# Interval Enclosures of Upper Bounds of Roundoff Errors using Semidefinite Programming

VICTOR MAGRON, CNRS Verimag

A longstanding problem related to floating-point implementation of numerical programs is to provide efficient yet precise analysis of output errors.

We present a framework to compute lower bounds on largest absolute roundoff errors, for a particular rounding model. This method applies to numerical programs implementing polynomial functions with box constrained input variables. Our study is based on three different hierarchies, relying respectively on generalized eigenvalue problems, elementary computations and semidefinite programming (SDP) relaxations. This is complementary of over-approximation frameworks, consisting of obtaining upper bounds on the largest absolute roundoff error. Combining the results of both frameworks allows to get enclosures for upper bounds on roundoff errors.

The under-approximation framework provided by the third hierarchy is based on a new sequence of convergent robust SDP approximations for certain classes of polynomial optimization problems. Each problem in this hierarchy can be solved exactly via SDP. By using this hierarchy, one can provide a monotone non-decreasing sequence of lower bounds converging to the absolute roundoff error of a program implementing a polynomial function, applying for a particular rounding model.

We investigate the efficiency and precision of our method on non-trivial polynomial programs coming from space control, optimization and computational biology.

CCS Concepts: •Design and analysis of algorithms  $\rightarrow$  Approximation algorithms analysis; Numeric approximation algorithms; Mathematical optimization; Continuous optimization; Semidefinite programming; Convex optimization; •Logic  $\rightarrow$  Automated reasoning;

Additional Key Words and Phrases: roundoff error, polynomial optimization, semidefinite programming, floating-point arithmetic, generalized eigenvalues, robust optimization

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# 1. INTRODUCTION

Over the last four decades, numerical programs have extensively been written and executed with finite precision implementations [Dekker 1971], often relying on single or double floating-point numbers to perform fast computation. A ubiquitous related issue, especially in the context of critical system modeling, is to precisely analyze the absolute gap between the real and floating-point output of such programs. The existence of a possibly high roundoff error gap is a consequence of multiple rounding occurrences, happening most likely while performing operations with finite precision systems, such as IEEE 754 standard arithmetic [IEEE 2008].

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© YYYY ACM. 0098-3500/YYYY/01-ARTA \$15.00 DOI: http://dx.doi.org/10.1145/0000000.0000000 The present study focuses on computing a lower bound on the largest absolute roundoff error for a particular rounding model, while executing a program implementing a multivariate polynomial function f, with a priorly fixed bracketing. For these programs, each input variable takes a value within a given closed interval. We consider a simple (multiplicative) rounding model of the variables and elementary operations involved in f. Executing the program in floating-point precision leads to the computation of a rounded expression  $\hat{f}(\mathbf{x}, \mathbf{e})$  depending on the input variables  $\mathbf{X}$  and additional roundoff error variables  $\mathbf{e} := (e_1, \dots, e_m)$ . With unit roundoff  $\varepsilon$ , the value of each  $e_j$  can take values in  $[-\varepsilon, \varepsilon]$ . Stated formally, our goal is to compute a lower bound on the largest absolute value of the expression  $\hat{f}(\mathbf{x}, \mathbf{e}) - f(\mathbf{x})$  for all possible values of  $\mathbf{x}$  in  $\mathbf{X}$  and each possible value of  $e_j$  in  $[-\varepsilon, \varepsilon]$ . Exact resolution of this problem is nontrivial as it requires to compute the maximum of a polynomial, which is known to be NP-hard [Laurent 2009] in general.

Several existing methods allow to obtain *lower bounds* of roundoff errors. The easiest way to obtain such a bound on the maximum of a given function is to evaluate this function at several points within the function input domain before taking the minimum over all evaluations. Testing approaches aim at finding the inputs causing the worst error. Such techniques often rely on guided random testing as in s3fp [Chiang et al. 2014], or heuristic search as in PRECIMONIOUS [Rubio-González et al. 2013], CORAL [Borges et al. 2012].

The s3fp tool implements the so-called Binary Guided Random Testing (BGRT) method. BGRT relies on shadow value executions and configuration evaluations. A shadow value execution is the execution of a program under certain precision settings to compute either absolute or relative roundoff errors. A configuration is a mapping from program inputs to corresponding range of values. An initial configuration is splitted recursively into tighter configurations, where tightness is determined thanks to shadow value executions. The BGRT algorithm starts with a configuration, enumerates a sub-part of its tighter configurations, in order to pick the set of inputs causing the (locally) maximial high floating-point errors. For more details, we refer the interested reader to [Chiang et al. 2014, Section 3].

The Precimonious tool aims at assisting users to execute numerical programs with priorly prescribed accuracy in a more efficient way. For this, the tool performs automated tuning of the floating-point precision related to the elementary operations involved in the program. In the best scenario, Precimonious outputs a program with an optimial configuration, that is the setting which uses the least bit precisions resulting in the best performance improvement over all complying configurations. This goal is pursued by performing local search over a subset of the input variables provided by the user. Precimonious relies on the delta-debugging algorithm for local search. This consists of dividing the sets of possible configuration changes and increasing the number of subsets inductively when no improvement occurs. While Precimonious does not take into account the correlation between variables, CORAL relies on meta-heuristic solvers based on genetic algorithms [Goldberg 1989] and particle-swarm optimization [Kennedy and Eberhart 1995] to handle complex mathematical constraints.

Lower bound computed with testing are complementary with tools providing validated upper bounds. These tools are mainly based on interval arithmetic (e.g. Gappa [Daumas and Melquiond 2010], Fluctuat [Delmas et al. 2009], Rosa [Darulova and Kuncak 2014]) or methods coming from global optimization such as Taylor approximation in FPTaylor by [Solovyev et al. 2015], Bernstein expansion in FPBern by [Rocca et al. 2016]. The recent framework by [Magron et al. 2016], related to the Real2Float software package, employs semidefinite programming (SDP) to obtain a hierarchy of upper bounds converging to the absolute roundoff error. This hierarchy is derived from the general moment-sum-of-squares hierarchy (also called Lasserre's hierarchy) initially provided by [Lasserre 2001] in the context of polynomial optimization. At each step of Lasserre's hierarchy, one can either rely on moments or sum-of-squares (SOS) to compute a certified upper bound on the maximum (or

similarly a lower bound on the minimum) for a given objective polynomial function f under a set of polynomial inequality constraints K. In the unconstrained case, the underlying idea is that if one can decompose f into a sum of squares (SOS) then it is straighforward to prove that this polynomial is nonnegative. In the constrained case, the idea is to write fas a weighted SOS decomposition, where the weights are the polynomials involved in set of constraints K. This also proves that this polynomial is nonnegative on K. After fixing the maximal degree of the SOS polynomials, computing their coefficients boils down to solving a semidefinite program (SDP). An SDP problem involves a linear objective function with constraints over symmetric matrices with nonnegative eigenvalues. It can be solved with interior-point methods, yielding polynomial time algorithms at prescribed accuracy. For more details about applications of SDP together with complexity estimates, we refer to [Nesterov and Nemirovski 1994; Vandenberghe and Boyd 1994; de Klerk and Vallentin 2016. A well-known limitation of the Lasserre's hierarchy is due to the size of the SDP matrices involved in the SOS decompositions. For a system involving polynomials with nvariables of maximal degree k, this size grows rapidly as it is proportional to  $\binom{n+k}{n}$ . To overcome these limitations, several research efforts have been pursued to take into account the properties of certain classes of structured systems, e.g. sparsity [Waki et al. 2006; Lasserre 2006] or symmetry [Riener et al. 2013]. In particular, previous work by the author [Magron et al. 2016] exploits the special structure of the roundoff error function by applying the sparse variant of the first Lasserre's hierarchy to the linear part.

While the first SDP hierarchy allows to approximate from above the maximum of a polynomial, [Lasserre 2011] provides a second complementary SDP hierarchy, yielding a sequence of converging lower bounds. At each step of the second hierarchy, the lower bound on the maximum of a given polynomial is computed by solving a so-called generalized eigenvalue problem. Given two symmetric matrices A and B with known entries, this consists of finding the smallest value of  $\lambda$  such that the matrix  $\lambda \mathbf{A} - \mathbf{B}$  has only nonnegative eigenvalues. In our context, the two matrices encode certain information regarding the moments of some probability measure  $\mu$  supported on the set of constraints **K**. For instance, in the bivariate case, the entries of these matrices at the second step of the hierarchy necessarily depend on the value of the integrals  $\int_{\mathbf{K}} y_1 d\mu$ ,  $\int_{\mathbf{K}} y_2 d\mu$ ,  $\int_{\mathbf{K}} y_1^2 d\mu$ ,  $\int_{\mathbf{K}} y_1 y_2 d\mu$  and  $\int_{\mathbf{K}} y_2^2 d\mu$ . In several cases, the value of these integrals are available analytically. This includes the case where  $\mu$  is the uniform (also called Lebesgue) measure and **K** is the unit box  $[0,1]^n$  (or any product of real closed intervals), the simplex or the euclidean ball. The interested reader can find more details about these closed formula in [Grundmann and Moller 1978; Jean B. Lasserre 2016; de Klerk et al. 2016]. By contrast with the first Lasserre's hierarchy, the second one cannot easily handle the case where  $\mathbf{K}$  is defined by a general set of polynomial inequality constraints. For instance, computing the moments of the uniform measure on a polytope is NP-hard (see e.g. [De Loera et al. 2010]). This is still an open problem to design a hierarchy yielding certified lower bounds in the general case. To the best of our knowledge, there is also no variant of the second Lasserre's hierarchy exploiting the properties of special problems.

Several efforts have been made to provide convergence rates for the two hierarchies. For the first hierarchy, the theoretical estimates from [Nie and Schweighofer 2007] yield convergence rates of  $O(1/\sqrt[c]{\log(2k/c)})$ , where c is a constant depending only on K (but not explicitly known) and k is the selected step of the hierarchy. This yields very pessimistic bounds by contrast with the results obtained for practical case studies. The situation is rather different for the complexity analysis [de Klerk et al. 2016] of the second Lasserre's hierarchy as the convergence rates are no worse than  $O(1/\sqrt{r})$  and often match practical experiments. Investigating the gap between the two hierarchies, either from a theoretical or practical point of view, could provide insights on how to improve the estimates of the first Lasserre's hierarchy.

Following this line of research, the motivation of this paper relates both to the roundoff error analysis and the use of SDP relaxations dedicated to sparse polynomial problems. On the one hand, we focus on deriving a sparse variant of the second Lasserre's hierarchy for the special case of roundoff error computation. On the other hand, we aim at providing insights regarding the gap between this variant and the hierarchy of upper bounds from [Magron et al. 2016].

Contributions. We provide an SDP hierarchy inspired from [Lasserre 2011] to obtain a sequence of converging lower bounds on the largest absolute roundoff error obtained with a particular rounding model. This hierarchy and the one developed in [Magron et al. 2016] complement each other as the combination of both now allows to enclose the largest absolute roundoff error in smaller and smaller intervals.

We release a software package called FPSDP<sup>1</sup> implementing this SDP hierarchy.

The rest of the article is organized as follows: in Section 2 we provide preliminary background about floating-point arithmetic and SDP, allowing to state the considered problem of roundoff error. This problem is then addressed in Section 3 with our SDP hierarchy of converging lower bounds. Section 4 is devoted to numerical experiments in order to compare the performance of our FPSDP software with existing tools.

#### 2. FLOATING-POINT ARITHMETIC AND SEMIDEFINITE PROGRAMMING

#### 2.1. Floating-Point Arithmetic and Problem Statement

Let us denote by  $\varepsilon$  the machine epsilon or unit roundoff,  $\mathbb R$  the field of real numbers and  $\mathbb F$  the set of binary floating-point numbers. Both overflow and subnormal range values are neglected. Under this assumption, any real number  $x \in \mathbb R$  is approximated with its closest floating-point representation  $\hat x = x(1+e)$ , with  $|e| \le \varepsilon$  and  $\hat \cdot$  being the rounding operator. This can be selected among either rounding toward zero, rounding toward  $\pm \infty$  or rounding to nearest. From now on, for the sake of simplicity we only consider rounding to nearest. We refer to [Higham 2002] for related background.

The number  $\varepsilon := 2^{-\mathrm{prec}}$  bounds from above the relative floating-point error, with prec

The number  $\varepsilon := 2^{-\text{prec}}$  bounds from above the relative floating-point error, with prec being called the *precision*. For single (resp. double) precision floating-point, the value of the unit roundoff is  $\varepsilon = 2^{-24}$  (resp.  $\varepsilon = 2^{-53}$ ).

To comply with IEEE 754 standard arithmetic [IEEE 2008], for each real-valued operation  $\mathsf{bop}_{\mathbb{R}} \in \{+,-,\times,/\}$ , the result of the corresponding floating-point operation  $\mathsf{bop}_{\mathbb{F}} \in \{\oplus,\ominus,\otimes,\oslash\}$  satisfies:

$$\mathrm{bop}_{\mathbb{F}}\left(\hat{x},\hat{y}\right) = \mathrm{bop}_{\mathbb{R}}\left(\hat{x},\hat{y}\right)\left(1+e\right) \ , \quad |e| \leq \varepsilon = 2^{-\mathrm{prec}} \ . \tag{1}$$

**Semantics**. Our program semantics is based on the encoding of polynomial expressions in the Real2Float software [Magron et al. 2016]. The input variables of the program are constrained within interval floating-point bounds.

We denote by C the type for numerical constants, being chosen between double precision floating-point and arbitrary-size rational numbers. This type C is used for the interval bounds and for the polynomial coefficients.

As in [Magron et al. 2016, Section 2.1], the type pexprC of polynomial expressions is the following inductive type:

The constructor Px allows to represent any input variable  $x_i$  with the positive integer i.

<sup>&</sup>lt;sup>1</sup>https://github.com/magronv/FPSDP

**Input:** input variables  $\mathbf{x}$ , input box  $\mathbf{X}$ , polynomial  $\hat{f}$ , rounded polynomial  $\hat{f}$ , error variables  $\mathbf{e}$ , error box  $\mathbf{E}$ , relaxation procedure  $\mathsf{sdp\_bound}$ , relaxation order k

**Output:** lower bound on the largest absolute roundoff error  $|\hat{f} - f|$  over  $\mathbf{K} := \mathbf{X} \times \mathbf{E}$ 

- 1: Define the absolute error  $r(\mathbf{x}, \mathbf{e}) := \hat{f}(\mathbf{x}, \mathbf{e}) f(\mathbf{x})$ 2: Compute  $l(\mathbf{x}, \mathbf{e}) := \sum_{j=1}^{m} \frac{\partial r(\mathbf{x}, \mathbf{e})}{\partial e_j}(\mathbf{x}, 0) e_j$  and h := r l
- 3: Compute an upper bound on  $h^*$ :  $\overline{h} := ia\_bound(h, \mathbf{K})$
- 4: Compute a lower bound on  $\bar{l}:\bar{l}_k:=\mathtt{sdp\_bound}(l,\mathbf{K},k)$
- 5: Compute an upper bound on  $\underline{l}$ :  $\underline{l}_k := -sdp\_bound(-l, \mathbf{K}, k)$
- 6: Compute a lower bound on  $l^*: l_k := \max\{-\underline{l}_k, \overline{l}_k\}$
- 7: **return**  $\max\{l_k \overline{h}, 0\}$

Fig. 1. fpsdp: our algorithm to compute lower bounds of absolute roundoff errors for polynomial programs.

Interval enclosures for bounds of roundoff errors. Let us consider a program implementing a polynomial function  $f(\mathbf{x})$  of type pexprC (with the above semantics), which depends on input variables  $\mathbf{x} := (x_1, \dots, x_n)$  constrained in a box, i.e. a product of closed (real) intervals  $\mathbf{X} := [\underline{x_1}, \overline{x_1}] \times \dots [\underline{x_n}, \overline{x_n}]$ . After rounding each coefficient and elementary operation involved in f, we obtain a polynomial rounded expression denoted by  $\hat{f}(\mathbf{x}, \mathbf{e})$ , which depends on the input variables  $\mathbf{x}$  as well as additional roundoff error variables  $\mathbf{e} :=$  $(e_1,\ldots,e_m)$ . Following (1), each variable  $e_i$  belongs to the interval  $[-\varepsilon,\varepsilon]$ , thus **e** belongs to  $\mathbf{E} := [-\varepsilon, \varepsilon]^m$ .

Here, we are interested in bounding from below the absolute roundoff error  $|r(\mathbf{x}, \mathbf{e})| :=$  $|\hat{f}(\mathbf{x}, \mathbf{e}) - f(\mathbf{x})|$  over all possible input variables  $\mathbf{x} \in \mathbf{X}$  and roundoff error variables  $\mathbf{e} \in \mathbf{E}$ . Let us define  $\mathbf{K} := \mathbf{X} \times \mathbf{E}$  and let  $r^*$  stands for the maximum of  $|r(\mathbf{x}, \mathbf{e})|$  over  $\mathbf{K}$ , that is  $r^* := \max_{(\mathbf{x}, \mathbf{e}) \in \mathbf{K}} |r(\mathbf{x}, \mathbf{e})|.$ 

Note that when  $\mathbf{e} = 0$  and  $\mathbf{x}$  corresponds to floating-point input, the value of  $|\hat{f}(\mathbf{x}, \mathbf{e})|$  $f(\mathbf{x})$  is zero, yielding the valid lower bound  $0 \le r^*$ . For instance, let us consider f(x) = x/2, for  $x \in [1,2]$ . When x is a floating-point number, this function has no roundoff error. However, higher lower bounds can be obtained when such conditions are not fulfilled, e.g. for non floating-point input values.

Following the same idea used in [Solovyev et al. 2015; Magron et al. 2016], we first decompose the error term r as the sum of a term  $l(\mathbf{x}, \mathbf{e})$ , which is linear w.r.t.  $\mathbf{e}$ , and a nonlinear term  $h(\mathbf{x}, \mathbf{e}) := r(\mathbf{x}, \mathbf{e}) - l(\mathbf{x}, \mathbf{e})$ . Then a valid lower bound on  $r^*$  can be derived by using the reverse triangular inequality:

$$r^* \ge \max_{(\mathbf{x}, \mathbf{e}) \in \mathbf{K}} |l(\mathbf{x}, \mathbf{e})| - \max_{(\mathbf{x}, \mathbf{e}) \in \mathbf{K}} |h(\mathbf{x}, \mathbf{e})| =: l^* - h^*$$
 (2)

We emphasize the fact that  $h^*$  is in general negligible compared to  $l^*$  since h contains products of error terms with degree at least 2 (such as  $e_i e_j$ ), thus can be bounded by  $O(\epsilon^2)$ . This bound is likely much smaller than the roundoff error induced by the linear term l. To compute a bound on  $h^*$ , it is enough in practice to compute second-order derivatives of r w.r.t. e then use Taylor-Lagrange inequality to get an interval enclosure of h as in Solovyev et al. 2015] or [Magron et al. 2016]. Doing so, one obtains an upper bound on  $h^*$ . In practice, we compute  $h^*$  with the implementation of ia bound available in the Real2Float software package.

Then, subtracting this upper bound to any lower bound on  $l^*$  yields a valid lower bound on  $r^*$  for the particular multiplicative rounding model which is considered here. Hence, from now on, we focus on approximating the bound  $l^*$  of the linear term. The framework [Magron et al. 2016 allows to obtain a hierarchy of converging upper bounds of  $l^*$  using SDP relaxations. By contrast with [Magron et al. 2016], our goal is to compute a hierarchy of

converging lower bounds on  $l^*$ . For the sake of clarity, we define  $\underline{l} := \min_{(\mathbf{x}, \mathbf{e}) \in \mathbf{K}} l(\mathbf{x}, \mathbf{e})$  and  $l := \max_{(\mathbf{x}, \mathbf{e}) \in \mathbf{K}} l(\mathbf{x}, \mathbf{e})$ . Computing  $l^*$  can then be cast as follows:

$$l^* := \max_{(\mathbf{x}, \mathbf{e}) \in \mathbf{K}} |l(\mathbf{x}, \mathbf{e})| = \max\{|\underline{l}|, |\overline{l}|\}.$$
(3)

Note that the computation of l can be formulated as a maximization problem since l := $\min_{(\mathbf{x},\mathbf{e})\in\mathbf{K}}l(\mathbf{x},\mathbf{e}) = -\max_{(\mathbf{x},\mathbf{e})\in\mathbf{K}}-l(\mathbf{x},\mathbf{e})$ . Thus, any method providing lower bounds on lcan also provide upper bounds on  $\underline{l}$ , eventually yielding lower bounds on  $l^*$ .

We now present our main fpsdp algorithm, given in Figure 1. This procedure is similar to the algorithm implemented in the upper bound tool Real2Float [Magron et al. 2016], except that we obtain lower bounds on absolute roundoff errors. Given a program implementing a polynomial f with input variables  $\mathbf{x}$  being constrained in the box  $\mathbf{X}$ , the fpsdp algorithm takes as input x, X, f, the rounded expression  $\hat{f}$  of f, the error variables e as well as the set **E** of bound constraints over **e**. The roundoff error  $r := \hat{f} - f$  (Line 1) is decomposed as the sum of a polynomial l which is linear w.r.t. the error variables e and a remainder h. As in [Magron et al. 2016; Solovyev et al. 2015], we obtain l by computing the partial derivatives of r w.r.t. e (Line 2). The computation of the upper bound on  $h^*$  (Line 3) is performed as explained earlier on, with the so-called procedure ia bound relying on basic interval arithmetic. Our algorithm also takes as input a sdp\_bound procedure, which computes lower bounds of the maximum of polynomials. In our case, we use sdp\_bound in Line 4 (resp. Line 5) to compute a lower (resp. upper) bound on  $\bar{l}$  (resp. l). Since zero is a valid lower bound for the largest absolute roundoff error, we return in Line 7 the maximal value between zero and the bound provided by sdp\_bound. This ensures that the fpsdp algorithm cannot return wrong results even if sdp\_bound returns bad error estimates.

In the sequel, we describe three possible instances of sdp\_bound: the first one relies on a hierarchy of generalized eigenvalue problems, the second one provides a hierarchy of bounds using only elementary computations and the third one is based on a hierarchy of semidefinite programming (SDP) relaxations. These three methods are described respectively in Section 2.2.1, Section 2.2.2 and Section 3. Each step of these hierarchies is indexed by an integer k, called relaxation order and given as input to fpsdp.

## 2.2. Existing Hierarchies of Lower Bounds for Polynomial Maximization

Here, we recall mandatory background explaining how to obtain hierarchies of lower bounds for a given polynomial maximization problem [Lasserre 2011]. Given  $p \in \mathbb{R}[\mathbf{y}]$  a multivariate polynomial in N variables  $y_1, \ldots, y_N$  and a box  $\mathbf{K} := [y_1, \overline{y_1}] \times \cdots \times [y_N, \overline{y_N}]$ , one considers the following polynomial maximization problem:

$$p^* := \max_{\mathbf{y} \in \mathbf{K}} p(\mathbf{y}). \tag{4}$$

The set of box constraints  $\mathbf{K} \subseteq \mathbb{R}^N$  is encoded by

$$\mathbf{K} := \{ \mathbf{y} \in \mathbb{R}^N : g_1(\mathbf{y}) \ge 0, \dots, g_N(\mathbf{y}) \ge 0 \} ,$$

for polynomials  $g_1 := (y_1 - \underline{y_1})(\overline{y_1} - y_1), \dots, g_N := (y_N - \underline{y_N})(\overline{y_N} - y_N).$ 

For a given vector of N nonnegative integers  $\alpha \in \mathbb{N}^N$ , we use the notation  $\mathbf{y}^{\alpha} := y_1^{\alpha_1} \cdots y_N^{\alpha_N}$  and  $|\alpha| := \sum_{i=1}^N \alpha_i$ . Any polynomial  $p \in \mathbb{R}[\mathbf{y}]$  of total degree at most k can then be written as  $p(\mathbf{y}) = \sum_{|\alpha| \le k} p_{\alpha} \mathbf{y}^{\alpha}$ . We write  $\mathbb{N}_k^N := \{\alpha \in \mathbb{N}^N : |\alpha| \le k\}$ . The cardinal of this set is equal to  $\binom{N+k}{k} = \frac{(N+k)!}{N!k!}$ .

We recall that a finite Borel measure  $\mu$  on  $\mathbb{R}^N$  is a nonnegative set function such that  $\mu(\emptyset) = 0$ ,  $\mu(\mathbb{R}^N)$  is finite, and  $\mu$  is countably sub-additive. The support of  $\mu$  is the smallest closed set  $\mathbf{K} \subseteq \mathbb{R}^N$  such that  $\mu(\mathbb{R}^N \backslash \mathbf{K}) = 0$  (see [Royden 1988] for more details).

Let  $\mu$  be a given finite Borel measure supported on  $\mathbf{K}$  and  $\mathbf{z}$  be the sequence of moments of  $\mu$ , given by  $z_{\alpha} := \int_{\mathbf{K}} \mathbf{y}^{\alpha} d\mu(\mathbf{y})$  for all  $\alpha \in \mathbb{N}^N$ . In some cases, one can explicitly compute  $z_{\alpha}$  for each  $\alpha \in \mathbb{N}^N$ . This includes the case when  $\mu$  is the uniform measure with density 1, i.e.  $d\mu(\mathbf{y}) = d\mathbf{y}$ , as  $\mathbf{K}$  is a product of closed intervals. For instance with N = 2,  $\mathbf{K} = [0, 1]^2$  and  $\alpha = (1, 0)$ , one has  $z_{1,0} = \int_{\mathbf{K}} y_1 d\mathbf{y} = \frac{1}{2}$ . With  $\alpha = (2, 1)$ , one has  $z_{2,1} = \int_{\mathbf{K}} y_1^2 y_2 d\mathbf{y} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ .

 $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}.$  Given a real sequence  $\mathbf{z} = (z_{\alpha})$ , we define the multivariate linear functional  $L_{\mathbf{z}} : \mathbb{R}[\mathbf{y}] \to \mathbb{R}$  by  $L_{\mathbf{z}}(p) := \sum_{\alpha} p_{\alpha} z_{\alpha}$ , for all  $p \in \mathbb{R}[\mathbf{y}]$ . For instance if  $p(\mathbf{y}) := y_1^2 y_2 + 3y_1 - \frac{2}{3}$ ,  $\mathbf{K} = [0, 1]^2$  then  $L_{\mathbf{z}}(p) = z_{2,1} + 3z_{1,0} - \frac{2}{3}z_{0,0} = \frac{1}{6} + \frac{3}{2} - \frac{2}{3} = 1$ .

Moment matrix. The moment matrix  $\mathbf{M}_k(\mathbf{z})$  is the real symmetric matrix with rows and columns indexed by  $\mathbb{N}_k^N$  associated with a sequence  $\mathbf{z} = (z_\alpha)$ , whose entries are defined by:

$$\mathbf{M}_k(\mathbf{z})(\beta, \gamma) := L_{\mathbf{z}}(\mathbf{y}^{\beta+\gamma}), \quad \forall \beta, \gamma \in \mathbb{N}_k^N.$$

We rely on the graded lexicographic order to compare the elements of  $\mathbb{N}_k^N$ . That is, we use the order which first compares the total degree (sum of all entries), and in case of a tie apply lexicographic order. For N=k=2, this gives (0,0)<(1,0)<(0,1)<(2,0)<(1,1)<(0,2).

Localizing matrix. The localizing matrix associated with a sequence  $\mathbf{z} = (z_{\alpha})$  and a polynomial  $p \in \mathbb{R}[\mathbf{y}]$  (with  $p(\mathbf{y}) = \sum_{\alpha} p_{\alpha} \mathbf{y}^{\alpha}$ ) is the real symmetric matrix  $\mathbf{M}_k(p\mathbf{z})$  with rows and columns indexed by  $\mathbb{N}_k^N$ , and whose entries are defined by:

$$\mathbf{M}_k(p\,\mathbf{z})(\beta,\gamma) := L_{\mathbf{z}}(p(\mathbf{y})\,\mathbf{y}^{\beta+\gamma}), \quad \forall \beta, \gamma \in \mathbb{N}_k^N.$$

The size of  $\mathbf{M}_k(p\mathbf{z})$  is equal to the cardinal of  $\mathbb{N}_k^N$ , i.e.  $\binom{N+k}{k}$ . Note that when p=1, one retrieves the moment matrix as a special case of localizing matrix.

Example 2.1. With  $p(\mathbf{y}) := y_1^2 y_2 + 3y_1 - \frac{2}{3}$ ,  $\mathbf{K} = [0,1]^2$  and k = 1, one has  $\mathbf{M}_1(\mathbf{z}) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \end{pmatrix}$  and  $\mathbf{M}_1(p\,\mathbf{z}) = \begin{pmatrix} 1 & \frac{19}{24} & \frac{19}{36} \\ \frac{19}{24} & \frac{1130}{52} & \frac{5}{12} \\ \frac{19}{36} & \frac{5}{2} & \frac{133}{36} \end{pmatrix}$ . Here the elements of  $\mathbb{N}_1^2$  indexing the rows and columns of both matrices are (0,0), (1,0) and (0,1), corresponding to the monomials  $1,y_1$  and  $y_2$ , respectively. For instance, the bottom-right corner of the localizing matrix  $\mathbf{M}_1(\mathbf{z})$  is obtained by computing  $L_{\mathbf{z}}(p(\mathbf{y})\,y_2^2) = z_{2,3} + 3z_{1,2} - \frac{2}{3}z_{0,2} = \frac{1}{12} + \frac{1}{2} - \frac{2}{3} \times \frac{1}{3} = \frac{13}{36}$ .

Next, we briefly recall two existing methods to compute lower bounds of  $p^*$  as defined in (4).

2.2.1. Hierarchies of generalized eigenvalue problems. Let us denote by  $\mathbb{R}^{n\times n}$  the vector space of  $n\times n$  real matrices. For a symmetric matrix  $\mathbf{M}\in\mathcal{M}_n(\mathbb{R})$ , the notation  $\mathbf{M}\succeq 0$  means that  $\mathbf{M}$  is semidefinite positive (SDP), i.e. has only nonnegative eigenvalues. The notation  $\mathbf{A}\succeq \mathbf{B}$  stands for  $\mathbf{A}-\mathbf{B}\succeq 0$ . A semidefinite optimization problem is an optimization problem where the cost is a linear function and the constraints state that some given matrices are semidefinite positive (see [Vandenberghe and Boyd 1994] for more details about SDP).

The following sequence of SDP programs can be derived from [Lasserre 2011], for each  $k \in \mathbb{N}$ :

$$\lambda_k(p) := \min_{\lambda} \quad \lambda$$
s.t.  $\lambda \mathbf{M}_k(\mathbf{z}) \succeq \mathbf{M}_k(p \mathbf{z}),$ 

$$\lambda \in \mathbb{R}.$$
(5)

The only variable of Problem (5) is  $\lambda$  together with a single SDP constraint of size  $\binom{N+k}{N}$ . This constraint can be rewritten as  $\mathbf{M}_k((\lambda-p)\mathbf{z}) \succeq 0$  by linearity of the localizing matrices. Solving Problem (5) allows to obtain a non-decreasing sequence of lower bounds which

converges to the global maximum  $p^*$  of the polynomial p. Problem (5) is a generalized eigenvalue problem. As mentioned in [de Klerk et al. 2015, Section 2.3], the computation of the number  $\lambda_k(p)$  requires at most  $O(\binom{N+k}{k}^3)$  floating-point operations (flops).

THEOREM 2.2. ([Lasserre 2011, Theorem 4.1]) For each  $k \in \mathbb{N}$ , Problem (5) admits an optimal solution  $\lambda_k(p)$ . Furthermore, the sequence  $(\lambda_k(p))$  is monotone non-decreasing and  $\lambda_k(p) \uparrow p^*$  as  $k \to \infty$ .

The convergence rate have been studied later on in [de Klerk et al. 2016], which states that  $p^* - \lambda_k(p) = O(\frac{1}{\sqrt{k}})$ .

Example 2.3. With  $p(\mathbf{y}) := y_1^2 y_2 + 3y_1 - \frac{2}{3}$ ,  $\mathbf{K} = [0,1]^2$ , we obtain the following sequence of lower bounds:  $\lambda_1(p) = 0.82 \le \lambda_2(p) = 1.43 \le \lambda_3(p) = 1.83 \le \cdots \le \lambda_{20}(p) = 2.72 \le p^* = \frac{10}{3}$ . The computation takes 16.2s on an Intel(R) Core(TM) i5-4590 CPU @ 3.30GHz. Here, we notice that the convergence to the maximal value  $p^*$  is slow in practice, confirming what the theory suggests.

2.2.2. Hierarchies of bounds using elementary computations. By contrast with the above method, further work by [de Klerk et al. 2015] provides a second method only requiring elementary computations. This method also yields a monotone non-decreasing sequence of lower bounds converging to the global maximum of a polynomial p while considering for each  $k \in \mathbb{N}$ :

$$p_k^H := \min_{(\eta,\beta) \in \mathbb{N}_{2k}^{2N}} \sum_{|\alpha| \le d} p_\alpha \frac{\gamma_{\eta + \alpha,\beta}}{\gamma_{\eta,\beta}} , \qquad (6)$$

where, for each  $(\eta, \beta) \in \mathbb{N}_{2k}^{2N}$  the scalar  $\gamma_{\eta,\beta}$  is the corresponding moment of the measure whose density is the multivariate beta distribution:

$$\gamma_{\eta,\beta} := \int_{\mathbf{K}} \mathbf{y}^{\eta} (\mathbf{1} - \mathbf{y})^{\beta} d\mathbf{y} = \int_{\mathbf{K}} y_1^{\eta_1} \cdots y_N^{\eta_N} (1 - y_1)^{\beta_1} \cdots (1 - y_N)^{\beta_N} d\mathbf{y}.$$
 (7)

As mentioned in [de Klerk et al. 2015, Section 2.3], the computation of the number  $p_k^H$  requires at most  $O\left(\binom{2N+2k-1}{2k}\right)$  floating-point operations (flops).

THEOREM 2.4. ( [de Klerk et al. 2015, Lemma 2.4, Theorem 3.1]) The sequence  $(p_k^H)$  is monotone non-decreasing and  $p_k^H \uparrow p^*$  as  $k \to \infty$ .

As for the sequence  $(\lambda_k(p))$ , the convergence rate is also in  $O(\frac{1}{\sqrt{k}})$  (see [de Klerk et al. 2015, Theorem 4.9]).

Example 2.5. With  $p(\mathbf{y}) := y_1^2 y_2 + 3y_1 - \frac{2}{3}$ ,  $\mathbf{K} = [0,1]^2$ , we obtain the following sequence of 20 lower bounds:  $p_1^H = 0.52 \le p_2^H = 0.95 \le p_3^H = 1.25 \le \cdots \le p_{20}^H = 2.42 \le p^\star = \frac{10}{3}$ . The computation takes 17.1s on the same machine as Example 2.3. For small order values ( $k \le 10$ ), this method happens to be more efficient than the one previously used in Example 2.3 but yields coarser bounds. At higher order ( $k \ge 10$ ), both methods happen to yield similar accuracy and performance with a slow rate of convergence. Note that performance could be improved in both cases by vectorizing our implementation code.

#### 3. A SPARSE SDP HIERARCHY FOR LOWER BOUNDS OF ROUNDOFF ERRORS

This section is dedicated to our main theoretical contribution, that is a new SDP hierarchy of converging lower bounds on the absolute roundoff error of polynomial programs. This hierarchy exploits the sparsity pattern occurring in the definition of l.

The two existing SDP hierarchies presented in Section 2.2 can be directly applied to solve Problem (3), that is the computation of lower bounds on  $l^* := \max_{(\mathbf{x}, \mathbf{e}) \in \mathbf{K}} |l(\mathbf{x}, \mathbf{e})|$ . In our case, N = n + m is the sum of the number of input and error variables, p = l and

 $\mathbf{y} = (\mathbf{x}, \mathbf{e}) \in \mathbf{K} = \mathbf{X} \times \mathbf{E}$ . At order k, a first relaxation procedure, denoted by geneig, returns the number  $\lambda_k(l)$  by solving Problem (5). A second relaxation procedure, denoted by mvbeta, returns the number  $l_k^H$  by solving Problem (6). In other words, this already gives two implementations geneig and mvbeta for the relaxation procedure sdp\_bound in the algorithm fpsdp presented in Figure 1.

However, these two procedures can be computationally demanding to get precise bounds for programs with larger number of variables, i.e. for either higher values of  $k~(\simeq 10)$  or  $N=n+m~(\simeq 50)$ . Experimental comparisons performed in Section 4 will support this claim. The design of a third implementation is motivated by the fact that both geneig and mvbeta do not take directly into account the special sparse structure of the polynomial l, that is the linearity w.r.t. e.

We first note that  $l(\mathbf{x}, \mathbf{e}) = \sum_{j=1}^{m} e_j s_j(\mathbf{x})$ , for polynomials  $s_1, \ldots, s_m \in \mathbb{R}[\mathbf{x}]$ . The maximization problem  $\bar{l} := \max_{(\mathbf{x}, \mathbf{e}) \in \mathbf{K}} l(\mathbf{x}, \mathbf{e})$  can then be written as follows:

$$\begin{split} \bar{l} := \min_{\lambda} \quad \lambda \\ \text{s.t.} \quad \lambda \ge \sum_{j=1}^{m} e_{j} \, s_{j}(\mathbf{x}) \,, \quad \forall \mathbf{x} \in \mathbf{X} \,, \forall \mathbf{e} \in \mathbf{E} \,, \\ \lambda \in \mathbb{R} \end{split}$$

From now on, we denote by  $(\mathbf{z}^{\mathbf{X}})$  the moment sequence associated with the uniform measure on  $\mathbf{X}$ . We first recall the following useful property of the localizing matrices associated to  $\mathbf{z}^{\mathbf{X}}$ :

PROPERTY 3.1. Let  $f \in \mathbb{R}[\mathbf{x}]$  be a polynomial. Then f is nonnegative over  $\mathbf{X}$  if and only if  $\mathbf{M}_k(f \mathbf{z}^{\mathbf{X}}) \succeq 0$ , for all  $k \in \mathbb{N}$ .

PROOF. This is a special case of [Lasserre 2011, Theorem 3.2 (a)] applied to the uniform measure supported on X with moment sequence  $z^X$ .  $\square$ 

In particular for f = 1, Property 3.1 states that the moment matrix  $\mathbf{M}_k(\mathbf{z}^{\mathbf{X}})$  is semidefinite positive, for all  $k \in \mathbb{N}$ . Let us now consider the following hierarchy of optimization programs, indexed by  $k \in \mathbb{N}$ :

$$\lambda'_{k}(l) := \min_{\lambda} \quad \lambda$$
s.t. 
$$\lambda \mathbf{M}_{k}(\mathbf{z}^{\mathbf{X}}) \succeq \sum_{j=1}^{m} e_{j} \mathbf{M}_{k}(s_{j} \mathbf{z}^{\mathbf{X}}), \quad \forall \mathbf{e} \in \mathbf{E},$$

$$\lambda \in \mathbb{R}.$$
(8)

Problem (8) is called a robust SDP program as it consists of minimizing the (worst-case) cost while satisfying SDP constraints for each possible value of the parameters  $\mathbf{e}$  within the box  $\mathbf{E}$ .

LEMMA 3.2. For each  $k \in \mathbb{N}$ , Problem (8) admits a finite optimal solution  $\lambda'_k(l)$ . Furthermore, the sequence  $(\lambda'_k(l))$  is monotone non-decreasing and  $\lambda'_k(l) \uparrow \bar{l}$  as  $k \to \infty$ .

PROOF. The proof is inspired from the one of [Lasserre 2011, Theorem 4.1] since Problem (8) is a robust variant of Problem (5).

First, let us define for all  $\mathbf{e} \in \mathbf{E}$  the polynomial  $l_{\mathbf{e}}(\mathbf{x}) := l(\mathbf{x}, \mathbf{e})$  in  $\mathbb{R}[\mathbf{x}]$ . The polynomial  $\bar{l} - l$  is nonnegative over  $\mathbf{X} \times \mathbf{E}$ , thus for all  $\mathbf{e} \in \mathbf{E}$ , the polynomial  $\bar{l} - l_{\mathbf{e}}$  is nonnegative over  $\mathbf{X}$ . By using Property 3.1, all localizing matrices of  $\bar{l} - l_{\mathbf{e}}$  are semidefinite positive. This yields  $\mathbf{M}_k((\bar{l} - l_{\mathbf{e}})\mathbf{z}^{\mathbf{X}}) \succeq 0$ , for all  $\mathbf{e} \in \mathbf{E}$ . By linearity of the localizing matrix, we get

 $\bar{l} \mathbf{M}_k(\mathbf{z}^{\mathbf{X}}) \succeq \sum_{j=1}^m e_j \mathbf{M}_k(s_j \mathbf{z}^{\mathbf{X}})$ , for all  $\mathbf{e} \in \mathbf{E}$ . For all  $k \in \mathbb{N}$ , this proves that  $\bar{l}$  is feasible for Problem (8) and  $\lambda_k'(l) \leq \bar{l}$ . Next, let us fix  $k \in \mathbb{N}$  and an arbitrary feasible solution  $\lambda$  for Problem (8). Since for all  $\mathbf{e} \in \mathbf{E}$ , one has  $\mathbf{M}_k((\lambda - l_{\mathbf{e}})\mathbf{z}^{\mathbf{X}}) \succeq 0$ , this is in particular the case for  $\mathbf{e} = 0$ , which yields  $\lambda \mathbf{M}_k(\mathbf{z}^{\mathbf{X}}) \succeq 0$ . Since the moment matrix  $\mathbf{M}_k(\mathbf{z}^{\mathbf{X}})$  is semidefinite positive, one has  $\lambda \geq 0$ . Thus the feasible set of Problem (8) is nonempty and bounded, which proves the existence of a finite optimal solution  $\lambda_k'(l)$ .

Next, let us fix  $k \in \mathbb{N}$ . For all  $\mathbf{e} \in \mathbf{E}$ ,  $\mathbf{M}_k((\lambda - l_{\mathbf{e}})\mathbf{z}^{\mathbf{X}})$  is a sub-matrix of  $\mathbf{M}_{k+1}((\lambda - l_{\mathbf{e}})\mathbf{z}^{\mathbf{X}})$ , thus  $\mathbf{M}_{k+1}((\lambda - l_{\mathbf{e}})\mathbf{z}^{\mathbf{X}}) \succeq 0$  implies that  $\mathbf{M}_k((\lambda - l_{\mathbf{e}})\mathbf{z}^{\mathbf{X}}) \succeq 0$ , yielding  $\lambda_k'(l) \leq \lambda_{k+1}'(l)$ . Hence, the sequence  $(\lambda_k'(l))$  is monotone non-decreasing. Since for all  $k \in \mathbb{N}$ ,  $\lambda_k'(l) \leq \overline{l}$ , one has  $(\lambda_k'(l))$  converges to  $\lambda'(l) \leq \overline{l}$  as  $k \to \infty$ . For all  $\mathbf{e} \in \mathbf{E}$ , for all  $k \in \mathbb{N}$ , one has  $\mathbf{M}_k((\lambda_k'(l) - l_{\mathbf{e}})\mathbf{z}^{\mathbf{X}}) \succeq \mathbf{M}_k((\lambda_k'(l) - l_{\mathbf{e}})\mathbf{z}^{\mathbf{X}}) \succeq 0$ . By using again Property 3.1, this shows that for all  $\mathbf{e} \in \mathbf{E}$ , the polynomial  $\lambda'(l) - l_{\mathbf{e}}$  is nonnegative over  $\mathbf{X}$ , yielding  $\lambda'(l) \geq \overline{l}$  and the desired result  $\lambda'(l) = \overline{l}$ .  $\square$ 

Next, we use the framework developed in [Ghaoui et al. 1998] to prove that for all  $k \in \mathbb{N}$ , Problem (8) is equivalent to the following SDP involving the additional real variable  $\tau$ :

$$\lambda_{k}''(l) := \min_{\lambda, \tau} \quad \lambda$$
s.t. 
$$\begin{pmatrix} \lambda \mathbf{M}_{k}(\mathbf{z}^{\mathbf{X}}) - \tau \mathbf{L}_{k} \mathbf{L}_{k}^{\mathrm{T}} & \mathbf{R}_{k}^{\mathrm{T}} \\ \mathbf{R}_{k} & \tau \mathbf{I} \end{pmatrix} \succeq 0,$$

$$\lambda, \tau \in \mathbb{R}.$$
(9)

Both matrices  $\mathbf{L}_k = [\mathbf{L}_k^1 \cdots \mathbf{L}_k^m]$  and  $\mathbf{R}_k = [\mathbf{R}_k^1 \dots \mathbf{R}_k^m]^{\mathrm{T}}$  are obtained by performing a full rank factorization of the localizing matrix  $\mathbf{M}_k(s_j \mathbf{z}^{\mathbf{X}})$  for each  $j = 1, \dots, m$ . This can be done e.g. with the PLDL<sup>T</sup>P<sup>T</sup> decomposition [Golub and Van Loan 1996, Section 4.2.9] and is equivalent to finding two matrices  $\mathbf{L}_k^j$  and  $\mathbf{R}_k^j$  such that  $\mathbf{M}_k(s_j \mathbf{z}^{\mathbf{X}}) = 2\mathbf{L}_k^j \mathbf{R}_k^j$ . For the sake of clarity we use the notations  $\mathbf{L}_k$  and  $\mathbf{R}_k$  while omitting the dependency of both matrices w.r.t.  $\mathbf{z}^{\mathbf{X}}$ .

For each  $j=1,\ldots,m$ , the matrix  $\mathbf{L}_k^j$  (resp.  $\mathbf{R}_k^j$ ) has the same number of lines (resp. columns) as  $\mathbf{M}_k(s_j\mathbf{z}^{\mathbf{X}})$ , i.e.  $\binom{n+k}{k}$ , and the same number of columns (resp. lines) as the rank  $r_j$  of  $\mathbf{M}_k(s_j\mathbf{z}^{\mathbf{X}})$ . The size of the identity matrix  $\mathbf{I}$  is  $m\binom{n+k}{k}$ .

THEOREM 3.3. For each  $k \in \mathbb{N}$ , Problem (9) admits a finite optimal solution  $\lambda_k''(l) = \lambda_k'(l)$ , where  $\lambda_k'(l)$  is the solution of Problem (8). Furthermore, the sequence  $\lambda_k''(l)$  is monotone non-decreasing and  $\lambda_k''(l) \uparrow \bar{l}$  as  $k \to \infty$ .

PROOF. It is enough to prove the equivalence between Problem (9) and Problem (8) since then the result follows directly from Lemma 3.2. Let us note  $\mathbf{0} := (0, \dots, 0) \in \mathbb{R}^n$ . Problem (8) can be cast as Problem (4) in [Ghaoui et al. 1998] with  $x = (\lambda, \mathbf{0})$ ,  $F(x) = \lambda \mathbf{M}_k(\mathbf{z}^{\mathbf{X}})$ ,  $\Delta = \operatorname{diag}(0, \mathbf{e})$ ,  $c = (1, \mathbf{0})$ ,  $\mathcal{D} = \mathbb{R}^{(m+1)\times(m+1)}$ ,  $\rho = 1$ .

In addition, the robust SDP constraint of Problem(8) can be rewritten as the linear matrix inequality (8) in Section 3.1 of [Ghaoui et al. 1998], i.e.  $F + L \Delta (I - D \Delta)^{-1} R + R^{T} \Delta (I - D \Delta)^{-T} L^{T} \succeq 0$ , with  $L = \mathbf{L}_{k}$ ,  $R = \mathbf{R}_{k}$  and D = 0. Then, the desired equivalence result follows from [Ghaoui et al. 1998, Theorem 3.1].  $\square$ 

This procedure provides a third choice, called robsdp, for the relaxation procedure sdp\_bound in the algorithm fpsdp presented in Figure 1.

Computational considerations. As for the geneig procedure, one also obtains the convergence rate  $\bar{l} - \lambda_k''(l) = O(\frac{1}{\sqrt{k}})$ . However, the resolution cost of Problem (9) can be smaller.

Indeed, from [Golub and Van Loan 1996, Section 4.2.9], the cost of each full rank factorization is cubic in each localizing matrix size, yielding a total factorization cost of  $O\left(m\binom{n+k}{k}^3\right)$  flops. From [Nesterov and Nemirovskii 1994, Section 11.3] the SDP solving cost is proportional to the cube of the matrix size, yielding  $O\left(m^3\binom{n+k}{k}^3\right)$  flops for Problem (9). Hence, the overall cost of the robsdp procedure is bounded by  $O\left(m^3\binom{n+k}{k}^3\right)$  flops. This is in contrast with the cost of  $O\left(\binom{n+m+k}{k}^3\right)$  flops for geneig as well as the cost of  $O\left(\binom{2n+2m+2k-1}{2k}\right)$  flops for mybeta. In the sequel, we compare these expected costs for several values of n, m and k.

#### 4. EXPERIMENTAL RESULTS AND DISCUSSION

Now, we present experimental results obtained by applying our algorithm fpsdp (see Figure 1) with the three relaxation procedures geneig, mvbeta and robsdp to various examples coming from physics, biology, space control, and optimization. The procedures geneig, mvbeta and robsdp provide lower bounds of a polynomial maximum by solving Problem (5), Problem (6) and Problem (9), respectively. The fpsdp algorithm is implemented as a software package written in MATLAB, called FPSDP. Setup and usage of FPSDP are described on the dedicated web-page<sup>2</sup> with specific instructions<sup>3</sup>. The three procedures are implemented using YALMIP [Löfberg 2004] which is a toolbox for advanced modeling and solution of (non-)convex optimization problems in MATLAB.

The only procedure which requires an SDP solver is robsdp. Indeed, geneig requires to solve a particular SDP instance which can be handled with a generalized eigenvalue solver. In practice, geneig relies on the function eig from MATLAB to solve generalized eigenvalue problems. Full rank factorization within the robsdp procedure is performed with the function rref from MATLAB.

For solving SDP problems, we rely on Mosek 7.0 [Andersen and Andersen 2000]. For more details about the installation, usage and setup instructions of Yalmip and Mosek, we refer to the respective dedicated web-pages  $^{4}$   $^{5}$ .

Note that the solution of Problem (9) is computed with a numerical solver implemented with finite-precision. Hence, we have to check the bounds obtained with Mosek by verifying that the matrix involed in (9) has nonnegative eigenvalues. We compute the eigenvalues of the matrices provided by Mosek with eig. If the minimal eigenvalue returned by eig is negative, then we add it to the lower bound and round the result towards  $-\infty$ . We emphasize that this procedure has an insignificant impact on the value of lower bounds computed for all benchmarks. To perform rigorous checking of the positive definitess of the matrices, one could rely e.g. on verified floating-point Cholesky's decomposition, as proposed in [Siegfried M. Rump 2006], but this is beyond the scope of this paper.

## 4.1. Benchmark Presentation

All examples are displayed in the appendix. Our results have been obtained on an Intel(R) Core(TM) i5-4590 CPU @ 3.30GHz. For the sake of further presentation, we associate an alphabet character (from a to i) to identify each of the 9 polynomial nonlinear programs which implement polynomial functions: a and b come from physics, c, d and e are derived from expressions involved in the proof of Kepler Conjecture [Hales 2006] and f, g and h implement polynomial approximations of the sine and square root functions. All programs are used for similar upper bound comparison in [Magron et al. 2016, Section 4.1]. Each

<sup>&</sup>lt;sup>2</sup>https://github.com/magronv/FPSDP

<sup>&</sup>lt;sup>3</sup>see the README.md file in the top level directory

<sup>&</sup>lt;sup>4</sup>https://docs.mosek.com/7.0/toolbox/index.html

<sup>&</sup>lt;sup>5</sup>http://users.isv.liu.se/johanl/valmip/pmwiki.php?n=Main.WhatIsYALMIP

program implements a polynomial f with n input variables  $\mathbf{x} \in \mathbf{X}$  and yields after rounding m error variables  $\mathbf{e} \in \mathbf{E} = [-\varepsilon, \varepsilon]^m$ .

Example 4.1. The program c (see the appendix) implements the polynomial expression

$$f(\mathbf{x}) := x_2 \times x_5 + x_3 \times x_6 - x_2 \times x_3 - x_5 \times x_6 + x_1 \times (-x_1 + x_2 + x_3 - x_4 + x_5 + x_6),$$

and the program input is the six-variable vector  $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$ . The set  $\mathbf{X}$  of possible input values is a product of closed intervals:  $\mathbf{X} = [4.00, 6.36]^6$ . The polynomial f is obtained by performing 15 basic operations (1 negation, 3 subtractions, 6 additions and 5 multiplications). When executing this program with a set  $\hat{\mathbf{x}}$  of floating-point numbers defined by  $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5, \hat{x}_6) \in \mathbf{X}$ , one obtains the floating-point result  $\hat{f}$ . The error variables are  $e_1, \ldots, e_{21} \in [-\varepsilon, \varepsilon]$  and  $\mathbf{E} = [-\varepsilon, \varepsilon]^{21}$ .

For the sake of conciseness, we only considered to compare the performance of FPSDP on programs implemented in double ( $\varepsilon = 2^{-53}$ ) precision floating point. We use two preprocessing features embedded in the Real2Float software package: first, we rely on the parser of Real2Float to get the expression of the linear part l of the roundoff error  $|f(\mathbf{x}) - \hat{f}(\mathbf{x}, \mathbf{e})| = |l(\mathbf{x}, \mathbf{e}) + h(\mathbf{x}, \mathbf{e})|$ . Second, we compute an interval enclosure  $h^*$  of h with the sub-rountine ia bound available in Real2Float (as recalled in Section 2.1).

Table II compares expected magnitudes of flop counts for geneig, mvbeta and robsdp, following from the study at the end of Section 3. For each program, we show the cost for the initial relaxation order k=1 as well as for the highest one used for error computation in Table I. The results indicate that we can expect the procedure geneig to be more efficient at low relaxation orders while being outperformed by robsdp at higher orders. Besides, the mvbeta procedure is likely to have performance lying in between the two others. We mention that the interested reader can find more detailed experimental comparisons between the two relaxation procedures geneig and mvbeta in [de Klerk et al. 2015].

#### 4.2. Numerical Evaluation

For each benchmark, Table I displays the quality of the roundoff error bounds with corresponding execution times. We emphasize that our FPSDP tool relies on the simple rounding model described in Section 2.1

Note that a head-to-head comparison between the three SDP relaxation procedures and s3fp [Chiang et al. 2014] would be more difficult. Indeed, s3fp relies on several possible heuristic search algorithms and measures output errors after executing programs written in C++ with certain input values. The rounding occurring while executing such programs is more likely to fit with an improved model, based for instance on a piecewise constant absolute error bound (see e.g. [Magron et al. 2016, Section 1.2] for more explanation about such models).

We also compare the three procedures based on SDP with the competitive NLopt [Steven G. Johnson 2008] software<sup>6</sup>. Since  $l(\mathbf{x}, \mathbf{e}) = \sum_{j=1}^m s_j(\mathbf{x})e_j$ , the maximal absolute value of l on  $\mathbf{X} \times [-\varepsilon, \varepsilon]^m$  is equal to the maximum of the function  $a(\mathbf{x}) := \varepsilon \sum_{j=1}^m |s_j(\mathbf{x})|$  on  $\mathbf{X}$ . For the sake of efficiency, we execute NLopt on a with the optimization algorithm NLOPT\_GN\_DIRECT, which is derivative-free so that it can optimize functions with absolute values. The results returned by NLopt are not necessarily upper nor lower bounds of a but when the solver stops, it returns a point  $\mathbf{x}^* \in \mathbf{X}$ . Then we obtain an interval enclosure of  $a(\mathbf{x}^*)$  in MATLAB which is a valid lower bound of a, thus a valid lower bound of |l|. The lower bounds on the absolute error are obtained as in [Darulova and Kuncak 2014], by executing each program on several random inputs satisfying the input restrictions. For

<sup>&</sup>lt;sup>6</sup>http://nlopt.readthedocs.io/en/latest

Table I. Comparison results of lower bounds and execution times (in seconds) for largest absolute roundoff errors among geneig, mvbeta, robsdp and NLopt.

| id | k             | geneig               |                     | mvbet                        | a            | robsdp               |                     | NLopt           | lower     | upper    |  |
|----|---------------|----------------------|---------------------|------------------------------|--------------|----------------------|---------------------|-----------------|-----------|----------|--|
|    |               | bound                | $_{ m time}$        | bound                        | $_{ m time}$ | bound                | time                | bound           | bound     | bound    |  |
| a  | 1             | 1.05e-15             | 0.63                | $1.85e{-}16$                 | 0.31         | 3.30e-14             | 0.75                |                 |           |          |  |
|    | 2             | 2.84e-14             | 1.69                | $4.07\mathrm{e}{-15}$        | 20.9         | 7.52e-14             | 0.79                |                 |           |          |  |
|    | 3             | 5.83e-14             | 29.7                | $8.88e{-15}$                 | 645.         | 1.10e-13             | 1.06                | 4.80e-13        | 2.28e-13  | 5.33e-13 |  |
|    | 4             | 8.72e-14             | >1e4                | 1.73e-14                     | >1e5         | 1.62e-13             | 2.09                |                 |           |          |  |
|    | 8             | _                    | _                   | _                            | _            | $3.55e{-}13$         | 164.                |                 |           |          |  |
| b  | 1             | 8.40e-14             | 1.10                | 4.99e-14                     | 1.83         | 3.56e-12             | 0.31                |                 |           |          |  |
|    | 2             | 1.31e-12             | 2.75                | $2.41e{-13}$                 | 226.         | 5.31e-12             | 0.42                |                 |           |          |  |
|    | 3             | 2.89e-12             | 288.                | 5.08e-13                     | >1e5         | 8.04e-12             | 1.04                | 6.40e-11        | 2.19e-11  | 6.48e-11 |  |
|    | 4             | _                    | _                   | _                            | _            | 1.13e-11             | 4.11                |                 |           |          |  |
|    | 7             | -                    | _                   | _                            | _            | 2.60e-11             | 152.                |                 |           |          |  |
|    | 1             | 9.45e-15             | 1.25                | $3.95e{-}15$                 | 4.34         | 9.68e-15             | 0.73                |                 |           |          |  |
| С  | 2             | 1.64e-14             | 37.3                | $7.89e{-15}$                 | >1e4         | 1.48e-14             | 6.69                | 1.02e-13        | 2.23e–14  | 1.18e–13 |  |
|    | 3             | _                    | _                   | _                            | _            | 2.62e-14             | 172.                | 1.020 10        |           |          |  |
|    | 4             | _                    | _                   | _                            | _            | _                    |                     |                 |           |          |  |
| d  | 1             | 3.01e-14             | 1.63                | $1.41e{-14}$                 | 13.6         | 1.49e-13             | 1.67                |                 |           |          |  |
|    | 2             | 5.38e-14             | 163.                | 2.45e-14                     | >4e4         | 2.22e-13             | 3.73                | 3.93e-13        | 7.58e-14  | 4.47e-13 |  |
|    | 3             | _                    | _                   | _                            | _            | 3.04e-13             | 33.3                | 0.000 10        | 1.000 11  | 1.110 10 |  |
|    | 4             |                      | _                   |                              | _            | 4.06e-13             | 275.                |                 |           |          |  |
|    | 1             | 9.72e-28             | 5.74                | 5.55e-14                     | 86.1         | 2.88e-13             | 2.53                |                 |           |          |  |
| е  | 2             | _                    | _                   | _                            | _            | 4.48e-13             | 55.2                | 2.01e-12        | 3.03e-13  | 2.09e-12 |  |
|    | 3             | _                    | _                   | _                            | _            | _                    | _                   | 2.010 12        | 0.000 10  | 2.000 12 |  |
|    | 4             | _                    | _                   | _                            | _            | _                    | _                   |                 |           |          |  |
|    | 1             | 8.34e-17             | 0.89                | 1.50e-17                     | 0.59         | 1.98e-16             | 1.28                |                 |           |          |  |
| _  | 2             | 1.52e–16             | 2.24                | 4.50e-17                     | 45.9         | 2.31e-16             | 1.29                | <b>* *</b> 0 10 | 4 45 10   | 0.00 10  |  |
| f  | 3             | 2.07e-16             | 65.5                | 7.95e-17                     | > 1e4        | 2.72e-16             | 1.30                | $5.50e{-}16$    | 4.45e–16  | 6.03e-16 |  |
|    | 4             | _                    | _                   | _                            | _            | 3.04e-16             | 1.30                |                 |           |          |  |
|    | 8             | -                    | _                   |                              | _            | 4.43e-16             | 1.40                |                 |           |          |  |
|    | 1             | 1.09e-16             | 0.33                | 4.93e-17                     | 0.04         | 3.99e-16             | 1.22                |                 |           |          |  |
|    | 2             | 2.43e-16             | 0.97                | 1.18e-16                     | 0.94         | 4.83e-16             | 1.22                |                 |           |          |  |
| g  | 3             | 3.68e-16             | 1.71                | 1.78e–16                     | 10.6         | 5.62e-16             | 1.23                | 1.00e-15        | 3.34e-16  | 1.19e-15 |  |
| Ū  | 4             | 4.72e–16             | 7.21                | 2.33e-16                     | 79.9         | 6.37e-16             | 1.24                |                 |           |          |  |
|    | 6             | 6.28e-16             | 646.                | 2.87e-16                     | > 1e4        | 7.85e-16             | 1.25                |                 |           |          |  |
|    | 8             | 2.30e-16             | $\frac{-}{1.52}$    | -<br>1.29e-16                | 0.28         | 9.30e-16             | 1.28                |                 |           |          |  |
| h  | 1<br>2        | 2.30e-16<br>4.00e-16 |                     |                              |              | 4.83e-16             | 1.34                |                 |           |          |  |
|    | 3             |                      | 2.87 $161$ .        | $2.55e{-}16$<br>$3.28e{-}16$ | 28.4         | 5.40e–16             | 1.35                | 7.10e–16        | 4 45 - 10 | 1.00- 15 |  |
|    | 3<br>4        | 5.36e–16             |                     |                              | > 1e4        | 5.78e–16<br>6.13e–16 | 1.39                | 1.10e-10        | 4.45e–16  | 1.29e–15 |  |
|    | $\frac{4}{7}$ | _                    | _                   | _                            | _            |                      | $\frac{1.40}{1.50}$ |                 |           |          |  |
|    | 1             | 3.07e–14             | 0.58                | -<br>1.60e-14                | 0.61         | 7.00e–16<br>1.56e–13 | 1.32                |                 |           |          |  |
| i  | $\frac{1}{2}$ | 6.71e-14             | $\frac{0.58}{2.48}$ | 1.60e-14<br>2.68e-14         | 42.5         | 1.50e-13<br>2.13e-13 | 1.32 $1.39$         |                 |           |          |  |
|    | 3             | 0.71e-14<br>1.25e-13 | $\frac{2.48}{47.1}$ | 3.72e-14                     | >1e4         | 2.13e-13<br>2.84e-13 | 1.39 $1.43$         | 1.42e-12        | 1.47e-13  | 1.43e-12 |  |
|    | 3<br>4        | 1.25e-15<br>1.92e-13 | >1e4                | 5.72e-14<br>5.35e-14         | >1e4<br>>2e5 | 3.63e-13             | 1.43 $1.62$         | 1.42e-12        | 1.476-13  | 1.45e-12 |  |
|    | 8             | 1.92e-13             | >1e4                | 5.55e-14                     | >2eo<br>_    |                      |                     |                 |           |          |  |
|    | 0             | _                    |                     |                              |              | 7.67e-13             | 7.34                |                 |           |          |  |

comparison purpose, we also provide the best known upper bounds computed with SDP from [Magron et al. 2016, Table II].

As shown in Table I, the robsdp procedure is the most accurate among our three hierarchies and provides the tightest enclosure bounds for all programs. The NLopt tool is always faster than all other methods. Our robsdp procedure is more precise than NLopt for the benchmark d. Even though NLopt is the most accurate procedure for all other benchmarks,

| mvbeta and robsdp. |          |   |    |   |            |            |            |   |  |
|--------------------|----------|---|----|---|------------|------------|------------|---|--|
| Benchmark          | id       | n | m  | k | geneig     | mvbeta     | robsdp     | _ |  |
| mi mi dDodre1      |          | 9 | 10 | 1 | 2.75e + 03 |            | 6.40e + 04 |   |  |
| 1 1g 1dbody 1      | dBody1 a | J | 10 | 8 | 8.43e + 15 | 1.04e + 12 | 4.50e + 09 |   |  |

Table II. Expected magnitudes of flops counts for the three procedures geneig,

| Benchmark   | id | n | m  | k | geneig     | mvbeta       | robsdp       |  |
|-------------|----|---|----|---|------------|--------------|--------------|--|
| rigidBody1  | a  | 3 | 10 | 1 | 2.75e + 03 | 3.50e + 03   | 6.40e + 04   |  |
| тідіавоауі  | а  |   |    | 8 | 8.43e + 15 | 1.04e + 12   | 4.50e + 09   |  |
| rigidBody2  | b  | 3 | 15 | 1 | 6.86e + 03 | 9.99e + 03   | 2.16e + 05   |  |
| 11g1db0dy2  |    |   |    | 7 | 1.12e + 17 | 1.02e + 13   | 5.84e + 09   |  |
| kepler0     | С  | 6 | 21 | 1 | 2.20e+04   | 3.12e+04     | 3.18e + 06   |  |
| Kebielo     |    |   |    | 4 | 3.12e+13   | 6.19e + 10   | $8.58e{+10}$ |  |
| kepler1     | d  | 4 | 28 | 1 | 3.60e+04   | 5.83e + 04   | 2.75e + 06   |  |
| Kebieii     |    |   |    | 4 | 2.05e+14   | $2.98e{+11}$ | 7.53e + 09   |  |
| kepler2     | e  | 6 | 42 | 1 | 1.18e + 05 | 1.96e + 05   | 2.55e + 07   |  |
| kebiei z    | е  |   |    | 4 | 1.99e + 16 | 9.99e + 12   | 6.87e + 11   |  |
| sineTaylor  | f  | 1 | 13 | 1 | 3.38e+03   | 5.28e + 03   | 1.76e + 04   |  |
| Sillerayion |    |   |    | 8 | 3.27e + 16 | 3.45e + 12   | 1.61e + 06   |  |
| sineOrder3  | ~  | 1 | 6  | 1 | 5.12e+02   | 6.30e + 02   | 1.73e + 03   |  |
| Silleorders | g  | 1 |    | 8 | 2.67e + 11 | 4.08e + 08   | 1.58e + 05   |  |
| aamaat      | h  | 1 | 15 | 1 | 4.92e + 03 | 7.92e + 03   | 2.70e + 04   |  |
| sqroot      | 11 |   |    | 7 | 1.48e + 16 | $2.51e{+12}$ | 1.73e + 06   |  |
| himmilbeau  | i  | 2 | 11 | 1 | 2.75e + 03 | 3.87e + 03   | 3.60e + 04   |  |
| nimiiibeau  | 1  |   |    | 8 | 8.43e + 15 | $1.14e{+12}$ | 1.22e + 08   |  |

robsdp often provides similar bounds. In particular, robsdp is respectively 7 % and 2 % less precise than NLopt for the benchmarks g and h.

For relaxations order greater than 2, robsdp is faster than the two other hierarchies for all programs. Except for program c, either geneig or mvbeta yields better performance at the first relaxation order. The symbol "—" in a column entry means that we aborted the execution of the corresponding procedure after running more than 1e6 seconds. Note that such a behavior systematically occurs when analyzing programs f-i with geneig and mvbeta at maximal relaxation orders. This confirms the expectation results from Table II, as robsdp yields more tractable SDP relaxations. Note that for these benchmarks, we performed experiments for each intermediate order k between 4 and the maximal indicated one. For conciseness, we have not displayed all intermediate results in the table but use them later on in Figure 2. One way to increase the performance of the FPSDP tool would be to vectorize the current code which creates moment/localizing matrices, instead of writing loop-based code.

The purpose of Figure 2 is to emphasize the ability of FPSDP to make a compromise between accuracy and performance. All program results (except e due to the lack of experimental data) are reported in Figure 2. Each value of k corresponds to a circled integer point. For each experiment, we define the three execution times  $t_{geneig}$ ,  $t_{mvbeta}$  and  $t_{robsdp}$ and the minimum t among the three values. The x-axis coordinate of the circled point is  $\ln(\frac{t_{\text{geneig}}}{t})$  for the geneig procedure (and similarly for the other procedures). The corresponding lower bounds are denoted by  $\varepsilon_{geneig}$ ,  $\varepsilon_{mvbeta}$  and  $\varepsilon_{robsdp}$ . With  $\overline{\varepsilon}$  being the reference upper bound, the y-axis coordinate of the circled point is the relative error gap for geneig, i.e.  $r_{\text{geneig}} := 1 - \frac{\hat{\varepsilon}_{\text{geneig}}}{\bar{\varepsilon}}$  and similarly for the other procedures.

For each k, the relative location of the corresponding circled integers indicate which procedure either performs better (by being faster) or provides more accurate bounds. In particular for i, robsdp is less efficient than the two other procedures for first relaxation orders (relative execution time less than 1.5) then outperforms the other procedures. We also observe that mvbeta is more efficient at low relaxation orders for programs a and g-i as well as geneig for programs c-d, f and i. The procedure geneig is always more accurate than mvbeta.

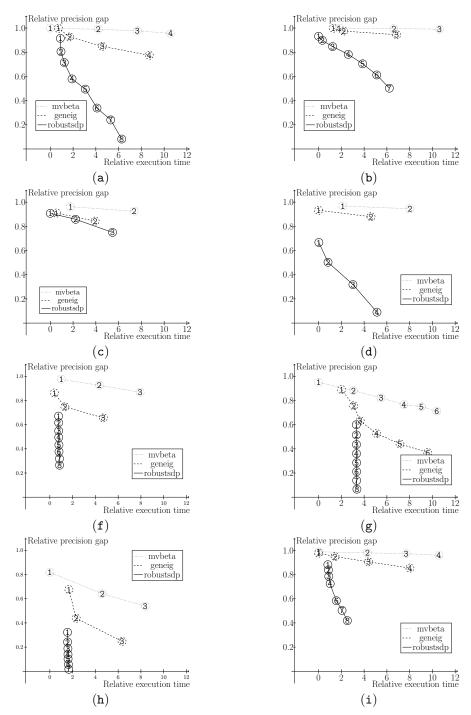


Fig. 2. Relative gap and execution time results for benchmarks.

#### 5. CONCLUSION AND PERSPECTIVES

We have presented three procedures, respectively based on hierarchies of generalized eigenvalue problems, elementary computations and semidefinite programming (SDP) relaxations. These three methods allow to compute lower bounds of roundoff errors for programs implementing polynomials with input variables being box constrained. While the first two procedures are direct applications of existing methods in the context of polynomial optimization, the third one relies on a new hierarchy of robust SDP relaxations, allowing to tackle specifically the roundoff error problem. Experimental results obtained with our FPSDP tool, implementing these three procedures, prove that SDP relaxations are able to provide accurate lower bounds in an efficient way.

A first direction of further research is the extension of the SDP relaxation framework to programs implementing non-polynomial functions, with either finite or infinite loops. This requires to derive a hierarchy of inner converging SDP approximations for reachable sets of discrete-time polynomial systems in either finite or infinite horizon. Another topic of interest is the formal verification of lower bounds with a proof assistant such as Coq [Coq 2016]. To achieve this goal, we could benefit from recent formal libraries [Dénès et al. 2012] in computational algebra.

#### APPENDIX: POLYNOMIAL PROGRAM BENCHMARKS

- a rigibody1 :  $(x_1,x_2,x_3)\mapsto -x_1x_2-2x_2x_3-x_1-x_3$  defined on  $[-15,15]^3$ . b rigibody2 :  $(x_1,x_2,x_3)\mapsto 2x_1x_2x_3+6x_3^2-x_2^2x_1x_3-x_2$  defined on  $[-15,15]^3$ . c kepler0 :  $(x_1,x_2,x_3,x_4,x_5,x_6)\mapsto x_2x_5+x_3x_6-x_2x_3-x_5x_6+x_1(-x_1+x_2+x_3-x_4+x_5))=0$  $x_5 + x_6$ ) defined on  $[4, 6.36]^6$ .
- d kepler1:  $(x_1, x_2, x_3, x_4) \mapsto x_1 x_4 (-x_1 + x_2 + x_3 x_4) + x_2 (x_1 x_2 + x_3 + x_4) + x_3 (x_1 + x_2 x_3 + x_4) x_2 x_3 x_4 x_1 x_3 x_1 x_2 x_4$  defined on  $[4, 6.36]^4$ . e kepler2:  $(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto x_1 x_4 (-x_1 + x_2 + x_3 x_4 + x_5 + x_6) + x_2 x_5 (x_1 x_2 + x_3 + x_4 x_5 + x_6) + x_3 x_6 (x_1 + x_2 x_3 + x_4 + x_5 x_6) x_2 x_3 x_4 x_1 x_3 x_5 x_1 x_2 x_6 x_4 x_5 x_6$ defined on  $[4, 6.36]^6$ .
- f sineTaylor :  $x \mapsto x \frac{x^3}{6.0} + \frac{x^5}{120.0} \frac{x^7}{5040.0}$  defined on  $[-\frac{\tilde{x}}{2}, \frac{\tilde{x}}{2}]$ , with  $\frac{\tilde{x}}{2} := 1.57079632679$ . g sineOrder3 :  $x \mapsto 0.954929658551372x 0.12900613773279798x^3$  defined on [-2, 2]. h sqroot :  $x \mapsto 1.0 + 0.5x 0.125x^2 + 0.0625x^3 0.0390625x^4$  defined on [0, 1]. i himmilbeau :  $(x_1, x_2) \mapsto (x_1^2 + x_2 11)^2 + (x_1 + x_2^2 7)^2$  defined on  $[-5, 5]^2$ .

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