A Note on the Midpoint Method of Integration*

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When integrating numerically systems of ordinary differential equations of the standard form

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1}$$

by difference methods, probably the simplest stable procedure that can be used is Euler's method [1]:

$$y_{n+1} = y_n + hy'_n, \qquad h = x_{n-1} - x_n. \tag{2}$$

This method requires only one substitution into the differential equation, and has a truncation error of the order $O(h^2)$.

A procedure of slightly higher accuracy is due to Heun [2]:

$$\begin{aligned} \bar{y}_{n+1} &= y_n + hy'_n \\ \bar{y}'_{n+1} &= f(x_{n+1}, \bar{y}_{n+1}) \\ y_{n+1} &= y_n + \tfrac{1}{2}h(y'_n + \bar{y}'_{n+1}). \end{aligned} \tag{3}$$

Its application requires two substitutions into eq. (1), but its truncation error is $O(h^3)$.

An alternate method of great simplicity, namely the "midpoint" method, which seems to be in favor in some places, makes use of the following relationships:

$$y_{n+\frac{1}{2}} = y_n + \tfrac{1}{2}(y_n - y_{n-1}) \tag{4}$$

$$y'_{n+\frac{1}{2}} = f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) \tag{5}$$

$$y_{n+1} = y_n + hy'_{n+\frac{1}{2}}. \tag{6}$$

Thus it necessitates only one substitution into eq. (1), and consequently requires essentially the same amount of computational labor that is involved in Euler's method. However, it has an accuracy that is comparable with that of Heun's method.

To substantiate this claim, we assume that the numerical information available at x_{n-1}, x_n is correct. Let us denote the exact solution by $y(x)$. Then we put

$$\begin{aligned} y_{n+\frac{1}{2}} &= y(x_n + \tfrac{1}{2}h) + T_1 \\ y'_{n+\frac{1}{2}} &= f(x_n + \tfrac{1}{2}h, y(x_n + \tfrac{1}{2}h)) + T_2 \\ y_{n+1} &= y(x_{n+1}) + T_M. \end{aligned}$$

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Now, by (4),

$$y_{n+\frac{1}{2}} = y(x_n) + \tfrac{1}{2}[y(x_n) - y(x_{n-1})],$$

and since

$$y(x_n) - y(x_{n-1}) = hy'(x_n) - \tfrac{1}{2}h^2y''(x_n) + O_1(h^3),$$

it follows that

$$y_{n+\frac{1}{2}} = y(x_n) + \tfrac{1}{2}hy'(x_n) - \tfrac{1}{4}h^2y''(x_n) + O_2(h^3).$$

On the other hand, the exact expansion is

$$y(x_n + \tfrac{1}{2}h) = y(x_n) + \tfrac{1}{2}hy'(x_n) + \tfrac{1}{8}h^2y''(x_n) + O_3(h^3).$$

Thus

$$T_1 = -\tfrac{3}{8}h^2y''(x_n) + O_4(h^3). \tag{7}$$

Consequently, by (5),

$$\begin{aligned} y'_{n+\frac{1}{2}} &= f(x_n + \tfrac{1}{2}h,\ y(x_n + \tfrac{1}{2}h) + T_1) \\ &= f(x_n + \tfrac{1}{2}h,\ y(x_n + \tfrac{1}{2}h)) + f_yT_1 + O_5(h^4), \end{aligned}$$

with the partial derivative f_y to be evaluated at $x_n + \frac{1}{2}h$, $y(x_n + \frac{1}{2}h)$. Finally, then, by (6),

$$y_{n+1} = y(x_n) + hy'(x_n + \tfrac{1}{2}h) + hf_yT_1 + O_5(h^5).$$

However,

$$y'(x_n + \tfrac{1}{2}h) = y'(x_n) + \tfrac{1}{2}hy''(x_n) + \tfrac{1}{8}h^2y'''(x_n) + O_6(h^3).$$

Therefore,

$$y_{n+1} = y(x_n) + hy'(x_n) + \tfrac{1}{2}h^2y''(x_n) + \tfrac{1}{8}h^3y'''(x_n) + hf_yT_1 + O_6(h^4).$$

This is to be compared with the exact expansion

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \tfrac{1}{2}h^2y''(x_n) + \tfrac{1}{6}h^3y'''(x_n) + O_7(h^4).$$

It follows that

$$T_M = -\tfrac{1}{24}h^3[y''' + 9y''f_y]_n + O_M(h^4). \tag{8}$$

A similar discussion of Heun's method (3) shows that

$$T_H = -\tfrac{1}{24}h^3[-2y''' + 6y''f_y]_n + O_H(h^4). \tag{9}$$

Thus both methods are exact for differential equations (1) whose solutions are polynomials of degree not exceeding two.

It is not too difficult to think of cases where the midpoint method is more accurate than Heun's method. Such a case is

$$y' = (1 + y^2)^{-1}, \qquad y(0) = 0, \tag{10}$$

in the interval $0 \leqq x \leqq 1.0$. The integrations, carried out for $h = 0.1$, are shown in Table 1. Results due to Euler's, Heun's, and the midpoint method are listed in columns (1), (2), (3), respectively. The exact solution $y^3 + 3y - 3x = 0$ is tabulated in the next column. The resulting errors $E_E = y_E - y(x)$, $E_H = y_H - y(x)$, and $E_M = y_M - y(x)$ are exhibited in columns (5), (6), (7); they bear out the claim made above.

Of interest is also the closeness of the leading terms in the expressions (8) and (9). In the case of equation (10) it turns out that

$$T_H \approx -\tfrac{1}{6}h^3(y')^4$$
$$T_M \approx -\tfrac{1}{12}h^3(y')^4[-1 + 24y^2y'].$$

Integrations starting with $y(1.0)$ yead to the values at $x = 1.1$ shown in Table 1 below the line. Therefore,

$$T_H \approx -2.1 \times 10^{-5}, \qquad T_M \approx -9.5 \times 10^{-5},$$

while the exact values for truncation (and rounding) are

$$T_H = -1.8 \times 10^{-5}, \qquad T_M = -9.1 \times 10^{-5}.$$

The truncation expressions in (8) and (9) are thus sufficiently close to be of practical utility.

In conclusion it might be stated that the midpoint method defined by equations (4), (5), (6) is only a special case of a whole class of such methods:

$$y_{n+\Theta} = y_n + \Theta(y_n - y_{n-1}), \qquad 0 < \Theta < 1$$
$$y'_{n+\Theta} = f(x_{n+\Theta}, y_{n+\Theta})$$
$$y_{n+1} = y_n + \alpha h y'_{n+\Theta},$$

TABLE 1

Comparison of Methods

(0) X	(1) Euler	(2) Heun	(3) Midpoint	(4) Exact Sol.	(5) E_E	(6) E_H	(7) E_M
0	.000 00	.000 00	.000 00	.000 00	0.10^{-5}	0.10^{-5}	0.10^{-5}
.1	.100 00	.099 50	.099 75	.099 67	33	−17	8
.2	.199 01	.197 12	.197 56	.197 44	157	−32	12
.3	.295 20	.291 29	.291 84	.291 72	348	−43	12
.4	.387 18	.380 97	.381 53	.381 49	569	−52	4
.5	.474 15	.465 64	.466 15	.466 22	793	−58	−7
.6	.555 79	.545 18	.545 60	.545 80	999	−62	−20
.7	.632 19	.619 77	.620 09	.620 40	1179	−63	−31
.8	.703 64	.689 70	.689 92	.690 33	1331	−63	−41
.9	.770 52	.755 35	.755 47	.755 97	1455	−62	−50
1.0	.833 27	.817 12	.817 15	.817 73	1554	−61	−58
1.1		.875 940	.875 867	.875 958			

with α and Θ denoting constants. For $\alpha = 1$ and $\Theta = \frac{1}{2}$ this method has a truncation error of the order $O(h^3)$; for other values of α or Θ the truncation error is of lower order.

REFERENCES

1. See, for example, F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw Hill, New York, 1956, Chapter 6.
2. See HILDEBRAND, loc. cit. 1.