hypergeometric solutions of linear recurrence relations ([14]) to our parameterized situation. We first describe a general method available for any recurrence relation (section 5.1). Then we study a particular class of equations whose recurrence relation has three terms and is of order two (section 5.2).

5.1 A general method

Let us assume that the recurrence relation associated to $L(y) = 0$ (in a suitable basis) is (1) : $\alpha_0(i)y_i + \cdots + \alpha_b(i)y_{i+b} = 0$, where α_0 and α_b are both non constant polynomials. The idea is to construct a finite set T of conditions and a two terms recurrence relation such that : for each $(m_1, \ldots, m_s)$ in T this two terms recurrence relation has finite solutions which also satisfy (1) and whose degrees may yet depend on the parameters.

Remark 1 *If there exists a positive integer d such that $(\ldots, 0, y_0, \ldots, y_d, 0, \ldots)$ is a solution to the recurrence relation (1) (where d may depend on some parameters), then there exists a polynomial R such that*

$$\forall i \in \mathbb{Z},\ (i+1)Q(i)y_{i+1} = (i-d)\,R(i)y_i$$

where

$$Q(i) = \begin{cases} \prod_{j \in J}(i-j) & \text{if } J \neq \emptyset \\ 1 & \text{if } J = \emptyset \end{cases}$$

$$J = \{j \in \{0, \ldots, d-1\} / y_j = 0\}$$

$$deg(R) = d.$$

It suffices to compute the coefficients of the polynomial $R(i)$ by interpolation in the coefficients $y_0, \ldots, y_d$.

Our aim is to construct a two terms recurrence relation

$$\beta_0(i)y_i + \beta_1(i)y_{i+1} = 0$$

where β_0 and β_1 are both polynomials whose degrees are numerically fixed. That is what we perform generalizing Petkovsek's algorithm.

Theorem 3 *Let us assume that $\alpha_b(-b) = 0$ and that there exists d in $\mathbb{N}$ such that $\alpha_0(d) = 0$. Let $(y_i)_{i\in\mathbb{Z}}$ be a sequence such that :*

$$\begin{cases} y_{i+1} = c\frac{A(i)C(i+1)}{B(i)C(i)}y_i & \text{if } i \in \{0, \ldots, d\} \\ y_i = 0 & \text{if } i \notin \{0, \ldots, d\} \end{cases}$$

where c is in $K^*_{\underline{M}}$, A, B, C are polynomials and where the following conditions are satisfied :

$$\begin{cases} (C_1)\ A(d) = B(-1) = 0, \\ (C_2)\ \forall i \in \{-b+1, \ldots, -1\}, A(i) \neq 0 \\ (C_3)\ \forall i \in \{0, \ldots, d+b-2\}, B(i) \neq 0 \\ (C_4)\ \forall i \in \{0, \ldots, d\}, C(i) \neq 0 \\ (C_5)\ \sum_{k=0}^{b} P_k(i)C(i+k) = 0 \end{cases}$$

and

$$P_k(i) = c^k\alpha_k(i)\prod_{j=0}^{k-1}A(i+j)\prod_{j=k}^{b-1}B(i+j).$$

Then the sequence $(y_i)_{i\in\mathbb{Z}}$ satisfies (1) :

$$\alpha_0(i)y_i + \cdots + \alpha_b(i)y_{i+b} = 0.$$

Proof
Let us notice first that the sequence $(y_i)_{i\in\mathbb{Z}}$ is well defined according to the conditions (C_2), (C_3) and (C_4).
As y_i cancels whenever i is not in $\{0, \ldots, d\}$, we can first notice that

$$\sum_{k=0}^{b}\alpha_k(i)y_{i+k} = \sum_{k=k_1}^{k_2}\alpha_k(i)y_{i+k}$$

where $k_1 = max(0, -i)$ and $k_2 = min(b, d-i)$.

If $i < -b$ or $i > d$ then $\sum_{k=0}^{b}\alpha_k(i)y_{i+k} = 0.$

If $i = d$ then $\sum_{k=0}^{b}\alpha_k(i)y_{i+k} = \alpha_0(d)y_d = 0.$

If $i = -b$, then $\sum_{k=0}^{b}\alpha_k(i)y_{i+k} = \alpha_b(-b)y_0 = 0.$

Let i be in $\{-b+1, \ldots, d-1\}$. Then $0 \leq i+k_1 \leq i+k_2 \leq d$, so $\forall k \in \{k_1, \ldots, k_2\}$,

$$y_{i+k} = c^kC(i+k)\prod_{j=0}^{k-1}A(i+j)\prod_{j=k}^{b-1}B(i+j)\frac{y_{i+k_1}}{F(i,k_1)}$$

where $F(i, k_1) = C(i+k_1)c^{k_1}\prod_{j=0}^{k_1-1}A(i+j)\prod_{j=k_1}^{b-1}B(i+j)$

and $F(i, k_1)$ does not cancel according to the conditions (C_2), (C_3) and (C_4).
So

$$\sum_{k=0}^{b}\alpha_k(i)y_{i+k} = \frac{y_{i+k_1}}{F(i,k_1)}\sum_{k=k_1}^{k_2}P_k(i)C(i+k).$$

Furthermore, as $B(-1)$ cancels, $P_k(i)$ also cancels whenever $-1-i$ is in $\{k, \ldots, b-1\}$. But $i \geq -b+1$ so $\forall k \in \{0, \ldots, k_1-1\}, P_k(i) = 0$. In the same way, as $A(d)$ cancels, one proves $\forall k \in \{k_2+1, \ldots, b\}, P_k(i) = 0$. To conclude,

$$\sum_{k=0}^{b}\alpha_k(i)y_{i+k} = \frac{y_{i+k_1}}{F(i,k_1)}\sum_{k=0}^{b}P_k(i)C(i+k) = 0$$

□

If the polynomials A, B and C satisfy the hypotheses of Gosper's lemma ([6]) (A and B monic; $\forall k \in \mathbb{N}, gcd(A(i), B(i+k)) = gcd(A(i), C(i)) = gcd(B(i+1), C(i)) = 1$), then one proves that $A(i)$ divides $\alpha_0(i)$ and $B(i)$ divides $\alpha_b(i-b+1)$ (see [14]). So in practice, we choose the polynomial $A(i)$ (resp. $B(i)$) among the divisors of $\alpha_0(i)$ (resp. $\alpha_b(i-b)$). The computation of the polynomial C remains. We encounter here the same type of problem as the one of the first section : the degree N may depend on the parameters. However if we fix a nonnegative integer N, we

can construct, as in lemma 1, a finite set of algebraic conditions on the parameters leading to a polynomial solution C of degree N.
We describe below a method to find a set of sufficient conditions leading to a polynomial solution of $L(y) = 0$.

1. **The recurrence relation.**
   Compute the recurrence relation associated to the operator L in a suitable basis $(P_i)_{i\in\mathbb{N}}$:
   $$\alpha_0(i)y_i + \cdots + \alpha_b(i)y_{i+b} = 0$$
   Assume that you know the factorization of α_0 and α_b. Construct the set I of the nonnegative integer roots of α_0 (which may depend on the parameters).

2. **The set of the possible degrees of (A, B).**
   Construct the set $\tilde{E}$ of the elements (d_A, d_B) of $\{0, \dots, e_0\} \times \{0, \dots, e_b\}$ such that :
   $\exists k_1, k_2 \in \{0, \dots, b\}, k_1 \neq k_2, \delta_{k_1} = \delta_{k_2} = \delta$,
   where $\delta_k = kd_A + (b-k)d_B + e_k = deg(P_k)$,
   $e_k = deg(\alpha_k)$, $\delta = max\{\delta_k, 0 \leq k \leq b\}$.
   If $\tilde{E}$ is empty then for each (A, B) with degree in $\{0, \dots, e_0\} \times \{0, \dots, e_b\}$, there is no polynomial C satisfying the condition (C_5).

3. **The set of the possible (A, B).**
   Construct the set E of the couples $(A(i), B(i))$ such that
   $$\begin{cases} A(i) \text{ divides } \alpha_0(i) \\ \exists d \in I/\alpha_0(d) = A(d) = 0 \\ B(i) \text{ divides } \alpha_b(i+1-b) \\ B(-1) = 0 \\ \exists (d_A, d_B) \in \tilde{E}/(deg(A), deg(B)) = (d_A, d_B) \end{cases}$$

4. **The set of the sufficient conditions on the parameters.**
   For each $(A(i), B(i))$ in E, fix a nonnegative integer N and set $C(i) = i^N + \sum_{k=0}^{N-1} c_k i^k$. Construct the set T_N of the parameters such that (C_2), (C_3), (C_4) and (C_5) are satisfied.

Remark 2 *If the set $\tilde{E}$ of the second step is empty then one can compute the recurrence relation associated to L in another basis of polynomials using the tools of [16].*

Example 4 *Let us consider the linear differential homogeneous equation*

$$xy''(x) + (M_2 + M_3x - 2x^2)y'(x) + (1 + 2M_1)x)y(x) = 0.$$

Its associated recurrence relation in the basis $(x^i)_{i\in\mathbb{N}}$ is :

$$-2(i-M_1)y_i + (M_3i+1+M_3)y_{i+1} + (i+2)(i+1+M_2)y_{i+2} = 0.$$

The set $\tilde{E}$ is empty. If you compute the recurrence relation in the basis of the Hermite polynomials, then you get :

$$(M_1 - i)y_i + (1 + M_3(i+1))y_{i+1}$$
$$-2(i+2)(i+2-M_1-M_2)y_{i+2} + 2M_3(i+2)(i+3)y_{i+3} = 0$$

If m_3 is zero then the set $\tilde{E}$ is empty else it is equal to $\{(0,0), (1,1)\}$. So the set E is reduced to $\{(i - m_1, i+1)\}$. Let us choose $N = 1$ and set $C(i) = i + c_0$. Then the condition (C_5) is satisfied if and only if $c = \frac{1}{m_3}$, $m_2 = 1$, $m_3 = -2m_1 - 1$, $c_0 = 2m_1^2$. One easily checks that the conditions (C_2), (C_3) and (C_4) are also satisfied.
Let $T_1 = \{(m_1, m_2, m_3)/\ m_1 \in \mathbb{N}, m_3 = 2m_1 - 1, m_2 = 1, m_3 \neq 0\}$. For each (m_1, m_2, m_3) in T_1 there is a polynomial solution of degree m_1 (which does not have any fixed value) to the initial differential equation.

The question that one can ask is : when can we construct non empty sets T_N such that the degree d of the finite hypergeometric sequence (y_i) (or of the polynomial y) does not depend on N ? We answer this question for a particular class of equations whose recurrence relation has three terms and is of order two.

5.2 An application to a particular family of equations

Let us consider the recurrence relation (2) :

$$\alpha_0(i)y_i + \alpha_1(i)y_{i+1} + \alpha_2(i)y_{i+2} = 0$$

and the recurrence relation

$$Q_0(i)C(i) + Q_1(i)C(i+1) + Q_2(i)C(i+2) = 0 \qquad (4)$$

where

$$\begin{cases} Q_0(i) &= \alpha_2(i-1) \\ Q_1(i) &= c\alpha_1(i) \\ Q_2(i) &= c^2\alpha_0(i+1) \end{cases}$$

and c is in K^*.

Proposition 3 *Assume*

$$(H_1): \begin{cases} \alpha_0(i) = i - M_1 \\ \alpha_2(i) = (i+2)\tilde{\alpha_2}(i) \\ \tilde{\alpha_2}(i) \text{ is monic and without root in } \{-1, 0, \dots\} \end{cases}$$

Let T be the set of all the $(m_1, \dots, m_s)$ such that there exists a nonzero c such that the equation (4) has got a polynomial sequence solution $(C(i))_{i\in\mathbb{Z}}$ satisfying

$$\forall i \in \{0, \dots, m_1\}, C(i) \neq 0.$$

Then for each $(m_1, \dots, m_s)$ in T such that m_1 is in $\mathbb{N}$, the homogeneous linear recurrence (2) has got a finite solution which is defined in the following way:

$$(\natural): \begin{cases} y_0 \neq 0 \\ \forall i \in \{0, \dots, m_1\},\ y_{i+1} = c\frac{\alpha_0(i)}{\alpha_2(i-1)}\frac{C(i+1)}{C(i)}y_i \\ \forall i \in \mathbb{Z}/\mathbb{N}, y_i = 0 \end{cases}$$

Proof
It suffices to apply the previous theorem with $A(i) = \alpha_0(i)$ and $B(i) = \alpha_2(i-1)$ and notice that $P_0(i)C(i) + P_1(i)C(i+1) + P_2(i)C(i+2) = \alpha_0(i)\alpha_2(i)(Q_0(i)C(i) + Q_1(i)C(i+1) + Q_2(i)C(i+2)) = 0$. □

Remark 3 *The hypothesis '$\alpha_2(i)$ monic' could be avoided; it just enables a simpler exposition. Furthermore, as the polynomial $\tilde{\alpha_2}$ has no root in $\{-1, 0, \dots\}$, for any integer i_0 in $\{-1, 0, \dots\}$ the set V_{i_0} constructed in the section 4.1 is empty. Indeed there is no parameter satisfying $\alpha_2(i_0) = 0$.*

We have constructed here a two terms recurrence relation :

$$c\alpha_0(i)C(i+1)y_i - \alpha_2(i-1)C(i)y_{i+1} = 0.$$

The question that remains is : can we find a polynomial C which satisfies (4) and whose degree is independent of the degree m_1 of the possible polynomial solution of $L(y) = 0$? The following lemma gives an answer to this question under some new hypotheses.

Lemma 3 *Let us assume that (H_1) is satisfied and let assume*

$$(H_2) : \begin{cases} Q_0(i) = i^2 + w_1 i + w_2 \\ Q_1(i) = c(v_0 i^2 + v_1 i + v_2) \\ v_0 \in \mathbb{Q}^* \\ v_1, w_1 \in \bar{\mathbb{Q}}(M_2, \ldots, M_s) \end{cases}$$

Let $(m_1, \ldots, m_s)$ such that the recurrence relation (4) has got a nonzero polynomial solution, then its degree does not depend upon m_1.

Proof
Let $C(i) = i^N + c_{N-1}i^{N-1} + \cdots$. Then $Q_0(i)C(i) + Q_1(i)C(i+1) + Q_2(i)C(i+2) = (1+cv_0)i^{N+2} + (c_{N-1}(1+cv_0) + (Ncv_0 + c^2 + cv_1 + w_1))i^{N+1} + \cdots$, so necessarily, $cv_0 + 1 = 0$ and $(cv_0)N + c^2 + cv_1 + w_1 = 0$.
As u_0, v_0, u_1, v_1 and c do not depend upon m_1, N also does not depend upon m_1. □

The following proposition gives a matrix characterization of the polynomial C.

Proposition 4 *Let us assume that the hypotheses (H_1) and (H_2) are satisfied. Let N be a numerically fixed integer and $(m_1, \ldots, m_s, c)$ such that $(cv_0)N + c^2 + cv_1 + w_1 = 1 + cv_0 = 0$. A polynomial C of degree N satisfies the recurrence (4) if and only if*

$$\mathcal{M}_C \; {}^t(C(0) \cdots C(N)) = 0$$

where

$$\mathcal{M}_C = \begin{pmatrix} \mathcal{M}_{N,N+1} \\ \mathcal{L}_{1,N+1} \end{pmatrix},$$

$$\mathcal{M}_{N,N+1} = \begin{bmatrix} Q_1(-1) & Q_2(-1) & & & \\ Q_0(0) & Q_1(0) & Q_2(0) & & \\ & \ddots & \ddots & \ddots & \\ & & Q_0(N-2) & Q_1(N-2) & Q_2(N-2) \end{bmatrix}$$

$$\mathcal{L}_{1,N+1} = \begin{bmatrix} l_1 & \ldots & l_i & \ldots & l_{N+1} \end{bmatrix}$$

with

$$\begin{cases} \forall i \in \{1, \ldots, N-1\},\ l_i = Q_2(N-1)\phi(N+1, i-1) \\ l_N = Q_0(N-1) + Q_2(N-1)\phi(N+1, N-1) \\ l_{N+1} = Q_1(N-1) + Q_2(N-1)\phi(N+1, N) \end{cases}$$

where $\phi(i,k) = \prod_{j=0, j\neq k}^{N} \frac{i-j}{k-j}$.

Proof
Let C be a polynomial of degree N.
Let us assume that C is a solution of (4). Then for i in $\{-1, \ldots, N-2\}$, we get a linear system which can be written

$$\mathcal{M}_{N,N+1} \; {}^t(C(0) \cdots C(N)) = 0.$$

If $i = N-1$ then

$$Q_0(N-1)C(N-1) + Q_1(N-1)C(N) + Q_2(N-1)C(N+1) = 0$$

now $C(N+1) = \sum_{k=0}^{N} C(k)\phi(N+1, k)$. So we get

$$\mathcal{L}_{1,N+1} \; {}^t(C(0) \cdots C(N)) = 0.$$

Let us assume now that

$$\mathcal{M}_C \; {}^t(C(0) \cdots C(N)) = 0.$$

then for each i in $\{-1, \ldots, N-1\}$, the relation (4) is satisfied. One can easily see that the degree of the polynomial $Q_0(i)\,C(i) + Q_1(i)\,C(i+1) + Q_2(i)\,C(i+2)$ is N (using the hypothesis $1 + cv_0 = (cv_0)N + c^2 + cv_1 + w_1 = 0$ which enables to cancel the terms of degree $N+2$ and $N+1$) but it has $N+1$ roots so it is identically zero. To conclude

$$\forall i \in \mathbb{Z}, Q_0(i)C(i) + Q_1(i)C(i+1) + Q_2(i)C(i+2) = 0$$

□

From the two last propositions and the last lemma one deduces the following proposition :

Proposition 5 *Let us assume that (H_1) and (H_2) are satisfied. Let N be in $\mathbb{N}$ and let T_N be the set of all the $(m_1, \ldots, m_s)$ such that*

$$1 + cv_0 = -v_0^2 N + 1 - v_0 v_1 + v_0^2 w_1 = 0 \text{ and } \det(\mathcal{M}_C) = 0.$$

For $(m_1, \ldots, m_s)$ in T_N, let C be a polynomial of degree N such that

$$\mathcal{M}_C \; {}^t(C(0) \cdots C(N)) = 0$$

Then for each $(m_1, \ldots, m_s)$ in T_N such that m_1 is in $\mathbb{N}$ and $C(i)$ has no integer root in $\{0, \ldots, m_1\}$, there exists a nonzero finite solution to the recurrence relation (2).

Example 5 *We handle the recurrence relation defined in the third example (section 4.2) by the conditions $(*)$. We assume that β is not an integer. We notice that $c = 1$. Let us choose $N = 2$. Then*

$$v_0^2 N + 1 - v_0 v_1 + v_0^2 w_1 = 0 \Leftrightarrow \gamma = 3.$$

The matrix $\mathcal{M}_C$ is

$$\begin{bmatrix} 1 - \nu - 1/2\,\beta & -\mu & 0 \\ 1 - \beta & 1 - \nu + 1/2\,\beta & -\mu + 1 \\ 2 - \mu & -2 - 2\beta + 3\,\mu & 5 - \nu + 3/2\,\beta - 3\,\mu \end{bmatrix}$$

Let T_2 be the set of all $(\beta, \gamma, \mu, \nu)$ such that $\det(\mathcal{M}_C) = 0$ and there exists a polynomial C of degree 2 with no nonnegative root satisfying $\mathcal{M}_C \; {}^t(C(0), C(1), C(2)) = 0$.
Then for each $(\beta, \gamma, \mu, \nu)$ in T_2 such that μ is an integer, the equation $L_h = 0$ has got a nonzero polynomial solution.
Note that we can find this last condition applying the method of the section 4.2 to the recurrence relation associated to L_h in the basis $(x+1)^i$:

$$(i - \mu)y_i - 1/2(3 + 2\nu - 3\gamma - 4\beta + \beta\gamma + 2\mu + (4 - 2\beta - 2\gamma)i + 2i^2)y_{i+1} - (i+2)(i+2-\gamma)y_{i+2} = 0$$

Conclusion

The following problem 'deciding for which values of the parameters a given linear differential equation depending on parameters has got polynomial solutions' is undecidable. When the degree does not depend on the parameters then the problem can be solved. Otherwise, even when we work modulo the Diophantine problems, the question is still open. No machine can handle this problem in its generality. However, we can provide computer tools (in maple) that can help such a study. Any two terms recursion can be treated. During the study of the hypergeometric solutions of three terms recurrence relations, we have provided finite sets of conditions on the parameters. Each of these sets leads to a polynomial solution provided that the left indicial equation has integer solutions; this last condition which was only necessary in the general case becomes then a necessary and sufficient condition.
We have seen in this article that our tools enable to characterize some liouvillian solutions of equations of order two given in [5]. Combined to the computation of the exponential parts, the study of the polynomial solutions also enables to give necessary conditions for the integrability of Hamiltonian systems([13], [4]).
The study of the scalar two terms recursions can also be partially generalized to some matrix two terms recurrence relations. Lastly, the use of the orthogonal polynomials as new basis of $K[x]$ may be fruitful for the search of hypergeometric finite solutions to linear recurrence relations.

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