

Tight Kernel Bounds for Problems on Graphs with Small Degeneracy*

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Abstract. Kernelization is a strong and widely-applied technique in parameterized complexity. In a nutshell, a kernelization algorithm for a parameterized problem transforms a given instance of the problem into an equivalent instance whose size depends solely on the parameter. Recent years have seen major advances in the study of both upper and lower bound techniques for kernelization, and by now this area has become one of the major research threads in parameterized complexity.

In this paper we consider kernelization for problems on d -degenerate graphs, i.e. graphs such that any subgraph contains a vertex of degree at most d . This graph class generalizes many classes of graphs for which effective kernelization is known to exist, e.g. planar graphs, H -minor free graphs, and H -topological-minor free graphs. We show that for several natural problems on d -degenerate graphs the best known kernelization upper bounds are essentially tight. In particular, using intricate constructions of weak compositions, we prove that unless $\text{coNP} \subseteq \text{NP/poly}$:

- DOMINATING SET has no kernels of size $O(k^{(d-1)(d-3)-\varepsilon})$ for any $\varepsilon > 0$. The current best upper bound is $O(k^{(d+1)^2})$.
- INDEPENDENT DOMINATING SET has no kernels of size $O(k^{d-4-\varepsilon})$ for any $\varepsilon > 0$. The current best upper bound is $O(k^{d+1})$.
- INDUCED MATCHING has no kernels of size $O(k^{d-3-\varepsilon})$ for any $\varepsilon > 0$. The current best upper bound is $O(k^d)$.

To the best of our knowledge, the result on DOMINATING SET is the first example of a lower bound with a super-linear dependence on d in the exponent.

In the last section of the paper, we also give simple kernels for CONNECTED VERTEX COVER and CAPACITATED VERTEX COVER of size $O(k^d)$ and $O(k^{d+1})$ respectively. We show that the latter problem has no kernels of size $O(k^{d-\varepsilon})$ unless $\text{coNP} \subseteq \text{NP/poly}$ by a simple reduction from d -SET COVER (a similar lower bound for CONNECTED VERTEX COVER is already known).

1 Introduction

Parameterized complexity is a two-dimensional refinement of classical complexity theory introduced by Downey and Fellows [13] where one takes into account not only the total input length n , but also other aspects of the problem quantified in a numerical parameter $k \in \mathbb{N}$. The main goal of the field is to determine which problems have algorithms whose exponential running time is confined strictly to the parameter. In this way, algorithms running in $f(k) \cdot n^{O(1)}$ time for some computable function $f()$ are considered feasible, and parameterized problems that admit feasible algorithms are said to be *fixed-parameter tractable*. This notion has proven extremely useful in identifying tractable instances for generally hard problems, and in explaining why some theoretically hard problems are solved routinely in practice.

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A closely related notion to fixed-parameter tractability is that of kernelization. A *kernelization algorithm* (or *kernel*) for a parameterized problem $L \subseteq \{0, 1\}^* \times \mathbb{N}$ is a polynomial time algorithm that transforms a given instance (x, k) to an instance (x', k') such that: (i) $(x, k) \in L \iff (x', k') \in L$, and (ii) $|x'| + k' \leq f(k)$ for some computable function f . In other words, a kernelization algorithm is a polynomial-time reduction from a problem to itself that shrinks the problem instance to an instance with size depending only on the parameter. Appropriately, the function f above is called the *size* of the kernel.

Kernelization is a notion that was developed in parameterized complexity, but it is also useful in other areas of computer science such as cryptography [22] and approximation algorithms [28]. In parameterized complexity, not only is it one of the most successful techniques for showing positive results, it also provides an equivalent way of defining fixed-parameter tractability: A decidable parameterized problem is solvable in $f(k) \cdot n^{O(1)}$ time iff it has a kernel [6]. From practical point of view, compression algorithms often lead to efficient preprocessing rules which can significantly simplify real life instances [16, 20]. For these reasons, the study of kernelization is one of the leading research frontiers in parameterized complexity. This research endeavor has been fueled by recent tools for showing lower bounds on kernel sizes [2, 4, 5, 7, 9, 10, 12, 23, 27] which rely on the standard complexity-theoretic assumption of $\text{coNP} \not\subseteq \text{NP}/\text{poly}$.

Since a parameterized problem is fixed-parameter tractable iff it is kernelizable, it is natural to ask which fixed-parameter problems admit kernels of reasonably small size. In recent years there has been significant advances in this area. One particularly prominent line of research in this context is the development of *meta-kernelization* algorithms for problems on sparse graphs. Such algorithms typically provide compressions of either linear or quadratic size to a wide range of problems at once, by identifying certain generic problem properties that allow for good compressions. The first work in this line of research is due to Guo and Niedermeier [21], which extended the ideas used in the classical linear kernel for DS in planar graphs [1] to linear kernels for several other planar graph problems. This result was subsumed by the seminal paper of Bodlaender *et al.* [3], which provided meta-kernelization algorithms for problems on graphs of bounded genus, a generalization of planar graphs. Later Fomin *et al.* [18] provided a meta-kernel for problems on H -minor free graphs which include all bounded genus graphs. Finally, a recent manuscript by Langer *et al.* [26] provides a meta-kernelization algorithm for problems on H -topological-minor free graphs. All meta-kernelizations above have either linear or quadratic size.

How far can these meta-kernelization results be extended? A natural class of sparse graphs which generalizes all graph classes handled by the meta-kernelizations discussed above is the class of d -degenerate graphs. A graph is called *d -degenerate* if each of its subgraphs has a vertex of degree at most d . This is equivalent to requiring that the vertices of the graph can be linearly ordered such that each vertex has at most d neighbors to its right in this ordering. For example, any planar graph is 5-degenerate, and for any H -minor (resp. H -topological-minor) free graph class there exists a constant $d(H)$ such that all graphs in this class are $d(H)$ -degenerate (see *e.g.* [11]). Note that the INDEPENDENT SET problem has a trivial linear kernel in d -degenerate graphs, which gives some hope that a meta-kernelization result yielding small degree polynomial kernels might be attainable for this graph class.

Arguably the most important kernelization result in d -degenerate graphs is due to Philip *et al.* [29] who showed a $O(k^{(d+1)^2})$ size kernel for DOMINATING SET, and an $O(k^{d+1})$ size kernel for INDEPENDENT DOMINATING SET. Erman *et al.* [14] and Kanj *et al.* [24] independently gave a $O(k^d)$ kernel for the INDUCED MATCHING problem, while Cygan *et al.* [8] showed a $O(k^{d+1})$

kernel is for CONNECTED VERTEX COVER. While all these results give polynomial kernels, the exponent of the polynomial depends on d , leaving open the question of kernels of polynomial size with a fixed constant degree. This question was answered negatively for CONVC in [8] using the standard reduction from d -SET COVER. It is also shown in [8] that other problems such as CONNECTED DOMINATING SET and CONNECTED FEEDBACK VERTEX SET do not admit a kernel of any polynomial size unless $\text{coNP} \subseteq \text{NP/poly}$. Furthermore, the results in [9, 23] can be easily used to show exclude a $O(k^{d-\varepsilon})$ -size kernel for DOMINATING SET, for some small positive constant ε .

Our results: In this paper, we show that all remaining kernelization upper bounds for d -degenerate graphs mentioned above have matching lower bounds up to some small additive constant. Perhaps the most surprising result we obtain is the exclusion of $O(k^{(d-3)(d-1)-\varepsilon})$ size kernels for DOMINATING SET for any $\varepsilon > 0$, under the assumption of $\text{coNP} \not\subseteq \text{NP/poly}$. This result is obtained by an intricate application of *weak compositions* which were introduced by [10], and further applied in [9, 23]. What makes this result surprising is that it implies that INDEPENDENT DOMINATING SET is fundamentally easier than DOMINATING SET in d -degenerate graphs. We also show a $O(k^{d-4-\varepsilon})$ lower bound for INDEPENDENT DOMINATING SET, and an $O(k^{d-3-\varepsilon})$ lower bound for INDUCED MATCHING. The latter result is also somewhat surprising when one considers the trivial linear kernel for the closely related INDEPENDENT SET problem. Finally, we slightly improve the $O(k^{d+1})$ kernel for CONNECTED VERTEX COVER of [8] to $O(k^d)$, and show that the related CAPACITATED VERTEX COVER problem has a kernel of size $O(k^{d+1})$, but no kernel of size $O(k^{d-\varepsilon})$ unless $\text{coNP} \subseteq \text{NP/poly}$. Table 1 summarizes the currently known state of the art of kernel sizes for the problems considered in this paper.

	Lower Bound	Upper Bound
DOMINATING SET	$(d-3)(d-1)-\varepsilon$	$(d+1)^2$ [29]
INDEPENDENT DOMINATING SET	$d-4-\varepsilon$	$d+1$ [29]
INDUCED MATCHING	$d-3-\varepsilon$	d [14, 24]
CONNECTED VERTEX COVER	$d-1-\varepsilon$ [8]	d
CAPACITATED VERTEX COVER	$d-\varepsilon$	$d+1$

Table 1. Lower and upper bounds for kernel sizes for problems in d -degenerate graphs. Only the exponent of the polynomial in k is given. Results without a citation are obtained in this paper.

2 Kernelization Lower Bounds

In the following section we quickly review the main tool that we will be using for showing our kernelization lower bounds, namely compositions. A composition algorithm is typically a transformation from a classical NP-hard problem L_1 to a parameterized problem L_2 . It takes as input a sequence of T instances of L_1 , each of size n , and outputs in polynomial time an instance of L_2 such that (i) the output is a YES-instance iff one of the inputs is a YES-instance, and (ii) the parameter of the output is polynomially bounded by n and has only “small” dependency on T . Thus, a composition may intuitively be thought of as an “OR-gate” with a guarantee bound on the parameter of the output. More formally, for an integer $d \geq 1$, a weak d -composition is defined as follows:

Definition 1 (weak d -composition). Let $d \geq 1$ be an integer constant, let $L_1 \subseteq \{0, 1\}^*$ be a classical (non-parameterized) problem, and let $L_2 \subseteq \{0, 1\}^* \times \mathbb{N}$ be a parameterized problem. A weak d -composition from L_1 to L_2 is a polynomial time algorithm that on input $x_1, \dots, x_{td} \in \{0, 1\}^n$ outputs an instance $(y, k') \in \{0, 1\}^* \times \mathbb{N}$ such that:

- $(y, k') \in L_2 \iff x_i \in L_1$ for some i , and
- $k' \leq t \cdot n^{O(1)}$.

The connection between compositions and kernelization lower bounds was discovered by [2] using ideas from [22] and a complexity theoretic lemma of [19]. The following particular connection was first observed in [10].

Lemma 1 ([10]). Let $d \geq 1$ be an integer, let $L_1 \subseteq \{0, 1\}^*$ be a classical NP-hard problem, and let $L_2 \subseteq \{0, 1\}^* \times \mathbb{N}$ be a parameterized problem. A weak- d -composition from L_1 to L_2 implies that L_2 has no kernel of size $O(k^{d-\varepsilon})$ for any $\varepsilon > 0$, unless $\text{coNP} \subseteq \text{NP/poly}$.

Remark 1. Lemma 1 also holds for *compressions*, a stronger notion of kernelization, in which the reduction is not necessarily from the problem to itself, but rather from the problem to some arbitrary set.

3 Dominating Set

We begin by considering the DOMINATING SET (DS) problem. In this problem, we are given an undirected graph $G = (V, E)$ together with an integer k , and we are asked whether there exists a set S of at most k vertices such that each vertex of G either belongs to S or has a neighbor in S (i.e. $N[S] = V$). The main result of this section is stated in Theorem 1 below.

Theorem 1. Let $d \geq 4$. The DOMINATING SET problem in d -degenerate graphs has no kernel of size $O(k^{(d-1)(d-3)-\varepsilon})$ for any constant $\varepsilon > 0$ unless $\text{NP} \subseteq \text{coNP/poly}$.

In order to prove Theorem 1, we show a lower bound for a similar problem called the RED BLUE DOMINATING SET problem (RBDS): Given a bipartite graph $G = (R \cup B, E)$ and an integer k , where R is the set of *red* vertices and B is the set of *blue* vertices, determine whether there exists a set $D \subseteq R$ of at most k red vertices which dominate all the blue vertices (i.e. $N(D) = B$). According to Remark 1, the lemma below shows that focusing on RBDS is sufficient for proving Theorem 1 above.

Lemma 2. There is a polynomial time algorithm, which given a d -degenerate instance $I = (G = (R \cup B, E), k)$ of RBDS creates a $(d + 1)$ -degenerate instance $I' = (G', k')$ of DS, such that $k' = k + 1$ and I is YES-instance iff I' is a YES-instance.

Proof. As the graph G' we initially take $G = (R \cup B, E)$ and then we add two vertices r, r' and make r adjacent to all the vertices in $R \cup \{r'\}$. Clearly G' is $(d + 1)$ -degenerate. Note that if $S \subseteq R$ is a solution in I , then $S \cup \{r\}$ is a dominating set in I' . In the reverse direction, observe that w.l.o.g. we may assume that a solution S' for I' contains r and moreover contains no vertex of B . Therefore I is a YES-instance iff I' is a YES-instance. \square

We next describe a weak $d(d+2)$ -composition from MULTICOLORED PERFECT MATCHING to RBDS in $(d+2)$ -degenerate graphs. The MULTICOLORED PERFECT MATCHING problem (MPM) is as follows: Given an undirected graph $G = (V, E)$ with even number n of vertices, together with a color function $\text{col} : E \rightarrow \{0, \dots, n/2 - 1\}$, determine whether there exists a perfect matching in G with all the edges having distinct colors. A simple reduction from 3-DIMENSIONAL PERFECT MATCHING, which is NP-complete due to Karp [25], where we encode one coordinate using colors, shows that MPM is NP-complete when we consider multigraphs. In the following lemma we show that MPM is NP-complete even for simple graphs.

Lemma 3. *The MPM problem is NP-complete.*

Proof. We show a polynomial time reduction from MPM in multigraphs, where several parallel edges are allowed. Let $(G = (V, E), \text{col} : V \rightarrow \{0, \dots, |V|/2 - 1\})$ be an instance of MPM where G is a multigraph, $|V|$ is even and $\pi : E \rightarrow \{0, \dots, |E| - 1\}$ is an arbitrary bijection between the multiset of edges and integers from 0 to $|E| - 1$. Moreover we assume that there is some fixed linear order on V . We create a graph G' as follows. The set of vertices of G' is $V' = \{x_v : v \in V\} \cup \{y_{v,e} : e \in E, v \in V\}$, that is we create a vertex in G' for each vertex of G and for each endpoint of an edge of G . The set of edges of G' is $E' = \{y_{u,e}y_{v,e} : e = uv \in E\} \cup \{x_vy_{v,e} : v \in V, e \in E, v \in e\}$. Finally, we define the color function $\text{col}' : E' \rightarrow \{0, \dots, n'/2 - 1\}$, where $n' = 2|E| + |V|$. For $y_{u,e}y_{v,e} \in E'$ we set $\text{col}'(y_{u,e}y_{v,e}) = \pi(e)$, for $x_vy_{v,e} \in E'$, where $e = uv$ and $u < v$ we set $\text{col}'(x_vy_{v,e}) = \pi(e)$, while for $x_vy_{v,e} \in E'$, where $e = uv$ and $u > v$ we set $\text{col}'(x_vy_{v,e}) = |E| + \text{col}(e)$.

Clearly the construction can be performed in polynomial time, and the graph G' is simple, hence it suffices to show that the instance (G', col') is a YES-instance iff (G, col) is a YES-instance. Let $M \subseteq E$ be a solution for (G, col) . Observe that the set $M' = \{y_{u,e}y_{v,e} : e = uv \in M\} \cup \{x_{v,e}y_{v,e} : e \in E \setminus M, v \in e\}$ is a solution for (G', col') .

In the other direction, assume that $M' \subseteq E'$ is a solution for (G', col') . Note that for each $e = uv \in E$ either we have $y_{u,e}y_{v,e} \in M'$ or both $x_u y_{u,e}$ and $x_v y_{v,e}$ belong to M' . Consequently we define $M \subseteq E$ to be the set of edges $e = uv$ of E such that $y_{u,e}y_{v,e} \notin M'$. It is easy to verify that M is a solution for (G, col) .

The construction of the weak composition is rather involved. We construct an instance graph H_{inst} which maps feasible solutions of each MPM instance into feasible solutions of the RBDS instance. Then we add an enforcement gadget (H_{enf}, E_{conn}) which prevents partial solutions of two or more MPM instances to form altogether a solution for the RBDS instance. The overall RBDS instance will be denoted by (H, k) , where H is the union of H_{inst} and H_{enf} along with the edges E_{conn} that connect between these graphs. The construction of the instance graph is relatively simple, while the enforcement gadget is rather complex. In the next subsection we describe H_{enf} and its crucial properties. In the following subsection we describe the rest of the construction, and prove the claimed lower bound on RBDS (and hence DS). Both H_{enf} and H_{inst} contain red and blue nodes. We will use the convention that R and B denote sets of red and blue nodes, respectively. We will use r and b to indicate red and blue nodes, respectively. A color is indicated by ℓ .

3.1 The Enforcement Graph

The enforcement graph $H_{enf} = (R_{enf} \cup B_{enf}, E_{enf})$ is a combination of 3 different gadgets: the *encoding gadget*, the *choice gadget*, and the *fillin gadget* (see also Fig. 1), i.e. $R_{enf} = R_{code} \cup R_{fill}$

and $B_{enf} = B_{code} \cup B_{choice} \cup B_{fill}$ (R_{choice} is empty).

Encoding gadget: The role of this gadget is to encode the indices of all the instances by different partial solutions. It consists of nodes $R_{code} \cup B_{code}$, plus the edges among them. The set R_{code} contains one node $r_{\delta,\lambda,\gamma}$ for all integers $0 \leq \delta < d+2$, $0 \leq \lambda < d$, and $0 \leq \gamma < t$. In particular, $|R_{code}| = (d+2)dt$. The set B_{code} is the union of sets B_{code}^ℓ for each color $0 \leq \ell < n/2$. In turn, B_{code}^ℓ contains a node b_a^ℓ for each integer $0 \leq a < (dt)^{d+2}$. We connect nodes $r_{\delta,\lambda,\gamma}$ and b_a^ℓ iff $a_\delta = \lambda \cdot t + \gamma$, where (a_0, \dots, a_{d+1}) is the expansion of a in base dt , i.e. $a = \sum_{0 \leq \delta < d+2} a_\delta (dt)^\delta$. There is a subtle reason behind this connection scheme, which hopefully will be clearer soon. Note that since $0 \leq \gamma < t$, pairs (λ, γ) are in one to one correspondence with possible values of digits a_δ .

Choice gadget: The role of the choice gadget is to guarantee the following *choice property*: Any feasible solution to the overall RBDS instance (H, k) contains all nodes R_{code} except possibly one node $r_{\delta,\lambda,\gamma_{\delta,\lambda}}$ for each pair (δ, λ) (hence at least $(d+2)d(t-1)$ nodes of R_{code} altogether are taken). Intuitively, the $\gamma_{\delta,\lambda}$'s will be used to identify the index of one MPM input instance. In order to do that, we introduce a set of nodes B_{choice} , containing a node $b_{\delta,\lambda,\gamma_1,\gamma_2}$ for every pair (δ, λ) and for every $0 \leq \gamma_1 < \gamma_2 < t$. We connect $b_{\delta,\lambda,\gamma_1,\gamma_2}$ to both $r_{\delta,\lambda,\gamma_1}$ and $r_{\delta,\lambda,\gamma_2}$. It is not hard to see that, in order to dominate B_{choice} , it is necessary and sufficient to select from R_{code} a subset of nodes with the choice property.

Fillin gadget: We will guarantee that, in any feasible solution, precisely $(d+2)d(t-1)$ nodes from R_{code} are selected. Given that, for each pair (δ, λ) , there will be precisely one node $r_{\delta,\lambda,\gamma_{\delta,\lambda}}$ which is not included in the solution. Consequently, as we will prove, for each $0 \leq \ell < n/2$ in B_{code}^ℓ there will be exactly d^{d+2} uncovered nodes, namely the nodes $b_a^\ell = b_{(a_0, \dots, a_{d+1})}^\ell$ such that for each $0 \leq \delta < d+2$ and $\lambda t \leq a_\delta < (\lambda+1)t$ one has $a_\delta = \lambda t + \gamma_{\delta,\lambda}$. Ideally, we would like to cover such nodes by means of red nodes in the instance graph H_{inst} (to be defined later), which encode a feasible solution to some MPM instance. However, the degeneracy of the overall graph would be too large. The role of the fillin gadget is to circumvent this problem, by leaving at most d uncovered nodes in each B_{code}^ℓ . The fillin gadget consists of nodes $R_{fill} \cup B_{fill}$, with some edges incident to them. The set R_{fill} is the union of sets R_{fill}^ℓ for each color ℓ . In turn R_{fill}^ℓ contains one node $r_{a,j}^\ell$ for each $1 \leq j \leq d^{d+2} - d$ and $0 \leq a < (dt)^{d+2}$. We connect each $r_{a,j}^\ell$ to b_a^ℓ . The set B_{fill} contains one node b_j^ℓ , for each color ℓ and for all $1 \leq j \leq d^{d+2} - d$. We connect b_j^ℓ to all nodes $\{r_{a,j}^\ell : 0 \leq a < (dt)^{d+2}\}$. Observe that, in order to cover B_{fill} , it is necessary and sufficient to select one node $r_{a,j}^\ell$ for each ℓ and j . Furthermore, there is a way to do that such that each selected $r_{a,j}^\ell$ covers one extra node in B_{code}^ℓ w.r.t. selected nodes in R_{code} . Note that we somewhat abuse notation as we denote by b_j^ℓ vertices of B_{fill} , while we use b_a^ℓ for vertices of B_{code} , hence the only distinction is by the variable name.

Lemma 4. *For any matrix $(\gamma_{\delta,\lambda})_{0 \leq \delta < d+2, 0 \leq \lambda < d}$ of size $(d+2) \times d$ with entries from $\{0, \dots, t-1\}$, there exists a set $\tilde{R}_{enf} \subseteq R_{enf}$ of size $k' := \frac{n}{2}(d^{d+2} - d) + (d+2)d(t-1)$, such that:*

- each vertex in $B_{choice} \cup B_{fill}$ has a neighbor in \tilde{R}_{enf} , and
- for every $0 \leq \ell < n/2$ we have $B_{code}^\ell \setminus N(\tilde{R}_{enf}) = \{b_a^\ell : 0 \leq \lambda < d, a = \sum_{0 \leq \delta < d+2} (\lambda t + \gamma_{\delta,\lambda})(dt)^\delta\}$.

Proof. For each $0 \leq \delta < d+2$ and $0 \leq \lambda < d$, add to \tilde{R}_{enf} the set $\{r_{\delta,\lambda,\gamma} : 0 \leq \gamma < t, \gamma \neq \gamma_{\delta,\lambda}\}$ containing $t-1$ vertices. Note that by construction \tilde{R}_{enf} dominates the whole set B_{choice} . Consider

a vertex $b_a^\ell \in B_{code}^\ell \setminus N(\tilde{R}_{enf})$ and observe that for each coordinate $0 \leq \delta < d+2$, there are exactly d values that a_δ can have, where (a_0, \dots, a_{d+1}) is the (dt) -ary representation of a . Indeed, for any $0 \leq \delta < d+2$, we have $a_\delta \in X_\delta = \{\lambda t + \gamma_{\delta,\lambda} : 0 \leq \lambda < d\}$, since otherwise b_a^ℓ would be covered by \tilde{R}_{enf} due to the δ -th coordinate. Moreover if we consider any $b_{(a_0, \dots, a_{d+1})}^\ell \in B_{code}^\ell$ such that $a_\delta \in X_\delta$ for $0 \leq \delta < d+2$, then $b_{(a_0, \dots, a_{d+1})}^\ell$ is not dominated by the vertices added to \tilde{R}_{enf} so far.

Next, for each ℓ define $M^\ell := \{b_a^\ell : 0 \leq \lambda < d, a = \sum_{0 \leq \delta < d+2} (\lambda t + \gamma_{\delta,\lambda})(dt)^\delta\}$ and observe that M^ℓ are not dominated by \tilde{R}_{enf} . For each $0 \leq \ell < n/2$, let Z^ℓ be the vertices of B_{code}^ℓ not yet covered by \tilde{R}_{enf} and for each $1 \leq j \leq d^{d+2} - d$ select exactly one distinct vertex $v_j \in Z^\ell \setminus M^\ell$, where $v_j = b_{a_j}^\ell$, and add to \tilde{R}_{enf} the vertex $r_{a_j, j}^\ell$. Observe that after this operation \tilde{R}_{enf} covers B_{fill} and moreover the only vertices of B_{code} not covered by \tilde{R}_{enf} are the vertices of $\bigcup_{0 \leq \ell < n/2} M^\ell$. Since the total size of \tilde{R}_{enf} equals $d(d+2)(t-1) + \frac{n}{2}(d^{d+2} - d)$, the lemma follows. \square

Lemma 5. *Consider an RBDS instance $(H = (R \cup B, E), k)$ containing $G_{enf} = (R_{enf} \cup B_{enf}, E_{enf})$ as an induced subgraph, with $R_{enf} \subseteq R$ and $B_{enf} \subseteq B$, such that no vertex of $B_{choice} \cup B_{fill}$ has a neighbor outside of R_{enf} . Then any feasible solution \tilde{R} to (H, k) contains at least $k' := \frac{n}{2}(d^{d+2} - d) + (d+2)d(t-1)$ nodes \tilde{R}_{enf} of R_{enf} . Furthermore, for any feasible solution \tilde{R} to (H, k) containing exactly k' vertices of R_{enf} , there exist a matrix $(\gamma_{\delta,\lambda})_{0 \leq \delta < d+2, 0 \leq \lambda < d}$ of size $(d+2) \times d$ with entries from $\{0, \dots, t-1\}$, such that for each $0 \leq \ell < n/2$:*

- (a) *there are at least d vertices in $U^\ell = B_{code}^\ell \setminus N(\tilde{R} \cap R_{enf})$, and*
- (b) *U^ℓ is a subset of the d^{d+2} nodes $b_a^\ell = b_{(a_0, \dots, a_{d+1})}^\ell$ such that for each $\delta \in \{0, \dots, d+1\}$ there exists $\lambda \in \{0, \dots, d-1\}$ with $a_\delta = \lambda t + \gamma_{\delta,\lambda}$.*

Proof. Let \tilde{R} be any feasible solution to (H, k) . Observe that since \tilde{R} dominates B_{choice} , for each $0 \leq \delta < d+2$ and $0 \leq \lambda < d$ we have $|\tilde{R} \cap \{r_{\delta,\lambda,\gamma} : 0 \leq \gamma < t\}| \geq t-1$. Moreover in order to dominate vertices of B_{fill} , the set \tilde{R} has to contain at least $n/2(d^{d+2} - d)$ vertices of R_{fill} . Consequently, if \tilde{R} contains exactly k' vertices of R_{enf} , then for each $0 \leq \delta < d+2$ and $0 \leq \lambda < d$, there is exactly one $\gamma_{\delta,\lambda}$ such that $r_{\delta,\lambda,\gamma_{\delta,\lambda}} \notin \tilde{R}$. By the same argument as in the proof of Lemma 4, we infer that for each ℓ , the set $B_{code}^\ell \setminus N(\tilde{R} \cap R_{code})$ contains exactly d^{d+2} vertices, and we denote them as U_0^ℓ . Observe that the set $\tilde{R} \setminus R_{code}$ dominates at most $d^{d+2} - d$ vertices of U_0^ℓ , for each $0 \leq \ell < n/2$, which proves properties (a) and (b) of the lemma. \square

3.2 The Overall Graph

The construction of $H_{inst} = (R_{inst} \cup B_{inst}, E_{inst})$ is rather simple. Let $(G_i = (V, E_i), \text{col}_i)$ be the input MPM instances, with $0 \leq i < T = t^{d(d+2)}$. By standard padding arguments we may assume that all the graphs G_i are defined over the same set V of even size n , i.e. $G_i = (V, E_i)$. For each $v \in V$, we create a blue node $b_v \in B_{inst}$. For each $e_i = \{u, v\} \in E_i$, we create a red node $r_{e_i, i} \in R_{inst}^i$ and connect it to both b_u and b_v . We let $R_{inst} := \bigcup_{0 \leq i < T} R_{inst}^i$. Intuitively, we desire that a RBDS solution, if any, selects exactly $n/2$ nodes from one set R_{inst}^i , corresponding to edges of different colors, which together dominate all nodes B_{inst} : This induces a feasible solution to MPM for the i -th instance.

It remains to describe the edges E_{conn} which connect H_{enf} with H_{inst} . This is the most delicate part of the entire construction. We map each index i , $0 \leq i < T$, into a distinct $(d+2) \times d$ matrix M_i with entries $M_i[\delta, \lambda] \in \{0, \dots, t-1\}$, for all possible values of δ and λ . Consider an instance G_i . We

connect $r_{e,i}$ to b_a^ℓ iff $\ell = \text{col}_i(e_i)$ and there exists $0 \leq \lambda < d$ such that the expansion (a_0, \dots, a_{d+1}) of a in base dt satisfies $a_\delta = M_i[\delta, \lambda] + \lambda \cdot t$ on each coordinate $0 \leq \delta < d + 2$. The final graph $H := (R \cup B, E)$ we construct for our instance RBDS is then given by $R := R_{inst} \cup R_{enf}$ and $E := E_{inst} \cup E_{enf} \cup E_{conn}$. See Fig. 1.

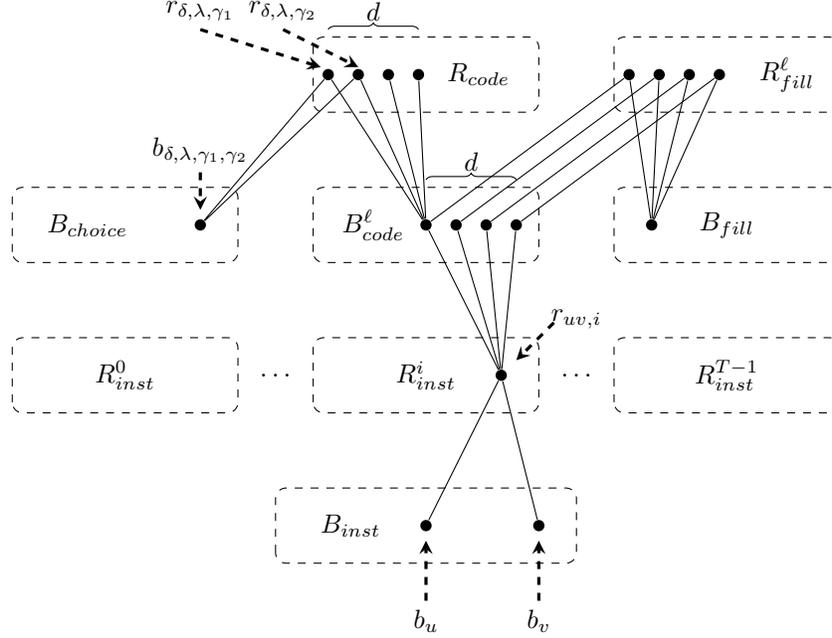


Fig. 1. Construction of the graph H . For simplicity the figure does not include sets $R_{fill}^{\ell'}$ and $B_{code}^{\ell'}$ for $\ell' \neq \ell$.

Lemma 6. H is $(d + 2)$ -degenerate.

Proof. Observe that each vertex of $\bigcup_{0 \leq i < T} R_{inst}^i$ is of degree exactly $d + 2$ in H , since it is adjacent to exactly two vertices of B_{inst} and exactly d vertices of the enforcement gadget, so we put all those vertices first to our ordering. Next, we take vertices of B_{inst} , as those have all neighbors already put into the ordering. Therefore it is enough to argue about the $(d + 2)$ -degeneracy of the enforcement gadget. We order vertices of $R_{fill} \cup B_{choice}$, since those are of degree exactly two in H . In $H \setminus R_{fill}$ the vertices of B_{fill} become isolated, so we put them next to our ordering. We are left with the vertices of the encoding gadget. Observe, that each blue vertex of the encoding gadget has exactly $d + 2$ neighbors in R_{code} , one due to each coordinate, hence we put the vertices of B_{code} next and finish the ordering with vertices of R_{code} . \square

Lemma 7. Let $k := (d + 2)d(t - 1) + n/2(d^{d+2} - d) + n/2 = k' + n/2$. Then (H, k) is a YES-instance of RBDS iff (G_i, col_i) is a YES-instance of MPM for some $i \in \{0, \dots, T - 1\}$.

Proof. Let us assume that for some i_0 the instance $(G_{i_0}, \text{col}_{i_0})$ is a YES-instance and $E' \subseteq E_{i_0}$ is the corresponding solution. We use Lemma 4 with the matrix M_{i_0} assigned to the instance i_0 to obtain the set \tilde{R}_{enf} of size $(d + 2)d(t - 1) + \frac{n}{2}(d^{d+2} - d)$. As the set \tilde{R} we take $\tilde{R}_{enf} \cup \{r_{e, i_0} : e \in E'\}$.

Clearly $|\tilde{R}| = k$. Since E' is a perfect matching, \tilde{R} dominates B_{inst} . By Lemma 4, \tilde{R} dominates $B_{fill} \cup B_{choice}$ and all but d vertices of each B_{code}^ℓ , so denote those d vertices by M^ℓ . Consider each $0 \leq \ell < n/2$, and observe that since E' is multicolored and by the construction of H , the set of neighbors of r_{e,i_0} in B_{code} is exactly $M^{\text{col}_{i_0}(e)}$; and hence \tilde{R} is a solution for (H, k) .

In the other direction, assume that (H, k) is a YES-instance and let \tilde{R} be a solution of size at most k . By Lemma 5, the set \tilde{R} contains at least $k' = \frac{n}{2}(d^{d+2} - d) + (d+2)d(t-1)$ vertices of R_{enf} and since \tilde{R} needs to dominate also B_{inst} it contains at least $\frac{n}{2}$ vertices of $\bigcup_{0 \leq i < T} R_{inst}^i$, since no vertex of H dominates more than two vertices of B_{inst} . Consequently $|\bigcup_{0 \leq i < T} R_{inst}^i \cap \tilde{R}| = n/2$ and $|R_{enf} \cap \tilde{R}| = k'$. We use Lemma 5 to obtain a matrix $M = (\gamma_{\delta,\lambda})$ of size $(d+2) \times d$. Moreover, by property (a) of Lemma 5, there are at least d vertices in U^ℓ , and consequently for each color ℓ the set \tilde{R} contains exactly one vertex of the set $\{r_{e,i} : 0 \leq i < T, \text{col}_i(e) = \ell\}$. Our goal is to show that for each $0 \leq i < T$, such that a matrix different than M is assigned to the i -th instance, we have $\tilde{R} \cap R_{inf}^i = \emptyset$, which is enough to finish the proof of the lemma. Consider any such i and assume that the matrices M_i and M differ in the entry $M_i[\delta', \lambda'] \neq \gamma_{\delta', \lambda'}$. Let ℓ be a color such that $r_{e,i} \in \tilde{R}$ and $\text{col}_i(e) = \ell$. By property (b) of Lemma 5, the set of at least d vertices of B_{code}^ℓ not dominated by $\tilde{R} \cap R_{enf}$ is contained in $U_0^\ell = \{b_{(a_0, \dots, a_{d+1})}^\ell : \forall 0 \leq \delta < d+2 \text{ if } \lambda t \leq a_\delta < (\lambda+1)t \text{ then } a_\delta = \lambda t + \gamma_{\delta, \lambda}\}$. However, by our construction of edges of H between R_{inst}^i and B_{code}^ℓ , we have $(N_H(r_{e,i}) \cap B_{code}^\ell) \not\subseteq U_0^\ell$ since the vertex $b_{(a_0, \dots, a_{d+1})}^\ell \in N_H(r_{e,i}) \cap B_{code}^\ell$ with $a_\delta = \lambda' t + M_i[\delta', \lambda']$ does not belong to U_0^ℓ and consequently does not belong to U^ℓ , which leaves at least one vertex of B_{code}^ℓ not dominated by \tilde{R} ; a contradiction. \square

Lemmas 6 and 7 imply that, for any $d \geq 1$, there exists a weak $d(d+2)$ -composition from MPM to RBDS in $(d+2)$ -degenerate graphs. The proof of Theorem 1 thus follows from Lemmas 1 and 2.

4 Independent Dominating Set

The INDEPENDENT DOMINATING SET (IDS) problem is the variant of DS where we require the dominating set S to induce an independent set (i.e. nodes in S have to be pairwise non-adjacent). We next describe a weak d -composition from 3-EXACT SET COVER, to IDS in $(d+4)$ -degenerate graphs. The input of 3-EXACT SET COVER is a set system (U, \mathcal{F}) , where each set in \mathcal{F} contains exactly three elements and the question is whether there is a collection $\mathcal{S} \subseteq \mathcal{F}$ of disjoint sets which partition U , i.e. $\bigcup \mathcal{S} = U$. The 3-EXACT SET COVER problem is NP-complete by a reduction from 3-DIMENSIONAL PERFECT MATCHING, which is NP-complete due to Karp [25].

Consider a fixed value of $d \geq 1$ and let $(\mathcal{F}_0, U_0), \dots, (\mathcal{F}_{T-1}, U_{T-1})$ be $T := t^d$ instances of 3-EXACT SET COVER. Without loss of generality we can assume that in each instance the same universe U of size n is used, such that $n \equiv 0 \pmod{3}$, since if for some i we have $|U_i| \not\equiv 0 \pmod{3}$, then (\mathcal{F}_i, U_i) is clearly a NO-instance, and moreover we can pad each universe with additional triples to make sure that each U_i has the same cardinality. Therefore we assume that for each $i = 0, \dots, T-1$ we have $U_i = U$ and $|U| = n$.

We construct an instance $(H = (V, E), k)$ of IDS for a properly chosen parameter k . Similar to construction in Section 3, the graph H will contain two graphs, the *enforcement graph* $H_{enf} = (V_{enf}, E_{enf})$ and the *instance graph* $H_{inst} = (V_{inst}, E_{inst})$, plus some edges E_{conn} connecting them (see Fig. 2). The graph H_{inst} contains the node set $V_{univ} := \{v_u : u \in U\}$ (i.e., one node per element of the universe). Furthermore, it contains a node set V_{triple} which includes one node $v_{i,S}$ for every instance index $0 \leq i < T$ and every triple of elements of the universe $S \in \binom{U}{3}$. We connect nodes

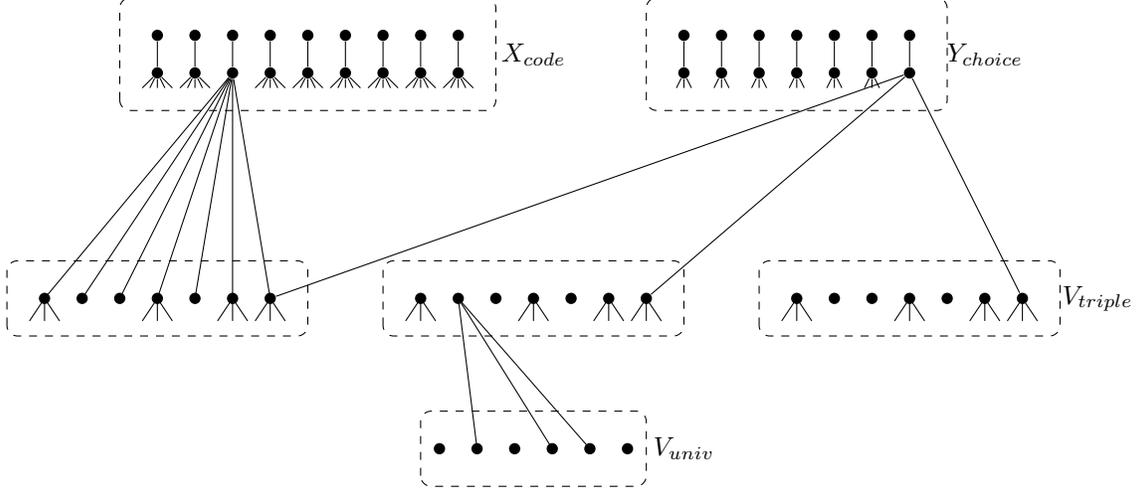


Fig. 2. The construction of the graph H in the composition for INDEPENDENT DOMINATING SET.

v_u and $v_{i,S}$ iff $S \in \mathcal{F}_i$ and $u \in S$. Observe that there might be some isolated nodes in V_{triple} . The graph H_{enf} consists of two induced matchings. It contains an *encoding matching* X_{code} with edges $\{x_{\gamma,\delta}, x'_{\gamma,\delta}\}$ for all integers $0 \leq \gamma < t$ and $0 \leq \delta < d$. Furthermore, it contains a *choice matching* Y_{choice} with edges $\{y_S, y'_S\}$ for all triples $S \in \binom{U}{3}$. It remains to describe E_{conn} . First of all, we connect each y_S to every $v_{i,S}$. Second, we connect $x_{\gamma,\delta}$ with $v_{i,S}$ iff $i_\delta = \gamma$ where (i_0, \dots, i_{d-1}) is the t -ary expansion of index i , i.e. $i = \sum_{\delta=0}^{d-1} i_\delta \cdot t^\delta$ with $0 \leq i_\delta < t$.

Lemma 8. *The graph H is $d + 4$ degenerate.*

Proof. Consider any ordering of vertices where we put vertices of V_{triple} first, and the remaining vertices. Observe, that each vertex of V_{triple} is of degree exactly $d + 4$ in H , since it has exactly d neighbors in X_{code} , exactly one neighbor in Y_{choice} and exactly three neighbors in V_{univ} . After removing V_{triple} , all vertices have degree at most 1. The claim follows. \square

Lemma 9. *Let $k := dt + \binom{n}{3} + n/3$. Then (H, k) is a YES-instance of IDS iff there exists $0 \leq j < T$, such that (\mathcal{F}_j, U) is a YES-instance.*

Proof. For the if part, let us assume that for some $0 \leq j < T$ the instance (\mathcal{F}_j, U) of 3-EXACT SET COVER is a YES-instance and let $\mathcal{S} \subseteq \mathcal{F}_j$ be a solution, i.e. a collection of $n/3$ disjoint sets. Construct an independent dominating set D of exactly k vertices as follows. Let (j_0, \dots, j_{d-1}) be the t -ary expansion of j . For $0 \leq \gamma < t, 0 \leq \delta < d$ if $j_\delta = \gamma$, then add $x'_{\gamma,\delta}$ to D and otherwise add to D the vertex $x_{\gamma,\delta}$. For $S \in \binom{U}{3}$, add y'_S to D if $S \in \mathcal{S}$ and add y_S to D otherwise. Finally, add to D the $n/3$ vertices $v_{j,S}$ with $S \in \mathcal{S}$. Clearly $|D| = k$. Moreover D is an independent set. In fact, $H[X_{code} \cup Y_{choice}]$ is a matching and we have taken exactly one endpoint of each one of its edges. Moreover for each $v_{j,S} \in D$ by construction there is no edge between $v_{j,S}$ and the remaining vertices of D . To prove that D is a dominating set observe that all the vertices of V_{univ} are dominated because \mathcal{S} is a solution for (\mathcal{F}_j, U) , and all the vertices of $X_{code} \cup Y_{choice}$ are dominated because D contains exactly one endpoint of each edge of the matching $H[X_{code} \cup Y_{choice}]$. Finally, each vertex

$v_{i,S}$ is dominated, since either $i \neq j$ and then $v_{i,S}$ is dominated by $X_{choice} \cap D$ due to the coordinate at which i and j differ, or $S \in \mathcal{S}$ and then $v_{j,S} \in D$ or $y_S \in D$.

For the only if part, let D be a dominating set in H of size at most k . Observe that since each vertex $x'_{\gamma,\delta}$ and y'_S is of degree exactly one in H , we have $|D \cap X_{code}| \geq dt$ and $|D \cap Y_{choice}| \geq \binom{n}{3}$. Moreover, since no vertex has more than three neighbors in V_{univ} we have $|D \cap (V_{univ} \cup V_{triple})| \geq n/3$. Therefore, all the three mentioned inequalities are tight. Moreover D contains exactly $n/3$ vertices of V_{triple} , since if D would contain a vertex of V_{univ} then $|D \cap (V_{univ} \cup V_{triple})|$ would be strictly greater than $n/3$. Define $\mathcal{S} = \{S \in \binom{U}{3} \mid \exists 0 \leq i < T : v_{i,S} \in D\}$. Observe, that since $|D \cap V_{triple}| = n/3$ we have $|\mathcal{S}| = n/3$. We want to show that there exists $0 \leq j < T$, such that $\mathcal{S} \subseteq \mathcal{F}_j$, which is enough to prove that (\mathcal{F}_j, U) is a YES-instance. Assume the contrary. Then there exist two indices $0 \leq i_1 < i_2 < T$, such that there exist $S_1, S_2 \in \mathcal{S}$ with $v_{i_1, S_1} \in D$ and $v_{i_2, S_2} \in D$. Since D is an independent set, it means that no vertex of $V' := \{v_{i,S} : i \in \{i_1, i_2\}, S \in \mathcal{S}\}$ has a neighbor in $D \cap X_{code}$ nor in $D \cap Y_{choice}$ and therefore $V' \subseteq D$. However $|V'| = 2|\mathcal{S}| = 2n/3 > n/3$, a contradiction. \square

Theorem 2. *Let $d \geq 4$. Then IDS has no kernel of size $O(k^{d-4-\varepsilon})$ for any constant $\varepsilon > 0$ unless $\text{coNP} \subseteq \text{NP/poly}$.*

Proof. From Lemmas 8 and 9, there exists a weak d -composition from 3-EXACT SET COVER to IDS in $(d+4)$ -degenerate graphs for any $d \geq 1$. The claim thus follows from Lemma 1.

5 Induced Matching

In this section we show a kernelization lower bound for the INDUCED MATCHING (IM) problem in d -degenerate graphs. In IM, the input is a graph G and an integer k , and the goal is to determine whether there exists a set of k edges e_1, \dots, e_k in G such that there is no edge in G connecting two endpoints of e_i and e_j for all $i \neq j \in \{1, \dots, k\}$. The main result of this section is given by the following theorem.

Theorem 3. *Let $d \geq 3$. Then IM has no kernel of size $O(k^{d-3-\varepsilon})$ for any constant $\varepsilon > 0$ unless $\text{coNP} \subseteq \text{NP/poly}$.*

For our kernel lower bound on IM, we present a weak d -composition from the MULTICOLORED CLIQUE problem in which the input is a graph $G := (V, E)$ and a vertex-coloring $\text{col} : V \rightarrow \{1, \dots, k\}$, and the goal is to determine whether there exists a multicolored clique of size k in G , that is, whether there exists a set of pairwise adjacent vertices v_1, \dots, v_k in G with $\text{col}(v_i) \neq \text{col}(v_j)$ for all $i \neq j \in \{1, \dots, k\}$. It is well known that MULTICOLORED CLIQUE is NP-hard [17].

Let $(G_i = (V_i, E_i), \text{col}_i)$, $0 \leq i < T := t^d$, be the input instances of MULTICOLORED CLIQUE. By standard padding and vertex-renaming arguments, we can assume that all graphs G_i are defined over the same vertex set V of size n , and that each vertex $v \in V$ is assigned the same color $\text{col}(v) \in \{1, \dots, k\}$ by all coloring functions col_i 's (note that this can be done even if the number of colors in each graph is different). We can further assume that for each $\{u, v\} \in E_i$ we have $\text{col}(u) \neq \text{col}(v)$, since all edges between vertices of the same color can never appear in any multicolored clique. Finally, we also assume that $\binom{k}{2} - k > d$, since otherwise a weak d -composition can trivially be constructed by solving each instance separately in polynomial time.

We next construct an instance $(H = (W, F), k')$ of IM for a proper parameter k' . As in previous sections, H consists of an *instance graph* $H_{inst} = (W_{inst}, F_{inst})$ and an *enforcement gadget*

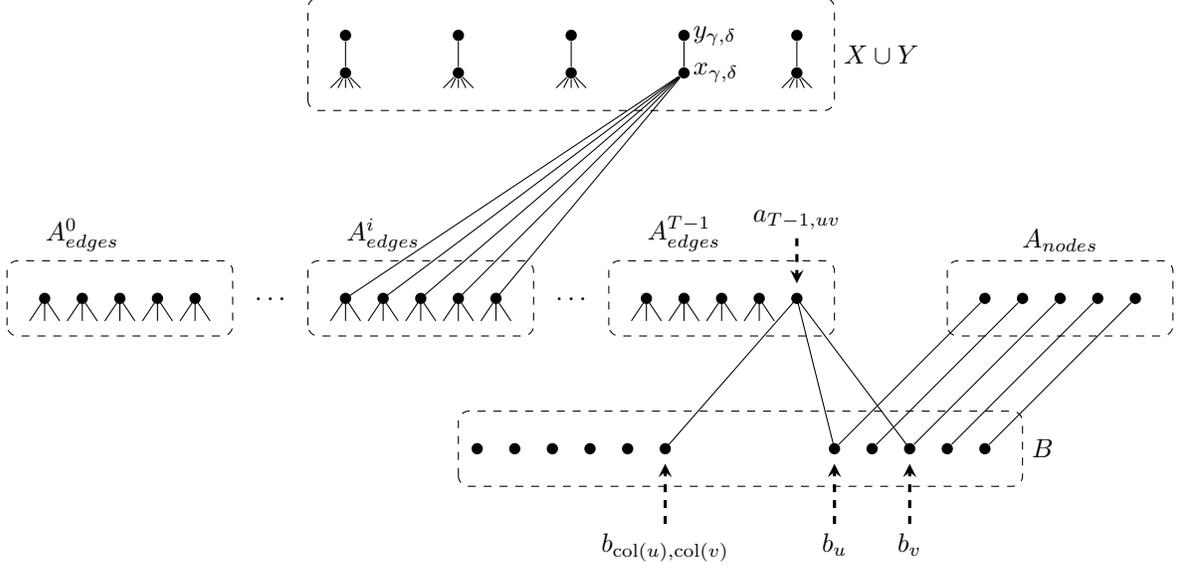


Fig. 3. The construction of the graph H in the composition for INDUCED MATCHING.

(H_{enf}, E_{conn}) , where H_{enf} is a graph and E_{conn} is a proper set of edges between H_{inst} and H_{enf} (see Fig. 3). The role of the instance graph is the guarantee that feasible solutions to any MULTICOLORED CLIQUE instance induce feasible solutions to the IM instance, and the enforcement gadget ensures that we cannot combine partial solutions of MULTICOLORED CLIQUE instances to obtain a feasible solution to the IM instance.

Graph $H_{inst} = (A \cup B, F_{inst})$ is bipartite, with $A = A_{nodes} \cup A_{edges}$ and $B = B_{nodes} \cup B_{col-pairs}$. Sets A_{nodes} and B_{nodes} contain a node a_v and b_v , respectively, for each $v \in V$. We have $A_{edges} = \bigcup_{0 \leq i < T} A_{edges}^i$, where the set A_{edges}^i contains a node $a_{i,e}$ for each instance i and $e_i \in E_i$. Set $B_{col-pairs}$ contains a node $b_{\alpha,\beta} \in B$ for every pair of colors $1 \leq \alpha < \beta \leq k$. Set F_{inst} contains all the edges of type $\{a_v, b_v\}$, plus edges between each $a_{i,e} = a_{i,uv}$ and nodes b_u, b_v and $b_{col(u),col(v)}$.

Lemma 10. *Let M' be an induced matching in H_{inst} . Then $|M'| \leq \binom{k}{2} + n - k$. Moreover, equality holds iff the graph with vertex set $V' = \{v \in V : b_v \in V(M')\}$ and edge set $E' := \{e : a_{i,e} \in V(M') \text{ for some } i \in \{0, \dots, T-1\}\}$ is a multicolored clique of the graph $(V, \cup_i E_i)$.*

Proof. Let M' be a maximum induced matching in H_{inst} . Clearly $n \leq |M'| \leq n + \binom{k}{2}$ (the lower bound comes from the matching induced by pairs (a_v, b_v)). If $|M'| = n$ then we are done, so assume $|M'| > n$. Since $\binom{k}{2} - k > d > 0$, we have $\binom{k}{2} + n - k > n$, and so some vertex $b_{\alpha,\beta}$ must be matched in M' . Consider some edge $m \in M'$ which includes some vertex $b_{\alpha,\beta}$. Then by construction, $m = \{b_{\alpha,\beta}, a_{i,e}\}$ where $a_{i,e}$ corresponds to an edge $e = \{u, v\} \in E_i$, for some $i \in \{0, \dots, T-1\}$, where $col(u) = \alpha$ and $col(v) = \beta$. As $\{b_u, a_{i,e}\}, \{b_v, a_{i,e}\} \in F_{inst}$, the vertices b_u and b_v cannot be matched in M' . Thus, for each vertex $b_{\alpha,\beta}$ that is matched in M' , a vertex of $\{b_u : u \in V, col(u) = \alpha\}$ is not matched in M' , and a vertex of $\{b_v : v \in V, col(v) = \beta\}$ is not matched in M' . Since a vertex can only appear in at most k color pairs, we have $|M'| \leq \binom{k}{2} + n - k$. Furthermore, if a vertex $v \in V$ is not contained in any edge of E' , then $M' \cup \{a_v, b_v\}$ is also an induced matching. It is now not

difficult to see that $|M'| = \binom{k}{2} + n - k$ iff $|V \setminus V'| = n - k$, and (V', E') is a multicolored clique of size k . \square

As Lemma 10 suggests, to ensure that maximum induced matchings in H_{inst} are meaningful to us, we need to guarantee that only nodes $c_{i,e}$ for a unique index i are matched. To this aim, we introduce the following enforcement gadget (H_{enf}, E_{conn}) . Graph $H_{enf} = (X \cup Y, E_{enf})$ is bipartite. Sets X and Y contain a node $x_{\gamma,\delta}$ and $y_{\gamma,\delta}$, respectively, for all integers $0 \leq \gamma < t$ and $0 \leq \delta < d$. We add edges between all pairs $\{x_{\gamma,\delta}, y_{\gamma,\delta}\}$. Finally, we connect $x_{\gamma,\delta}$ with $a_{i,e}$ iff $i_\delta = \gamma$ where $(i_0, i_1, \dots, i_{d-1})$ is the t -ary expansion of index i . Observe that each $a_{i,e}$ is adjacent to exactly d distinct vertices $x_{\gamma,\delta}$.

Lemma 11. *H is $(d+3)$ -degenerate.*

Proof. Consider any ordering of the vertices which places first nodes A_{nodes} and Y , then node A_{edges} , then nodes B , and finally nodes X . It is not difficult to check that each vertex is adjacent to at most $d+3$ vertices appearing to its right in this ordering. \square

Lemma 12. *Let $k' := t(d-1) + \binom{k}{2} + n - k$. Then (H, k') is a YES-instance of IM iff (G_i, col_i) is a YES-instance of MULTICOLORED CLIQUE for some index $i \in \{0, \dots, T-1\}$.*

Proof. Suppose some G_i has a multicolored clique of size k . Then by Lemma 10 we can find an induced matching M' of size $\binom{k}{2} + n - k$ in H_{inst} which matches only vertices of type $a_{i,e}$. Let us add all the edges $\{x_{\gamma,\delta}, y_{\gamma,\delta}\}$ such that $i_\delta \neq \gamma$. There are precisely $t(d-1)$ such edges, which together with the edges of M' , form an induced matching of size k' in H .

For the converse direction, suppose M is an induced matching of size k' in H . First observe that we can assume M does not contain any edges of $F(A_{edge}, X)$ since any such edge $\{a_{i,e}, x_{\gamma,\delta}\}$ can be safely replaced with the edge $\{x_{\gamma,\delta}, y_{\gamma,\delta}\}$. Now as $\binom{k}{2} - k > d$ w.l.o.g., the matching M must include some edge of $F(B, A_{edge})$ since there are $n + td < k'$ edges altogether in $F(A_{node}, B) \cup F(X, Y)$. So let $a_{i,e}$ be a vertex of A_{edge} which is matched in M . Then as $a_{i,e}$ is matched in M , this means that d vertices of X cannot be matched in M , precisely those vertices $x_{\gamma,\delta}$ with $i_\delta = \gamma$. Thus, $|F(X, Y) \cap M| \leq t(d-1)$. Since $M \setminus F(X, Y)$ is a subset of edges in H_{inst} , and since the maximum induced matching in H_{inst} is at most $\binom{k}{2} + n - k$ by Lemma 10, this implies that M contains exactly $\binom{k}{2} + n - k$ edges of H_{inst} and $t(d-1)$ edges of $F(X, Y)$. By construction of $F(A_{edge}, X)$, the latter assertion implies that there exists some i such that each vertex of A_{edge} that is matched by M is of the form $a_{i,e}$ for some $e \in E_i$. By Lemma 10, the second assertion implies that $\{v \in V : \{b_v, a_{i,e}\} \in M \text{ for some } e \in E_i\}$ is a multicolored clique of size k in G_i . \square

Theorem 3 now directly follows from Lemmas 11, 12, and 1.

6 Upper Bounds

Both upper bounds that we present rely on the following easy lemma.

Lemma 13. *Let $G := (A \cup B, E)$ be a bipartite d -degenerate graph where all vertices in B have degree greater than d . Then $|B| \leq d|A|$.*

Proof. By the d -degeneracy of G , we know that $|E| \leq d(|A| + |B|)$. Since each vertex of B has degree greater than d , we also know that $(d+1)|B| \leq |E|$. Subtracting $d|B|$ from both inequalities gives $|B| \leq d|A|$. \square

6.1 Dominating and independent dominating set

6.2 Connected and capacitated vertex cover

Let us begin with the kernel for CONVC. In CONVC, our goal is to determine whether a given graph G has a set of vertices A such that $G[V(G) \setminus A]$ is edgeless (A is a *vertex cover*) and $G[A]$ is connected (A is *connected*). Our kernelization algorithm uses two simple reduction rules which are given below, the second of which is a variant of the well-known *crown reduction rule* [15]. We say that a set of vertices $S \subseteq V(G)$ is a set of *twins* in G if $N(v) = N(u)$ for all $u, v \in S$ (note that this implies that S is an independent set in G). We let $N(S)$ denote the common set of neighbors of S .

Rule 1. *If G has an isolated vertex remove it.*

Rule 2. *If $S \subseteq V(G)$ is a subset of at most $d + 1$ twin vertices with $|N(S)| < |S|$, remove an arbitrary vertex of S from G .*

Lemma 14. *Let G be a graph, and let G' be a graph resulting from applying either Rule 1 or Rule 2 to G . Then for any integer k , the graph G has a connected vertex cover of size k iff G' has a connected vertex cover of size k .*

Proof. The lemma is obvious for Rule 1, so let us focus on Rule 2. Observe that there exists a minimal size connected vertex cover A in G with $N(S) \subseteq A$. Furthermore, A contains at most one vertex of S , and this vertex can be replaced by any other vertex of S to obtain another connected vertex cover for G of equal size. Replacing this vertex (if it exists) with a vertex of $S \cap V(G')$, we obtain a connected vertex cover A' for G' with $|A'| = |A|$. Conversely, using similar arguments one can transform a minimum size connected vertex cover for G' to an equal size connected vertex cover for G . \square

Theorem 4. *CONVC in d -degenerate graphs has a kernel of size $O(k^d)$.*

Proof. Our kernelization algorithm for CONNECTED VERTEX COVER in d -degenerate graphs exhaustively applies Rule 1 and Rule 2 until they no longer can be applied. Since Rule 1 can be implemented in linear time, and Rule 2 can be done in $n^{O(d)}$ time, this algorithm runs in polynomial time. Let G' be the graph resulting from the kernelization. Observe that both reduction rules that were used do not increase the degeneracy of the graph, and so G' is d -degenerate as well. Furthermore, due to Lemma 14, we know that G has a connected vertex cover of size k iff G' has one as well. We next show that $|V(G')| = O(k^d)$, or otherwise G' has no connected vertex cover of size k .

Suppose that G' has a connected vertex cover A of size k . Then as A is a vertex cover, the set $B := V(G) \setminus A$ is an independent set in G . For $i := 0, \dots, d$, define $B_i \subseteq B$ to be set of all vertices in B with degree i in G , and let $B_{>d} \subseteq B$ be the vertices in B with degree greater than d in G . Then $B := B_0 \cup \dots \cup B_d \cup B_{>d}$, and $|B_0| = 0$ since Rule 1 cannot be applied. Due to Rule 2, for each subset of i vertices $A' \subseteq A$, $1 \leq i \leq d$, there are at most i vertices $B' \subseteq B_i$ with $N(B') = A'$. We therefore have $|B_i| \leq i \binom{k}{i}$ for each $i \in \{1, \dots, d\}$, and $\sum_{i=0, \dots, d} |B_i| \leq dk^d$. Furthermore, we also have $|B_{>d}| \leq dk$ by applying Lemma 13 to the bipartite graph on A and $B_{>d}$. Accounting also for A , we get

$$|V(G')| = |A| + |B| \leq k + \sum_{i=0, \dots, d} |B_i| + |B_{>d}| \leq k + dk^d + dk = O(k^d),$$

and the theorem is proved. \square

Next we consider CAPVC. In this problem, we are given a graph G , an integer k , and a vertex capacity function $cap : V(G) \rightarrow \mathbb{N}$, and the goal is to determine whether there exists a vertex cover of size k where each vertex covers no more than its capacity. That is, whether there is a vertex cover C of size k and an injective mapping α mapping each edge of $E(G)$ to one of its endpoints such that $|\alpha^{-1}(v)| \leq cap(v)$ for every $v \in V(G)$. We may assume that $k + 1 > d$, since otherwise a kernel is trivially obtained by solving the problem in polynomial time.

Rule 3. *If $S \subseteq V(G)$ is a subset of twin vertices with a common neighborhood $N(S)$ such that $|S| = k + 2 > d \geq |N(S)|$, remove a vertex with minimum capacity in S from G , and decrease all the capacities of vertices in $N(S)$ by one.*

Lemma 15. *Let $k \geq 1$ be an arbitrary integer, let G be a vertex capacitated graph, and let G' be a vertex capacitated graph resulting from applying either Rule 1 or Rule 3 to G . Then G has a capacitated vertex cover of size k iff G' has a capacitated vertex cover of size k .*

Proof. The lemma is trivial for Rule 1. For Rule 3, let A be a capacitated vertex cover of size k in G , and let u be a vertex of minimum capacity in S . As $|S| > k$, there is some $v \in S \setminus A$, and moreover it must be that $N(S) \subseteq A$. Thus, if $u \notin A$, then A is also a capacitated vertex cover of G' . Otherwise, if $u \in A$, we can replace u with v in A . As $cap(u) \leq cap(v)$, this would result in another capacitated vertex cover for G which is also a solution for G' . Conversely, any capacitated vertex cover of size k for G' must also include all vertices of $N(S)$, and thus by increasing the capacities of all these vertices by one we obtain a solution of size k for G . \square

Theorem 5. *CAPVC has a kernel of size $O(k^{d+1})$ in d -degenerate graphs.*

Proof (sketch). The argument is similar to the one used in Theorem 4. The only difference is that now the size of sets B_i is bounded by $(k+1)\binom{k}{i}$ instead of $d\binom{k}{i}$, which yields a kernel size of $O(k^{d+1})$ instead of $O(k^d)$. \square

The following complimenting lower bound follows from a simple reduction d -SET COVER. In this problem we are given a d -regular hypergraph (V, \mathcal{E}) and an integer k and the question is whether there are $E_1, \dots, E_k \in \mathcal{E}$ with $V = \cup_i E_i$. Clearly we can assume that $|V| \leq dk$ since otherwise there is no solution, and the problem remains hard also when $|V| = dk$ (and the solution is a partition of V). Dell and Marx show that unless $\text{coNP} \subseteq \text{NP/poly}$, d -SET COVER has no kernel of size $O(k^{d-\varepsilon})$ for any $\varepsilon > 0$ [9]. Note that this lower bound holds even if the output of the kernel is an instance of another decidable problem; in particular, even if the output is an instance of CAPVC.

Theorem 6. *Let $d \geq 3$. Unless $\text{coNP} \subseteq \text{NP/poly}$, CAPVC in d -degenerate graphs has no kernel of size $O(k^{d-\varepsilon})$ for any $\varepsilon > 0$.*

Proof. Given an instance (V, \mathcal{E}, k) of d -SET COVER with $|V| = kd$, we construct a graph $H := (U, F)$ by initially taking the incidence bipartite graph on $V \cup \mathcal{E}$, and then connecting each $v \in V$ to new a leaf-vertex v' which is adjacent only to v . In this way, $U := V' \cup V \cup \mathcal{E}$, where $V' := \{v' : v \in V\}$ is a set of copies of V , and $F := \{\{v, E\} : v \in V, e \in \mathcal{E}, \text{ and } v \in e\} \cup \{\{v, v'\} : v \in V\}$. To complete our construction, we set the capacity of each vertex $u \in U$ to be its degree in H minus one. Note that H is d -degenerate.

It is not difficult to see that (V, \mathcal{E}, k) has a solution iff H has a capacitated vertex cover of size $k + |V| = (d + 1)k$. Indeed if E_1, \dots, E_k is a solution for (V, \mathcal{E}, k) , then $\{E_1, \dots, E_k\} \cup V$ is a

capacitated vertex cover of H . Conversely, any minimal capacitated vertex cover of H must include all vertices of V and none of V' , and hence if it is of size $|V| + k$, it must include k vertices which correspond to k edges of \mathcal{E} that cover V . Thus, combining the above construction with a $O(k^{d-\varepsilon})$ kernel for CAPVC in d -degenerate graphs shows that $\text{coNP} \subseteq \text{NP/poly}$ according to [9]. \square

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