Convergence Problems in Maehly's Second Method: Part II

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Abstract. Maehly's second method is a general algorithm for finding the best Chebyshev approximation to a continuous function on a finite interval. This paper examines the convergence properties of Maehly's second method, a modification, and the more commonly used algorithm of Remez, using both analytical and numerical results.

Introduction

This paper cannot be followed in detail if the reader is unfamiliar with the results and notation of the preceding note [1]. It is therefore best that the introductory section of [1], containing a statement of the approximation problem, a characterization of best approximations, and a description of Maehly's second method, serve also as an introduction to this note. It is assumed henceforth that the reader is familiar with the notation developed in that section.

Of fundamental interest in [1] and this paper are correction procedures for Maehly's second method, to be precise, procedures for calculating corrections $\delta z_i^{(k)}$ to the interpolating points $z_i^{(k)}$ in order to obtain new interpolating points $z_i^{(k+1)}$. If the corrections are obtained by solving a system of equations the system is called a corrector system, and similarly if a formula is used it is called a corrector formula. There are two good reasons for studying correction procedures.

First, two correction procedures may always yield the same results when an infinite number of figures are carried and so are equivalent in theory; however, they can differ considerably in the amount of calculation required and in numerical stability. For example, in the second section of [1] it is shown that the corrector formula (5) can be obtained by inverting the matrix of the corrector system (9) used by Maehly; the evaluation of the formula and the solution of the system are thus equivalent correction procedures, but the number of arithmetic operations required is shown to be quite different. Further, the stability of the formula (5) depends strongly on the nonzero constant μ used. Given a correction procedure, it is therefore desirable to find a correction procedure equivalent to it which requires a minimum of calculation and is numerically stable. A further reason for studying equivalent correction procedures is that it may be necessary to study several equivalent correction procedures to determine completely the behavior of Maehly's second method when any of them is used as a correction procedure. For example, Theorem 1 of this paper is obtained from formula (5) and Theorem 2 is obtained from system (9). The development of these particular results is quite intricate; it may be helpful to examine the development in [1] of similar results for the case of approximation by a constant in order to see the basic argument.

A second reason for studying correction procedures is that it may be desirable to modify Maehly's second method by using a correction procedure which is not equivalent to those used by Maehly. In this note, attention is placed on the correction procedure which uses the system (12) and which gives quadratic convergence

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n the polynomial case. In the last section of [1] it is proved that this correction pro-edure does not give global convergence. In the next section of this paper several nalytical results concerning the convergence of Maehly's second method are de-eloped. Unless stated otherwise, these apply to the method developed by Maehly ather than to any modified method. A discussion of numerical results follows.

Analytical Results

It will be convenient to consider the set of interpolating points $\{z_i \mid i = 0, \cdots, n\}$ is an element of an $(n+1)$-dimensional normed linear space and to denote this set by $\bar{z}$. Once $\bar{z}$ is given it determines in turn $\{f(z_i) \mid i = 0, \cdots, n\}$, the interpolant $F(\bar{a}, x)$, the error curve $\epsilon(x)$, its extreme points x_i and extreme values $\epsilon_i \equiv \epsilon(x_i)$, $i = 0, \cdots n+1$, and finally, by means of a correction procedure, the corrections $\{\delta z_i \mid i = 0, \cdots, n\}$ to the interpolating points. All these are, therefore, functions of $\bar{z}$, and it is henceforth assumed that there exists $\gamma > 0$ such that if g is any one of these functions of $\bar{z}$, then

$$g(\bar{z}) = g(\bar{z}^*) + \sum_{k=0}^{n} \frac{\partial g}{\partial z_k}\bigg]_* (z_k - z_k^*) + O(\| \bar{z} - \bar{z}^* \|^2) \text{ as } \bar{z} \to \bar{z}^* \tag{1}$$

if $\| \bar{z} - \bar{z}^* \| < \gamma$, where the subscript $*$ will denote evaluation of an expression with $\bar{z}^*$ substituted for $\bar{z}$.

Consider a case in which a formula of the form

$$\delta z_i = \sum_{j=0}^{n} c_{ij}(z_j^* - z_j) + O(\| \bar{z} - \bar{z}^* \|^2) \quad \text{as} \quad \bar{z} \to \bar{z}^* \tag{2}$$

holds for a correction procedure CP and all vectors $\bar{z}$. Let $C = (c_{ij})$, $\| C \|_m = \sup_{i=0,\cdots,n} \| (c_{i0}, \cdots, c_{in}) \|$, I equal the unit matrix, and $L = \| C - I \|_m$. The lemma below follows clearly from the identity

$$z_i^{(m+1)} - z_i^* = \delta z_i^{(m)} + z_i^{(m)} - z_i^*. \tag{3}$$

LEMMA 1. *When using correction procedure CP, Maehly's second method converges quadratically if and only if $L = 0$. If $L < 1$ there exists $\gamma > 0$ such that for all $\bar{z}^{(0)}$ in the neighborhood $\| \bar{z}^{(0)} - \bar{z}^* \| < \gamma$, $\| \bar{z}^{(k)} - \bar{z}^* \|$ converges to zero at least as rapidly as L^k, whereas if $L > 1$ no such neighborhood can exist and Maehly's second method may diverge.*

In the remainder of this section analytical results are obtained for approximation by polynomials. In this case, the approximation is given by the Lagrange formula, as a function of the interpolating points z_k :

$$F(\bar{a}, x) = \sum_{k=0}^{n} l_k(x) f(z_k), \qquad l_k(x) = Z(x)/(x - z_k)Z'(z_k),$$
$$Z(z) = \prod_{k=0}^{n} (z - z_k). \tag{4}$$

In [1], an explicit corrector formula was obtained for Maehly's second method, namely,

$$\delta z_i = \frac{X(z_i)}{Z'(z_i)} \sum_{j=0}^{n+1} \frac{1}{z_i - x_j} \left(\frac{Z(x)_j}{X'(x_j)} \ln \left| \frac{\epsilon_j}{\mu} \right| \right), \tag{5}$$

where $X(x) = \Pi_{k=0}^{n+1}(x - x_k)$ and μ can be any constant. By the remarks before the formula (1), the right-hand side of (5) is clearly a function of $\bar{z}$ and so (1) can be applied to δz_i. Since the logarithms all vanish when $|\epsilon_0| = |\epsilon_1| = \cdots = |\epsilon_{n+1}| = \mu$, one obtains with rearrangement

$$\delta z_i = \frac{X(z_i)}{Z'(z_i)} \sum_{k=0}^{n} \left[\sum_{j=0}^{n+1} \frac{1}{z_i - x_j} \cdot \frac{Z(x_j)}{X'(x_j)} \cdot \frac{\partial \ln |\epsilon_j|}{\partial z_k} \right]_* (z_k - z_k^*) + O(\| \bar{z} - \bar{z}^* \|^2). \quad (6)$$

For the case of polynomial approximation one can obtain, using (4),

$$\frac{\partial \ln |\epsilon_j|}{\partial z_k} = \left. \frac{\frac{\partial}{\partial z_k} \epsilon_j}{\epsilon_j} \right]_* = \frac{-l_k(x_j)\epsilon'(z_k)]_*}{(-1)^j \epsilon_0^*}$$

and (6) becomes an explicit formula of form (2),

$$\delta z_i = -\frac{X(z_i)}{Z'(z_i)} \sum_{k=0}^{n} \left[\sum_{j=0}^{n+1} \frac{(-1)^j}{z_i - x_j} \cdot \frac{Z(x_j)}{X'(x_j)} l_k(x_j) \right]_* \left. \frac{\epsilon'(z_k)}{\epsilon_0} \right]_* (z_k - z_k^*) + O(\| \bar{z} - \bar{z}^* \|^2). \quad (7)$$

Applying Lemma 1 one obtains

THEOREM 1. *Maehly's second method has quadratic convergence if and only if*

$$s_{ij} = 0 \qquad (i \neq j;\ i = 0, \cdots, n;\ j = 0, \cdots, n), \quad (8a)$$

$$\epsilon'(z_j)]_* = \left. \frac{\epsilon_0 Z'(z_j)}{X(z_j) s_{jj}} \right]_* \quad (8b)$$

where in the polynomial case

$$s_{ij} = \sum_{j=0}^{n+1} \left. \frac{(-1)^k}{(z_i - x_k)(z_j - x_k)} \cdot \frac{(Z(x_k))^2}{X'(x_k) Z'(z_j)} \right]_* . \quad (8c)$$

It will be shown later how (8c) can be modified for nonpolynomial approximations. If for any $j = 0, \cdots, n$ one has $\epsilon'(z_j)$ more than twice the value specified by condition (8b), then Maehly's second method can diverge. Condition (8a) is a constraint on the zeros and extrema of the best error curve.

Since condition (8a, 8b) is rarely satisfied, Maehly's second method does not have quadratic convergence. However, following Maehly, if $G(x)$ is defined for given $\bar{z}$ by

$$\epsilon(x) = G(x) Z(x),$$

then it can easily be shown that a system used by Maehly to obtain corrections,

$$\ln |\epsilon_i| = \ln |\lambda| + \sum_{k=0}^{n} \frac{1}{x_i - z_k} \delta z_k \qquad (i = 0, \cdots, n+1) \quad (9)$$

gives Maehly's second method quadratic convergence if

$$\left. \frac{\partial G(x_i)}{\partial z_k} \right]_* = 0 \qquad (i = 0, \cdots, n+1;\ k = 0, \cdots, n).$$

THEOREM 2. *If an $(n+1)$-st degree polynomial is approximated by a polynomial of degree n, Maehly's second method converges quadratically.*

PROOF. In this case $G(x)$ is the leading coefficient of the $(n+1)$-st degree polynomial.

The importance of this result lies in the fact that many error curves are similar to the error curve of such a case and convergence is therefore almost quadratic.

Applying (1) to ϵ_i, the ith extreme value, one obtains:

$$\epsilon_i = \epsilon_i^* + \sum_{k=0}^{n} \frac{\partial \epsilon_i}{\partial z_k}\bigg]_* (z_k - z_k^*) + O(\| \tilde{z} - \tilde{z}^* \|^2). \tag{10}$$

From this may be obtained an explicit formula for the polynomial case,

$$\epsilon_i = (-1)^i \epsilon_0^* - \sum_{k=0}^{n} l_k(x_i)\epsilon'(z_k)(z_k - z_k^*)]_* + O(\|\tilde{z} - \tilde{z}^* \|^2). \tag{11}$$

It is clear from identity (3) that the related system

$$\epsilon_i = (-1)^i\lambda + \sum_{k=0}^{n} l_k(x_i)\epsilon'(z_k)\delta z_k \quad (i = 0, \cdots, n+1) \tag{12}$$

will give quadratic convergence when used as a correction procedure for the case of polynomial approximation, provided the matrix is nonsingular when $\tilde{z} = \tilde{z}^*$, which happens if $\epsilon'(z_k)]_* \neq 0$, $(k = 0, \cdots, n)$. In the next section, the convergence of this modified second method is compared with that of Maehly's second method and the Remez algorithm.

Consider now the case when a nonpolynomial approximant $F(\tilde{a}, x)$ is used. Let $F(\tilde{a}, x)$ be written as a function $\phi(\tilde{z}, \tilde{f}, x)$ of the points (z_k, f_k), $k = 0, \cdots, n$ which it interpolates. Consider the partial derivative

$$\frac{\partial}{\partial f_k} \phi(\tilde{z}, \tilde{f}, x) \equiv \psi_k(\tilde{z}, \tilde{f}, x)$$

obtained by holding $\tilde{z}$, x, and f_i, $i \neq k$ fixed and differentiating with respect to f_k. In the case of polynomial approximation it is clear from the Lagrange formula (4) that $\psi_k(\tilde{z}, \tilde{f}, x) = l_k(x)$. For cases of nonpolynomial approximation satisfying hypothesis (1), it can be shown that $l_k(x)$ or an equivalent expression can be replaced by $\psi_k(\tilde{z}, \tilde{f}, x)$ in formulas (7, 8c, 11). Thus the replacement of $l_k(x)$ in the system (12) by $\psi_k(\tilde{z}, \tilde{f}, x)$ yields a correction system which gives Maehly's second method quadratic convergence in the case of a nonpolynomial approximant $F(\tilde{a}, x)$ if the system is nonsingular when $\tilde{z} = \tilde{z}^*$. It would be interesting to have such correction systems in explicit form and to know how useful they would be in practice.

Numerical Results and Computational Implications

In cases of practical interest it is not necessary to compute the best approximation; it is sufficient to obtain an approximation whose error extrema $\{\epsilon_i \mid i = 0, \cdots, n+1\}$ agree to a given number of figures, usually 2 to 4. In this note, the statement that the error extrema agree to k figures means precisely that $\sup_{i=0,\cdots,n+1} |\epsilon_i|$ agrees with $\inf_{i=0,\cdots,n+1} |\epsilon_i|$ to k significant decimal digits. What is therefore wanted is an algorithm obtaining such results in a minimum of calculation. Both Maehly's second method and the Remez algorithm are iterative schemes for obtaining a best approximation; in each iteration an approximation is determined and the extrema of its error curve are found. It can be shown that the amount of calculation required for an iteration is usually comparable for both methods. Hence, it is necessary only to choose the algorithm taking fewest iterations, provided the precision required for the algorithms is comparable.

Best approximations by polynomials and rationals were computed for several cases using Maehly's second method, the modified second method using correction system (12), the starting points for these two being the zeroes of the $(n+1)$-st Chebyshev polynomial on $[\alpha, \beta]$, and the Remez algorithm with starting points the extrema of the $(n+1)$-st Chebyshev polynomial. The number of figures agreement of the error extrema $\{\epsilon_i^{(k)} \mid i = 0, \cdots, n+1\}$ was tabulated against iteration number k. In Table I, the number of figures agreement is plotted against iteration number for the three algorithms. In Table II, the number of iterations required to have a given agreement of error extrema are shown. As an example of how the

TABLE I. NUMBER OF SIGNIFICANT DECIMAL DIGITS AGREEMENT OF ERROR EXTREMA FOR MAEHLY'S SECOND METHOD (M), THE MODIFIED SECOND METHOD (Q) AND THE REMEZ ALGORITHM (R) FOR SELECTED CASES OF RATIONAL APPROXIMATION OF DEGREE l/m

Function	e^x						$\ln x$					$\sqrt{x}$					
Interval	$[-1, 1]$						$[1/16, 1]$			$[1/4, 1]$		$[1/16, 1]$			$[1/2, 1]$		
Degree	5/0			4/1			5/0			4/2		5/0			2/1		
Algorithm	M	Q	R	M	Q	R	M	Q	R	M	R	M	Q	R	M	Q	R
Iteration Number																	
0	0	0	2	0	0	1	0	0	1	0	0	0	0	1	0	0	1
1	2	2	5	1	0	3	1	0	2	0	0	1	0	2	2	1	4
2	6	5	8+	4	2	7	2	1	5	1	2	2	1	5	4	3	8+
3	8+	8+		7	4	8+	3	2	8+	3	5	4	3	8+	6	4	
4				8+	5		5	4		4	8+	6	5		8+	6	
5					7		7	8+		6		7	8+			7	
6							8+			8+		8+				8+	

TABLE II. NUMBER OF ITERATIONS REQUIRED FOR AGREEMENT TO 2 AND 4 SIGNIFICANT DECIMAL DIGITS FOR MAEHLY'S SECOND METHOD AND THE REMEZ ALGORITHM

Number of Figures and Algorithm	Rational Approximation of e^x on $[-1, 1]$ of degree							
	1/0	3/0	5/0	2/1	4/1	1/2	3/2	2/3
$2M$	2	1	1	2	2	2	2	2
$2R$	1	1	0	1	1	1	1	2
$4M$	3	2	2	3	2	3	3	3
$4R$	1	1	1	2	2	2	2	2
	Rational Approximation of $\ln x$ on $[1/4, 1]$ of degree							
	1/1	2/1	3/1	4/1	1/2	2/2	3/2	4/2
$2M$	3	3	2	2	2	3	3	3
$2R$	1	2	1	2	1	2	2	2
$4M$	4	4	3	3	3	4	4	4
$4R$	2	2	2	2	2	3	3	3

tables may be used, in the approximation of e^x on $[-1, 1]$ by a polynomial of degree 5 using Maehly's second method, the error extrema agreed to 0 figures for the initial approximation (iteration 0) and to 2 figures for iteration 1; one iteration, therefore, being required to have 2 figures agreement.

By hypothesis (1), quadratic convergence in $\tilde{z}$ implies quadratic convergence in $\| \epsilon \|$, and thus the numerical results may be examined for convergence rates. These results confirm the analytical results, namely that the modified second method has quadratic convergence in the polynomial case and Maehly's second method has a convergence rate less than quadratic. It has been shown that the Remez algorithm converges quadratically in $\| \epsilon \|$ for the polynomial case and unpublished proofs generalize this result. The numerical results suggest, however, that algorithms with quadratic convergence may have no advantage in practice over algorithms without this property. In particular, in the polynomial case the modified second method is superior to Maehly's second method only for very low degrees or a high number of figures agreement, and is inferior for most cases of practical interest. The modified second method was also tried in the rational case and it gave slow convergence.

To conclude, for the polynomial case the Remez algorithm is undoubtedly superior, as it is superior in three respects to Maehly's second method; namely, its initial approximation is usually better, its convergence is quadratic and it never fails to converge. In the rational case, theory and numerical examples seem to favor the Remez algorithm. There are, however, theoretical and practical difficulties connected with the nonlinear system used by the Remez algorithm to obtain rational approximations and until these are better understood, the question of whether the Remez algorithm is superior will not be solved completely.

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