An Error Bound for a Numerical Filtering Technique*

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Abstract. Let the transfer function for the numerical filter be $H(w) = \sum_{n=-\infty}^{\infty} h_n e^{(iwn/f_s)}$ where h_n is the nth weight and f_s is the sampling frequency. The weights are given by $h_n = 1/2\pi f_s \int_{-\pi f_s}^{\pi f_s} H(w) e^{(-iwn/f_s)}\, dw$. If we assume that $H(w)$ has a continuous first derivative and the second derivative exists, an error bound is given by $\epsilon(N, f_s) \leqq 2f_s/\pi N \int_{w_c}^{w_t} \mid H''(w) \mid dw$, where w_c is the angular cut-off frequency, w_t the termination frequency, and $2N + 1$ is the number of weights used. This is a bound for the error in recovering the transfer function by using a finite number of weights.

Introduction

In 1961, Joseph Ormsby [1] proposed a numerical filter whose transfer function had the following form:

$$H(w) = \begin{cases} 0; & \mid w \mid \geqq w_t \\ f(w); & -w_t < w < -w_c, \quad w_c < w < w_t \\ 1; & \mid w \mid \leqq w_c \end{cases}$$

where w_c = cutoff frequency (angular) = $2\pi f_c$ and w_t = filter roll-off termination frequency.

In general, a polynomial type roll-off was used and there was no restriction on $f(w)$. An error term is given which appears in the form of a rather complicated integral and which will require some sort of numerical integration scheme for evaluation. This means that even though the error term is exact, there will be an error in the numerical integration method. If machine time is disregarded, this error term can be evaluated to give a very good bound. In this paper, we derive an error bound for a subclass of the above type of filter transfer functions. We will use that subclass of transfer functions having continuous first derivatives and whose second derivative exists.

It is true that in some specific cases better error bounds are available than the ones derived here. In particular, a few empirical bounds have been found which are better; however, the following bound is applicable to the entire class of such filters and also to the case of derivative filters; i.e. those filters which simultaneously filter and differentiate. The primary advantage of these bounds is that they can usually be found as a closed form of an integral. We might add as a note that these bounds can also be extended to the n-variable case.

We first find an error bound for the general class of filters given above. Then, as an example, we will find an error bound for a particular filter given by Graham [2].

The majority of this work was done under contract number NAS8-5164 with the Auburn Research Foundation, Auburn University, Auburn, Alabama.

* Presented at Southeastern Regional Meeting of ACM, September 20–21, 1963.

Journal of the Association for Computing Machinery, Vol. 12, No. 1 (January, 1965), pp. 137–140

Error Bound

THEOREM. *If* $H(w) = \sum_{n=-\infty}^{\infty} h_n e^{(iwn/f_s)}$ *is the transfer function of a numerical filter such that* $H(w)$ *has a continuous first derivative and whose second derivative exists, then an error bound in recovering the transfer function by a finite number of terms is given by*

$$\epsilon(N, f_s) \leqq \frac{2f_s}{\pi N} \int_{w_0}^{w_t} | H''(w) | \, dw,$$

where f_s *is the sampling frequency and* h_i, $-N \leqq i \leqq N$, *are the weights used for the recovery.*

PROOF. We may express the proposed transfer function as

$$H(w) = \sum_{n=-\infty}^{\infty} h_n e^{(iwn/f_s)} \tag{1}$$

and our actual transfer function as

$$\hat{H}(w) = \sum_{n=-N}^{N} h_n e^{(iwn/f_s)}, \tag{2}$$

where

$$h_n = \frac{1}{2\pi f_s} \int_{-\pi f_s} H(w) e^{(-iwn/f_s)} \, dw. \tag{3}$$

In light of the above statements, the error between the two transfer functions is given by $\epsilon(w, N) = \hat{H}(w) - H(w)$. Since both $H(w)$ and h_n, $-N \leqq n \leqq N$, are real we may write (1), (2) and (3) as

$$H(w) = \sum_{n=-\infty}^{\infty} h_n \cos\left(\frac{wn}{f_s}\right), \qquad \hat{H}(w) = \sum_{n=-N}^{N} h_n \cos\left(\frac{wn}{f_s}\right)$$

and

$$h_n = \frac{1}{2\pi f_s} \int_{-\pi f_s}^{\pi f_s} H(w) \cos\left(\frac{wn}{f_s}\right) dw.$$

Using these expressions and taking the absolute value of the error, we get

$$| \epsilon(w, N) | = | \hat{H}(w) - H(w) | = \left| 2 \sum_{n=N+1}^{\infty} h_n \cos\left(\frac{wn}{f_s}\right) \right|$$
$$= \left| \frac{1}{\pi f_s} \sum_{n=N+1}^{\infty} \cos\left(\frac{wn}{f_s}\right) \int_{-\pi f_s}^{\pi f_s} H(\lambda) \cos\left(\frac{\lambda n}{f_s}\right) d\lambda \right|.$$

Interchanging summation and integration, we get

$$| \epsilon(w, N) | = \left| \frac{1}{\pi f_s} \int_{-\pi f_s}^{\pi f_s} H(\lambda) \sum_{n=N+1}^{\infty} \cos\left(\frac{wn}{f_s}\right) \cos\left(\frac{\lambda n}{f_s}\right) d\lambda \right|. \tag{4}$$

In (4), let us apply the partial integration formula

$$\int_a^b g \, df = f(b)g(b) - f(a)g(a) - \int_a^b f \, dg,$$

where $g = H(\lambda)$, $df = \cos(\lambda n/f_s)\, d\lambda$.

From the well-known Shannon Sampling Theorem, we know that f_s is no less than twice the highest frequency present in our data. Thus, πf_s will be considerably higher than the termination frequency. Hence, we note that $H(\pi f_s) = H(-\pi f_s) = 0$. Then

$$|\epsilon(w, N)| = \left| \frac{1}{\pi} \int_{-\pi f_s}^{\pi f_s} \sum_{n=N+1}^{\infty} H'(\lambda) \frac{\cos\left(\frac{wn}{f_s}\right)\sin\left(\frac{\lambda n}{f_s}\right)}{n} d\lambda \right|.$$

We also note that, except for the intervals $-w_t \leqq w \leqq -w_c$ and $w_c \leqq w \leqq w_t$, $H'(w) = 0$. Both of these intervals are contained in the interval $[-\pi f_s, \pi f_s]$; thus,

$$|\epsilon(w, N)| = \left| \frac{1}{\pi} \int_{-w_t}^{-w_c} \sum_{n=N+1}^{\infty} H'(\lambda) \frac{\cos\left(\frac{wn}{f_s}\right)\sin\left(\frac{\lambda n}{f_s}\right)}{n} d\lambda + \frac{1}{\pi} \int_{w_c}^{w_t} \sum_{n=N+1}^{\infty} H'(\lambda) \frac{\cos\left(\frac{wn}{f_s}\right)\sin\left(\frac{\lambda n}{f_s}\right)}{n} d\lambda \right|.$$

Using the partial integration formula again and rearranging terms, we get

$$\begin{aligned}
\epsilon(w, N) &= \left| \frac{f_s}{\pi} \left[\int_{-w_t}^{-w_c} H''(\lambda) \sum_{n=N+1}^{\infty} \frac{\cos\left(\frac{\lambda n}{f_s}\right)\cos\left(\frac{wn}{f_s}\right)}{n^2} d\lambda + \int_{w_c}^{w_t} H''(\lambda) \sum_{n=N+1}^{\infty} \frac{\cos\left(\frac{\lambda n}{f_s}\right)\cos\left(\frac{wn}{f_s}\right)}{n^2} d\lambda \right] \right| \\
&\leqq \left|\frac{f_s}{\pi}\right| \left[\left| \int_{-w_t}^{-w_c} H''(\lambda) \sum_{n=N+1}^{\infty} \frac{\cos\left(\frac{\lambda n}{f_s}\right)\cos\left(\frac{wn}{f_s}\right)}{n^2} d\lambda \right| + \left| \int_{w_c}^{w_t} H''(\lambda) \sum_{n=N+1}^{\infty} \frac{\cos\left(\frac{\lambda n}{f_s}\right)\cos\left(\frac{wn}{f_s}\right)}{n^2} d\lambda \right| \right] \\
&\leqq \left|\frac{f_s}{\pi}\right| \left[\int_{-w_t}^{-w_c} |H''(\lambda)| \sum_{n=N+1}^{\infty} \frac{1}{n^2} d\lambda + \int_{w_c}^{w_t} |H''(\lambda)| \sum_{n=N+1}^{\infty} \frac{1}{n^2} d\lambda \right] \\
&\leqq \frac{f_s}{\pi N} \left[\int_{-w_t}^{-w_c} |H''(\lambda)|\, d\lambda + \int_{w_c}^{w_t} |H''(\lambda)|\, d\lambda \right].
\end{aligned}$$

We see that this expression no longer involves w but does involve N and f_s as variables. Since the function is even, the two integrals will be equal. If $H''(w)$ is odd, as in the case of the first derivative filter, the absolute values will be the same; hence, we have

$$\epsilon(N, f_s) \leqq \frac{2f_s}{\pi N} \int_{w_c}^{w_t} |H''(\lambda)|\, d\lambda.$$

Thus, for a given filter whose transfer function satisfies the necessary differentiability requirements, we can determine an error bound simply by evaluating an integral. In most cases, this integral can be evaluated in closed form.

The bounds for derivative filters may be handled in the same way. As a matter of fact there are only two forms, as the transfer function is either real or pure imaginary. In general, the nth derivative filter will be of the form $kw^n H(w)$, where $H(w)$ is the transfer function for the 0-derivative filter and k is a constant. Hence, any filter satisfying our original differentiability assumptions will certainly satisfy the necessary conditions for derivative filters.

One other difference must be pointed out. The limits of integration must be examined in the light of the transfer function for each derivative filter. We would need to apply the partial integration techniques more than twice to arrive at limits of $-w_t$, $-w_c$, and w_c, w_t if we are using a derivative filter of order greater than one. However, we are usually safe in using two applications and evaluating the definite integral thus obtained, as, in general, the bounds are larger for higher derivatives.

One note of caution. The above derivation tacitly assumes that $H(w) = 1$, $-w_c \leqq w \leqq w_c$. If this is not the case, the above method is still applicable, but the limits of integration will require careful study.

A Particular Filter

As an illustration we use the filter proposed by Graham. The second derivative of the transfer function exists, hence we may proceed directly to the formula. We use the w-notation rather than the λ-notation of the formula.

$$H(w) = \begin{cases} 0; & |w| \geqq w_t \\ 1; & |w| \leqq w_c \\ \dfrac{1}{2}\left\{ \cos\left(\dfrac{w - w_c}{\Delta w}\pi\right) + 1 \right\}; & w_c < w < w_t \\ \dfrac{1}{2}\left\{ \cos\left(\dfrac{w + w_t}{\Delta w}\pi\right) + 1 \right\}; & -w_t < w < -w_c, \end{cases}$$

where $\Delta w = w_t - w_c$. Differentiating the roll-off twice, we get

$$H''(w) = -\frac{\pi^2}{2\Delta w^2}\cos\frac{w - w_c}{\Delta w}\pi, \qquad w_c < w < w_t.$$

Hence

$$\epsilon(N, f_s) \leqq \frac{2f_s}{\pi N}\int_{w_c}^{w_t}\left|\frac{\pi^2}{2\Delta w^2}\cos\frac{w - w_c}{\Delta w}\pi\right| dw = \frac{2f_s}{\Delta w}\frac{1}{N}.$$

Thus we have an error bound for recovering the function in terms of f_s, N and Δw. To illustrate the first derivative bound, we note that the bound is

$$\epsilon(w, N) \leqq \frac{2f_s}{\pi N}\int_{w_c}^{w_t} |H''(\lambda)|\, d\lambda, \tag{5}$$

where, in this case, $H(w)$ is the transfer function for the derivative filter. Thus,

$$H(w) = \begin{cases} 0\ ; & |w| \geqq w_t \\ iw; & |w| \leqq w_c \\ \dfrac{iw}{2}\left\{\cos\left(\dfrac{w - w_c}{\Delta w}\,\pi\right) + 1\right\}; & w_c < w < w_t \\ \dfrac{iw}{2}\left\{\cos\left(\dfrac{w + w_c}{\Delta w}\,\pi\right) + 1\right\}; & -w_t < w < -w_c\,. \end{cases}$$

For this bound, we can use the same limits of integration as above. Thus

$$H''(w) = -\frac{w\pi^2 i}{2\Delta w^2}\cos\frac{w - w_c}{\Delta w}\,\pi - \frac{\pi i}{\Delta w}\sin\frac{w - w_c}{\Delta w}\,\pi, \quad w_c < w < w_t\,.$$

If we use these expressions in inequality (5) we will arrive at a bound $\epsilon(w, N) \leqq 6f_s/\Delta wN$. This bound is, of course, considerably higher than in the former case. This is typical of the higher derivative bounds; hence, they are usually impractical after the first derivative.

RECEIVED NOVEMBER, 1963

REFERENCES

1. ORMSBY, JOSEPH F. A. Design of numerical filters with application to missile telemetry data processing. *J. ACM 8*, 3 (July 1961).
2. GRAHAM, RONALD J. Determination and Analysis of Numerical Smoothing Weights. Tech. Rep. R-179, NASA, MSFC, Huntsville, Alabama.