Tables of the Generalized Stirling Numbers of the First Kind*

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Abstract. The generalized Stirling numbers of the first kind are defined, certain of their basic properties are discussed, and tables are given for the square grid $k = 0(1)10$ and $j = 0(1)10$ with $l = -10(1)10$.

Many problems which arise in the solution of differential equations, in the evaluation of integrals or in interpolation can be treated using techniques which rest upon the construction of an approximating polynomial. When the function to be approximated can be computed over a set of equally spaced points, the approximation is readily constructed by means of a lozenge diagram [1]. However, the use of this method requires the evaluation and/or expansion of the factorial polynomials

$$(u - l)^{[k]} = (u - l)(u - l - 1) \cdots (u - l - k + 1) \tag{1}$$

where l and k are integers, k being non-negative and l unrestricted, and this can be both complicated and time-consuming if either of the two indices is large. The labor, and the possibility of blunder, involved in working with the factorial polynomials can be considerably reduced by dealing with $(u - l)^{[k]}$ in a series rather than in a product form:

$$(u - l)^{[k]} = \sum_{j=0}^{k} {}_lS_j^{\,k} u^{k-j}. \tag{2}$$

The coefficients ${}_lS_j^k$ can be called the generalized Stirling numbers of the first kind in analogy with the terminology used for the numbers ${}_0S_j^k$. They are conveniently determined by using the obvious identities,

$$ {}_lS_0^k = 1, \qquad k = 0, 1, 2, \cdots \tag{3a}$$

$$ {}_lS_j^0 = 0, \qquad j = 1, 2, 3, \cdots \tag{3b}$$

and the recursive relationship

$$ {}_lS_j^{k+1} = -(l + k)\,{}_lS_{j-1}^k + {}_lS_j^k, \qquad k = 0, 1, 2, \cdots, j = 1, 2, 3, \cdots. \tag{4}$$

(4) is readily derived by expanding

$$(u - l)^{[k+1]} = (u - l - k)(u - l)^{[k]} \tag{5}$$

and equating the coefficients of the several powers of u.

Only the ${}_lS_j^k$ for $l = 0$ have been extensively tabulated [2]; those for $l \neq 0$

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TABLE I. GENERALIZED STERLING NUMBERS OF THE FIRST KIND

$\ell = -10$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	10									
2	1	19	90								
3	1	27	242	720							
4	1	34	431	2414	5040						
5	1	40	635	5000	19524	30240					
6	1	45	835	8175	44524	127860	151200				
7	1	49	1015	11515	77224	305956	652640	604800			
8	1	52	1162	14560	111769	537628	1580508	2592720	1814400		
9	1	54	1266	16884	140889	761166	2655764	5753736	6999840	3628800	
10	1	55	1320	18150	157773	902055	3416930	8409500	12753576	10628640	3628800

$\ell = -9$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	9									
2	1	17	72								
3	1	24	191	504							
4	1	30	335	1650	3024						
5	1	35	485	3325	11274	15120					
6	1	39	625	5265	24574	60216	60480				
7	1	42	742	7140	40369	133938	241128	181440			
8	1	44	826	8624	54649	214676	509004	663696	362880		
9	1	45	870	9450	63273	269325	723680	1172700	1026576	362880	
10	1	45	870	9450	63273	269325	723680	1172700	1026576	362880	

$\ell = -8$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	8									
2	1	15	56								
3	1	21	146	336							
4	1	26	251	1066	1680						
5	1	30	355	2070	5944	6720					
6	1	33	445	3135	12154	24552	20160				
7	1	35	511	4025	18424	48860	69264	40320			
8	1	36	546	4536	22449	67284	118124	109584	40320		
9	1	36	546	4536	22449	67284	118124	109584	40320		
10	1	35	510	3990	17913	44835	50840	-8540	-69264	-40320	

TABLE I. CONTINUED

$\ell = -7$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	7									
2	1	13	42								
3	1	18	107	210							
4	1	22	179	638	840						
5	1	25	245	1175	2754	2520					
6	1	27	295	1665	5104	8028	5040				
7	1	28	322	1960	6769	13132	13068	5040			
8	1	28	322	1960	6769	13132	13068	5040			
9	1	27	294	1638	4809	6363	-64	-8028	-5040		
10	1	25	240	1050	1533	-3255	-12790	-7900	11016	10080	

$\ell = -6$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	6									
2	1	11	30								
3	1	15	74	120							
4	1	18	119	342	360						
5	1	20	155	580	1044	720					
6	1	21	175	735	1624	1764	720				
7	1	21	175	735	1624	1764	720				
8	1	20	154	560	889	140	-1044	-720			
9	1	18	114	252	-231	-1638	-1324	1368	1440		
10	1	15	60	-90	-987	-945	3590	5340	-2664	-4320	

$\ell = -5$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	5									
2	1	9	20								
3	1	12	47	60							
4	1	14	71	154	120						
5	1	15	85	225	274	120					
6	1	15	85	225	274	120					
7	1	14	70	140	49	-154	-120				
8	1	12	42		-231	-252	188	240			
9	1	9	6	-126	-231	441	944	-324	-720		
10	1	5	-30	-150	273	1365	-820	-4100	576	2880	

$\ell=-4$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	4									
2	1	7	12								
3	1	9	26	24							
4	1	10	35	50	24						
5	1	10	35	50	24						
6	1	9	25	15	-26	-24					
7	1	7	7	-35	-56	28	48				
8	1	4	-14	-56	49	196	-36	-144			
9	1		-30		273		-820		576		
10	1	-5	-30	150	273	-1365	-820	4100	576	-2880	

$\ell=-3$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	3									
2	1	5	6								
3	1	6	11	6							
4	1	6	11	6							
5	1	5	5	-5	-6						
6	1	3	-5	-15	4	12					
7	1		-14		49		-36				
8	1	-4	-14	56	49	-196	-36	144			
9	1	-9	6	126	-231	-441	944	324	-720		
10	1	-15	60	90	-987	945	3590	-5340	-2664	4320	

$\ell=-2$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	2									
2	1	3	2								
3	1	3	2								
4	1	2	-1	-2							
5	1		-5		4						
6	1	-3	-5	15	4	-12					
7	1	-7	7	35	-56	-28	48				
8	1	-12	42		-231	252	188	-240			
9	1	-18	114	-252	-231	1638	-1324	-1368	1440		
10	1	-25	240	-1050	1533	3255	-12790	7900	11016	-10080	

TABLE I. CONTINUED

$\ell = -1$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	1									
3	1		-1								
4	1	-2	-1	2							
5	1	-5	5	5	-6						
6	1	-9	25	-15	-26	24					
7	1	-14	70	-140	49	154	-120				
8	1	-20	154	-560	889	-140	-1044	720			
9	1	-27	294	-1638	4809	-6363	-64	8028	-5040		
10	1	-35	510	-3990	17913	-44835	50840	8540	-69264	40320	

$\ell = -0$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1										
2	1	-1									
3	1	-3	2								
4	1	-6	11	-6							
5	1	-10	35	-50	24						
6	1	-15	85	-225	274	-120					
7	1	-21	175	-735	1624	-1764	720				
8	1	-28	322	-1960	6769	-13132	13068	-5040			
9	1	-36	546	-4536	22449	-67284	118124	-109584	40320		
10	1	-45	870	-9450	63273	-269325	723680	-1172700	1026576	-362880	

$\ell = +1$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-1									
2	1	-3	2								
3	1	-6	11	-6							
4	1	-10	35	-50	24						
5	1	-15	85	-225	274	-120					
6	1	-21	175	-735	1624	-1764	720				
7	1	-28	322	-1960	6769	-13132	13068	-5040			
8	1	-36	546	-4536	22449	-67284	118124	-109584	40320		
9	1	-45	870	-9450	63273	-269325	723680	-1172700	1026576	-362880	
10	1	-55	1320	-18150	157773	-902055	3416930	-8409500	12753576	-10628640	3628800

$\ell = +2$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-2									
2	1	-5	6								
3	1	-9	26	-24							
4	1	-14	71	-154	120						
5	1	-20	155	-580	1044	-720					
6	1	-27	295	-1665	5104	-8028	5040				
7	1	-35	511	-4025	18424	-48860	69264	-40320			
8	1	-44	826	-8624	54649	-214676	509004	-663696	362880		
9	1	-54	1266	-16884	140889	-761166	2655764	-5753736	6999840	-3628800	
10	1	-65	1860	-30810	326613	-2310945	11028590	-34967140	70290936	-80627040	39916800

$\ell = +3$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-3									
2	1	-7	12								
3	1	-12	47	-60							
4	1	-18	119	-342	360						
5	1	-25	245	-1175	2754	-2520					
6	1	-33	445	-3135	12154	-24552	20160				
7	1	-42	742	-7140	40369	-133938	241128	-181440			
8	1	-52	1162	-14560	111769	-537628	1580508	-2592720	1814400		
9	1	-63	1734	-27342	271929	-1767087	7494416	-19978308	30334320	-19958400	
10	1	-75	2490	-48150	600033	-5030235	28699460	-109911300	270074016	-383970240	239500800

$\ell = +4$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-4									
2	1	-9	20								
3	1	-15	74	-120							
4	1	-22	179	-638	840						
5	1	-30	355	-2070	5944	-6720					
6	1	-39	625	-5265	24574	-60216	60480				
7	1	-49	1015	-11515	77224	-305956	662640	-604800			
8	1	-60	1554	-22680	203889	-1155420	4028156	-7893840	6652800		
9	1	-72	2274	-41328	476049	-3602088	17893196	-56231712	101378880	-79833600	
10	1	-85	3210	-70890	1013313	-9790725	64720340	-288843260	832391136	-1397759040	1037836800

TABLE I. CONTINUED

$\ell = +5$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-5									
2	1	-11	30								
3	1	-18	107	-210							
4	1	-26	251	-1066	1680						
5	1	-35	485	-3325	11274	-15120					
6	1	-45	835	-8175	44524	-127860	151200				
7	1	-56	1330	-17360	134449	-617624	1557660	-1663200			
8	1	-68	2002	-33320	342769	-2231012	8969148	-20355120	19958400		
9	1	-81	2886	-59346	775929	-6687009	37972304	-136954044	284574960	-259459200	
10	1	-95	4020	-99750	1606773	-17550015	131590430	-668566300	2201931576	-4243508640	3632428800

$\ell = +6$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-6									
2	1	-13	42								
3	1	-21	146	-336							
4	1	-30	335	-1650	3024						
5	1	-40	635	-5000	19524	-30240					
6	1	-51	1075	-11985	74524	-245004	332640				
7	1	-63	1687	-24885	218344	-1139292	3272688	-3991680			
8	1	-76	2506	-46816	541849	-3977764	18083484	-46536624	51891840		
9	1	-90	3570	-81900	1197273	-11563650	73772180	-299705400	703404576	-726485760	
10	1	-105	4920	-135450	2425773	-29522745	247226930	-1406288100	5198985576	-11277554400	10897286400

$\ell = +7$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-7									
2	1	-15	56								
3	1	-24	191	-504							
4	1	-34	431	-2414	5040						
5	1	-45	805	-7155	31594	-55440					
6	1	-57	1345	-16815	117454	-434568	665280				
7	1	-70	2086	-34300	336049	-1961470	6314664	-8648640			
8	1	-84	3066	-63504	816249	-6666156	33775244	-97053936	121080960		
9	1	-99	4326	-109494	1768809	-18909891	133767584	-603682596	1576890000	-1816214400	
10	1	-115	5910	-178710	3520713	-47210835	436325840	-2743963940	11235811536	-27046454400	29059430400

$\ell = +8$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-8									
2	1	-17	72								
3	1	-27	242	-720							
4	1	-36	539	-3382	7920						
5	1	-50	995	-9850	48504	-95040					
6	1	-63	1645	-22785	176554	-725592	1235520				
7	1	-77	2527	-45815	495544	-3197348	11393808	-17297280			
8	1	-92	3682	-83720	1182769	-10630508	59354028	-188204400	259459200		
9	1	-108	5154	-142632	2522289	-29554812	229442156	-1137868848	3270729600	-4151347200	
10	1	-125	6990	-230250	4947033	-72433725	731873960	-5038385500	22614500016	-59753750400	70572902400

$\ell = +9$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-9									
2	1	-19	90								
3	1	-30	299	-990							
4	1	-42	659	-4578	11880						
5	1	-55	1205	-13145	71394	-154440					
6	1	-69	1975	-30015	255424	-1153956	2162160				
7	1	-84	3010	-59640	705649	-4985316	19471500	-32432400			
8	1	-100	4354	-107800	1659889	-16275700	99236556	-343976400	518918400		
9	1	-117	6054	-181818	3492489	-44493813	375923456	-2030997852	6366517200	-8821612800	
10	1	-135	8160	-290790	6765213	-107358615	1176812090	-8797620060	42924478536	-123418922400	158789030400

$\ell = +10$

k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-10									
2	1	-21	110								
3	1	-33	362	-1320							
4	1	-46	791	-6026	17160						
5	1	-60	1435	-17100	101524	-240240					
6	1	-75	2335	-38625	358024	-1763100	3603600				
7	1	-91	3535	-75985	976024	-7491484	31813200	-57657600			
8	1	-108	5082	-136080	2267769	-24083892	159168428	-598482000	980179200		
9	1	-126	7026	-227556	4717209	-64903734	592678484	-3463513704	11752855200	-17643225600	
10	1	-145	9420	-361050	9040773	-154530705	1825849430	-14724404900	77559615576	-240947474400	335221286400

seem to have been neglected. To facilitate the construction of formulas from the lozenge diagram, the IBM 7090 at the Harvard University Computing Center was utilized to calculate the ${}_lS_j^k$ over the square grid $k = 0(1)10$, $j = 0(1)10$ with $l = -10(1)10$. The results of these calculations are presented in Table I. The ${}_lS_j^k$ given here enable one to construct a variety of polynomials of degree less than or equal to ten. This will be adequate for virtually all problems to which lozenge diagram methods can be applied: the practical limitations in accuracy imposed by modern computing machinery and/or by uncertainties in the input data make polynomials of degree $k > 10$ of dubious usefulness since, as k increases beyond 10, the growth of computational error will almost surely vitiate any reduction in truncation error achieved by increasing k.

REFERENCES

1. Kunz, K. S. *Numerical Analysis*, Ch. 4. McGraw-Hill, New York, 1957.
2. Fletcher, A. Miller, J. C. P., Rosenhead, L., and Comrie, L. J. *An Index of Mathematical Tables*. Second Ed., Addison-Wesley, Reading, 1962, Sec. 4.9231.